Chapter Three
Modal Semantics for Logics of Formal Inconsistency

This chapter collects three papers: **3.1** brings 'Logics of essence and accident', henceforth LEA; **3.2** brings 'Modality and paraconsistency', henceforth MODPAR; **3.3** brings 'Nearly every normal modal logic is paranormal', henceforth PARANORMAL.

Resumo de LEA

Dizemos que as coisas ocorrem acidentalmente quando elas de fa[c]to ocorrem, mas apenas por acaso. Na situação oposta, uma ocorrência essencial é inescapável, a sua inevitabilidade constituindo o sine qua non de sua própria concretização. Este artigo investigará lógicas modais numa linguagem desenhada para falar acerca de enunciados essenciais e acidentais. A completude de alguns dentre os sistemas mais fracos e mais fortes desta classe de lógicas é alcançada. Chama-se a atenção para o fraco poder expressivo da linguagem proposicional clássica enriquecida pelos operadores modais não-normais da essência e do acidente, e ilustra-se este fa[c]to tanto com relação à definibilidade dos operadores modais mais usuais quanto com relação à caracterizabilidade de classes de enquadramentos. Vários problemas interessantes e direções de investigação em aberto são sugeridos para investigação futura.

Resumo de MODPAR

A lógica paraconsistente nasceu na vizinhança da lógica modal. Além do mais, como quaisquer outros lógicos não-clássicos, os paraconsistentistas frequentemente privaram com as modalidades. O primeiro sistema conhecido de lógica paraconsistente foi de fa[c]to definido como um fragmento de $S5$, no final dos anos 40. Mas um fragmento de um sistema modal não é necessariamente um sistema modal. Mostrarei aqui, com efeito, que a lógica **D2** de Jaśkowski não é uma lógica modal, no sentido contemporâneo usual do termo. Em contraste, mostrarei também, em seguida, que qualquer sistema modal não-degenerado é inerentemente paraconsistente.

Resumo de PARANORMAL

Uma lógica *extracompleta* é uma lógica que 'deixa de fazer a diferença': de acordo com uma tal lógica, todas as inferências valem independentemente da natureza dos enunciados envolvidos. Uma lógica *negação-inconsistente* é uma lógica que possui ao menos um modelo que tanto satisfaz um certo enunciado quanto a sua negação. Uma lógica *negação-incompleta* possui ao menos um modelo que não satisfaz um certo enunciado nem a sua negação. Lógicas *paraconsistentes* são negação-inconsistentes mas não-extracompletas; lógicas *paracompletas* são negação-incompletas mas não-extracompletas. Uma lógica *paranormal* é tão-somente uma lógica que é tanto paraconsistente quanto paracompleta.

Apesar de ser perfeitamente consistente e completa com relação à negação clássica, praticamente toda lógica modal normal, na sua linguagem e interpretação usuais, admite uma paranormalidade latente: ela é paracompleta com relação a uma negação definida como um operador de impossibilidade, e paraconsistente com relação a uma negação definida como não-necessidade. Com efeito, como aqui mostraremos, mesmo em linguagens desprovidas de uma negação clássica primitiva, as lógicas modais normais podem frequentemente ser caracterizadas alternativamente dire[c]tamente através de suas negações paranormais e operadores relacionados. Assim, ao invés de lógicas que falam sobre 'necessidade', 'possibilidade', e assim por diante, as lógicas modais podem ser vistas apenas como dispositivos forjados para o estudo da negação (modal). Este artigo mostra como e até que ponto tal caracterização alternativa das lógicas modais pode ser levada a bom efeito.

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> It would be madness, and inconsistency, to suppose that things which have never yet been performed, can be performed without employing some hitherto untried means.
> —Francis Bacon, *Instauratio Magna*, 1620.

This chapter explores the links between Logics of Formal Inconsistency and usual modal logics. I will in the following introduce the herein contained papers, Lea (cf. [49]), ModPar (cf. [50]) and Paranormal (cf. [51]).

Some metaphysics

In **Chapter 1** a precise abstract characterization is provided for the Logics of Formal Inconsistency (**LFI**s), and with them a particular notion of consistency is formalized and put into use. A very inclusive class of logics in which this notion is internalized in terms of a single unary connective is then illustrated, mostly through Hilbertian axiomatizations, but also through some many-valued truth-functional interpretations. **Chapter 2.2** offers non-truth-functional interpretations to some of those logics (all of which are non-characterizable by finite matrices) and shows how those same logics can be interpreted as combinations of 3-valued scenarios by way of possible-translations semantics. Many people will find the previous semantic accounts in terms of many-valued or possible-translations semantics not very compelling, and this circumstance by itself would already justify the search for interpretations for **LFI**s in terms of more well-established semantics, such as possible-worlds semantics. The fact that many such logics fail the replacement property, as it has been noticed several times before, certainly does not help in our intention to characterize them from a modal viewpoint. Are there some **LFI**s that do satisfy replacement and, moreover, behave straightforwardly as ordinary modal logics? What kind of modal interpretations for negation could help us in the search for such **LFI**s? Can the consistency and the inconsistency connectives be given sensible modal interpretations at all?

As a methodological policy, it is often helpful to deal with one problem at a time. Let's try and find thus a solution for each part of our problem in separate before trying to find a solution to all problems. Of course, even if each part of the problem turns out to have a solution, it does not follow in general that there will be a solution for the whole problem at once. But in the present case, fortunately, each part of the solution has no reason to cancel the solution to any other part. What I would like to do in the following is to give a feeling of the heuristics involved in this kind of problem-solving activity —but you might as well have a look at [64], an authoritative guide to this territory. In case that might be found instructive, here are Pólya's four theoretical steps to problem-solving and how I dealt with them, in practice, in this particular case.

1. Understanding the problem. We want to find proper modal interpretations for **LFI**s. To do that, we will look for a way of modelling a paraconsistent negation and its consistency connective companion using the idiom of possible-worlds. Moreover, for the sake of simplicity, we will try to keep all other aspects of our logic as 'classical' as possible.

The idea of enriching paraconsistent logics with modal operators is anything but new. An early reference on this strategy is the paper [70], intended to investigate paraconsistent logics with alethic and deontic operators. More recently, **LFI**s with alethic and epistemic operators were investigated in [21]. In both approaches, the paraconsistent aspect was realized by forcing a possible-worlds semantics to be based on inconsistent worlds, and then adding modal operators to the language in the usual way. Our strategy here, though, will be in a sense a strategy of 'minimal deviation'. What we want to do is to keep the worlds entirely classical, considering instead a paraconsistent negation and a consistency connective as the sole primitive modal operators, that is, the only primitive operators whose interpretation would require looking at other (classical) worlds. In theory, that intent should not be too hard to fulfill, if we first think of paraconsistent negation as dual to intuitionistic negation and recall the general lines of the well-known modal interpretation of the latter in terms of $S4$, and if we look next for an operator that satisfies the conditions for behaving as a consistency operator is supposed to behave, from the very definition of **LFI**s. In the paper Lea, however, we want to deal with only one aspect of the problem, throwing away the paraconsistent negation and working directly with the consistency operator as added to an entirely classical language.

