

# Symmetric Solutions of Einstein-Yang-Mills Equations <sup>1</sup>

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## Abstract

Symmetric gauge fields and invariant metrics in homogeneous spaces are found. Their use for finding exact solutions of the Einstein-Yang-Mills (EYM) equations is discussed.

## 1. Introduction

The search for exact solutions in general relativity can usually be divided in two parts. The first part, which we call "kinematical", consists in the restriction to a subset of fields in the space of field configurations (i.e. in the making of appropriate ansätze).

This restriction is made in accordance with the physical problem in consideration and is achieved by considering only the configurations which are invariant (or symmetric) under the action of a symmetry group  $S$ . For example in the case of closed, flat and open Friedmann-Robertson-Walker cosmological models,  $S$  is respectively  $SO(4)$ ,  $E^3$ ,  $SO(1,3)$  and plays the role of group of spatial homogeneity and isotropy.

In the second (or "dynamical") part of the search for exact solutions in General Relativity we study the equations implied by the original theory in the set of  $S$ -symmetric configurations. The number of independent variables in these equations is equal to the dimensionality of space-time minus the dimensionality of the orbits of the symmetry group.

Here we will describe, following [1,2], a method of finding invariant metrics and symmetric gauge fields used in the kinematical part of the search for solutions of the EYM equations (see also [3,4]).

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<sup>1</sup>to appear in the Proceedings of the *First Iberian Meeting on Gravity*, Évora, Portugal, September, 1992; World Scientific, 1993

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## 2. Invariant Metrics and Symmetric Gauge Fields

Consider a space-time of the form

$$M = \tilde{M} \times S/R \ , \quad (1)$$

where  $S$  and  $R$  are Lie groups, and assume that we are interested in  $S$ -symmetric solutions of the EYM equations. The gauge group  $K$  is assumed to be a compact Lie group.

Let us for further simplicity restrict ourselves to a homogeneous space-time

$$M = S/R \ . \quad (2)$$

The reader interested in the general case (1) is referred to the contribution by Kapetanakis and Zoupanos in the present volume and to the literature [1-5]. Notice that the  $S$ -invariant ansatz for the metric and the  $S$ -symmetric (i.e. invariant up to a gauge transformation) ansatz for the gauge field, discussed here, are always present in the general case.

We first chose a preferred moving frame in  $S/R$  and describe the  $S$ -invariant ansatz for the metric in  $S/R$ .

Consider the canonical ( $\mathcal{S} \equiv Lie(S)$ -valued) left-invariant one-form on the group manifold of  $S$

$$\theta = \theta^\alpha T_\alpha = s^{-1} ds \ ,$$

where the  $T_\alpha$  are the generators of  $S$  and  $\theta^\alpha$  are left-invariant one-forms on  $S$ . The form  $\theta$  is very important for the study of the invariant geometry of the Lie group  $S$  and its pull back to  $S/R$  plays, as we shall see below, an analogous role in the study of the invariant geometry of the coset space  $S/R$ . The group  $S$  can be considered as a principal bundle over  $S/R$  with structure group  $R$  (we assume  $R$  to be a closed subgroup of  $S$  and  $S/R$  to be a reductive coset space). Let us chose a (local) section  $\sigma$  of this bundle in a neighborhood of the origin  $o \equiv [e] \equiv R$

$$\sigma : \mathcal{U} \subset S/R \longrightarrow S$$

and pull the form  $\theta$  back to  $S/R$  with respect to  $\sigma$ :

$$\bar{\theta}_x = (\sigma^* \theta)_x = \sigma^{-1}(x) d\sigma(x) = \bar{\theta}^\alpha T_\alpha \ . \quad (3)$$

The  $\mathcal{S}$ -valued one-form  $\bar{\theta}$  in (3) is called the Maurer-Cartan one-form in  $S/R$ . A local moving coframe can be obtained from the Maurer Cartan one form as follows. Let

$$\mathcal{S} = \mathcal{R} + \mathcal{M} \quad (4)$$

be a reductive decomposition of  $\mathcal{S}$ , where  $\mathcal{R} \equiv Lie(R)$  and  $\mathcal{M}$  is a reductive (i.e.  $[\mathcal{R}, \mathcal{M}] \subset \mathcal{M}$ ) complementary subspace and

$$\{T_\alpha\} = \{T_a, T_{\dot{a}}\} \ , T_a \in \mathcal{M}, T_{\dot{a}} \in \mathcal{R} \quad (5)$$

be an adapted basis in  $\mathcal{S}$ . We have the natural decomposition of the Maurer-Cartan one-form associated with (5)

$$\bar{\theta} = \bar{\theta}_{\mathcal{M}} + \bar{\theta}_{\mathcal{R}} \equiv \bar{\theta}^a T_a + \bar{\theta}^{\dot{a}} T_{\dot{a}} \quad (6)$$

