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## On the cosmology of massive vector fields with $SO(3)$ global symmetry

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**Abstract.** We study the dynamics of flat Friedmann–Robertson–Walker (FRW) cosmologies in the presence of a triplet of massive vector fields with  $SO(3)$  global symmetry. We find an  $E^3$ -symmetric ansatz for the vector fields that is compatible with the  $E^3$ -invariant FRW metric and propose a method to make an invariant ansatz for more general cosmological models. We use the techniques of dynamical systems to study qualitatively the behaviour of the model and find, in particular, that the effective equation of state of the system changes gradually from a radiation-dominated to a matter-dominated form and that the scale of the transition depends on the mass of the gauge fields.

### 1. Introduction

Cosmological solutions to the Einstein–Yang–Mills (EYM) equations have been known for some time. In particular, solutions corresponding to open, closed and flat Friedmann–Robertson–Walker (FRW) universes, in the case where the gauge group is  $SU(2)$ , were studied in [1, 2]. The generalization of these results to arbitrary gauge groups, for closed FRW models, was done in [3, 4] (the Euclidian case), where an  $SO(4)$ -symmetric ansatz is derived from the theory of symmetric fields on homogeneous spaces [5]. This  $SO(4)$ -symmetric ansatz has also been used for constructing the wavefunction of a universe dominated by radiation [6] and to study the cosmological implications of generalized Kaluza–Klein theories [7]. The case of a flat FRW inflationary model in the presence of  $E^3$ -symmetric gauge and scalar fields was studied in [8].

In treating the classical dynamics of the very early universe with the help of a variational principle applied to actions derived from particle physics theories we hope to obtain a better understanding of the effective equations of state valid in the classical regime. On the other hand, this approach may be useful in order to bridge

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the gap between the distinct descriptions of the classical and quantum regimes that have been used so far. In this context it is quite natural from the particle physics viewpoint to study non-Abelian gauge theories in a cosmological setting. In realistic particle-physics-motivated cosmological models, however, the vector bosons normally acquire a mass thereby breaking the gauge symmetry. As this process takes place in the very early universe, it is of interest to investigate cosmological solutions of the resulting theory.

The purpose of this paper is to investigate homogeneous and isotropic solutions to the EYM system after symmetry breaking. We shall assume that the initial gauge symmetry is  $SO_1(3)$ , which subsequently breaks down to a global  $SO_1(3)$  (through a mechanism which we choose not to specify but can be thought of as the usual Higgs mechanism) and that, in this process, the gauge fields acquire a mass. We derive an  $E^3$ -symmetric ansatz for the triplet of massive vector fields with  $SO_1(3)$  global symmetry that is compatible with the  $E^3$ -invariant (flat) FRW ansatz for the metric. We compare the dynamical behaviour of the system before and after symmetry breaking and find, in particular, that in the latter phase the equation of state changes with time, gradually from a radiation-dominated to a matter-dominated form and that the scale of the transition depends on the mass of the vector field.

Homogeneous and isotropic solutions exist for other gauge groups and symmetry breaking patterns (see the appendix). For the case of electroweak symmetry breaking,  $SU(2) \otimes U(1)$  down to a  $U(1)$  however, at least for the following ansatz, homogeneity and isotropy cannot be achieved as the vector fields acquire different masses after symmetry breaking thus the energy-momentum tensor (EMT) cannot have the perfect fluid form.

Recently there has been great interest in models where inflation is driven by massive vector fields [9]. In these works the vector field is a  $U(1)$  gauge field which, upon symmetry breaking, acquires a potential leading to inflation. Such a vector field is not allowed by an isotropic spacetime and consequently the authors have to choose an anisotropic cosmological model (Bianchi I), which, at least in some cases, leads to excessive anisotropy at late times. The present work shows that if one starts from a non-Abelian gauge theory this problem may not arise since, in this case, it is, in general, possible to build models which are compatible with an FRW cosmology from the start. However, the presence of a mass term does not by itself lead to sufficient inflation and more involved potentials have to be used in order to achieve it [9,10]. In this paper, we shall only be interested in analysing the consequences of a mass term for the vector fields and therefore we do not find inflationary solutions.