A possible difficulty that might arise from the above plot is the following. If consistency is to be understood as what might be lacking to a paraconsistent negation in order to make it explosive, what sense can be given to consistency as a connective of a logic when a paraconsistent negation is *not* present from the start? While in a consistent logic a sentence cannot be true together with its negation, in a paraconsistent logic that is not the case. So, while in a consistent logic one can conclude from the truth of a sentence the falsity of the negation of this sentence, in a paraconsistent logic it would be convenient that, in general, the negation of a sentence is at most *possibly false* when this sentence is true. On that modal understanding of paraconsistent negations, an inconsistency will turn out to be *a sort of accident*: A sentence that proves to be inconsistent is one that is true in the current state of affairs but false in at least one possible alternative state of affairs. On the one hand, from a semantic perspective this could be criticized as too weak an interpretation for the notion of inconsistency. Is that all that paraconsistency is about, dealing with *this* kind of inconsistencies? On the other hand, from a more pragmatic perspective it seems quite appropriate: Has inconsistency ever been anything else than an (unfortunate) accident? Further discussion of such interpretation of negation is to be found in a later step of the present process.

2. Devising a plan. In logic it is often easier to attack the whole problem at once instead of attacking just part of it. The fact is hardly surprising: If you use a richer language you should be able to say more things, and if your device has more components then it is more likely that you will find a way of conforming to it the tools you have. In order to fix a modal understanding for our new connectives of consistency and inconsistency, it is worth looking at what happens when the full language of **LFI**s is considered, as in **Chapter 3.3**. So, here is how we will do it: We will steal those new connectives from the latter paper and try to investigate them in **Chapter 3.1** as added to an entirely classical language. This way we can also try to make sense of those connectives for their own sake: Can we justify their use independently of the presence of a paraconsistent negation? Is the alternate reading of these connectives as connectives of essence and accident justified? Should this study be an appendix of the paper PARANORMAL or is it sufficiently independent to deserve rather a paper for itself?

Here are some more practical questions one may ask. Has this problem been studied before? Or maybe a similar problem? The well-known philosophical notion of *contingency* seems strikingly similar to the notion of *accident* that I present here. But they're distinct: A contingent truth is one that, from the present state of affairs, *could* be true but could *also* be false; an accidental truth, I recall, is one that *is* true in the present state of affairs as a matter of fact, *yet* could be false had things been otherwise. Another usual reading for 'contingent', in a classical framework, is the one that calls contingent any formula that is neither a tautology nor an antilogy. The latter reading could in fact be related to the former one, but we might as well simply ignore that issue here. The relevant questions for us at this point are: Were the notions of contingency and of non-contingency formally studied in modal logic? Were the technical problems involved in that study completely solved there, and if so can we use a similar approach to help in devising our own solutions to similar problems?

The earliest paper in which a modal logic is investigated in a language containing no primitive boxes nor diamonds but only the (non-normal) modal connectives of contingency and non-contingency as added to the classical language was Montgomery & Routley's [54]. The problem of axiomatizing the usual modal logics using this alternative language was formulated there, but only a few cases of classes of frames in which boxes and diamonds *did* turn to be definable from the new language were investigated. This paper was followed by a number of similar studies (cf. [56, 55, 57, 58]), most of them quite shallow from a technical viewpoint, none of them solving the original problem in full, for arbitrary classes of frames. From a number of possible philosophical uses for that language, not many were really explored until in [73] Routley used the language of contingency to formulate the "radical conventionalist thesis that all assertions of modalities are contingent". Almost two decades went by before a really important technical contribution was made. The interesting paper [22], by Cresswell, came and offered some conditions for the non-definability of boxes and diamonds in the language of non-contingency. The technique is standard, yet very useful. It is based on proving that the geometries of the canonical models of certain non-contingency logics do not allow for the definition of box or diamond (in their usual interpretations). This method can be partly adapted for the language of essence and accident. Cresswell's paper also provided an example of a logic of non-contingency that does not require reflexiveness from its adequate class of frames yet allows for the definition of the usual modal connectives. To find an analogous example is still an open problem for our logics of essence and accident, as explored in my paper.

Another decade had to wait before the initial problem of (non-)contingency would be finally extensively solved, by Humberstone (cf. [38]), in a very readable and instructive paper. It should be noticed, however, that Humberstone's solution, an axiomatization for the minimal logic of non-contingency with an infinite number of primitive rules, is not as simple and elegant as one might expect. At any rate, his paper also proves some interesting results on the definability of classes of frames from the poor language of non-contingency, and these can be partly adapted for the language of essence and accident. Finite axiomatizations for the minimal logic of non-contingency were found immediately after that, by Kuhn (cf. [43]) and Pizzi (cf. [63]). Pizzi's paper studies several non-equivalent formulations of the notion of contingency, and solves the problem for the received notion by way of the construction of a canonical model whose accessibility relation connects pairs of worlds to pairs of worlds. Kuhn's solution is the simplest one, despite its somewhat mysterious rationale. It does work well, however, and it can be adapted for the language of essence and accident.

All that said and done, the paper Lea was not intended to dwell on matters related to non-contingency, but it sought instead to explore a very precise modal definition of 'consistency' on its behavior as a connective for 'essence'. One thing that is not mentioned there, but will be noticed by any good reader, is that, even when the modal definition of paraconsistent negation $\neg$ is kept fixed as it is (namely, as the possibility of a classical negation), there are of course other modal definitions of consistency that will allow for the resulting logic to be characterized as an **LFI**. I explored only the definition that seemed more novel and more general. But in some cases even the modal connective $\triangle$ of 'non-contingency', as discussed above, will serve quite well in order to define a connective $\circ$ of 'consistency' (not by coincidence, the definition of consistency in **D2**, as exposed in the **Errata** to the **Chapter 1**, mocked the form of a non-contingency connective). As a matter of fact, in any class of reflexive frames we can check that both $(\triangle p \wedge p \wedge {\sim}p)$ and $(\circ p \wedge p \wedge {\sim}p)$ are explosive, while none of $(\triangle p \wedge p)$, $(\triangle p \wedge {\sim}p)$, $(\circ p \wedge p)$ and $(\circ p \wedge {\sim}p)$ are explosive. Moreover, as it was pointed out in the final section of the paper, in such classes of frames the formulas

$\triangle p$ and $(\circ p \wedge \circ{\sim}p)$ are equivalent —as predicted indeed by Walt Whitman in the quote that opens the paper.

3. Carrying out the plan. I can dissert about it, but you can also read the paper and check what was done, and how. Section 1 sets the stage, defining the language of essence and accident. Section 2 axiomatizes the minimal logic of essence and accident, to wit, the set of theorems and inferences validated by the above language with no restriction presumed over the set of frames. The strategy for the 'desessentialization' of a world, used in the construction of the canonical model, is adapted from Kuhn's strategy, mentioned above. Obviously, some changes are in order. Our Lemma 2.4 proves properties of the canonical model, and it is fundamental for the completeness result. This lemma clearly had to be adapted so as to conform to the properties of the present language, different from those of the language of non-contingency. Section 3 discusses the definability of the more usual modal connectives from the language of essence and accident. A certain definition is shown to work only for extensions of KT, and some of Cresswell's results about non-definability in the language of non-contingency are adapted to the present language. Not much more is proved there than the straightforward. Section 4 modifies the approach by Humberstone to the logics of non-contingency to show that the logics of essence and accident are even less expressive than the former, in a sense, in not being able to characterize many more usual classes of frames. Section 5 shows some connections between the earlier language of non-contingency and the new language of essence and accident and discusses some viable philosophical uses for the new notions.

