Nearly every normal modal logic is paranormal

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> The principal interest is philosophical: not to confine oneself to what is necessary for (current) practice, but to see what is possible by way of theoretical analysis.
> —Kreisel (1970).

An *overcomplete* logic is a logic that 'ceases to make the difference': According to such a logic, all inferences hold independently of the nature of the statements involved. A *negation-inconsistent* logic is a logic having at least one model that satisfies both some statement and its negation. A *negation-incomplete* logic has at least one model according to which neither some statement nor its negation are satisfied. *Paraconsistent* logics are negation-inconsistent yet non-overcomplete; *paracomplete* logics are negation-incomplete yet non-overcomplete. A *paranormal* logic is simply a logic that is both paraconsistent and paracomplete.

Despite being perfectly consistent and complete with respect to classical negation, nearly every normal modal logic, in its ordinary language and interpretation, admits to some latent paranormality: It is paracomplete with respect to a negation defined as an impossibility operator, and paraconsistent with respect to a negation defined as non-necessity. In fact, as it will be shown here, even in languages without a primitive classical negation, normal modal logics can often be alternatively characterized directly by way of their paranormal negations and related operators. So, instead of talking about 'necessity', 'possibility', and so on, modal logics could be seen just as devices tailored for the study of (modal) negation. This paper shows how and to what extent this alternative characterization of modal logics can be realized.

1 Affirmative and negative modalities

In the course of the last hundred years or so, traditional modal logic was extraordinarily reinvigorated, at the outset with the firsthand assistance of symbolic logic, then by the successful development of both its algebraic and relational semantics. Of all adverbs which have been formalized with the help of modal languages, the most popular turned out to be a certain '$\Box$-like' modality with a universal character and its '$\Diamond$-like' existential dual, irrespective of their circumstantial readings —alethic, deontic, doxastic, temporal, etc— on each particular application field. The gate to possible worlds (and to some bad science fiction) was opened by the tacit assumption that the usual classical connectives should be interpreted locally, while $\Box$ and $\Diamond$ were supposed to have a global scope.

To be perfectly fair, not all modal semantics conform to the above pattern. The traditional modal interpretation of intuitionistic and intermediate logics, for example, as well as the ternary relations of relevance logics, end up with a global interpretation of both the implication and the negation connectives, all other connectives being interpreted classically and locally. Other modal logics go farther, and are themselves built over non-empty sets of non-classical worlds, be they many-valued, incomplete or even inconsistent. On the other hand, several other linguistic modal bases have also been tried at a few occasions. To mention just a particularly meaningful one, I recall the contingency / non-contingency logics explored by several authors since Montgomery and Routley (1966), trading $\Box$ and $\Diamond$ for the non-normal modal connectives $\triangledown$ and $\vartriangle$, with which the former are interdefinable only in the case of sufficiently convoluted classes of frames.

Traditional literature on modal logic such as Hughes and Cresswell (1968) has it that a 'modality' is just an arbitrary finite sequence of $\Box$'s, $\Diamond$'s and $\sim$'s, where $\sim$ is a symbol for classical negation. Aristotle had a picture of a 'Square of Oppositions' (SOO) involving negation and quantification. An analogous picture (see Figure 1) for the basic case involving modalities can be found in Łukasiewicz (1953) —and probably even earlier.

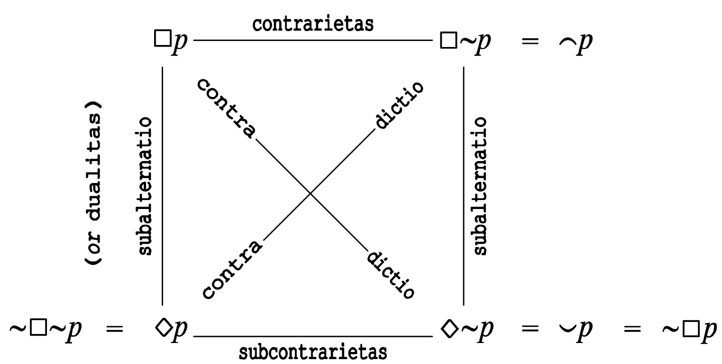

Figure 1: *Square of Modalities* (SOM)

The four modal corners from the above SOM were not really treated on an equal footing in the recent literature of modal logic. To be sure, that circumstance alone should not count against any of the modalities thereby contained, as no one still nowadays knows even what modal logic *is*, in general abstract terms. In a brilliant book originated from a frustrated attempt at such a definition, Segerberg (1982), p.128, the following comment can be found:

> Among the many possible operators that have never been proposed by anyone, there is one that should be mentioned here, the unary $\smile$, with $\smile\alpha$ bearing the intuitive reading 'it is not necessary that α' or 'α is non-necessary'. The concept of non-necessity does not appear to equal in intuitive significance that of impossibility, let alone those of necessity or possibility. But from a theoretical point of view, $\smile$ is on a par with $\frown$ as well as with $\Box$ and $\Diamond$. [the symbols for $\frown$ and $\smile$ are mine]

On that matter, according to Horn (1989), linguistic researches attest that, at least for pragmatic reasons, the bottom-right corners of both the SoO and the SoM seem not to have exact natural language equivalents in any of the world's living natural languages (but it should be noticed that this is no longer true if one considers artificially constructed languages such as *Lojban*, check Cowan (1997)). The noted asymmetry does not seem to have a convincing semantic explanation, and one can indeed find authors like Béziau in a series of papers culminating recently at Béziau (2004), preaching the study of the 'nameless corner of the square of oppositions and modalities' as an utterly intuitive enterprise. On what concerns the upper-right corner of the SoM, one should note that, alongside the classical connectives and a binary modality of strict implication, impossibility ($\frown$) was in fact the *only* primitive unary modality appearing in the cornerstone study that marked the contemporary revival of modal logic, the book of Lewis (1918).

In the philosophical literature (and only there!), modal logics are still often seen simply as the study of operators 'used to qualify the truth of a judgement' (check, for instance, Garson (2003)). Of course, such truth-qualifying operators can analogously be used to qualify falsehood, and if the left-hand side of the SoM can be seen as displaying operators that qualify *affirmation*, the right-hand side can similarly purport to display operators that qualify *negation*. But does that interpretation really make sense? Can $\frown$ and $\smile$ be seriously proposed as proxies for a negation operator? The answer is very often YES, but to understand that 'very often' it is useful to fix first some terminology.