The paper is organized as follows. In section 2 we derive the  $E^3$ -invariant ansatz for a triplet of massive vector fields with  $SO_1(3)$  global symmetry. In section 3 we briefly discuss non-trivial solutions to the coupled EYM equations with gauge group  $SO_1(3)$  which correspond to the spatially flat FRW universe. We then examine the case where the vector fields become massive and derive the corresponding equations of motion from an effective action built upon substitution of the  $E^3$ -symmetric ansatz in the initial action. In section 4 we analyse the solutions of the equations of motion using methods of the theory of dynamical systems and in section 5 we present our conclusions. Finally, in the appendix, we present a method to find  $G$ -symmetric ansatz (where  $G$  is the isometry group) for massive vector fields with an arbitrary global internal symmetry.

**2. Ansatz for vector fields with  $SO(3)$  global internal symmetry**

The spacetime in flat FRW universes has the form

$$M^4 = R^4 = R \times E^3/SO(3) \tag{1}$$

where the six-dimensional Euclidean group  $E^3$  is the isometry group of the spatial hypersurfaces

$$M^3 = R^3 = E^3/SO(3). \tag{2}$$

The isotropy group  $SO(3)$  leaves the origin  $o \in R^3 = E^3/SO(3)$  invariant but rotates the curves which pass through  $o$  and therefore defines a representation of  $SO(3)$  in the tangent space at the origin  $T_o R^3$ . This is the so-called isotropy representation which, in this case, is just the vector representation  $\underline{3}$  of  $SO(3)$ .

The fact that there are no singlets in the isotropy representation has far reaching implications. Indeed, if a vector field  $X$  in  $R^3$  is  $E^3$ -invariant then, in particular, it must be invariant at  $o$ , i.e. a singlet under the action of  $SO(3)$  on  $T_o R^3$ . Since this is not possible, we conclude that there are no  $E^3$ -invariant vector fields in  $R^3$ .

Being physically obvious for  $R^3$  (as it just means that there are no homogeneous and isotropic vector fields) this result is important as it applies for any homogeneous space  $G/H$  for which the isotropy representation has no trivial singlets. This is also the case for the spatial hypersurfaces  $S^3$  and  $H^3$  of closed and open FRW models which, therefore, do not have vector fields which are invariant under the action of the corresponding isometry group either. This explains why a single vector field is incompatible with FRW geometry; indeed, being non-spatially homogeneous and isotropic, the vector fields generate anisotropic energy-momentum tensors which therefore cannot be proportional to the Einstein tensor of the FRW metric.

In this section we shall study the case of flat FRW models, leaving the analysis of the more general case of a (not necessarily four-dimensional) spacetime of the form  $M^d = R \times G/H$  to the appendix. Models with  $d > 4$  are relevant to the study of the cosmological implications of Kaluza-Klein theories [7].

If, instead of a single vector field, we consider a theory with a multiplet of (covariant) vector fields  $A_\mu^a, a = 1, \dots, N$ , where  $a$  is an internal index, we may hope that, although the EMT corresponding to each field  $A_\mu^a$  is not  $E^3$ -invariant, it would be possible to relate the asymmetries of the different vector fields in such a way that the total EMT is  $E^3$ -invariant. Let us then consider the case of a triplet of massive vector fields with global  $SO_1(3)$  symmetry. The generators of the isometry group  $G = E^3$  satisfy the following commutation relations

$$\begin{aligned} [T_i, T_j] &= 0 \\ [Q_i, Q_j] &= \epsilon_{ijk} Q_k \\ [Q_i, T_j] &= \epsilon_{ijk} T_k \quad i, j, k = 1, 2, 3 \end{aligned} \tag{3}$$

so that the  $T_i$  span an (Abelian) invariant subalgebra of  $\text{Lie}(E^3)$ —the algebra of translations. The corresponding Killing vector fields are

$$X_i = \frac{\partial}{\partial x^i} \tag{4}$$

where  $x^i$  are the Cartesian coordinates in  $R^3$  and  $\partial/\partial x^i$  together with  $\partial/\partial t$  form a (global) moving frame in  $R^4 = R \times E^3/SO(3)$ ,  $\{X_\mu\} = \{\partial/\partial t, \partial/\partial x^i\}$ , with the dual coframe being given by  $\{\omega^\mu\} = \{dt, dx^i\}$  ( $\mu = 0, 1, 2, 3$ ). The remaining Killing vector fields, corresponding to the rotations  $Q_i$ , read