The short paper Lea is packed with novel ideas and results. Some of them, however, have barely scratched the surface of the space of possibilities. This does not mean, though, that they are 'underdeveloped' in any respect. I believe that what was already done was much more than just a good start. That many destinies are left open for exploration is a sign that many roads were paved. They can only be improved for the traffic as the signals get installed, from now on. The connections of this paper to the subsequent one, Paranormal, are not overemphasized in the Lea for two main reasons. First, because it was unnecessary to do it, and even inadvisable for the the sake of relative independence of the two lines of investigation. Second, because the connections will be obvious anyway for the perspicacious reader. As it will be seen, the paper Paranormal could be understood, as a matter of fact, as a natural continuation of Lea in which the modal language of the latter is enriched with specific modal connectives for (non-classical) negation, and 'essence' is reinterpreted (abusively? naturally?) as 'consistency'.

4. Looking back. Because we're not angry young men (anymore), we'll now look back, but not in anger —we'll learn instead to accept the rituals of our society, as long as our society accepts us. Let us appreciate what was accomplished so far, to the extent that it relates to some interesting possible directions of continuation for this work, and then briefly discuss the reach and the significance of the present study.

Looking at the more recent literature on non-contingency, one should heed a few promising lines of investigation that have been trodden by Zolin. In [90], the author looks at the counterparts of some usual modal axioms in the language of non-contingency and finds the logics axiomatized with the help of such axioms to have first-order definable classes of frames. Yet he shows that such classes of frames do not coincide with the classes of frames that are characteristic of the related axioms in the more expressive language of boxes and diamonds. In particular, any class of frames definable in the non-contingency language can be shown to contain the class of functional frames. Moreover, on the path of completeness results, the canonical frame of the non-contingency version of any logic containing the seriality axiom is proven to be non-serial. Results parallel to these are still to be sought for the present language of essence and accident. Other papers by Zolin extend the use of non-contingency even further. A number of non-contingency logics receives in [92] (*apud* the reviewer) sequent-style versions that enjoy the Craig interpolation property but not cut-elimination. And in [91] (again, *apud* the reviewer), the meaning of non-contingency in the context of provability logics is investigated: Non-contingency is there interpreted very naturally as 'formal decidability', and completeness with respect to finite irreflexive transitive frames (as in Gödel-Löb's logic of provability) is attained. Similarly, in the language of /essence/consistency/ and /accident/inconsistency/, the meaning of 'formal consistency' and related connectives should still be studied in the same context of provability interpretations. In [48] I venture a first step in this direction: Assuming proofs to be defeasible, a paraconsistent negation is used to represent the notion of 'admissible falsehood' and a paracomplete negation is used to represent the notion of 'refutable truth'. Check it out.

Another interesting line of investigation seems to be the following. In the paper Lea, my approach to the modalities of essence and accident was indirect. I tried always to axiomatize the set of theorems and axioms determined by some class of frames derived from some known normal modal logic. But those new modalities are *not* normal. It would be only natural to investigate them instead using some semantics that is more suitable for non-normal modal logics, as the one based on 'minimal models' (cf. [19]). As pointed out in [31], the main property presupposed by these models about the logic in question is that it should be 'congruential', that is, that the replacement property should hold for it. And that much we can count on.

At last, let me make some considerations on the philosophical aspects of 'essence' and 'accident'. In Aristotle's Metaphysics (cf. [20]), the notion of essence is nothing short than fundamental. Roughly speaking, according to Aristotle, to characterize an entity one would have to characterize the sort of things that individuate its substance, that is, one would have to say what it is to be that entity, what are its essential features. In fact, the very term '*essentia*' has been coined by the Roman translators in order to avoid repeating the peculiar yet frequent Aristotelian expression '*to ti ên einai*' (τὸ τί ἦν εἶναι), literally 'the what-it-is-to-be'. In Aristotle's Logic (cf. [80]), the essence of an entity would typically be fixed by a definition, where by definition one does not mean a set of words explaining the meaning of a term that denotes that entity, but some sort of account (λόγος) which signifies the what-it-is-to-be for that entity. As I understand it, a modern realization of such an account might take us away from definite descriptions and closer to grammar and game-theoretical interpretations. But I had better drop the coin here, and move on.

Among all subsequent philosophers, Leibniz arguably had one of the richest modal idiolects. Necessity, contingency, essence, all the modal operators that we here discuss seem to appear in Leibniz's writings, and they frequently appear in fact in their attributive reading. In that reading, essential features are typically used as a justification for necessity ('what is necessary is so by its essence, since the opposite implies a contradiction' —V letter to Clarke, cf. [1]). This interpretation was applied, among other things, to update the Ontological Argument of St. Anselm: The proposition expressing God's existence would, in our present terminology, constitute an 'essential truth' —if God exists, It exists by way of necessity.

Jumping now to contemporary times, Wittgenstein in the *Tractatus* makes yet another use of the language of essence and accident (*Wesen* and *Zufall*, cf. [88]). Propositions are said to have essential and accidental features (verse 3.34): Essential features of a proposition are exactly those that are needed for it to express its sense, and what it has in common with other propositions sharing the same sense. Moreover, the world is constituted of atomic facts (verses 1 and 2), and those facts are essential combinations of things (verse 2.011). Logical facts are non-accidental (verse 2.012). Yet the facts of the world are wholly accidental, and the sense of the world lies outside the world (verse 6.41). In spite of the use of many modal terms and modal figures (to the point of having influenced Carnap on his approach to modal logic), Wittgenstein seems willing to shut the door to the employment of a fully modal language, in insisting that propositions are very specific kinds of truth-functions (verses 5 and 6) —though it has been modernly argued that the Tractarian semantics can be very naturally reconstructed in modal terms (cf. [44]). With a poor language and a bold intention, no wonder that the philosopher should conclude the book by asserting that there are many things about which he should stay silent (verse 7).

The reader will notice from the above historic examples that there is nothing like a standard use of the modal idiom in traditional discursive philosophy (I use 'discursive' here as opposed to a more formal, or 'technically informed', kind of philosophy). While an extensive philosophical literature was dedicated, and with a good reason, to the investigation of sentences with an essential content, for a long long time no good soul would make an effort so as to guarantee that there was a sufficiently precise and rich language adequate to the expression of quiddity. The paper LEA was aimed as a contribution to the logico-metaphysical legitimation of basic modal languages capable of expressing essence and accident in their assertoric uses (as discussed in the final section of LEA.