1.1 Basic modal semantics

Consider the standard *language* (or *signature*) of classical propositional logic, with binary connectives for conjunction ($\wedge$), disjunction ($\vee$), implication ($\supset$), and a unary connective for negation ($\sim$). Let $\mathcal{S}_{\mathbf{CPL}}$ or $\mathcal{S}_{\wedge\vee\supset\sim}$ denote the set of *formulas* freely generated by a denumerable set of sentential variables, $\mathcal{P}$, over the above signature (the subscripts will be dropped when clear from the context). A *frame* here will be given by a non-empty set of *worlds*, $\mathcal{W}$, and a *model* over a given frame will be obtained by coupling it with a *(bi)valuation* $V : \mathcal{P} \times \mathcal{W} \to \{0, 1\}$. Valuations can be used to define a canonical notion of *satisfiability*, $\models_x^{\mathcal{M}} \subseteq \mathsf{Pow}(\mathcal{S}) \times \mathsf{Pow}(\mathcal{S})$, for each world x of a model $\mathcal{M}$, with the help of the following clauses that tell us how each connective should be understood:

$$\begin{array}{lll}
\models_x^{\mathcal{M}} p & \text{iff} & V(p, x) = 1, \text{ for } p \in \mathcal{P} \\
\models_x^{\mathcal{M}} \alpha \wedge \beta & \text{iff} & \models_x^{\mathcal{M}} \alpha \text{ and } \models_x^{\mathcal{M}} \beta \\
\models_x^{\mathcal{M}} \alpha \vee \beta & \text{iff} & \models_x^{\mathcal{M}} \alpha \text{ or } \models_x^{\mathcal{M}} \beta \\
\models_x^{\mathcal{M}} \alpha \supset \beta & \text{iff} & \models_x^{\mathcal{M}} \alpha \text{ implies } \models_x^{\mathcal{M}} \beta \\
\models_x^{\mathcal{M}} {\sim}\alpha & \text{iff} & \not\models_x^{\mathcal{M}} \alpha
\end{array}$$

To write $\not\models_x^{\mathcal{M}} \alpha$ is to say that $\models_x^{\mathcal{M}} \alpha$ does *not* hold. I will also denote that, alternatively, by writing $\alpha \models_x^{\mathcal{M}}$. In general, for a given world x of a model $\mathcal{M}$ of a given frame, I will assume that:

$$\Gamma \models_x^{\mathcal{M}} \Delta \;\text{ iff }\; (\exists\gamma \in \Gamma)\, \gamma \models_x^{\mathcal{M}} \;\text{ or } (\exists\delta \in \Delta) \models_x^{\mathcal{M}} \delta$$

The notion of a *valid inference* and the corresponding entailment (semantic global consequence relation) $\models_{\mathbf{CPL}} \subseteq \mathsf{Pow}(\mathcal{S}) \times \mathsf{Pow}(\mathcal{S})$ associated to classical propositional logic is fixed by setting $\Gamma \models_{\mathbf{CPL}} \Delta$ iff $\Gamma \models_x^{\mathcal{M}} \Delta$ for every world x of every model $\mathcal{M}$ of an arbitrary frame. Of course, in the case of **CPL**, the recourse to a set of worlds $\mathcal{W}$ does not help that much, as all the connectives of this logic are evaluated locally, that is, evaluated inside each (classical) world.

The expressive power of **CPL** is well-known: The logic has an adequate 2-valued functional semantics, and in fact every 2-valued n-ary truth-function can be written with the help of the above connectives. Some other particular connectives that are often used in the literature and that will be mentioned in the text below include the 0-ary connectives top ($\top$) and bottom ($\bot$), and the binary connectives for equivalence ($\equiv$) and coimplication ($\not\subset$, the 'dual' to implication in a precise sense to be specified in Subsection 2.1). Here is the intended interpretation of these connectives, together with some possible ways of defining them in terms of the connectives taken above as primitive or defined earlier on:

Definitions			*Characterizing properties*
$\alpha \not\subset \beta$	$\stackrel{\text{def}}{=}$	${\sim}\alpha \wedge \beta$	$\models_x^{\mathcal{M}} \alpha \not\subset \beta$ iff $\beta \supset \alpha \models_x^{\mathcal{M}}$
$\alpha \equiv \beta$	$\stackrel{\text{def}}{=}$	$(\alpha \supset \beta) \wedge {\sim}(\alpha \not\subset \beta)$	$\models_x^{\mathcal{M}} \alpha \equiv \beta$ iff $\models_x^{\mathcal{M}} \alpha \supset \beta$ and $\models_x^{\mathcal{M}} \beta \supset \alpha$
$\top$	$\stackrel{\text{def}}{=}$	$\alpha \supset \alpha$, for any α	$\models_x^{\mathcal{M}} \top$
$\bot$	$\stackrel{\text{def}}{=}$	$\alpha \not\subset \alpha$, for any α	$\bot \models_x^{\mathcal{M}}$

In the case of ordinary normal modal logics, I will consider again a frame based on non-empty set of classical worlds but now I will enrich it with an *accessibility* relation $\mathsf{R} \subseteq \mathcal{W} \times \mathcal{W}$ between the worlds, and read $x\mathsf{R}y$ as 'x sees y' or 'y is accessible to x'. A model based on such a frame, as before, will be assembled from a given valuation over the sentences and worlds, and a corresponding inductive definition of the interpretation for the whole set of formulas. This time the signature will contain two further unary connectives, box ($\Box$, often read as 'necessity') and diamond ($\Diamond$, often read as 'possibility'), and be denoted by $\mathcal{S}_{\mathbf{NML}}$ or $\mathcal{S}_{\wedge\vee\supset\sim\Box\Diamond}$. The interpretation of the new connectives is given by the following clauses (where $\Rightarrow$ substitutes 'implies' and & is used for 'and'):

$$\models_x^{\mathcal{M}} \Box\alpha \quad \text{iff} \quad (\forall y \in \mathcal{W})(x\mathsf{R}y \Rightarrow \models_y^{\mathcal{M}} \alpha)$$
$$\models_x^{\mathcal{M}} \Diamond\alpha \quad \text{iff} \quad (\exists y \in \mathcal{W})(x\mathsf{R}y \;\&\; \models_y^{\mathcal{M}} \alpha)$$

All other definitions are similar to the classical case. Several different modal logics (that is, several different relations of global entailment) can be defined in the above signature, according to the restrictions set over the accessibility relations in each case. In fact, when talking about a logic, from here on, I will always make sure that its set of formulas and an associated consequence relation are clearly defined, be it in proof-theoretical, in semantical or in abstract terms. The minimal normal modal logic, K, where NO restrictions are made over R, can be axiomatized by adding to any complete set of axioms and rules for **CPL** any of the three following sets of further axioms and rules:

(1.1) $\vdash \Box(\alpha \supset \beta) \supset (\Box\alpha \supset \Box\beta)$
(1.2) $\vdash \alpha \;\Rightarrow\; \vdash \Box\alpha$

(2.1) $\vdash \alpha_0 \wedge \ldots \wedge \alpha_n \supset \alpha \;\Rightarrow\; \vdash \Box\alpha_0 \wedge \ldots \wedge \Box\alpha_n \supset \Box\alpha$, where this rule reduces to (1.2) in case $n = 0$

(3.1) $\vdash \Box\top$
(3.2) $\vdash \Box\alpha \wedge \Box\beta \supset \Box(\alpha \wedge \beta)$
(3.3) $\vdash \alpha \supset \beta \;\Rightarrow\; \vdash \Box\alpha \supset \Box\beta$

(The axioms for $\Diamond$ are dual. For the purposes of this section, $\Diamond\alpha$ may be defined as $\sim\Box\sim\alpha$.)