Unlike the group case  $S$ , the one-forms  $\bar{\theta}^\alpha$  in  $S/R$  are, in general, not  $S$ -invariant. Nevertheless they have simple transformation laws under the action of  $S$ . For given  $s \in S, x \in \mathcal{U}$  and  $\sigma$ , let  $r_s(x; \sigma)$  be the element of the isotropy group  $R$  defined by

$$\sigma(sx) = s\sigma(x)r_s(x; \sigma) \quad .$$

Then the transformation law of  $\bar{\theta}^a$  and  $\bar{\theta}^{\dot{a}}$  under the action of  $s \in S$  can be easily obtained from

$$s^* \bar{\theta}_{\mathcal{M}} = ad r_s^{-1}(x) \bar{\theta}_{\mathcal{M}} \quad (7a)$$

$$s^* \bar{\theta}_{\mathcal{R}} = ad r_s^{-1}(x) \bar{\theta}_{\mathcal{R}} + r_s^{-1}(x) dr_s(x) \quad , \quad (7b)$$

where  $ad$  denotes the adjoint representation of  $S$ . The reductiveness of the decomposition (4) implies that the restriction of  $ad$  to the subgroup  $R$  has  $\mathcal{M}$  as an invariant subspace. The representation of  $R$  in  $\mathcal{M}$  obtained in this way is called the isotropy representation and is denoted by  $ad_R$ . Let  $\{X_a\}$  be the frame dual to  $\{\bar{\theta}^a\}$ . It is clear from (7a) that under  $s_*$  the vectors  $X_a$  in this frame transform in the same way the  $T_a$  do under the isotropy action of  $r_s^{-1}(x)$  (i.e.  $ad r_s^{-1}(x)$ ). Let  $B(\cdot, \cdot)$ , be an  $ad_R$ -invariant scalar product in  $\mathcal{M}$  with components  $B_{ab} = B(T_a, T_b)$ . Then we conclude [6] that the metric in  $S/R$  with the same components  $B_{ab}$  in the frame  $\{X_a\}$ ,

$$\gamma = B_{ab} \bar{\theta}^a \otimes \bar{\theta}^b \quad (8)$$

is  $S$ -invariant. Moreover any  $S$ -invariant metric in  $S/R$  can be constructed in this way. Therefore we have succeeded in reducing the (differential geometry) problem of finding  $S$ -invariant metrics in the coset space  $S/R$  to the purely algebraic problem of finding  $ad_R$ -invariant scalar products in  $\mathcal{M}$ . The equation (8) with  $B_{ab}$  being an  $ad_R$ -invariant scalar product in  $\mathcal{M}$  gives the ansatz for  $S$ -invariant metrics in  $S/R$ .

In an analogous way [1-5] the problem of finding  $S$ -symmetric gauge fields with gauge group  $K$  in  $S/R$  is reduced by Wang's theorem [6] to the algebraic problem of intertwining equivalent representations of the isotropy group. To see this recall that, according to Wang's theorem, the most general form of a  $S$ -symmetric gauge field in  $S/R$  is given by

$$A = \lambda(\bar{\theta}_{\mathcal{R}}) + \Lambda(\bar{\theta}_{\mathcal{M}}) \quad , \quad (9)$$

where

$$\lambda : \mathcal{R} \longrightarrow \mathcal{K} = Lie(K) \quad (10a)$$

is an homomorphism from the isotropy algebra  $\mathcal{R}$  to the Lie algebra of the gauge group  $\mathcal{K}$  and  $\Lambda$  is a mapping from  $\mathcal{M}$  to  $\mathcal{K}$

$$\Lambda : \mathcal{M} \longrightarrow \mathcal{K} \quad (10b)$$

satisfying the linear property

$$\Lambda(ad(r)u) = ad(\lambda(r))\Lambda(u) \quad , \quad u \in \mathcal{M} \quad , \quad r \in R \quad (10c)$$

This property means that the mapp  $\Lambda$  intertwines the representation of  $R$  in  $\mathcal{M}$  (i.e. the isotropy representation) with the representation of  $\lambda(R)$  in  $\mathcal{K}$  (obtained by restricting the adjoint representation of  $K$  to  $\lambda(R)$ ). The solutions of (10c) are therefore given by Schur's lemma.