Substantial work should still be carried out in order to illustrate the advantages of the present use of the notions of essence and accident in philosophy. I find particularly promising their use in investigating Kripke's notion of 'rigid designation', if only to clarify by appeal to some clear-cut sort of essentialism how and in which conditions some identity propositions could be both necessary and *a posteriori* (cf. [42]), or to apply the same idea to more general propositions involving the characterization of proper names, general terms, or of natural kinds. Kripke's theory was an update on the anti-rationalist Humean theory about the existence of *a posteriori* truths, as recovered by the linguistic conventionalism of the logical positivism, and a reaction *both* to the collapse of the metaphysical and epistemological modalities promoted by Quine and Barcan-Marcus in their understanding of the received doctrine of essentialism *and* to the theories of resemblance and counterfactuality defended by David Lewis and other people that took the idiom of 'possible worlds' a bit too serious. Again, this is no place to detail the above proposals any further. At any rate, it is interesting to check how, based on Kripke's ideas, Murcho (cf. [60]) has defended a version of 'naturalized essentialism' according to which there would be essential properties of particulars that do not constitute neither logical nor conceptual necessities. Many questions though are left open by such an analysis: Would there be essentialist assertions known only *a posteriori*? Beyond essence, which would be the sufficient conditions for the individuation of existents? Would some form of 'accidentalism' be as important as essentialism for the description of universals? Can this in fact be related to the negative characterization of logics and logical constants that I propose in **Chapter 4.1**? Does that give us a hint about the possibility of providing negative characterizations for general terms or for natural kinds? Or would the 'anti-essentialist' posture be much more coherent, after all? A formal approach to these matters might help in settling a few answers, by allowing us to more easily evaluate the consequences of each philosophical stance.

Such issues shall here be left as matter for future work. The reader is invited to contribute.

Some esoterism

In spite of the impression left by the work of some algebraists, the research on algebraic logic is intended to make logics (and life) *simpler*. It will be somewhat disappointing, however, that some logics like da Costa's C_1 should turn out to be just *too* simple. However, instead of subscribing to the received wisdom according to which "logic can (and perhaps should) be viewed from an algebraic perspective" (cf. [37]), one could well oppose that reductionist strategy and use the very simplicity of some logics to argue for a non-algebraic study of them (cf. [4]).

1. Of algebraization. Let me expand on that. In mathematics, a *congruence* relation over a given structure is simply an equivalence relation over its domain that is /closed under/compatible with/ the operations of this structure. There are at least two obvious ways of quotienting a given structure so as to produce degenerate related algebraic structures, namely, through the roughest congruence relation, that puts every element of the domain in the same class, or through the finest congruence relation, the identity, that makes of every element a class of its own. Both ways are pretty fruitless: The former cannot discern an element from any other element, the latter understands that no two elements can ever be identified. An algebra that only admits of these two degenerate congruences is called *simple*. In the same spirit, a logical matrix for a non-overcomplete logic[1] is said to be simple (cf. [89], p.198) if the identity relation is the only congruence that it admits of, and a simple logic is a non-overcomplete logic that only admits of a single congruence relation, the identity (cf. [4]).

When we define a congruence over a given (tarskian) logic, we expect it to divide the set of formulas into classes whose elements are all *indiscernible* with respect to that congruence. The most straightforward style of algebraization for a logic is the so-called Lindenbaum-Tarski procedure, that makes use of the associated consequence relation: Whenever two formulas are interderivable they are put in the same class. If that procedure defines a congruence and associates a non-degenerate quotient algebra to the logic, the logic is said to be LT-algebraizable. In that case, the logic will also automatically respect the so-called *replacement property*, according to which the derivability of a formula φ from a theory Γ is preserved when subformulas of φ are replaced by equivalent formulas. For LT-algebraizable logics, equivalent formulas are thus indiscernible.

Now, there are other logics for which the replacement property holds good and there are also logics for which non-degenerate congruence relations can be associated, but that still fail to be LT-algebraizable (recall subsection 3.12 of the paper TAXONOMY, in **Chapter 1.0**). Nonetheless, adequate algebraic semantics can often still be associated to such logics, and several other styles of algebraization might apply to them, making them qualify for instance as 'protoalgebraizable', 'weakly algebraizable', 'equivalential', or 'Blok-Pigozzi-algebraizable' (cf. [11]). Simple logics, however, are hardly ever considered to be 'algebraizable' in any interesting sense of the word.

It had been known since long (cf. [27]) that the logic C_1 and its early companions fail the replacement property. Two decades had to pass though before Mortensen (cf. [59]) added to that result the observation that these

[1]Check the next chapter for a comprehensive definition of 'overcompleteness'.

logics are simple. So, in spite of early attempts by da Costa to provide adequate 'algebraic' counterparts for the logic C_1 (cf. [25]), following a recipe of Curry (cf. [23]), and some late attempts by other authors (cf. [18, 76]) to resurrect and update that approach, the logic C_1 was not to admit of any other congruence relation than the identity.

Contrary to anecdotal evidence, people do not algebraize just because that's what they do in life. From the point of view of the logician, the questions one should bear in mind are related to the applications of the algebraic tools, as guaranteed by the 'bridge theorems' that connect algebraic properties to (meta)logical properties. Any new algebraic tools that are proposed for the use in logic should substantiate their claim for deserving any attention from the community by providing material for bridges to be built. So, the really interesting question in the end is not so much whether algebraic counterparts can be associated to given logics, but what contribution the former structures can give to the latter once you have designed them.

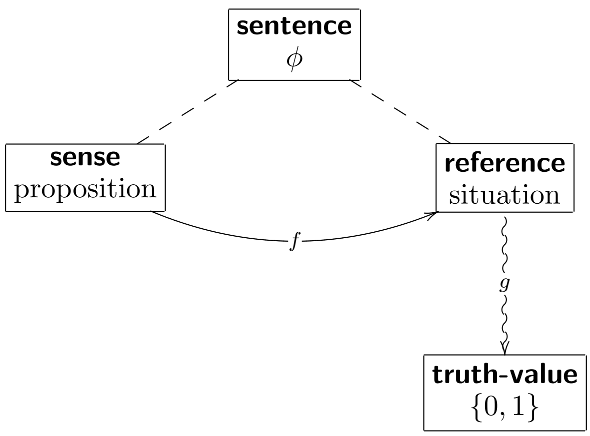
Some people think that logics that do not have a 'normal' algebraic counterpart are unworthy of consideration. Others have argued that 'normal' logics always have some built-in linguistic-algebraic element, say, in the way they build their set of formulas (cf. [46]). The Bourbakian architecture of mathematics (cf. [12]) proposed a structuralist *division of labor* according to which every mathematical structure would be characterized either as an algebraic structure, as an order structure, as a topological structure, or else as a 'multiple structure' that combine characteristics of more than one of the preceding 'mother-structures'. According to the modern Polish approach to 'logical calculi' (cf. [89]), logics are to be seen as hybrids including multiple elements of the Bourbakian mother-structures, yet it should be acknowledged that some profane authors have recently strived to eliminate the linguistic aspect of that approach (cf. [52]). An independent approach that has proposed to regard logics as a fourth class of mother-structures in their own right is that of *Universal Logic* (cf. [3]).[2] Free from the algebraizing impetus that would use algebra to justify investigations in logic and substantiate the very recognition of logical structures as real mathematical structures, universal logicians could still criticize a logic for failing the replacement property or for not being algebraizable in any usual sense, but they would not expel a logic from the realm of mathematics just because it turned out to be a simple logic. Quite to the contrary, as argued by Béziau in [4], such examples of logics that "cannot be reduced to algebra" could

[2]In defense of the great Poldavian mathematician, one should admit that the paper [12] had already left some space open for new mothers like Universal Logic to emerge. The old Nicholas, in spite of anathemazing the 'lifeless skeleton' of formal logic, praises the 'axiomatic method' and adverts that his 'rapid sketch' of the 'whole of the mathematical universe' is but *frozen*, as he writes that: "The structures are not immutable, neither in number nor in their essential contents. It is quite possible that the future development of mathematics may increase the number of fundamental structures, revealing the fruitfulness of new axioms, or of new combinations of axioms."

even be claimed to furnish some evidence for the thesis that the algebraic perspective cannot exhaust the wealth of logical investigations.