The explicit definability of all 'admissible modal operators' from the basic modal language was investigated, for instance, in Wansing (1996), with respect to their associated 'proof-theoretic semantics'. Among the many new connectives that can now be defined in every **NML**, one could pinpoint contingency (∇) and non-contingency ($\triangle$), definable for instance by setting $\nabla\alpha \stackrel{\text{def}}{=} \Diamond\alpha \vee \Diamond\sim\alpha$ and $\triangle\alpha \stackrel{\text{def}}{=} \Diamond\alpha \supset \Box\alpha$, besides, of course, the two new modalities at the right-hand side of the SOM, $\frown$ and $\smile$, definable as in Figure 1.

1.2 Modal negations?

Some particular restrictions on the accessibility relation R will produce *degenerate* examples of modal logics. Call a world *autistic* in case there is no world accessible to it according to R, and call it *narcissistic* in case it can only see itself. The collection of all autistic frames (that is, frames whose worlds are all autistic) determines a logic called *Ver*, and can be axiomatized by the addition of the axiom $\vdash \Box\alpha$ to the axioms and rules of K. The collection of all narcissistic frames (that is, frames whose worlds are all narcissistic) determines a logic known as $Triv$, or $KT!$ as in Chellas (1980), and can be axiomatized by the addition to the axioms and rules of K of the axiom $\vdash \Box\alpha \equiv \alpha$. It is easy to see that both *Ver* and $Triv$ are but thin disguises for classical propositional logic: In the first, $\Box$ and $\Diamond$ are unary operators that produce tops, in the second, $\Box$ and $\Diamond$ behave like identity operators. The logic that I will call TV and that is situated exactly midway in between $Triv$ and *Ver* is also important in the present story. It is determined by the class of all frames that are either narcissistic or autistic, and axiomatized by the addition to K of the axiom $\vdash \alpha \supset \Box\alpha$.

In what follows it will be helpful to use $\circledcirc^n$ as an abbreviation for n iterations of a given unary connective $\circledcirc$. I will be saying that a logic $\mathcal{L}_2$ is a (deductive) *fragment* of a logic $\mathcal{L}_1$ (and $\mathcal{L}_1$ is an *extension* of $\mathcal{L}_2$) if $\mathcal{L}_1$ can be written in a signature containing all the symbols from the signature of $\mathcal{L}_2$ and if, in that case, all valid inferences of $\mathcal{L}_2$ are also valid in $\mathcal{L}_1$.

Makinson (1971) proved that every normal modal logic is a fragment of either *Ver* or $Triv$ (and possibly of both, that is, of TV). For instance, the modal logic KT, determined by the class of reflexive frames and axiomatized by the addition to K of the axiom $\vdash \Box\alpha \supset \alpha$, is only a fragment of $Triv$ but not of *Ver*; on the other hand, the logic of provability GL, determined by the class of transitive and reversely well-founded frames and axiomatized by the addition to K of the axiom $\vdash \Box(\Box\alpha \supset \alpha) \supset \Box\alpha$, is only a fragment of *Ver* but not of $Triv$; finally, $K5$, determined by the class of euclidean frames and axiomatized by the addition to K of the axiom $\vdash \Diamond\alpha \supset \Box\Diamond\alpha$, is a fragment of TV. More importantly, every extension of K obtained by the sole addition of axioms of the form $\vdash \Diamond^i\Box^j\alpha \supset \Box^k\Diamond^l\alpha$, for $i, j, k, l \in \mathbb{N}$, complete with respect to a convenient combination of the so-called confluential (Church-Rosser) frames, *is* a fragment of $Triv$ —and in fact, very few of the most widely known modal logics fail to be a fragment of $Triv$.

Can $\frown$ and $\smile$ be understood as 'negations' inside all of the above logics? For one thing, inside of *Ver* it seems already difficult to accept that reading: All formulas of the form $\frown\alpha$ and $\smile\alpha$ would be theorems of this logic... But what connectives are to count as 'negations', to start with? First of all, it must be cleared up that there is NO general —nor even partial— agreement in the literature on an answer to that. As we will see, this is not to say, however, that the very concept of negation is unruly!

Consider from this point on a (non-overcomplete)[1] logic $\mathcal{L}_1$ endowed with some symbol $\neg$ intended to denote 'negation'. Even if we consider no other circumstantial symbols from $\mathcal{L}_1$'s signature and its corresponding set of formulas $\mathcal{S}_1$, there is a number of *pure* positive meta-rules that might be considered to govern the behavior of negation with respect to $\Vdash_1$, the consequence relation associated to $\mathcal{L}_1$. For instance, the following two rules can fully characterize classical negation inside a non-overcomplete logic:

(Explosion) $(\forall\Gamma,\Delta\subseteq\mathcal{S}_1)(\forall\alpha\in\mathcal{S}_1)\ \Gamma,\alpha,\neg\alpha\Vdash_1\Delta$
(Implosion) $(\forall\Gamma,\Delta\subseteq\mathcal{S}_1)(\forall\alpha\in\mathcal{S}_1)\ \Gamma\Vdash_1\alpha,\neg\alpha,\Delta$

Any non-classical negation will have to fail one of the above rules, and possibly both. In that case, what are the 'stable' rules of negation, if any, i.e. the rules that *every* negation ought to obey? This is the very issue about which each author will have his preferred answer, and it seems that there is little hope for any sort of agreement to be expected to settle around that. However, there is some possibility of agreement, I submit, if one only turns the attention to a small set of pure *negative* rules, such as:

(*n-verificatio*) $(\exists\Gamma,\Delta\subseteq\mathcal{S}_1)(\exists\alpha\in\mathcal{S}_1)\ \Gamma,\neg^{n+1}\alpha\nVdash_1\neg^{n}\alpha,\Delta$
(*n-falsificatio*) $(\exists\Gamma,\Delta\subseteq\mathcal{S}_1)(\exists\alpha\in\mathcal{S}_1)\ \Gamma,\neg^{n}\alpha\nVdash_1\neg^{n+1}\alpha,\Delta$

In the present environment, the above rules have at least 3 immediate pleasant consequences for the behavior of $\neg^{n+1}$ over $\neg^{n}$: If $\neg^{n+1}$ is to obey those rules, it cannot produce only bottoms, it cannot produce only tops, and it cannot be an identity operator. Seems sensible enough: Is anyone prepared to accept or propose as a 'real negation' any symbol failing the above rules? On the one hand, those rules are sufficient to confirm already our intuition that the logic *Ver* should be ruled out as a system interpreting $\frown$ and $\smile$ as negation operators. What will we be able to say, however, about its fragments? On the other hand, the last rules are clearly respected by classical negation, and thus also by $\frown$ and $\smile$ inside the logic $Triv$. With that criterion in mind, from here on, I will assume, as in Marcos (2005d), that a *decent* negation should respect (*n-verificatio*) and (*n-falsificatio*), for all $n\in\mathbb{N}$.

