Chapter Four
An Abstract Perspective on Negation

This chapter is composed of two papers: Part **4.1** contains the 'On negation: Pure local rules', henceforth PureLocal; part **4.2** contains the 'Ineffable inconsistencies', henceforth Ineffable.

Resumo de PureLocal

Este é um estudo inicial sistemático das propriedades da negação do ponto de vista dos sistemas dedutivos abstra[c]tos. Ado[p]ta-se um arcabouço unificador de relações de consequência com conclusão múltipla de modo a nos permitir explorar a simetria na exposição e na comparação de um grande número de regras subclássicas contextuais positivas envolvendo esta constante lógica —dentre as quais, formas bem conhecidas de demonstração por casos, *consequentia mirabilis* e redução ao absurdo. Definições mais finas de paraconsistência e da paracompletude dual podem assim ser formuladas, permitindo a diferenciação das regras do pseudo-escoto e *ex contradictione*, e a apresentação de uma versão abrangente do Princípio da Não-Trivialidade. Uma proposta final é feita de tal sorte que —dada a falibilidade frequente das regras positivas puras envolvendo a negação— uma caracterização do que a maior parte das negações da literatura tem em comum deveria envolver, na realidade, um conjunto reduzido de regras negativas.

Resumo de Ineffable

Para cada lógica tarskiana consistente dada é possível encontrar outra lógica não-trivial que admite um modelo inconsistente e mesmo assim coincide com a lógica inicial dada do ponto de vista de suas relações de consequência com conclusão única associadas.
Um paradoxo? Esta breve nota lhe mostra como isso funciona.
Isto pode ser lido como uma descrição de uma expedição a regiões inexploradas da lógica abstra[c]ta, da teoria das valorações e da paraconsistência.

Contents

General and abstract ideas are the source of the greatest errors of mankind.
—Jean-Jacques Rousseau, *The Creed of a Savoyard Priest*, 1762.

This chapter enters the play almost as an appendix. It shows how I would have approached the main problems of this thesis, had I known then what I know now. Coming back to the initial theme of the first chapter of the thesis, the paper [11] does some investigation on General Abstract Logic (a.k.a. Universal Logic), but this time in the framework of multiple-premise-multiple-conclusion consequence relations. The choice of framework is claimed to make a significant difference, and many examples are brought forward to illustrate this claim. The paper [10] provides yet a further illustration of that, mixing abstraction and semantics. It questions the received notion of 'explosion', and entertains a definition of 'consistency' that does not depend on negation.

Here is how you should do it

One of the things that philosophy, mathematics and logic have in common is their concern for fine conceptual distinctions. Abstraction should be pursued, however, only to the point that it does not coalesce notions that had better stay apart. Many a time, the choice of framework can help either in disguising or otherwise in displaying a specific property of a formal system. Of course, there is no perfect framework —it all depends pretty much on your objectives and on your object of research at the time.

In the paper PureLocal I show some advantages of the choice of a multiple-conclusion framework for the study of logics and of logical constants (that is, logical connectives), in allowing us to draw some novel and insightful distinctions, to perform some upgrades on received theories, and to exploit all the themes of this thesis over a common background. The key ideas depend on making heavy use of the symmetry promptly provided by the new framework. Among the successes of the study, in the above mentioned directions, I could count the following: The inference rule known as *consequentia mirabilis* is shown to be misidentified by some authors with the rule of *reductio ad absurdum*; the rules of *pseudo-scotus* and of *ex contradictione sequitur quodlibet* are shown to be distinguishable; rules that are dual to *ex contradictione*, *consequentia mirabilis*, proof-by-cases, and *reductio ad absurdum* are all acknowledged in the paper; the *reductio* rules are shown not to derive all the other rules, as it has been claimed by other authors. As a matter of fact, the last section of the paper brings already a more extensive list of contributions.

I am certainly not the first to consider an abstract study of logics based on a multiple-conclusion framework. The roots of multiple-conclusion can indeed be traced as far back as to Gentzen's [4], Carnap's [1] and Kneale's [7]. It is somewhat unfortunate that the main source books that explore multiple-conclusion in obtaining results for Universal Logic, such as [19] and [25], are not known or accessible to a wider audience. The semantic aspect of such logics is quite simple. Under the canonical notion of entailment, not only truth must be preserved from the set of premises to the set of concluding alternatives, but falsehood must also be preserved from the alternatives to the assumed premises. The development of the syntactic aspects of such logics have been less of a consensus. It seems, though, that the situation has been changing, in recent years, with the advent of multiple-conclusion versions of natural deduction (cf. [24]) and of more geometrical outlooks on the development of proofs (cf. [3]).