$$Y_i = -\epsilon_{ijk} x^j \partial/\partial x^k \quad (5)$$

so that

$$\mathcal{L}_{X_i} X_j = [X_i, X_j] = 0 \quad (6a)$$

$$\mathcal{L}_{Y_i} X_j = -\mathcal{L}_{X_j} Y_i = [Y_i, X_j] = \epsilon_{ijk} X_k \quad (6b)$$

$$\mathcal{L}_{Y_i} Y_j = [Y_i, Y_j] = \epsilon_{ijk} Y_k. \quad (6c)$$

The conditions of spatial homogeneity and isotropy read, infinitesimally

$$\mathcal{L}_{X_i} A = 0 \quad (7a)$$

$$\mathcal{L}_{Y_i} A = 0 \quad (7b)$$

where  $A = A_\mu^\alpha(t, x) \omega^\mu L_\alpha$ , the  $L_\alpha$  being the generators of the internal group  $SO_1(3)$

$$[L_a, L_b] = \epsilon_{abc} L_c. \quad (8)$$

In agreement with our previous considerations, we conclude that the only solution of (7) is the trivial one, i.e. such that only the zeroth component of the covector fields is different from zero:  $A = A_0^\alpha(t) dt L_\alpha$ . Condition (7b) is, however, too restrictive and we now consider a weaker version of it by allowing the field  $A$  to be such that infinitesimal spatial rotations generate internal  $SO_1(3)$  rotations

$$\mathcal{L}_{X_i} A = 0 \quad (9a)$$

$$\mathcal{L}_{Y_i} A = -[L_i, A]. \quad (9b)$$

It is clear that if (9) takes place, then, although  $A$  is not  $E^3$ -invariant, it generates an EMT that is invariant under the action of  $E^3$ . Indeed, the action of  $E^3$  in the EMT is equivalent to an internal rotation with respect to which the EMT is invariant

$$\mathcal{L}_{Y_i} \text{EMT}(A) = -\delta_{L_i} \text{EMT}(A) = 0. \quad (10)$$

The conditions for spatial homogeneity, (9a), imply that the components  $A_\mu^\alpha$  depend only on the time variable  $t$ . In order to solve (9b), let us consider the linear space of vector fields with constant coefficients:  $C = \{X = a^\mu X_\mu, a^\mu \in R\}$ . Equation (6b) implies that  $X \in C$  transforms as  $\underline{1} \oplus \underline{3}$  under  $SO(3)$  spatial rotations. On the other hand, the one-form  $A$  can be considered, for any  $t$ , as a linear mapping from  $C$  to the internal space (i.e. to the Lie algebra of  $SO_1(3)$ )

$$A(X) = A(a^\mu X_\mu) = A_\mu^\alpha(t) a^\mu L_\alpha \quad (11)$$

in which the triplet representation  $\underline{3}$  of  $SO_1(3)$  is realized. Rewriting (9b) in the form

$$A(\mathcal{L}_Y X) = [L_i, A(X)] \tag{12}$$

we see that  $A$  intertwines the representations  $\underline{1} \oplus \underline{3}$  and  $\underline{3}$  of  $SO(3)$  and  $SO_1(3) \stackrel{\text{isom}}{\cong} SO(3)$ , respectively. Schur's lemma then implies that  $A$  vanishes between subspaces of non-equivalent representations

$$A\left(\frac{\partial}{\partial t}\right) = 0 \iff A_0 = 0 \tag{13}$$

and is proportional to the identity operator between subspaces with equivalent representations

$$A\left(\frac{\partial}{\partial x_i}\right) = \chi_0(t)L_i \iff A_i^a(t) = \chi_0(t)\delta_i^a \tag{14}$$

where  $\chi_0(t)$  is an arbitrary function. Alternatively, the  $E^3$ -symmetric ansatz for  $A$  satisfying (9) and therefore generating  $E^3$ -invariant EMT, can be written as

$$A = \chi_0(t) dx^i L_i. \tag{15}$$

Some comments are in order here. Notice that (7) and (9) define  $E^3$ -symmetric fields in the case we are considering, i.e. a spacetime of the form (1). For a general definition of symmetric vector fields and the related one for symmetric gauge fields we refer the reader to the appendix and [4, 5, 11].