Consider now a fragment $\mathcal{L}_2$ of $\mathcal{L}_1$, such that $\mathcal{L}_2$ is directly embeddable in $\mathcal{L}_1$ by way of an identity translation, that is, $\Vdash_2\subseteq\Vdash_1$, where $\Vdash_2$ is the consequence relation associated to $\mathcal{L}_2$. In case the signature of $\mathcal{L}_2$ also contains $\neg$ then it is clear that $\neg$ will in $\mathcal{L}_2$ respect at most as many positive rules as it did in the case of $\mathcal{L}_1$, never more. One might say in that case that $\neg$ in $\mathcal{L}_2$ is *sub-*$\mathcal{L}_1$; if $\mathcal{L}_1$ is classical logic one might simply say that $\neg$ in $\mathcal{L}_2$ is *subclassical*. So, now one can at least ask the question: In which normal modal logics the operators $\frown$ and $\smile$ produce subclassical operators? It is not difficult to check for instance that GL is not one of such logics: As shown in Vakarelov (1989), the characterizing axiom of GL can be rewritten

[1] For a semantic account of that concept, check Section 2.

in terms of $\frown$ as $\vdash \frown(\alpha \wedge \frown\alpha) \supset \frown\alpha$, and this is not a valid formula in **CPL**. One can count though on the following straightforward answer to the above question:

The operators $\frown$ and $\smile$ constitute subclassical negations inside a given normal modal logic if and only if this logic is a fragment of $Triv$.
Indeed, we already know that $\frown$ and $\smile$ coincide with classical negation inside $Triv$. As a consequence, those symbols define decent subclassical negations, *a fortiori*, also in the fragments of $Triv$. On the other hand, a logic with a subclassical negation is by definition a fragment of classical logic, as long as both logics are written in the same language. But recall that $Triv$ *is* classical logic in disguise, possibly with some extra boxes and diamonds coloring its inferences but behaving just like identity operators. This proves our case. (Alternatively, suppose that you erase the boxes and diamonds from any normal modal logic that is *not* a fragment of $Triv$. Then you clearly transform $\frown$ and $\smile$, taken to be defined as in Figure 1, into non-subclassical negations.) QED

Still and all, the reader should not imagine that all decent negations are subclassical. Post's cyclic many-valued negations, for instance, are counterexamples to that. This paper will concentrate, in one way or another, exclusively on the more usual subclassical negations.

The next sections will show which properties *are* enjoyed by $\frown$ and $\smile$, and to what classes of negations they belong to. It will also show how normal modal logics can be naturally reconstructed on other signatures based on $\frown$, $\smile$ and related connectives.

2 Varieties of paranormality

For the sake of the following discussion, let $\mathcal{L}$ be an arbitrary logic with an entailment relation $\models$ (recall Section 1.1) defined over a set of formulas $\mathcal{S}$ of a language that contains a negation symbol $\neg$ with a decent interpretation (that is, respecting rules *verificatio* and *falsificatio* from the last section). For all we know, such logic might turn out to have some queer models, such as:

(Dadaistic) $(\forall\alpha \in \mathcal{S})(\forall x \in \mathcal{W})\ \models^{\mathcal{M}}_{x} \alpha$
(Nihilistic) $(\forall\alpha \in \mathcal{S})(\forall x \in \mathcal{W})\ \alpha \models^{\mathcal{M}}_{x}$

(To simplify notation, I will from this section on drop the contexts Γ's and Δ's from the inferences.) From the above definitions, everything is true for a dadaistic model, and everything is false for a nihilistic model. Following Marcos (2005d), I will say that the logic $\mathcal{L}$ is *overcomplete* in case all of its models are either dadaistic or nihilistic. Thus, for a non-overcomplete logic, $(\exists\alpha, \beta \in \mathcal{S})\ \alpha \not\models \beta$. Now, even in the case of such a logic, it might still happen that negation has some funny models such as:

(¬-inconsistent) $\quad (\exists\alpha \in \mathcal{S})(\exists x \in \mathcal{W})\ \models^{\mathcal{M}}_{x} \alpha$ and $\models^{\mathcal{M}}_{x} \neg\alpha$

(¬-incomplete, or ¬-undetermined) $\quad (\exists\alpha \in \mathcal{S})(\exists x \in \mathcal{W})\ \alpha \models^{\mathcal{M}}_{x}$ and $\neg\alpha \models^{\mathcal{M}}_{x}$

So, a ¬-inconsistent model allows for some formula to be satisfied together with its negation, and a ¬-undetermined model allows instead for both formulas to be non-satisfied. Obviously, a dadaistic model is simply an extreme case of an inconsistent model, and a nihilistic model an extreme case of an undetermined model. In the present framework, and following Marcos (2005b), $\mathcal{L}$ will be called a *decent ¬-paraconsistent* logic if it allows for non-dadaistic ¬-inconsistent models, that is, if $(\exists\alpha, \beta \in \mathcal{S})\ \alpha, \neg\alpha \not\models \beta$. Dually, $\mathcal{L}$ will be called a *decent ¬-paracomplete* logic if it allows for non-nihilistic ¬-undetermined models, that is, if $(\exists\alpha, \beta \in \mathcal{S})\ \beta \not\models \alpha, \neg\alpha$. In particular, a paraconsistent logic will be non-explosive, and a paracomplete logic will be non-implosive (recall the definitions of those properties from Section 1.2). Following da Costa and Béziau (1997) and Béziau (1999), I will call $\mathcal{L}$ *paranormal* if it is both paraconsistent and paracomplete.

Paranormality comes in several brands. Explosion or implosion might be lost, but maybe it is possible to recover them, 'with gentleness and time'. Maybe there is something that we can *say* about a formula so as to guarantee that it behaves consistently / determinedly? Here is a way of realizing this intuition. Let $\underline{\overline{\bigcirc}}(p)$ be a (possibly empty) set of formulas on one single variable such that:

$$(\exists\alpha \in \mathcal{S})\ \ \underline{\overline{\bigcirc}}(\alpha), \alpha \not\models \ \text{ and }\ \underline{\overline{\bigcirc}}(\alpha), \neg\alpha \not\models,$$

and yet

$$(\forall\alpha \in \mathcal{S})\ \ \underline{\overline{\bigcirc}}(\alpha), \alpha, \neg\alpha \models$$

Following Carnielli and Marcos (2002), any logic containing such a schema of formulas is called *¬-gently explosive*. A *logic of formal inconsistency* (**LFI**) is a paraconsistent yet gently explosive logic. In such a logic, $\underline{\overline{\bigcirc}}$ is said to express *¬-consistency*.