To be sure, the multiple-conclusion framework of the paper PureLocal is not even a novelty in the context of the present thesis. Such a framework already had a role to play in **Chapters 2.1** and **3.3**. But this paper came earlier and went deeper into the subject. While it might have been possible to circumvent the more symmetricalist approach in the earlier chapters, that movement here would have done nothing but seriously cripple the paper. The paper, moreover, insists on the use of this framework for an abstract study of the properties of connectives.[1] I treat the latter much in the same way one treats 'natural kinds' in scientific discourse (cf. [13]), though the language I employ is not per force the language of 'essentialism'. To the contrary, my proposal is to characterize logics and their connectives, in general, not by the features they *have* in common, but by the features they 'lack in common'. Once the paper PureLocal formulates, examines and compares a number of abstract properties of logics, and a number of abstract properties of negation, one could quite naturally expect from it yet another answer to the big questions: 'What is a logic?' / 'What is negation?' Numerous papers and books have been written about that. What is novel in my proposal is the emphasis put on *negative properties*: In spite of the little chance of agreement that should be expected when people set their preferred set of positive properties about what such-and-such *is*, I claim that a more prolific and unifying approach would be one that looked for the properties that are *not* enjoyed by such-and-such, under its many possible guises. The best I can offer thus as a response to the big questions are a set of criteria for 'minimal decency'. There is no definitive canon to be found, but only a few guidelines for the logic-designer that wants to avoid degenerate examples of 'logics', and degenerate examples of 'negations'. Interestingly, the last criteria were also put into practice in **Chapter 3.3** in order to dodge a number of entities that we did not want to consider as candidates for '(modal paraconsistent) negations'.

On what concerns the above mentioned problems, the paper PureLocal also contains a substantive survey of the related literature. Several mistakes by other authors are localized and eliminated, when that is the case. (I might of course have left my own mistakes, as an uncalculated gift for attentive readers of the future.) As it has been remarked elsewhere, sym-

[1] Another approach to that study along similar lines can be found in Koslow's book, [8].

metry facilitates the work on duality. A byproduct of the paper is that many definitions that had been set on matters related to paraconsistency can be straightforwardly restated in terms of its dual paracompleteness. Some of the causes and the effects of a choice for 'paranormality' (recall the previous chapter) are also illustrated in this paper.

Here is how you should not do it

The paper INEFFABLE explores some of the issues raised by the preceding paper. More specifically, it stresses the semantic rationale behind what had been called 'Principle of Non-Overcompleteness', as a generalization of the 'Principle of Non-Triviality' proposed in **Chapter 1.0**, and it also explores the novel distinction that had been delineated between the rules of *pseudo-scotus* and of *ex contradictione sequitur quodlibet*, in order to show that degenerate examples of 'paraconsistency' (namely, those logics that disrespect only the former rule, but not the latter) are possible and should also be avoided thus through 'minimal decency'.

I assume paraconsistency to have been born when the first logical systems were developed with the professed intention of allowing for some inconsistencies at a local level while avoiding triviality at a global level (cf. [5, 12, 2]). Because allowing for inconsistencies meant that there could be a surplus of truths in the interpretation of a given logic, it was certainly natural that those who toiled over the design of the first paraconsistent logics became worried about there being too many truths around. They worried about that particular variety of 'overcompleteness'[2] in which every sentence of the logic turned out to be a thesis / a theorem / a tautology. They seem to be justified in their worry: If one proposes an approach to inconsistencies that lets some of them stay in our theories and perhaps even fructify, one will certainly not want to be so liberal as to let all inconsistencies become hopelessly indiscernible —a deranged state of affairs that I, in **Chapter 3.3** and in the papers composing the present chapter, call 'dadaism'.