Conditions (7) and (9) are the infinitesimal form of the condition of  $G$ -symmetry defined in the appendix (equation (A.2)). We see from (A.2) that  $G$ -symmetric covector fields transforming according to the adjoint representation of the group  $K$  (i.e.  $\Psi \equiv Ad$ ) have, under the action of  $G$ , the same transformation properties as  $G$ -symmetric gauge fields with gauge group  $K$  (see [4, 5, 11]) but with the property that the gauge transformation which compensates a given action  $g^*$  of the symmetry group is a global gauge transformation. In the appendix we show that solutions of (A.2) can be found for special cases of homogeneous spaces (condition (A.4)), which include the spatial hypersurfaces of closed and flat FRW models.

In the principal bundle formulation (see [1, 5, 11, 12])  $G$ -symmetric gauge fields are pull-backs of  $G$ -invariant connections. The latter are completely classified by Wang's theorem (see [12]). Therefore, a  $G$ -symmetric gauge field  $A$ , in the base manifold  $M^d$ , is given by  $A = \sigma^*\omega$ , where  $\omega$  is a  $G$ -invariant connection in the principal  $K$ -bundle over  $M^d$  and  $\sigma$  is a local section of the bundle. The condition (A.2) is implied by the  $G$ -invariance of the connection  $\omega$ :  $g^*\omega = \omega$  (see [1, 5, 11]).

### 3. Effective action and field equations

#### 3.1. Massless case

We start with the following action for the coupled EYM system with  $SO_1(3)$  gauge group

$$S = \int_{M^4} d^4x \sqrt{-g} \left( \frac{1}{2k^2} R + \frac{1}{8e^2} \text{Tr} [F_{\mu\nu} F^{\mu\nu}] \right) \tag{16}$$

where  $k^2 = 8\pi G$  and  $e$  is the gauge coupling. Our conventions and notation for the gravitational part of the action correspond to those of [3,4]. The gauge field strength is  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ , where  $A_\mu = A_\mu^a L_a$ . We assume that the spacetime manifold  $M^4$  is topologically of the type (1), where  $\{t_o\} \times E^3/SO(3) = \{t_0\} \times R^3$ ,  $t_o \in R$ , are the spatial orbits of the isometry group  $G = E^3 = R^3SO(3)$ .

The FRW ansatz, for flat models, gives the most general form of an  $E^3$ -invariant metric

$$ds^2 = -N^2(t) dt^2 + a^2(t) ((dx^1)^2 + (dx^2)^2 + (dx^3)^2) \quad (17)$$

where the lapse function  $N(t)$  and the scale factor  $a(t)$  are arbitrary non-vanishing functions.

Cosmological models in the presence of gauge fields have been investigated in some detail in the literature [1-3, 7, 8]; for these models the local symmetry of the action allows the use of the theory of symmetric gauge fields [5] in order to make the necessary ansatz. The  $G$ -symmetric ansatz corresponds to fields which, under the action of the isometry group  $G$ , undergo a local internal rotation; however, in theories with global symmetry such fields do not, in general, lead to  $G$ -invariant EMTs, as can be seen from (10) (for more details, the reader is referred to the appendix). Ansatz (15), for (covariant) vector fields with global  $SO_1(3)$  internal symmetry is also valid in the case of action (16), with local  $SO_1(3)$  symmetry, and we shall use it in what follows. It is easy to check that this ansatz leads to a (traceless) EMT with the required  $E^3$ -invariance

$$T_o^0 = -\hat{\rho}(t) \quad T_i^j = \hat{p}(t) \delta_i^j \quad (18)$$

i.e. it corresponds to the EMT of a perfect comoving fluid with energy density  $\hat{\rho}$  and pressure  $\hat{p}$  given by

$$\begin{aligned} \hat{\rho} &= \frac{3}{4a^2 e^2 N} \left[ \frac{\dot{\chi}_0^2}{2} + \frac{N^2}{a^2} V_{gf} \right] \\ \hat{p} &= \frac{1}{4a^2 e^2 N} \left[ \frac{\dot{\chi}_0^2}{2} + \frac{N^2}{a^2} V_{gf} \right] \end{aligned} \quad (19)$$

$$V_{gf} = \frac{1}{8} \chi_0^4$$

where the dot denotes the derivative with respect to  $t$ ; moreover, the equation of state for the system is clearly that of a radiation fluid;  $\hat{\rho} = 3\hat{p}$ .