Similarly, let $\underline{\overline{\star}}(p)$ be a (possibly empty) set of formulas on one single variable such that:

$$(\exists\alpha \in \mathcal{S})\ \not\models \alpha, \underline{\overline{\star}}(\alpha)\ \text{ and }\ \not\models \neg\alpha, \underline{\overline{\star}}(\alpha),$$

and yet

$$(\forall\alpha \in \mathcal{S})\ \models \neg\alpha, \alpha, \underline{\overline{\star}}(\alpha)$$

Any logic containing such a schema of formulas is called *¬-gently implosive*. A *logic of formal undeterminedness* (**LFU**) is a paracomplete yet gently implosive logic. In such a logic, $\underline{\overline{\star}}$ is said to express *¬-determinedness*, or *¬-completeness*.

The following lines are very rough, but should suffice to inform the reader about what **LFI**s and **LFU**s are good for. As the reader might have suspected, $\neg$-consistency and $\neg$-determinedness in paranormal logics serve as sorts of 'normalizing connectives'. In fact, I will from here on call them 'perfect'. From the original meaning of the word, in Latin, we know that something is *perfect* when it is 'done to the end', when it is somehow 'complete', and 'nothing essential is lacking'. In case a logic has a negation lacking the 'consistency presupposition', if one adds to it the power to express consistency then one can somehow recover what had been lost: Consistency in this case is the sought perfection. To put it in a different and semi-formal way, consider a logic $\mathcal{L}_1$ in which explosion holds good for a decent negation $\neg$, that is, a logic that validates, in particular, $(\forall\alpha \in \mathcal{S}_1)\ \alpha, \neg\alpha \vDash_1$. Let $\mathcal{L}_2$ now be some other logic written in the same signature as $\mathcal{L}_1$ such that: (i) $\mathcal{L}_2$ is a proper fragment of $\mathcal{L}_1$ that validates many or most inferences of $\mathcal{L}_1$ that are compatible with the *failure* of explosion; (ii) $\mathcal{L}_2$ is *expressive* enough so as to be an **LFI**, thus, in particular, there will be in $\mathcal{L}_2$ a set of formulas $\overline{\underline{\bigcirc}}(p)$ such that $(\forall\alpha \in \mathcal{S}_2)\ \overline{\underline{\bigcirc}}(\alpha), \alpha, \neg\alpha \vDash_2$ holds good; (iii) $\mathcal{L}_1$ can in fact be *recovered* from $\mathcal{L}_2$ by the addition of $\overline{\underline{\bigcirc}}(p)$ as a new set of valid schemas / axioms. These constraints alone suggest that the reasoning of $\mathcal{L}_1$ might somehow be recovered from inside $\mathcal{L}_2$, if only a sufficient number of 'consistency assumptions' are added in each case. Thus, typically the following *derivability adjustment theorem* (**DAT**) can be proved:

$$(\forall\Gamma\forall\Delta\exists\Sigma)\ \ \Gamma \vDash_1 \Delta \ \text{ iff } \ \overline{\underline{\bigcirc}}(\Sigma), \Gamma \vDash_2 \Delta.$$

The essentials behind such sort of **DAT**s were highlighted in Batens (1989), but some very specific instances of **DAT**s could already be found in one of the forerunning formal studies on paraconsistent logic, da Costa (1963). It is no exaggeration to say that such theorems constitute the fundamental idea behind both the 'Brazilian approach' to paraconsistency (**C**-systems and **LFI**s) and the 'Belgian approach' (inconsistency-adaptive logics). As I see it, the main difference between the two approaches is in fact methodological (but also a bit ideological). As I argued in Marcos (2001), while retaining in a paraconsistent logic 'most rules and schemata of classical logic' was a desideratum laid down already in da Costa (1974), it was never really systematically pursued by the 'Brazilian school'. The approach favored by Batens (1989) and the 'Belgian school', in contrast, took the motto to the letter: Assuming consistency *by default*, maximality is pursued by way of allowing for non-monotonic reasoning to take place. Another remarkable peculiarity is that in an **LFI**, by its very design, the clauses in the above theorem can in fact be *internalized* at the object-level language, making its statement more convenient and language-independent. Sometimes, moreover, there are yet other ways of reproducing classical reasoning inside an **LFI** through a direct translation, without the addition of further premises.

For its importance, I will dub the ability of recovering consistent reasoning in one way or another the **Fundamental Feature of LFIs**.

Clearly, all that was said for consistency and **LFI**s in the previous paragraph can be easily dualized for determinedness and **LFU**s.

I will from here on consider only the simpler case in which $\overline{\underline{\bigcirc}}$ reduces to a single schema, thus to a consistency connective $\bigcirc$ whose contradictory opposite (its classical negation), will represent an inconsistency connective to be denoted by $\bullet$. A similar thing will be done for $\overline{\underline{\star}}$, in that I will be working from here on more simply with a unary determinedness connective ☆ and the accompanying undeterminedness connective ★.

2.1 Duality, at last

I have been mentioning *duality* all along, with a strong semantic intuition, but in a very loose way. Let me here make a short digression to explain precisely what that is supposed to mean.

Given an arbitrary connective $\circledcirc$, let its *dual* be denoted by $\circledcirc^d$. Given a set of formulas Γ, let Γ^d denote the result of substituting all connectives of Γ by their duals. Given a logic $\mathcal{L}_1$ with a consequence relation $\Vdash_1$ over a set of formulas $\mathcal{S}_1$, the dual logic $\mathcal{L}_2$ will be defined by setting $\mathcal{S}_2 = \mathcal{S}_1^d$ and $\Gamma^d \Vdash_2 \Delta^d$ iff $\Delta \Vdash_1 \Gamma$. So, semantically, all we have to do, somehow, is to read the original inferences from right to left, instead of reading them from left to right, and change the names of the logical constants whenever necessary (some connectives can of course be self-dual inside a given logic).

This little trick is just enough for conjunction to be characterizable as dual to disjunction (even more, each elimination rule for conjunction will be dual to a corresponding introduction rule for disjunction, and so on), implication as dual to coimplication (and this coincides in fact with the algebraic intuition about duality explored already in Rauszer (7374)), box as dual to diamond, explosive negation as dual to implosive negation, (para)consistency as dual to (para)completeness, **LFI**s as dual to **LFU**s, dadaism as dual to nihilism, and so on.