Interestingly enough, the concern about dadaism (as a specific variety of overcompleteness) was already to be found in one of the papers on the notion of logical consequence proposed by Tarski himself.[3] To explain this point, I had better make a brief digression first. The early Polish tradition used to think about logic from a topological point of view, and to construe

[2]From the Polish 'przepełnienie'. 'Overcompleteness' was Jaśkowski's term according to the first English translation of his 1948 Polish paper. Nelson refers to this paper in 1959 (with the title in French, as it had in fact been published with a summary in French) and uses the term 'overcompleteness' eight years before Jaśkowski's paper was to receive its first published English translation. More or less at the same time, da Costa was starting to use the word 'triviality' in his papers published in Portuguese, even before he wrote his 1963 thesis, also in Portuguese. The second English translation of Jaśkowski's paper employs the term 'overfilling'.

[3]He certainly did not share, however, the motivations that moved the paraconsistentists in their concern. Recall indeed from section 1 of the TAXONOMY (**Chapter 1.0**) the passage about Tarski's hostility towards inconsistent theories.

logical consequence as a closure operator.[4] Later on, logic was turned into an animal of another breed, when the topological outlook on consequence remained but the set of formulas itself started to be presented as an algebra. Tarski presented most of his ideas on logical consequence in terms of closure operators, but it is not too difficult in general to translate the clauses governing the behavior of closure *operators* into similar clauses governing the behavior of (single-conclusion) consequence *relations*, a framework that became more common nowadays. The axioms proposed by Tarski for the notion of logical consequence varied a lot over the years. So, what I call 'tarskian logic' in some of my papers is in reality quite stipulative. On that matter I usually try to settle around the semantic notion of derivability presented by Tarski in [23]. For that notion a nice suitable adequacy theorem is available (recall **Chapter 2.1**), according to which a single-conclusion consequence relation is to be axiomatized in abstract terms by:

(CR1s)	$\Gamma, \varphi, \Delta \Vdash \varphi$	(overlap)
(CR2s)	if $\Delta \Vdash \varphi$, then $\Gamma, \Delta \Vdash \varphi$	(dilution)
(CR3s)	if $(\forall \delta \in \Delta)(\Gamma \Vdash \delta)$ and $\Delta \Vdash \varphi$, then $\Gamma \Vdash \varphi$	(cut for sets)

Before that, however, Tarski had already published a much more detailed account of 'some fundamental concepts of Metamathematics' (cf. [22]),[5] where to the above axioms he added the requirements that logics should be compact, and their underlying languages should be denumerable and should contain a bottom particle (that is, logics should respect, in particular, our 'Principle of *Ex Falso*', from **Chapter 1.0**). In [21], the first study towards the latter paper and Tarski's first published note on the theme of logical consequence, logic was running closer to topology, and to the above three axioms the requirement was added that closure operators should preserve arbitrary unions. Moreover, in that initial paper Tarski[6] also considered a number of specializations of the above defined structures, by the addition of further axioms. Among such possible axioms there is one that brings us back to the initial theme of this paragraph, namely:

(CR0s)	$(\exists \varphi) \nVdash \varphi$	(compatibility)

This compatibility condition —that seems to have passed unnoticed and to have been completely forgotten in later years— corresponds neatly, in the presence of dilution, to Jaśkowski-Nelson-da Costa's concerns about avoiding dadaism. It is easy to see that Tarski's compatibility corresponds, in the logics we here consider, to the already traditional 'Principle of Non-Triviality', and my present version of the 'Principle of Non-Overcompleteness'

[4]As it had been shown by Kuratowski, topologies can be seen as particular cases of closure operators, for which you require the empty set to have an empty closure and the union of closed sets to be identical to the closure of their union.

[5]First presented by Jan Łukasiewicz to the Warsaw Scientific Society on 27 March 1930.

[6]Or whoever wrote the paper for him. The paper is part of a 'comptes-rendus' where someone is supposed to write down a summary of the main contents of the lectures presented in a meeting of the Polish Mathematical Society. In the resulting text, Tarski is referred to in the third person.

introduces a generalization of the former principle, in abstract, to the point of regulating also three other examples of degenerate logics.

