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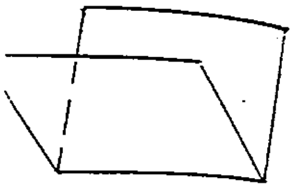

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GRAPHIC SCHEMES OF ROOT ORDERING IN SIMPLE LIE ALGEBRAS

Yuri A. Kubyshin*, Jose M. Mourão#, Igor P. Volobujev*
*Nuclear Physics Institute, Moscow State University, Moscow 117899, USSR
#Centro de Fisica Nuclear, Av. Prof. Gama Pinto 2, 1699 Lisboa Codex, Portugal

1 . All information about roots, root vectors and commutation relations of complex simple Lie algebras is coded in the Dynkin diagrams [1] (see also [2-3]). Analogously, one can derive a system of weights and weight vectors of finite dimensional irreducible representations (irreps) of the algebras using schemes of the representations proposed by Dynkin [1]. However, for numerous problems, which one encounters in theoretical physics, it is very useful to have more detailed schemes, providing a graphic image of the structure of the algebra or representation. In the present paper we will consider such ordering schemes representing the positive roots of the simple Lie algebras or the weights of their irreps. We call them root lattices (RL) and weight lattices (WL) correspondingly. An example of their application will be given in the last section.

2 . Let $\mathfrak{g}$ be a simple complex Lie algebra and ϕ be its finite dimensional irrep. We denote the set of the simple roots of the algebra by Π, the root vector in $\mathfrak{g}$ corresponding to a root α by e_{α}, the set of the weights of the representation ϕ by $\Delta(\phi)$ and the maximal weight of ϕ by λ.

The WL of ϕ is constructed in the following way. First draw a dot (we call it vertex) for each weight $\xi \in \Delta(\phi)$. Then, if two weights ξ and ξ' differ by a simple root $\alpha \in \Pi$, i.e. $\xi - \xi' = \mp\alpha$, join the corresponding vertices on the scheme by a line (called link). For many applications it is also useful to indicate the Dynkin coordinates $(\xi_{\alpha_1}, \xi_{\alpha_2}, \ldots, \xi_{\alpha_n})$ [1,4] of the weight ξ near the vertex corresponding to it, and the corresponding simple root α near each link. We follow a convention (introduced by Dynkin) to draw the maximal weight λ at the very bottom of WL (zero level), the weights obtained from λ by subtracting one simple root one level above (first level) and so on. A weight $\xi = \lambda - \sum_{\alpha\in\Pi} k_\alpha \alpha$ (k_α are non-negative integers) is placed on the level number

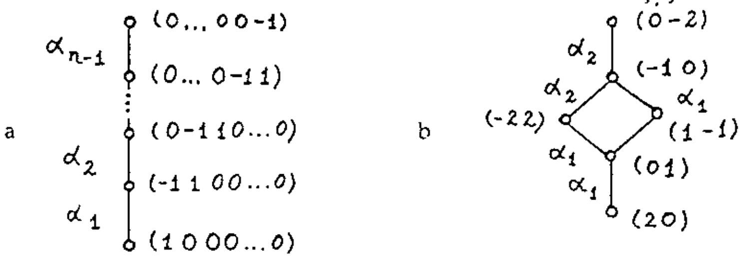
$$\gamma(\xi) = \sum_{\alpha\in\Pi} k_\alpha .$$

According to the general theory of algebras the WLs must be of a spindle form. In Fig. 1 we present two examples of the WL: a) for the fundamental representation $\underline{n}$ of A_{n-1}, and b) for $\underline{6}$ of A_2. We should mention that the WLs are very close to the idea of etages of weights by Dynkin [1] and to weight diagrams described in [4].

3 . The Lie algebra g can be viewed as a vector space of the adjoint representation which is realized by operators ad$x$, $x \in g$: ad$x(y) = [x, y]$, $y \in g$. Since the negative roots form exactly the same pattern as positive ones, it suffices for this representation to draw only a "half" of the WL and represent positive roots only. We omit also the vertices of zero weights corresponding to the elements of the Cartan subalgebra of g. Such modification of the WL for the adjoint representation ad$g$ is called RL. Note, that all roots (non-zero weights) are non-degenerate, while the zero weights have the degeneracy equal to n = rank g. Traditionally, the top level of the RL, representing the simple roots, is drawn in the same way as the Dynkin diagram with all standard conventions about the vertices and the connecting lines [1-3].

The RLs for the classical Lie algebras were described in [5] (see also [6]). As an example we will discuss here the RL for A_n (Fig. 2). The vertex lying at the intersection of segments starting from α_k and α_j corresponds to the root $\alpha(k, j) = \alpha_k + \alpha_{k+1} + \dots + \alpha_j$. The non-trivial action of ade_{α_k} corresponds to shifts of roots on the RL, these shifts are indicated by arrows in Fig. 2.

4 . The RL can be analyzed as a graph, using results from graph theory. For example, one can prove that the RL for the algebras D_n, $n \geq 5$ are non-planar. This follows from the Kuratowski theorem [7] and the fact that these lattices contain the graph $K_{5,5}$ as a subgraph.