2. Some puzzles. With the same paper, [34], Frege helped at the same time to found analytic philosophy and to puzzle generations of logicians with his proposal of splitting the meaning of a sentence into its 'sense' (*Sinn*) and its 'reference' (*Bedeutung*). Modern first-order logic was also born with Frege (as well as with Peirce, and Schröder) in his *Begriffsschrift*, a few years later (1879), as "a formula language, modelled on that of arithmetic, of pure thought". This book was supposed by his author to be fundamentally different from the other more famous study of the 'Laws of Thought', Boole's landmark piece that preceded Frege's by more than 4 decades (1854). Using loan words from the Leibnizian vocabulary, Frege repeatedly stated (against Schröder) that his own approach was intended to provide for a *characteristica universalis*, a universal language to be applied first to mathematics and then to real world problems, while Boole's was a mere *calculus ratiocinator* intended for mechanically deducing all possible truths of (propositional) logic from the list of simple thoughts in a purely syntactical fashion (cf. [78]).

For the good or for the bad, modern tradition in Abstract Algebraic Logic has associated Frege's name to a particular class of logics for which a property somewhat stronger than replacement holds good, namely a sort of 'contextual replacement' according to which, for each theory Γ of a logic $\mathbf{L}$, the class of formulas that are equivalent in the presence of Γ define a congruence relation over $\mathbf{L}$ (see [62, 32]). In [24] the authors explain this property by saying that a logic is Fregean "if interderivability is compositional". They claim that this idea is based on Roman Suszko's formal reading of Frege. I am not sure though that this is a good reading of Suszko. I explain. The issue here concerns the Fregean notion of 'sense'. While Czelakowski and Pigozzi might well be right, in [24], in saying that Frege "viewed this concept as extra-linguistic and did not attempt to incorporate it in his formal system", that is clearly not how Suszko himself chooses to formalize Frege. Schematically, this is how Suszko reconstructs the Fregean picture, in [81]:



According to the Suszkian Frege, a sentence's meaning can only be understood in terms of its sense and its reference, the latter being a function of the former. In addition, claimed Suszko, the picture would only be complete after the 2 classical truth-values were added as a function of the reference, for it is in them that resides the "genuine definition of logic". The reference of a sentence is maintained as an intermediary step in between the sentence and its truth-value, a step that one in general cannot get rid of, "unless one agrees that thought is about nothing, or, rather, stops talking with sentences". The 'algebraic truth-values' of many-valued logics were to play thus a referential role, while only two 'logical truth-values' would really exist. No further logical values were possible, for "obviously any multiplication of logical values is a mad idea" (cf. [82]). In his typical grandiloquent style, in this summary of a talk given in 1976 to the 22nd Conference on the History of Logic, Suszko complains that "after 50 years we still face an illogical paradise of many truths and falsehoods" (the 'truths' being the designated values of many-valued logic, and the 'falsehoods' the undesignated values). But he knows all too well who is to blame for that, as he adds: "Łukasiewicz is the chief perpetrator of a magnificent conceptual deceipt lasting out in mathematical logic to the present day". Suszko would enter the history of logic, as I showed in detail in the **Chapter 2.1**, for the idea that any semantics of a tarskian logic could in principle be reduced to a 2-valued semantics. So, the trade-off would in general involve the decision of retaining either truth-functionality or bivalence in our semantics, as we have discussed in our research note [17], the paper [14], and the forthcoming paper [15]. Finally, it should be remarked that, considering what has been said above about 'contextual replacement', for Suszko *congruent* formulas are simply formulas that have *synonymous senses* according to a given theory (cf. [81], supplement III) —and on that Suszko claims to be following Quine's notion of 'cognitive synonymy', proposed in order to explain the notion of 'analyticity' (cf. [71]).

A related puzzle left to us by Suszko should here be mentioned, namely, his own notion of a 'Fregean logic'. According to Suszko's [81], the main objective of his analysis is to get us rid of the 'intensional ghosts of modality' (a term he partly borrows from Herman Weyl's [87]). Suszko clearly dislikes traditional possible-worlds semantics. Apart from that, he drops only here and there some hints about what it means to be 'non-Fregean', as in: "The construction of [the] so-called many-valued logics by Jan Łukasiewicz was the effective abolition of the Fregean Axiom" (cf. [82]). It is unfortunate that Suszko does not appear to be willing to formally clear up beyond any doubt what he means by the 'Fregean Axiom'. For one thing —and recalling that Suszko regards truth-functionality and replacement as negotiable properties of a logic, while he takes structurality for granted at least since [46]— this 'Fregean axiom' was supposed by Suszko to be the main responsible for the format in which '2-valued extensional logic' came to be consacrated. "How was it possible that the humbug of many logical values persisted over the last fifty years?”, he asks us in [82]. In this paper, and in [81], Suszko says, on the one hand, that the ‘Fregean axiom’ is ‘equivalent’ to a restriction in the possible referents of a sentence to the two elements from its set of possible truth-values and, on the other hand, Suszko also asserts in the latter paper that “because of the Fregean axiom, the replacement property of logically equivalent formulas holds in Fregean logic”. These comments are certainly intriguing, given that 2-valuedness and replacement are completely independent properties of a logic! For a one-sided attempt to resolve this puzzle through partial replacement the reader is invited again to consult Czelakowski & Pigozzi’s [24].

3. Of replacement. Where did we stop? I recollect and continue. As we have seen, the replacement property needs to be satisfied for a logic to be LT-algebraizable. For such a logic, equivalent formulas are indiscernible: If it has top particles, for instance, they will group into the same congruence class; similarly for bottom particles. From the point of view of the quotient algebra, any member of a congruence class behaves just like any other of its synonymous companions, and it can legitimately represent them for all operative purposes. This simplifies the initial task of working with the whole language of the logic. Not all logics are prone to such an algebraization procedure, however. Some logics, like C_1 (our **Cila**), cannot be further simplified. They are, so to speak, *anarchistic*: No formula has the right to represent any other formula. In case a non-degenerate congruence can be defined over a given logic, though, this logic gives hope for an algebraic treatment. Indeed, as we have seen in Section 3.12 of the paper Taxonomy, in **Chapter 1**, there are extensions and alternatives to **Cila** that are not simple: The logic **Cilo**, for instance, is ‘(finitely) equivalential’, and the 8K 3-valued maximal logics that extend **Cia** and were mentioned in that paper are all ‘Blok-Pigozzi-algebraizable’. Some form or another of the replacement property always play a role in this process of defining classes of indiscernible formulas. Now, if replacement alone is not capable of guaranteeing that a logic falls into one of the main classes of algebraizability,[3] it does at least help a good deal in guaranteeing that the logic is not simple and not completely degenerate from an algebraic point of view. All that said and done, from this point on, in this section, I will be concentrating exclusively on logics satisfying the replacement property in its usual formulation.