By making the ansätze (8) and (9, 10) we have concluded the kinematical part of the search for  $S$ -symmetric solutions of the EYM equations.

Analogous symmetric ansätze for linear connections with torsion have been made in the framework of generalized Einstein-Cartan theories [7].

### 3. Acknowledgements

It is a pleasure to thank M.C. Bento, O. Bertolami, A.B. Henriques, Yu. A. Kubyshin, P.V. Moniz, R. Picken, P. Sá, J.N. Tavares and I.P. Volobuev for collaboration over the years and Yu. A. Kubyshin for usefull suggestions.

### 4. References

1. I.P. Volobuev and Yu.A. Kubyshin, *JETP Lett.* **45** (1987) 581; *Theor. Math. Phys.* **75** (1988) 509;  
 Yu.A. Kubyshin, J.M. Mourão, G. Rudolph and I.P. Volobuev, *Lect. Notes in Physics* **349** (Springer Verlag, 1988);  
 Yu.A. Kubyshin, J.M. Mourão and I.P. Volobuev, *Phys. Lett.* **B203** (1988) 349; *Nucl. Phys.* **B322** (1989) 531
2. O. Bertolami, J.M. Mourão, R. Picken and I.P. Volobuev, *Int. J. Mod. Phys.* **A6** (1991) 4149;  
 P.V. Moniz and J.M. Mourão, *Class. Quant. Grav.* **8** (1991) 1815;  
 O. Bertolami and J.M. Mourão, *Class. Quant. Grav.* **8** (1991) 1271;  
 O. Bertolami, Yu.A. Kubyshin and J.M. Mourão, *Phys. Rev. D* **45** (1992) 3405;  
 P.V. Moniz, J.M. Mourão and P.M. Sá, *Class. Quant. Grav.* in Press;  
 M.C. Bento, O. Bertolami, P.V. Moniz, J.M. Mourão and P.M. Sá, *Class. Quant. Grav.* in Press

3. M. Henneaux, *J. Math. Phys.* **23** (1982) 830;  
Yu.A. Kubyshin, V.A. Rubakov and I.I. Tkachev, *Int. J. Mod. Phys.* **A4** (1989) 1409;  
R. Coquereaux and A. Jadczyk, *Lect. Notes in Physics* **16** (World Scientific, 1988);  
K. Yoshida, S. Hirenzaki and K. Shiraishi, *Phys. Rev. D* **42** (1990) 1973  
Yu. A. Kubyshin, J.I. Perez Cadenas, *Preprint Nuclear Phys. Inst., Moscow State University*, Moscow (1993)
4. H.P. Kunzle, *Class. Quant. Grav.* **8** (1991) 2283
5. P. Forgacs and N.S. Manton, *Comm. Math. Phys.* **72** (1980) 15  
I.P. Volobuev and Yu.A. Kubyshin, *Theor. Math. Phys.* **68** (1986) 788; **68** (1986) 885  
F.A. Bais, K.J. Barnes, P. Forgacs and G. Zoupanos *Nucl. Phys.* **B291** (1986) 557;  
G. Rudolph and I.P. Volobuev, *Nucl. Phys.* **B313** (1989) 95;  
D. Kapetanakis and G. Zoupanos, *Phys. Lett.* **B249** (1990) 73; *Phys. Rep.* **C 219** (1992) 1;  
S. Shabanov, *Phys. Lett.* **B272** (1992) 11
6. S. Kobayashi and K. Nomizu, "Foundations of Differential Geometry" (Interscience Publishers, 1969)
7. Yu. A. Kubyshin, O. Richter and G. Rudolph, *Rep. Math. Phys.* **30** (1991) 355; *Leipzig Univ. Preprint*, Leipzig (1992)  
Yu. A. Kubyshin, V. Malysenko and D. Marin Ricoy, *Preprint Nuclear Phys. Inst., Moscow State University*, Moscow (1993)