As it has been noted, the single-conclusion framework, under the above axioms, guarantees that truth is preserved forwards, from premises to conclusion. Full symmetry is only installed though with a multiple-conclusion framework, where falsehood is likewise preserved, but backwards. Where $\mathsf{Ptn}(\Sigma)$ denotes the set of all partitions of the set Σ, here are the 'tarskian' axioms in a multiple-conclusion fashion:

(CR1m)	$\Gamma, \varphi, \Delta \Vdash \Sigma, \varphi, \Pi$	(overlap)
(CR2m)	if $\Delta \Vdash \Sigma$, then $\Gamma, \Delta \Vdash \Sigma, \Pi$	(dilution)
(CR3m)	if $(\forall \langle \Sigma_1, \Sigma_2 \rangle \in \mathsf{Ptn}(\Sigma))(\Gamma, \Sigma_1 \Vdash \Sigma_2, \Delta)$, then $\Gamma \Vdash \Delta$	(cut for sets)

Sometimes one finds other conditions in the place of cut for sets, as for instance the following 'contextual' forms of cut for formulas:

(CR3cp)	if $\Gamma, \varphi \Vdash \psi$ and $\Gamma \Vdash \varphi$, then $\Gamma \Vdash \psi$
(CR3cc)	if $\varphi \Vdash \psi, \Delta$ and $\Vdash \varphi, \Delta$, then $\Vdash \psi, \Delta$
(CR3c)	if $\Gamma, \varphi \Vdash \Delta$ and $\Gamma \Vdash \varphi, \Delta$, then $\Gamma \Vdash \Delta$

It should be noted, however, that such alternative versions of cut can be shown to be strictly weaker than the initial formulation above, the 'cut for sets' (cf. chap. 2 of [19]). Again, the reason for the stronger choice of axioms for multiple-conclusion consequence relations, as an extension of the original approach by Tarski to consequence operators, is the availability of a nice suitable adequacy theorem (recall **Chapter 2.1**), maintaining something very much like the original semantic intuition.[7]

The multiple-conclusion framework has some further advantages. In the paper INEFFABLE I show that a single-conclusion framework cannot see the difference between a consistent logic and this 'same' logic when added of an extra dadaistic model. That a multiple-conclusion framework *can* see the difference should not really come as a great surprise. To explain that, let me first make yet another brief digression to recall a few concepts. Roughly speaking, the term *gap* is customarily used to mark a situation in which there is a 'paucity of truths'. For instance, in the semantics of paracomplete

[7] An authoritative referee has called my attention to the 'mistake' of calling 'tarskian' the class of logics whose consequence relation is multiple-conclusion and is axiomatized through clauses (CR1m)–(CR3m). He claimed that this is "what the literature calls 'Scott Consequence Relations'", and advised me to "see e.g. Gabbay's book on intuitionistic logic". Well, if there is a mistake involved in my decision, it is certainly not *my* mistake, and maybe not even of Gabbay's book (which book?). Dana Scott has indeed proposed the study of multiple-conclusion versions of the preceding tarskian axioms, initially formulated in terms single-conclusion consequence relations. Typically, [16, 15, 17] are the papers published by Scott that are cited by those who claim that 'multiple-conclusion logics are scottian'. I know that too well —I have made that confusion myself. Nonetheless, axiom (CR3m) is never to be found in those papers; at best one can find the weaker (CR3c) in its place. Scott's approach in these papers, in fact, always seems quite tentative, and it shows no hint of a deep underlying semantic motivation. Not surprisingly, nowhere has Scott an adequacy theorem to offer about the weaker notion of consequence relation he proposes. My own approach, thus, cannot be 'scottian'. It is based instead on the work of Shoesmith & Smiley (cf. [18, 19, 25]).

tarskian logics, the circumstance that a formula α and its negation $\sim\alpha$ both are given non-designated values can be reformulated by simply saying that α is 'neither true nor false' or that there is a truth-value gap in α. A similar account can be given about *gluts* and situations in which there is an 'excess of truths', as in the semantics of paraconsistent tarskian logics. Moreover, consequence relations are said to be *categorical* if distinct sets of valuations characterize distinct consequence relations. Now, in [14, 6] one is assured that, while single-conclusion tarskian consequence relations are in general *not* categorical, multiple-conclusion tarskian consequence relations for logics whose semantics contain either gaps or gluts *are* categorical.