There are, for this model, two equivalent methods with which to obtain the dynamical equations for the functions  $a(t)$  and  $\chi_0(t)$ ; one can either substitute the ansatz, (13), (14) and (17), into the field equations derived from action (16), or substitute the ansatz directly into the action and then get the dynamical equations from this effective action. The latter method has the advantage of simplifying the study of the equations of motion and we shall adopt it in what follows. The effective action is given by (disregarding the infinite volume of the spatial hypersurface)

$$S_{\text{eff}} = \int_{t_1}^{t_2} dt \left[ -\frac{3}{k^2} \frac{\dot{a}^2 a}{N} + \frac{3a}{4e^2 N} \left( \frac{\dot{\chi}_0^2}{2} - \frac{N^2}{a^2} V_{gf} \right) \right]. \quad (20)$$

The equations of motion are obtained upon variation of this action with respect to  $N(t)$ ,  $a(t)$  and  $\chi_0(t)$ . We get, respectively, in the 'gauge'  $N = 1$

$$H^2 = \frac{1}{3}k^2 \hat{\rho} \tag{21a}$$

$$\dot{H} + H^2 = -\frac{k^2}{6}(\hat{\rho} + 3\hat{p}) \tag{21b}$$

$$\ddot{\chi}_0 + \dot{\chi}_0 H = -\frac{1}{2a^2}\chi_0^3 \tag{21c}$$

with  $\hat{\rho}$  and  $\hat{p}$  given by (19) and  $H \equiv \dot{a}/a$ .

The Yang-Mills equation, (21c), takes a much simpler form in the conformal time [3, 4]

$$d\eta = dt/a(t) \tag{22}$$

since, in this case, this equation decouples from the remaining ones. Indeed, substituting (22) in (21c), we obtain

$$\chi_0'' = -\chi_0^3/2 \tag{23}$$

where the prime denotes the derivative with respect to the conformal time. The solution of (23) is given by

$$\chi_0 = f(\eta; E, \eta_0) \tag{24}$$

where  $f = f(\eta; E, \eta_0)$  is a function defined implicitly by

$$\eta - \eta_0 = (2E)^{-1/4} F(\gamma(f, E), r). \tag{25}$$

In (25)  $F(x, k)$  denotes the elliptic function of the first kind [13],

$$\begin{aligned} \gamma(f, E) &= \cos^{-1} \left[ (8E)^{-1/4} f \right] \\ r &= \frac{1}{2}\sqrt{2} \end{aligned} \tag{26}$$

and the constant  $E$  is the conserved 'mechanical energy' of the solution

$$E = \frac{1}{2}(\chi_0')^2 + V_{\text{gf}}. \tag{27}$$

Using a 'mechanical' analogy, we can consider equation (23) as describing the motion of a particle with kinetic energy  $\chi_0'^2/2$ , potential energy  $V_{\text{gf}}$  and total energy  $E$ . This solution therefore describes a particle oscillating in a potential well between the turning points:  $-(8E)^{1/4}$  and  $(8E)^{1/4}$ .

3.2. *Massive case*

Consider now the action which results from adding a mass term for the gauge fields (vector fields in the presence of gravity have been discussed in, for example, [14]) to the action of (16)

$$S' = \int_{M^4} d^4x \sqrt{-g} \left( \frac{1}{2k^2} R + \frac{1}{8e^2} \text{Tr} [F_{\mu\nu} F^{\mu\nu}] + \frac{1}{2} m^2 \text{Tr} [A_\mu A^\mu] \right). \tag{28}$$