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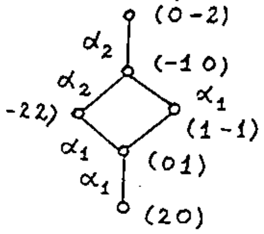

and the fact that these lattices contain the graph $K_{5,5}$ as a subgraph.

For a given algebra the RLs can be drawn in different but topologically equivalent (from the point of view of the theory of graphs) ways, and there is no natural pattern for presenting the positive roots of the algebra. This point is important indeed since the advantage of using the RL is based mainly on the apt way of arranging positive roots. For exceptional algebras the problem of "convenient arranging of roots" becomes even more important because of non-regular and complicated character of their root systems. So, we need a guiding principle for drawing the RLs. We propose here a recursive algorithm for building the RLs and will explain it for the case of A_n. Let us take the decomposition of ad$A_n \downarrow A_{n-1}$, the restriction of the adjoint representationof A_n to its subalgebra A_{n-1},

$$\mathrm{ad}\, A_n \downarrow A_{n-1} = \mathrm{ad}\, A_{n-1} + \underline{n} + \underline{n}^* + \underline{1}. \qquad (1)$$

Then the RL for A_n can be drawn as a combination of the RL for A_{n-1} and the WL for the fundamental representation $\underline{n}$ (see Fig. 1). Starting from the RL for A_1 (which is just one vertex) we can obtain the RLs for the whole series; the RL in Fig. 1 is drawn in accordance with this principle.

As the second application of the algorithm described we consider the exceptional algebra $g = E_6$. In this case we take the basic decomposition in the form

$$\mathrm{ad}E_6 \downarrow D_5 = \mathrm{ad}D_5 + \underline{16} + \underline{16}^* + \underline{1}. \qquad (2)$$

Taking the RL for D_5 [6] and completing it by the WL of the irrep $\underline{16}$ of D_5 (shown by filled-in dots) one gets the RL for E_6 presented in Fig. 3. Of course, the RL depends on the subalgebra in the basic decomposition (2) of ad$g$. We will discuss this ambiguity as well as

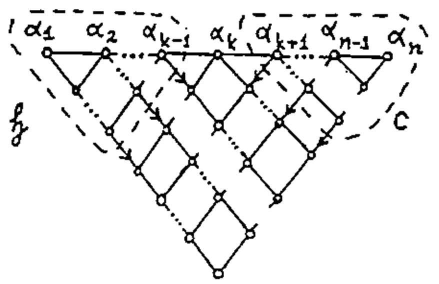

Fig. 2 :

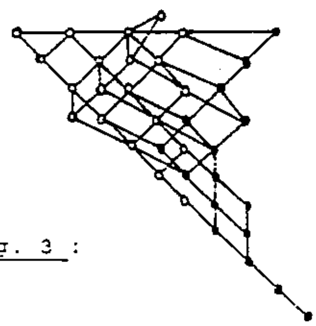

Fig. 3 :

the RLs for other exceptional algebras elsewhere.

5 . In conclusion we give an example of application of the RL. Let $\mathfrak{h}$ be a regular subalgebra of $\mathfrak{g}$, and we are interested in the decomposition of ad$\mathfrak{g}$ restricted to $\mathfrak{h}$ into irreps and the explicit realization of these irreps on root vectors of $\mathfrak{g}$. Problems of this type arise in calculations of the Lagrangians obtained by dimensional reduction of multidimensional gauge theories [5,6]. In the case $\mathfrak{g} = A_n$, $\mathfrak{h} = A_{k-1}$ Fig. 2 gives us the answer immediately. If $\mathfrak{h}$ is realized on the simple roots $\alpha_1,\ldots,\alpha_{k-1}$, then the algebra $\mathfrak{c}$ = A_{n-k}, realized on $\alpha_{k+1},\ldots,\alpha_n$, is the non-abelian part of the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$ (trivial representations of $\mathfrak{h}$). The roots, which belong neither to $\mathfrak{h}$ nor to $\mathfrak{c}$, lie on the segments $(\alpha_k,\ldots,\alpha(1,k))$, $(\alpha(k,k+1),\ldots,\alpha(1,k+1)),\ldots(\alpha(k,n),\ldots,\alpha(1,n))$. These segments represent irreps of $\mathfrak{h}$ of dimension k (fundamental representations), since ad$x$, $x \subset \mathfrak{h}$ shifts these roots along them. The transformations ad$\mathfrak{c}$ permute these invariant subspaces so that they form the irrep <u>n-k+1</u>. An element of the Cartan subalgebra orthogonal to the elements of $\mathfrak{h}$ and $\mathfrak{c}$ generate the 1-dimensional subspace of the trivial representation of $\mathfrak{h}$.

Other applications of the RLs for both regular and non-regular embeddings $\mathfrak{h} \subset \mathfrak{g}$ can be found in [8]. We would like to mention that graphic schemes similar to the RLs were used in [9] for representing the so-called non-compact positive roots corresponding to non-compact real forms of simple Lie algebras.

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