According to Wójcicki ([89], chap. 3.2.0):

> We are free to create as many logical calculi as we wish, which certainly does not mean that the outcome of our activity will eventually turn out to be of any interest. Although there is no generally accepted

[3] The class of ‘Fregean logics’, the most demanding logico-algebraic class from the so-called ‘Leibniz hierarchy’, in [33], generalizing the Lindenbaum-Tarski procedure, requires not only replacement but also contextual replacement to be satisfied.

> definition of a 'good' or 'interesting' logic, we incline to consider certain properties of logical calculi as desirable whereas some others are not.

Among the 'desirable' properties of logical calculi the author includes *self-extensionality*, which is tantamount to the above mentioned replacement property in abstract logic. Moreover, in chapter 5 of the same book, one can find a proof that a structural tarskian logic is self-extensional if, and only if, it has an adequate class of 2-valued 'frame interpretations'. This is intended to establish a link between the replacement property and the logics having a usual modal-like semantics. The confidence about the existence of such a link is in fact shared by modal logicians and sympathizers (henceforth, *modalists*). Indeed, replacement is sometimes taken to constitute the characterizing property of 'classical operators' (check Segerberg's [75]) and, together with the duality between $\Box$ and $\Diamond$, it is taken to characterize as well what is known as the 'classical systems of modal logic' (check Chellas's [19]). In opposition to that, Béziau has argued, in [8], that to satisfactorily capture the notion of intensionality a logic must be non-self-extensional, and that people think that a logic must be self-extensional "rather because this is a nice technical and practical property than for any precise philosophical reason". Given that no precise philosophical reasons are offered by this author for us to take the contrary position either, and given that replacement is a property of every normal system of modal logic, I will be assuming here, together with the modalist tradition, that a logic that does not respect the replacement property is simply *not modal*, in the usual contemporary sense of the term.

Most paraconsistent logics and Logics of Formal Inconsistency presented in the previous chapters fail the replacement property[4] —a quite comprehensive result in that respect is Theorem 3.51 from the TAXONOMY, which shows that there are many paraconsistent logics which cannot even be extended so as to originate other paraconsistent logics that would satisfy replacement. Many authors have seen such a failure as a major technical and philosophical defect of the **C**-systems; some have thought that this was an intrinsic defect of paraconsistent logic in general. As we now know, the latter were wrong, while the former were... hmmm... misoriented. Richard Sylvan (née Routley), in [83], asserts for instance that the more traditional daCostian logics "appear to lack natural and elegant algebraic and semantical formulations, largely because they fail to guarantee intersubstitutivity of equivalents". Béziau says in [6] that "from the philosophical point of view there must be an intuition supporting the non-self-extensional behaviour of a negation",[5] and he repeats basically this same complaint in [7], insisting

[4] In the TAXONOMY this property was studied mostly from a syntactical perspective and was called (IpE), an acronym standing for 'intersubstitutivity of provable equivalents'.

[5] The author was mistaken, however, in that paper, in suggesting Sette's logic $\mathbf{P}^1$ as an example of a self-extensional logic.

that in the cases of logics like C_1, LP, and $\mathbf{J}_3$ “no philosophical justification for this failure has been presented”. To that sort of analysis, da Costa and Otávio Bueno have retorted by comparing self-extensional logics with abelian groups (cf. [26]), saying that “from the perspective of pure logic, such a critique would be similar to that made by an algebraist who wishes that only commutative groups be studied”.

Is the solution to the replacement quagmire to be found in the universe of modal logics? And how do we get there, finding convenient modal interpretations for all the connectives of our logics of formal inconsistency? Parts **3.2** and **3.3** of the present chapter will fully answer such questions.

The oldest paraconsistent logic ever, Jaśkowski's logic **D2** (cf. [39, 40]), was introduced as a certain fragment of the modal logic $S5$. Does it point the way out of our pickle? No, it doesn't. As I show in the paper ModPar (part **3.2** of the present chapter), **D2** and its close relatives are all Logics of Formal Inconsistency, as a matter of fact, but they all fail replacement and do *not* constitute thus examples of modal logics, in spite of the impression one might get from the related literature. This result is obtained, by the way, as a direct application of the Theorem 3.51(v) from the Taxonomy, our **Chapter 1.0**.

4. Of duality and modality. Paraconsistent logics have quite often been thought of, and with a good reason, as dual to intuitionistic-like logics, be they intermediate logics or, more generally, *paracomplete* logics (cf. [45]). Duality issues will in fact guide us from now on, and they will be very much explored for the rest of this thesis. But I am a newcomer on that scene. In the 30s and the 40s, several years before Jaśkowski's founding work on paraconsistency hit the press, one could find Karl Popper criticizing dialecticians for failing to take into account the Principle of Explosion that would render the theories trivial and uninformative in the presence of what he called ‘embracing contradictions’. Later, though, after having engaged on a long dispute with Harold Jeffreys (recall note 10, in section 2.4 of the Taxonomy), and more or less at the same time in which da Costa was publishing his initial investigations in paraconsistency, Popper substantially updated, in [69], his first English-written paper ever, ‘What is dialectic?’ (cf. [65]), and added that he had in fact been thinking about a (paraconsistent) logic that would be dual to intuitionistic logic (cf. [66]), that he finally dismissed as too weak as to be useful (recall his argument about the failure of contraposition presented at section 3.3 of the Taxonomy). Popper was a distinguished critic of the logico-positivistic methodology and the verificationist approach to empirical sciences. To that he opposed a scheme according to which critical rationalism and falsificationism would take priority. The research in natural sciences would, accordingly, advance by the proposal of new theoretical conjectures and the attempt at refuting them through experimentation (cf. [67]). The strategy in that case —where some hypotheses and their negations could both temporarily be assumed to be unfalsified— would seem to appeal to a paraconsistent-like interpretation —where some propositions and their negations could both be assumed to be true. Not surprisingly, inasmuch as intuitionistic logic has been commonly given an interpretation as a verificationist logic of constructive truth, the falsificationist logic of constructive falsehood is expected to be paraconsistent.[6] The basic idea about the connections between falsificationism and paraconsistency has been explored by Popper's disciple, David Miller, in [53].[7] The constructive interpretation of the resulting logics can be found in Shramko's [77]. The obvious implications of the above ideas for Epistemology and for the Philosophy of Science make this an area of investigation that deserves a lot more attention.