Such a mass term would arise, e.g., via the Higgs mechanism, with a complex doublet of scalar particles acquiring a vacuum expectation value. This action is no longer gauge invariant and only a global  $SO_1(3)$  symmetry remains. Substituting the ansatz, (13), (14) and (17), into the EMT corresponding to the action (28), we obtain

$$\begin{aligned} T_o^0 &= -\hat{\rho} - \frac{3}{4} m^2 (\chi_0/a)^2 \equiv -\rho \\ T_i^j &= \left( \hat{p} - \frac{1}{4} m^2 (\chi_0/a)^2 \right) \delta_i^j \equiv p \delta_i^j \end{aligned} \tag{29}$$

with  $\hat{\rho}$  and  $\hat{p}$  given by (19), which shows that the perfect fluid form of the EMT is preserved in spite of the presence of the mass term. Notice that, if instead of a mass term we had a quartic potential, this could be absorbed in a redefinition of  $V_{gf}$ , which would lead us back to the results of the previous subsection.

We are interested in spatially homogeneous and isotropic cosmologies associated with the action (28). Hence, we restrict ourselves to the set of  $E^3$ -symmetric configurations for the fields  $g_{\mu\nu}$  and  $A_\mu$ . Substituting these ansatz into (28), we obtain for the effective action

$$S_{\text{eff}} = \int_{t_1}^{t_2} dt \left[ -\frac{3}{k^2} \frac{\dot{a}^2 a}{N} + \frac{3a}{4e^2 N} \left( \frac{\dot{\chi}_0^2}{2} - \frac{N^2}{a^2} V_{gf} \right) - \frac{3}{4} N m^2 \chi_0^2 a \right]. \tag{30}$$

The equations of motion, derived upon variation with respect to  $N(t)$ ,  $a(t)$  and  $\chi_0(t)$  are, respectively, in the ‘gauge’  $N = 1$ :

$$H^2 = \frac{1}{3} k^2 \rho \tag{31a}$$

$$\dot{H} + H^2 = -\frac{1}{6} k^2 (\rho + 3p) \tag{31b}$$

$$\ddot{\chi}_0 + \dot{\chi}_0 H = -\frac{1}{2a^2} \chi_0^3 - 2m^2 e^2 \chi_0 \tag{31c}$$

with  $\rho$  and  $p$  as given by (29). Consider the following change of variables

$$x = k \frac{\chi_0}{a} \quad y = \frac{k^2}{2\sqrt{2}} \frac{\dot{\chi}_0}{a} \quad z = kH \quad \tau = \frac{t}{k}. \tag{32}$$

The system of (31), in the new (dimensionless) variables, can be written as

$$x_\tau = 2\sqrt{2}y - xz \tag{33a}$$

$$y_\tau = -\frac{1}{2}\sqrt{2}\mu^2 x - 2yz - \frac{1}{8}\sqrt{2}x^3 \tag{33b}$$

$$z_\tau = -2z^2 + \frac{1}{4}\mu^2 x^2 \tag{33c}$$

$$z^2 = y^2 + \frac{1}{32}x^4 + \frac{1}{4}\mu^2 x^2 \tag{33d}$$

where  $\mu^2 = m^2 k^2$  and we have set  $e = 1$ ; the index  $\tau$  denotes the derivative with respect to  $\tau$ . Equations (33a)–(33c) define a three-dimensional dynamical system in the variables  $x, y, z$  and (33d) is a constraint equation which defines, in  $R^3$ , the phase space of the dynamical system. Substituting the constraint into (33a) and (33b), we are left with a two-dimensional dynamical system in the variables  $x, y$

$$\begin{aligned} x_\tau &= 2\sqrt{2}y - x \left( y^2 + \frac{1}{32}x^4 + \frac{1}{4}\mu^2 x^2 \right)^{1/2} \\ y_\tau &= -\frac{1}{2}\sqrt{2}\mu^2 x - 2y \left( y^2 + \frac{1}{32}x^4 + \frac{1}{4}\mu^2 x^2 \right)^{1/2} - \frac{1}{8}\sqrt{2}x^3 \end{aligned} \tag{34}$$

where we have considered the positive root of the constraint (33d) as we are interested in expanding models.

It is this system of differential equations (34) that we are going to study using the methods of the qualitative theory of dynamical systems [15].