Besides the typical inertia, and some ignorance, of scholars, it seems hard to find reasons, in fact, for single-conclusion consequence relations to remain so popular in the current literature on logic. Perhaps this can be explained by the everlasting influence of closure operators, or the appealing lopsidedness of the natural deduction formalism. Or maybe this is just because philosophers have accommodated around the notions of *theoremhood* and *truth*, to which study a single-conclusion framework seems tailored to fit. But that poses to me then yet another enigma. Why in the world do some metaphysicians seem to think nowadays that logic has anything to tell you about 'truth', in the first place? For all I know, they might be misled by the spell of language and the influence of old habits of thought. I do not understand why there is still such a persistent bias towards truth, anyway, when falsehood would in theory seem equally important. Why should truth be privileged over falsehood? Why to worry exclusively with dadaism when 'nihilism' —the situation in which every sentence of the logic turns out to be an antithesis / an antitheorem / an antilogy— should seem equally deranging? Nihilism, just like dadaism, makes everything indiscernible.

Other possible reasons that might have collaborated for the perpetuation of the single-conclusion framework in logic and of the related obsession about truth are: (a) that the most common logical systems enjoy compactness and some suitable form of deduction theorem (so that provable inferences are just as good as theorems); (b) that the most common systems of sequents are finitary and contain suitable conjunctions and disjunctions; (c) that the traditional study of constructiveness, or of (effective) provability, often emphasizes single-conclusion; (d) that some popular syntactical mechanisms correspond naturally to single-conclusion calculi. In contrast, multiple-conclusion can: (a) help implementing a natural notion of *duality*; (b) put *truth* and *falsehood* on equal footing, without requiring much from the underlying language; (c) provide a framework for empirical evidences to be collected directly about the *absence* of a given property; (d) internalize a primitive notion of *rejection*, alongside with *assertion*. On the latter point, special attention should be given to *refutative* systems, that dualize the Fregean primitive symbol of assertion as suggested by Brentano and studied by Łukasiewicz (for a natural deduction approach to the work of the latter, check for instance [20]).

The present chapter of the thesis illustrates on and on how the choice of underlying framework can make a difference for the purposes of Universal Logic, the abstract study of logical structures. As I have already mentioned, the paper Ineffable shows that a single-conclusion framework simply cannot detect some varieties of inconsistency that are allowed by an approach to Universal Logic based on the theory of valuations, even when overcompleteness is diligently shunned. Wariness on what concerns those latent possibilities is the least that should be expected if one aspires Universal Logic to be anything more than General Abstract Nonsense.

Brief history

An early version of the paper PureLocal was ready by June 2002, while I was working in Brazil under a CNPq doctoral grant. It was intended at the start to be presented at the Workshop on Paraconsistent Logic (WoPaLo), which I organized as part of the 14th Summer School in Logic, Language and Information (ESSLLI 2002), held in Trento (IT) in August 2002. However, the workshop was a big concert in need of an introduction, so I renounced to presenting the above paper and I wrote and presented there instead the overture [9]. A more thorough version of the above material was submitted nevertheless to the 'Proceedings of the WoPaLo' a few months later, and its ideas were criticized during a talk I gave at the Theory of Computation Seminar promoted by the Center for Logic and Computation of the IST (PT), in May 2003. The paper was corrected with extreme care while I was already working in Portugal under an FCT doctoral grant, and besides all the people acknowledged in the paper I should show my gratitude to the friend João Rasga for his continuous encouragement and his help with doing some important bibliographic research. The corrected final proofs of the paper, to be published soon by the Journal of Applied Logic, appeared on-line at Elsevier's ScienceDirect web-site in August 2004.

The ideas and the general critique contained in the paper Ineffable were first presented at the round-table 'Systems of Paraconsistent Logic', at the closure session of the III World Congress on Paraconsistency (WCP 3), held in Toulouse (FR) in July 2003.

A melange of ideas about multiple-conclusion and about the generalization of the Principle of Non-Trivialization propounded in both papers from this chapter constituted the kernel of a talk presented at the colloquium Logic, Ontology, Aesthetics: The Golden Age of Polish Philosophy, jointly promoted by the Université du Québec à Montréal, the Consulate of Poland, and the Concordia University (CA), in September 2004. I am very grateful for the reactions of the audience there, and specially for the historical clarifications made by Jan Woleński and Jean-Yves Béziau. Bog knows why, the organizers decided to prize my contribution as a 'best communication project'. I am confident that they did not regret their decision.

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