The place where duality will show up in the Square of Modalities (Figure 1) is in place of the relation of 'subalternation'. According to the traditional semantic intuition behind subalternation, the truth of each upper corner implies the truth of the corresponding bottom corner, but not the other way around. The application of this simple idea is not without problems: The subalternation in the SoO only works well once you grant existential import to the universal quantifier, the subalternation in the SoM works fine only if you are talking about normal modal logics extending *KD*, the 'deontic' system with the seriality presupposition (in which $\Diamond\top$ is provable). The above definitions of duality, however, suggest a full horizontal symmetry in the very same square, allowing for the mentioned provisos to be dispensed

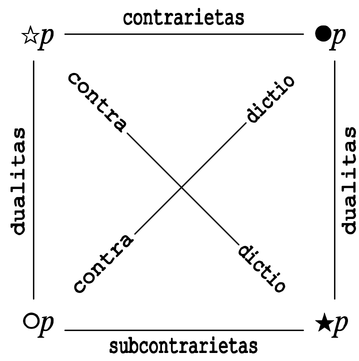

Figure 2: *Square Of Perfections* (SOP)

with. With that in mind, it does not really seem illuminating thus to think of diamond as subalternate to box (nor the other way around). That's why I proposed from the start the update of the (SOM) with the denomination '*dualitas*' in the place of '*subalternatio*'. Now, Figure 2 shows how the square would look like if rebuilt so as to apply to the perfect connectives introduced in Section 2. Notice that, according to the traditional semantic intuition of the square, ☆p and ●p are 'contrary' (they cannot both be simultaneously true), ○p and ★p are 'subcontrary' (they cannot both be simultaneously false).

2.2 The route from modality to paranormality, and the easy way back

Where $\mathbb{K}$ is some class of frames and sig is some propositional signature, let $(\mathcal{L})_{\mathsf{sig}}$ denote the logic whose set of formulas is $\mathcal{S}_{\mathsf{sig}}$ and whose set of valid inferences is determined with the help of the canonical interpretation of the connectives in sig. With this abbreviation, every normal modal logic $\mathcal{L}$, in its usual language with set of formulas $\mathcal{S}_{\mathbf{NML}}$, will here be denoted as $(\mathcal{L})_{\wedge\vee\supset\sim\Box\Diamond}$.

We already know from the above that the usual language of normal modal logics is expressive enough so as to be able to define a decent paraconsistent negation $\smile$ and a decent paracomplete negation $\frown$. It is not difficult now to see how the corresponding perfect connectives for consistency and inconsistency (○ and ●), and for determinedness and undeterminedness (☆ and ★) can also be produced. Of course, those connectives will only have their expected behavior under specific circumstances. Consider some normal modal logic $\mathcal{L}$ not extending TV (recall Section 1.2). Then, here is a possible set of definitions for the above connectives and the properties they should have in $\mathcal{L}$:

Definitions			*Properties enjoyed by them*
$\smile\alpha$	$\stackrel{\text{def}}{=}$	$\Diamond{\sim}\alpha$	$p, \smile p \not\models q$
$\bigcirc\alpha$	$\stackrel{\text{def}}{=}$	$\alpha \supset \Box\alpha$	$\bigcirc p, p \not\models$ and $\bigcirc p, \smile p \not\models$, and yet $\bigcirc p, p, \smile p \models$
$\bullet\alpha$	$\stackrel{\text{def}}{=}$	$\alpha \wedge \smile\alpha$	$\models \bullet\alpha$ iff $\bigcirc\alpha \models$, $\bullet\alpha \models \alpha$ and $\bullet\alpha \models \smile\alpha$
$\frown\alpha$	$\stackrel{\text{def}}{=}$	$\Box{\sim}\alpha$	$q \not\models \frown p, p$
$\star\alpha$	$\stackrel{\text{def}}{=}$	$\alpha \not\subset \Diamond\alpha$	$\not\models p, \star p$ and $\not\models \frown p, \star p$, and yet $\models \frown p, p, \star p$
$\bigstar\alpha$	$\stackrel{\text{def}}{=}$	$\frown\alpha \vee \alpha$	$\bigstar\alpha \models$ iff $\models \star\alpha$, $\alpha \models \bigstar\alpha$ and $\frown\alpha \models \bigstar\alpha$

Indeed, as a consequence of the above definitions:

A necessary and sufficient condition for
$(\mathcal{L})_{\wedge\vee\supset\smile\bigcirc\bullet}$ to characterize a modal LFI, and for
$(\mathcal{L})_{\wedge\vee\supset\frown\star\bigstar}$ to characterize a modal LFU
is that $\mathcal{L}$ does not extend TV.

It is obvious that the condition is necessary. Indeed, if $\mathcal{L}$ is TV, $Triv$ or Ver, then it is not paranormal with respect to the new connectives above. Conversely, to show that this restriction provides a sufficient condition to verify the expected properties of the new connectives, consider first the case of $\smile$ and $\bigcirc$, and define a model $\mathcal{M}_1$ such that $\mathcal{W} = \{x, y\}$, $V(p, x) = 1$, $V(p, y) = 0$ and $V(q, x) = 1$, and any R such that $(x, y) \in \mathsf{R} \subseteq \mathcal{W} \times \mathcal{W}$. Such models are always possible in logics that do not extend TV, and all you have to do is to vary the accessibility relation according to the strictures of each class of frames. But then, $p, \smile p \not\models_x^{\mathcal{M}_1} q$. Next, consider any model $\mathcal{M}_2$ based on a frame such that $\mathcal{W} = \{x\}$, $V(p, x) = 1$. Then, $\bigcirc p, p \not\models_x^{\mathcal{M}_2}$, once $\Box$ is not an operator producing only bottoms —and we know that it is not, from rule (1.2) or axiom (3.1) (recall Section 1.1). Finally, consider a model $\mathcal{M}_3$ exactly like $\mathcal{M}_1$, except that now $V(p, x) = 0$. In this model $\bigcirc p, \smile p \not\models_x^{\mathcal{M}_3}$, for every logic distinct from Ver. It is clear, moreover, that $\bigcirc p, p, \smile p \models$ for any normal modal logic.
The case of $\frown$ and $\star$ is similar. QED

Now, what if we start from a paranormal language and try to define the usual connectives of normal modal logics? Can that be done at all? Again, the answer is very often YES, but, as we will see below, to understand that 'very often' one had better pay a lot of attention to the initial choice of the language.

Consider first the connectives $\smile$, $\bigcirc$, $\frown$ and $\star$ to be primitively defined by the clauses:

$$\models_x^{\mathcal{M}} \smile\alpha \quad \text{iff} \quad (\exists y \in \mathcal{W})(x\mathsf{R}y \;\&\; \alpha \models_y^{\mathcal{M}})$$

$$\models_x^{\mathcal{M}} \bigcirc\alpha \quad \text{iff} \quad \models_x^{\mathcal{M}} \alpha \text{ implies } (\forall y \in \mathcal{W})(x\mathsf{R}y \Rightarrow \models_y^{\mathcal{M}} \alpha)$$

$$\models^{\mathcal{M}}_x \frown\alpha \quad \text{iff} \quad (\forall y \in \mathcal{W})(x\mathsf{R}y \Rightarrow \alpha \models^{\mathcal{M}}_y)$$

$$\models^{\mathcal{M}}_x \text{☆}\alpha \quad \text{iff} \quad \alpha \models^{\mathcal{M}}_x \text{ and } (\exists y \in \mathcal{W})(x\mathsf{R}y \;\&\; \models^{\mathcal{M}}_y \alpha)$$