Besides some previous scattered ideas and suggestions, *dual-intuitionistic* logic started to be developed only in the 80s, with the paper [36], by Nicholas Goodman, who called it 'anti-intuitionistic logic'. A Brouwer algebra, the dual to Heyting algebra, was the basic construct intended to represent the new logic, whose proof-theoretic presentation was based on single-premise-multiple-conclusioned inferences, dual to the multiple-premise-single-conclusioned inferences of intuitionistic logic in Gentzen's formulation of it. Several variations and extensions of that initial study of dual-intuitionistic logic were produced by Igor Urbas in [84]. Both intuitionistic logic and dual-intuitionistic logic have a constructive leaning, especially as reflected on the heredity condition of their Kripke semantics: While the former preserves truth towards the future, the latter preserves falsehood. Moreover, both logics satisfy the replacement property. On the other hand, the relational semantics of one of the earliest paraconsistent specimen, Nelson's logic (cf. [61]), takes both verification and falsification as primitive and equally important concepts, and takes both truth and falsehood as constructive notions, but the logic turns out to fail replacement. However, it should be pointed out that Nelson's logic admits of a non-degenerate congruence (for details, check chap. 6 of [86]) that from the point of view of abstract algebraic logic makes it qualify as a finitely equivalential logic (such congruence can be defined exactly like the non-trivial congruence defined for **Cilo** around the Fact 3.81 of the TAXONOMY). It would seem interesting to check whether a consistency connective could be naturally defined in this logic, to

[6]Curiously, in [13], a class of dual-intuitionistic logics called 'anti-constructive' are proposed with the following rationale: "This denomination can be understood taking into account that, as far as the intuitionistic philosophic program can be seen as committed to constructing truthood [sic], our anti-constructive logics can be seen as committed to eliminating falsehood". Such a purported 'anti-constructive falsehood elimination' remains at best unclear, however, as no interpretation is offered in the paper so as to justify it.

[7]Miller has even considered da Costa's C_1 as a possible 'logic of unfalsified hypotheses'. However, as he rightly recalls, the non-classical stance is not one that was favored by Popper himself, who insisted that "we should (in the empirical sciences) use the full or classical or two-valued logic" (cf. chap. 8 of [68]).

see if it also qualifies as a Logic of Formal Inconsistency. Another interesting line of investigation would seem to be the study of structures that are, in a sense, 'dual' to algebraic structures (but not necessarily in the sense of coalgebraic structures). For such structures, instead of privileging a certain notion of indiscernibility (or 'identity') given by congruence relations, a notion of 'apartness', 'discordance' or 'difference' would be expected to play a role. But I had better leave this point here as a somewhat vague suggestion, and move on.

A different modal-like interpretation of negation was contributed by the relevance logic community, starting with Routley & Routley's 'star operator' for the semantics of first-degree entailment (cf. [74]), greatly generalized later on by Mike Dunn (cf. [30]) in terms of a 'compatibility relation' that is added to Kripke frames. Roughly speaking, a negation sentence $\sim\alpha$ is true in a world x iff α is false in every world y compatible with x. The trick of course rests in defining the right conditions for compatibility in each case: In the case of Routley star, for instance, the compatibility relation is assumed to be symmetric, directed and convergent (cf. [72]). Yet another way of extending the notion of a modal semantics so as to apply it to larger classes of non-classical logics was proposed by Matthias Baaz in [2], as applied to da Costa's logic C_ω. In that paper, the structure of a Kripke model was enriched by a function T that associates to each world a set of negated sentences that, intuitively, are to be taken as true in that world independently of the truth-values of the subformulas. This results of course in a trick for making the underlying worlds non-classical without really touching them. Both the relevantist approach and the proposal by Baaz have the potential to enormously extend the scope of what can be called 'modal semantics'. Nonetheless, I will here be content with exploring the possibilities of the *usual* frame semantics —just a set of worlds and an accessibility relation connecting some of them.

A simpler way of defining a paraconsistent negation in the usual modal setting without committing oneself to the whole apparatus of dual-intuitionistic logic is by isolating and making use of its interpretation of negation —the dual interpretation of that of intuitionistic negation— independently of heredity conditions and of the particular semantics of implication. This of course entails the abandonment of usual 'constructive' interpretations of negation (cf. [86]), or at least the extension of the notion of 'proof' so as to take refutations and defeasible reasoning into consideration, as I propose in [48]. Let **not** denote a classical negation. Then, intuitionistic negation will be quite strong and demand for 'necessarily **not**', while paraconsistent negation will be more permissive and ask for 'possibly **not**'. Jean-Yves Béziau wrote a series of papers around the latter interpretation (cf. [5, 9, 10]), where this modal paraconsistent negation is explored inside the logic $S5$.[8]

[8] Béziau mentions a series of theorems and inferences validated or invalidated by this

In the 80s, however, such dual-intuitionistic negations had already been explored by Kosta Došen (for a survey, check [29]). Even earlier than that, this had actually been studied from a very general perspective, allowing for arbitrary conditions to be imposed over the accessibility relations. That study was done by Dimiter Vakarelov (cf. [85]), based on the (algebraic-related) thesis he had written on the theme in 1974.

Recall that in LEA, the first paper of this chapter, I will be showing how the language of classical logic can be enriched with a modal operator of consistency. If I now added to that language a modal paraconsistent negation, the resulting logics would obviously become **LFI**s. Moreover, one could then easily check that each normal modal logic can be recast in this new language, and vice-versa. But that would be too easy a solution, in fact, for we would have already started from a very rich language, including the whole set of classical connectives and a primitive classical negation. It would seem more interesting to check, instead, if the same solution could be attained if we started from the usual language of our **LFI**s, without a primitive classical negation, or even to check if the same could be done if we started from the language of positive classical logic plus the paraconsistent negation only. Section 4 of the paper MODPAR, the second paper in this chapter, hints at how both tasks can be successfully accomplished, and the paper PARANORMAL that closes this chapter shows in detail how they can be realized. The careful choice of initial language marks indeed the main difference between my present investigations and those of other authors. I base my study in the poor language of Béziau's logic **Z** (the paraconsistent version of $S5$), that is, from positive classical connectives plus connectives related to paraconsistency, and I show how to extend his proposal so as to cover all non-degenerate normal modal logics. In contrast, the papers by Došen start from the full language of intuitionistic logic and add to it extra non-classical negations, and the study by Vakarelov add those same non-classical negations either to the positive part of classical or to that of intuitionistic logic, but it also considers some further connectives to be always present, namely the 0-ary connectives denoting bottom and top particles.

Coda. Another attractive innovation from the paper PARANORMAL is the use of a multiple-premise-multiple-conclusion framework. That helps in easily characterizing an immediate notion of duality (reading inferences from left to right, or the other way around; changing 0's for 1's and vice-versa in any two-valued semantic characterization), that will be very important in the next, and final, chapter of the thesis. Paraconsistency can then be very easily understood as dual to paracompleteness (and such an idea was indeed

paraconsistent version of $S5$. In the paper MODPAR I point a mistake on his [10]'s list: The formulas $(\alpha \vee \beta) \to \neg(\neg\alpha \wedge \neg\beta)$, $(\alpha \vee \neg\beta) \to \neg(\neg\alpha \wedge \beta)$, and $(\neg\alpha \vee \beta) \to \neg(\alpha \wedge \neg\beta)$ are *not* theorems of $S5$, where $\neg$ denotes the modal paraconsistent negation. But there is also a related mistake is to be found in his [9]'s list: The formula $(\neg\alpha \vee \neg\beta) \to \neg(\alpha \wedge \beta)$ *is* a theorem of $S5$, contrary to what is affirmed there.

applied in Brunner & Carnielli's paper on 'anti-intuitionism', [13]). A natural question that might be entertained concerns the notions that are dual to consistency and inconsistency, notions that one might dub 'determinedness' and 'undeterminedness'. The related dual-**LFI**s, the class of logics that I dub **LFU**s, as proxies for *Logics of Formal Undeterminedness*, is then immediately characterizable. We gain thus a much better view of the world of *paranormality* —the world of both paraconsistency and paracompleteness.