Consider next an arbitrary normal modal logic $(\mathcal{L})_{\wedge\vee\supset\smile\bigcirc}$, where the non-classical connectives from the signature are interpreted as above. The question now is whether $(\mathcal{L})_{\wedge\vee\supset\sim\Box\Diamond}$ can be recovered from that. And the answer is that it can, if only the following definitions are set:

$$\bot \stackrel{\text{def}}{=} \alpha \wedge \smile\alpha \wedge \bigcirc\alpha, \text{ for any } \alpha \qquad \Box\alpha \stackrel{\text{def}}{=} \sim\smile\alpha$$

$$\sim\alpha \stackrel{\text{def}}{=} \alpha \supset \bot \qquad \Diamond\alpha \stackrel{\text{def}}{=} \smile\sim\alpha$$

Furthermore, to obtain an inconsistency connective one can obviously just set $\bullet\alpha \stackrel{\text{def}}{=} \sim\bigcirc\alpha$. It is not difficult to check, indeed, that even inside the minimal normal modal logic K the new connectives $\sim$, $\Box$ and $\Diamond$ behave exactly as they should. For instance, in K the following rules hold good: $(\alpha, \sim\alpha \models)$ and $(\models \sim\alpha, \alpha)$. As we know, those two rules fully characterize classical negation (recall Section 1.2). Therefore:

For every normal modal logic, $(\mathcal{L})_{\wedge\vee\supset\sim\Box\Diamond}$ and $(\mathcal{L})_{\wedge\vee\supset\smile\bigcirc\bullet}$ characterize the same logic under two different signatures.

Can the same be done if one starts from the language containing $\frown$ and ☆ instead of $\smile$ and $\bigcirc$? The answer now is not as immediate as one might expect. Indeed, consider an arbitrary modal logic $(\mathcal{L})_{\wedge\vee\supset\frown\text{☆}}$, where the non-classical connectives are interpreted as above. How can a classical negation now be defined so as to work as expected for all classes of frames? It is easy to see that the above definitions will not do. An alternative is to set:

$$\sim\alpha \stackrel{\text{def}}{=} \alpha \supset \frown\alpha \qquad \Box\alpha \stackrel{\text{def}}{=} \frown\sim\alpha \qquad \Diamond\alpha \stackrel{\text{def}}{=} \sim\frown\alpha$$

In this case, however, in spite of $(\models \sim\alpha, \alpha)$ holding good for every normal modal logic, $(\alpha, \sim\alpha \models)$ holds good only for extensions of KT. Therefore, all one can guarantee in general is that:

For every extension of KT, $(\mathcal{L})_{\wedge\vee\supset\sim\Box\Diamond}$ and $(\mathcal{L})_{\wedge\vee\supset\frown\text{☆}\bigstar}$ characterize the same logic under two different signatures.

To recover full generality and symmetry in the second result, the easiest solution is to change implication for coimplication (putting *both* implication and coimplication in the signature is too easy a solution, as those two connectives alone already provide a functionally complete set of connectives for classical logic). So, using the coimplication alone one can set:

$$\top \stackrel{\text{def}}{=} \text{☆}\alpha \vee \frown\alpha \vee \alpha, \text{ for any } \alpha \qquad \sim\alpha \stackrel{\text{def}}{=} \alpha \not\subset \top$$

This new negation behaves classically already in K, and with its help one can define box and diamond, again, exactly as in the preceding set of definitions. Obviously, a connective for undeterminedness can be defined by setting $\bigstar\alpha \stackrel{\text{def}}{=} \sim\text{☆}\alpha$. The last paragraph shows that:

For every normal modal logic, $(\mathcal{L})_{\wedge\vee\not\subset\sim\Box\Diamond}$ and $(\mathcal{L})_{\wedge\vee\not\subset\frown\text{☆}\bigstar}$ characterize the same logic under two different signatures.

3 Imagine there are no sea battles...

I argued in Marcos (2005d) that the development of a really good theory about 'what negation is', in logic, presupposes the previous development of a modern and comprehensive formal version of the received *theory of oppositions*.[2] This was nothing short than a big issue in ancient Greek philosophy. Even nowadays, though, if one looks in retrospect, it is difficult to get a feeling that the deep philosophical advances made on this topic have received the formal counterpart they deserved. If we are to trust Plato on his account of the pre-Socratic philosophy, Heraclitus of Ephesus has seemingly spent his whole life thinking about opposition, and Parmenides spent his own thinking about how he could oppose Heraclitus on that. The dispute was allegedly also fed by their respective disciples, Cratylus and Zeno of Elea. It has often been argued that Aristotle's theory of opposition, and the Square of Oppositions that would be polished from it along the following centuries,[3] was born from an attempt to reconcile the opponents and make sense of the above dispute. A sympathizer of Heraclitus (whom he dubbed 'the Obscure') in some respects and a strong critic in many others, Aristotle seems also to have been the first (later, Apuleius, Boethius and Peter of Spain were also not entirely without fault) to pervert the initial idea of a theory of oppositions into a long and problematic theory of modal syllogisms.

In the last section we have seen how the language of normal modal logic could have been alternatively chosen as the language of paranormal negations and related operators. Maybe, had Aristotle not been the tutor of Alexander, there would never have been so much talk about sea battles, the contingency of the future and the necessity of the past. Had modal logic and kripke-like semantics been developed with the objective of understanding negation and exploring the viability of reasoning under inconsistent situations, and maybe the reader would have been surprised to learn only here and now that YES, the same modal ideas and tools could be used to talk about boxes and diamonds!

The negative modalities $\smile$ and $\frown$ have received some attention in the last decades as legitimate interpretations of negation. From this point on, let $\rightarrow$ and $-$ denote intuitionistic implication and negation. In Došen (1984) and subsequent papers, Kosta Došen showed how to axiomatize the logics $(\mathcal{L})_{\wedge\vee\rightarrow-\frown}$ and $(\mathcal{L})_{\wedge\vee\rightarrow-\smile}$, for $\mathcal{L} = K$ and for many extensions of K. Those logics were treated as bi-modal, with one accessibility relation (reflexive and transitive) used to interpret the intuitionistic connectives and another accessibility relation (that of $\mathcal{L}$) used to interpret $\frown$ and $\smile$. A similar approach had in fact been undertaken a decade earlier by Dimiter Vakarelov, and was

[2] In particular, as argued in Section 2.1, it could be advantageous in such a theory to talk about 'duality' instead of 'subalternation'.

[3] For the historical development of the SOO, check Parsons (2004).

published in Vakarelov (1989), where the logics $(\mathcal{L})_{\wedge\vee\to\frown\top\bot}$ and $(\mathcal{L})_{\wedge\vee\to\smile\top\bot}$ were axiomatized, for $\mathcal{L} = K$ and for many extensions of K, and also for signatures containing classical instead of intuitionistic implication.

An interesting problem that was left open was that of axiomatizing such logics in the language containing only the usual positive classical connectives of normal modal logics $(\wedge, \vee, \supset, \Box, \Diamond)$, extended only by the paranormal negations $\frown$ or $\smile$, without recourse to the perfect connectives $(\bigcirc, \bullet, \star, \bigstar)$, as above. Consider the paraconsistent case and the set of formulas $\mathcal{S}_{\wedge\vee\supset\smile}$. (Recall that the case where the related signature is extended by the addition of the connective $\bigcirc$ was fully solved above, where the logics obtained were shown to provide just different versions of the usual normal modal logics.) Suppose someone might object to the addition of the connective $\bigcirc$ as a 'natural connective' of our logics. This person then should take equal care so as not to add neither a bottom, $\bot$, nor a classical negation, $\sim$, to the original signature: On the one hand, we have already seen how $\sim$ and $\bot$ can be defined from $\bigcirc$; on the other hand, from a primitive $\sim$ one could easily define $\bot \overset{\text{def}}{=} \alpha \wedge \sim\alpha$, for an arbitrary α, and from a primitive $\bot$ one could define $\sim\alpha \overset{\text{def}}{=} \alpha \supset \bot$, and in both cases $\bigcirc$ could be recovered by setting $\bigcirc\alpha \overset{\text{def}}{=} (\alpha \supset \bot) \vee (\smile\alpha \supset \bot)$. Notice also that, whenever a classical negation $\sim$ is present, the consistency connective $\bigcirc$ will be sufficient so as to define the remaining perfect connectives from Figure 2: Just set $\bigstar\alpha \overset{\text{def}}{=} \bigcirc\sim\alpha$, $\star\alpha \overset{\text{def}}{=} \sim\bigcirc\sim\alpha$, and $\bullet\alpha \overset{\text{def}}{=} \sim\bigcirc\alpha$.

On what concerns the above problem, vividly denounced in Béziau (2002) for the case of $S5$, an axiomatization of $(\mathcal{L})_{\wedge\vee\supset\smile}$ was offered in Béziau (9798) only for that extreme case in which $\mathcal{L} = S5$. As Jean-Yves Béziau confessed, the extension of this result to the case of other normal modal logics proved non-obvious. I have recently found a thorough solution to the problem, but for limitations of space I can only display here the corresponding axioms. For the case of $\mathcal{L} = K$, an adequate axiomatization is given by adding to any complete set of axioms and rules for positive classical propositional logic the following further axioms and rules:

(I.1) $\vdash \alpha \supset \beta \;\Rightarrow\; \vdash \smile\beta \supset \smile\alpha$
(I.2) $\vdash \alpha \;\Rightarrow\; \vdash \smile\alpha \supset \beta$
(I.3) $\vdash \smile(\alpha \wedge \beta) \supset (\smile\alpha \vee \smile\beta)$

It is not difficult to extend this axiomatization so as to cover other logics. Indeed, for $\mathcal{L} = KT$ you just have to add $\vdash \alpha \vee \smile\alpha$ as a new axiom, for $\mathcal{L} = KB$ it suffices to add $\vdash \smile\smile\alpha \supset \alpha$ as a new axiom, for $\mathcal{L} = K5$ the axiom $\vdash \smile\alpha \supset (\smile\smile\alpha \supset \beta)$ will do. In fact, and here comes the GREAT SURPRISE, again it is possible to recover all normal modal logics from this simpler signature, if we now define classical negation by setting $\sim\gamma \overset{\text{def}}{=} \gamma \supset \smile(\alpha \supset \alpha)$, for an arbitrary formula α. So:

For every normal modal logic, $(\mathcal{L})_{\wedge\vee\supset\sim\Box\Diamond}$ and $(\mathcal{L})_{\wedge\vee\supset\smile}$ characterize the same logic under two different signatures.

The paracomplete case is a bit more complicated (recall the need we had for a coimplication in Section 2.2), as it can be proved that there is *no* definable classical negation in $(K)_{\wedge\vee\supset\frown}$, but only in $(KT)_{\wedge\vee\supset\frown}$. But there *is* a classical negation in $(K)_{\wedge\vee\not\subset\frown}$. The difficulties and details of the above mentioned solutions are to be found in Marcos (2005a).

It should be highlighted that one of the most remarkable features of all the above mentioned paranormal logics is the validity of the *replacement property* (a.k.a. *self-extensionality*). A very common and desirable property of logical systems, and a typical property of the usual systems of normal modal logic, replacement is known to fail in the great majority of well-known systems of paraconsistent logic, and that often translates into trouble for the study of their algebraic counterparts (check for instance the section 3.12 of the survey paper Carnielli and Marcos (2002)). The above modal paraconsistent logics, by their very nature, shun such difficulties.

One last comment. I have hinted above to the reticence that is sometimes to be found about the use of consistency connectives and **LFI**s, notwithstanding the possibility they inaugurate of internalizing nice properties such as the **DAT**s (recall Section 2). I have also mentioned the unavoidability of such connectives as soon as we are talking about positive classical propositional logic extended by some paraconsistent negation and by either a classical negation or a bottom. But the question might still remain as to whether that consistency connective makes any sense if there is no paraconsistent negation around. Let us assume the above modal interpretation of this consistency connective and of the related inconsistency connective to be taken as primitive, and let us conservatively extend classical propositional logic by the addition of such connectives. It is not difficult to see that the resulting language has little expressive power: No diamonds nor boxes can in general be defined, and the new connectives are not even 'normal' modal connectives in the sense of the former. In the language whose formulas are $\mathcal{S}_{\wedge\vee\supset\equiv\sim\circ\bullet}$, however, one could read $\bullet\alpha$ as saying that 'α is the case, but could have been otherwise': It works as a kind of (local) connective for 'accidental truth'. Similarly, $\circ$ could be read as expressing a (local) notion of 'essential truth'. In Marcos (2005c) I have axiomatized the minimal such logic of essence and accident, $(K)_{\wedge\vee\supset\equiv\sim\circ\bullet}$, by extending positive classical propositional logic with the following axioms and rules:

(K0.1) $\vdash \varphi \equiv \psi \;\Rightarrow\; \vdash \circ\varphi \equiv \circ\psi$ (K0.2) $\vdash \varphi \;\Rightarrow\; \vdash \circ\varphi$
(K1.1) $\vdash (\circ\varphi \wedge \circ\psi) \supset \circ(\varphi \wedge \psi)$
(K1.2) $\vdash ((\varphi \wedge \circ\varphi) \vee (\psi \wedge \circ\psi)) \supset \circ(\varphi \vee \psi)$
(K1.3) $\vdash \bullet\varphi \supset \varphi$ (K1.4) $\vdash \bullet\varphi \equiv \sim\circ\varphi$

A similar interpretation could be proposed for the determinedness connective. One could read $\star\alpha$ as saying that 'α is not the case, but it could have been'. This suggests that $\star$ could work as a kind of (local) connective for 'counterfactual truth'. I will leave this here as a path that seems worth exploring. It is easy if you try.

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