A related contribution of the last paper of this chapter, applying the above mentioned notion of duality, is the proposal of a way of restoring and generalizing the classic-like 'square of oppositions', as a further step towards a more general 'theory of oppositions'. I will advance no more about this here, but recommend instead the reader to check the paper. The next chapter of the thesis will touch on this theme again.

A last important contribution of the Paranormal is the emphasis put on the so-called 'Derivability Adjustment Theorems' that show how **LFI**s and **LFU**s, in spite of constituting fragments of consistent and complete (or determined) logics, can recapture the full reasoning allowed by the latter. Thus, a gently explosive paraconsistent logic, for instance, fails the 'consistency presupposition' that is typical of many classic-like logics that extend it, but still such a paraconsistent logic can in principle recover the reasoning that depends on 'consistency assumptions', by directly adding such assumptions to the set of premises of a given inference that depends on them. This is, in a sense, *the* Fundamental Feature of **LFI**s, their most remarkable trait and essential virtue. And similarly for **LFU**s.

Brief history

I nurtured and cherished the idea of a modal approach to paraconsistent negations and the related perfect connectives of consistency and inconsistency for quite some time. While working on other more urgent issues I always kept that idea incubated on backlog and often suggested it as a research topic to colleagues. I started my own research on it by getting acquainted with the so-called 'Polish school of paraconsistency', which had allegedly produced logics with a modal flavor. I still remember being greatly surprised then to discover the amount of ambiguity and misunderstanding that exist in the literature concerning the so-called 'discussive logic(s)' generated by Jaśkowski's early approach to paraconsistent logic. Amazingly, I found most abuses and mistakes to be committed by an incredible sluggishness of the authors to simply go and read the original sources, the papers [39, 40]. I lectured about my own proposal of a modern reconstruction and generalization of discussive logic in July 2002 at the State University of São Paulo (BR) and in September 2002 at the Nicholas Copernicus University in Toruń (PL). As we know, and as the reader can recall from **Chapter 1** (and specially from its **Errata**), it turned out that Jaśkowski's logic **D2** can define a consistency connective but, despite the many claims to the contrary, this logic is certainly *not* modal in the usual sense of the word, as it fails replacement to start with. It was clear thus that it is not enough just to provide a modal interpretation to a non-modal paraconsistent logic by way of some handy trick. That kind of 'second-hand' interpretation is in fact not so hard to obtain, as it was shown in [28, 47], where translations were produced from usual 3-valued logics such as $\mathbf{P}^1$ and Ł$_3$ into usual modal logics such as T and $S5$. I discussed that interpretation in detail during a mini-workshop organized in Ghent (BE) in June 2001, together with Diderik Batens and Jean-Yves Béziau, on the multiple relations between paraconsistent and modal logics.

An early version of the paper Lea was put forward in mid-April 2004. It proposed modal interpretations for the consistency and the inconsistency connectives and investigated them independently of the presence of paraconsistent negations in the language. Its main results and some of their extensions had been tested in January 2004 on an audience from the XII Latin-American Symposium on Mathematical Logic (XII SLALM) at the University of San José (CR), and in March 2004 in a seminar of the Group for Pure and Applied Logic at the State University of Campinas (BR). Without a paraconsistent negation around, the consistency and inconsistency connectives appeared to have a compelling and reasonably original interpretation in terms of connectives for essential and accidental truth, and that philosophical interpretation was defended from the point of view of formal metaphysics in my contribution to the XI National Meeting on Philosophy (XI ANPOF Meeting), in October 2004 in Salvador (BR). The paper Lea was accepted for publication at the Bulletin of the Section of Logic of the University of Łódz (PL).

The two main forums I had for discussing and receiving lively feedback on my views related to Suszko's approach to abstract logic, structurality, algebraization, replacement and two-valued reduction of many-valued logics, as mentioned in the last section, were a talk given at the Seminaire de l'Institut de Logique et le Centre de Recherches Sémiologiques, on the occasion of a scientific visit to the Université de Neuchâtel (CH) in April 2003, and a contribution talk to the XII International Congress of Logic, Methodology and Philosophy of Science (XII LMPS), held in Oviedo (ES) in August 2003. My co-authors in [17] and [16] also had several occasions of presenting our related work on dyadic semantics, including two contributed talks to the III World Congress on Paraconsistency (WCP 3), held in Toulouse (FR) in July 2003.

The paper Paranormal was the natural complement to Lea, written in between the end of July and the beginning of September 2004, upgrading the language of Lea so as to obtain the full languages of **LFI**s and of their duals, the **LFU**s (Logics of Formal Undeterminedness). The main ideas of this paper were advanced in February 2004 in a seminar of the Center of Logic and Computation of the IST, in Lisbon (PT), and then presented as a contributed talk to the Logica 2004 Symposium, the XVIII international symposium promoted by the Institute of Philosophy of the Academy of Sciences of the Czech Republic, in Hejnice (CZ). A blend of ideas from this study and some ideas from the next chapter of the present thesis, together with a few further developments on the theme and their corresponding philosophical justifications, were presented under invitation at the International Workshop on Negation in Constructive Logic, promoted by the Professoriat for the Theory of Science and Logic of the Dresden University of Technology, held in Dresden (DE) in July 2004. Some related notes on modality, duality and natural language contained in the PARANORMAL had already been presented in a poster at the III World Congress on Paraconsistency (WCP 3), held in Toulouse (FR) in July 2003.

At last, the paper MODPAR was planned initially as an abridged version of PARANORMAL. When I finally wrote the former in November 2004, however, I wanted it to bring in something new (namely, the issue about Jaśkowski's **D2**), and it ended up thus just by having a partial intersection with the latter paper, but not really coinciding with it. This text is soon to appear at the Logica Yearbook 2004, published in Czech Republic by ΦΙΛΟΣΟΦΙΑ.

For the many occasions I had of presenting this material, and for all the feedback I had on it, the people who commented, asked, doubted, wailed and gnashed their teeth, or just stared at me in utter wonder are too numerous to thank in but a few lines. I would only like to express here my special acknowledgement to Jairo da Silva, Marcelo Finger and Frank Sautter for their philosophical and technical appreciations of the herein contained papers during the preliminary oral examinations of the thesis, and to Jean-Yves Béziau for some important terminological clarifications. Of course, that does not mean that they would necessarily endorse my views on paraconsistency or on modal logic. The first two papers from this chapter were entirely written in Portugal under an FCT doctoral grant. The third paper was written under the same grant while I was back to Brazil for a longer stay.

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