#### 4. Qualitative analysis

The dynamical system (34) has just one critical point in the finite region of variation of  $x, y$ , the origin  $F(0, 0)$  or simply  $F$ . The remaining critical points, if any, will lie at infinity. In order to study them, we complete the phase space with an infinitely distant boundary (for which  $x^2 + y^2 = +\infty$ ). It is also convenient to perform a change of variables to polar coordinates,  $(x, y) \rightarrow (r, \theta)$ , followed by a compactification of the entire phase space into a circle of unit radius. This can be achieved through the introduction of yet new radial and time coordinates  $\rho$  and  $\zeta$ , such that

$$r = \frac{\rho}{1 - \rho} \quad (0 \leq \rho \leq 1) \quad \frac{d\zeta}{d\tau} = \frac{1}{(1 - \rho)^2}. \tag{35}$$

In the variables  $(\rho, \theta, \zeta)$ , the system (34) becomes

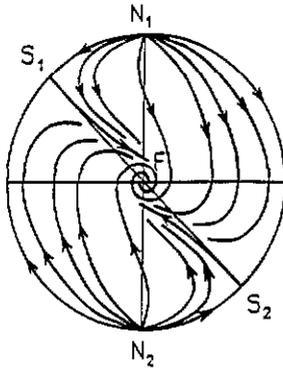
$$\begin{aligned} \rho_\zeta &= \sqrt{2} \left( 2 - \frac{1}{2}\mu^2 \right) \rho (1 - \rho)^3 \sin \theta \cos \theta - \rho^2 (1 - \rho) (1 - \sin^2 \theta) f(\rho, \theta) \\ &\quad - \frac{1}{8}\sqrt{2}\rho^3 (1 - \rho) \sin \theta \cos^3 \theta \equiv \Pi(\rho, \theta) \\ \theta_\zeta &= -\sqrt{2} (1 - \rho)^2 \left( 2 \sin^2 \theta + \frac{1}{2}\mu^2 \cos^2 \theta \right) - \frac{1}{8}\sqrt{2}\rho^2 \cos^4 \theta \\ &\quad - \rho \sin \theta \cos \theta f(\rho, \theta) \equiv \Psi(\rho, \theta) \end{aligned} \tag{36}$$

where  $f^2(\rho, \theta) = (1 - \rho)^2 \sin^2 \theta + \frac{1}{32}\rho^2 \cos^4 \theta + \frac{1}{4}\mu^2 (1 - \rho)^2 \cos^2 \theta$ .

The critical points in the infinitely distant boundary ( $\rho = 1$ ) are the solutions of the equations:  $\Pi(1, \theta) = 0$  and  $\Psi(1, \theta) = 0$ , namely  $N_1(1, \frac{1}{2}\pi)$ ,  $N_2(1, \frac{3}{2}\pi)$ ,  $S_1(1, \frac{3}{4}\pi)$ ,  $S_2(1, \frac{7}{4}\pi)$ —see figure 1.

We now turn to the analysis of the nature of the critical points of the system. First, we would like to point out that, since (36) is invariant under  $\theta \rightarrow \theta + \pi$ , we need only consider two of the four critical points at infinity; hence, we are left with three points to analyse,  $F(0, 0)$  and, e.g.,  $N_1(1, \frac{1}{2}\pi)$  and  $S_1(1, \frac{3}{4}\pi)$ .

Regarding  $F(0, 0)$ , it is easy to check that this point is degenerate (see table 1) and therefore the Hartman–Grobman theorem [15] does not apply. We resort to the method known as ‘blow-up’ [15], which allows us to establish that  $F$  is a focus.



**Figure 1.** All trajectories of the phase diagram of the model asymptotically approach the focus  $F$ ;  $N_1, N_2, S_1, S_2$  are critical points at infinity. Near the focus, trajectories change from oblate to prolate shape as the mass of the vector bosons increases.

**Table 1.** Classification of critical points from the analysis of the dynamical system of (34) and (36)

Point	Eigenvalues	Eigenvectors	Classification
$F(0, 0)$	$\pm i\sqrt{2}\mu$		Focus
$S_1(1, \frac{3}{4}\pi)$	$\frac{1}{16}\sqrt{2}$ $-\frac{1}{16}\sqrt{2}$	$\partial/\partial\rho$ $\partial/\partial\theta$	Saddle
$N_1(1, \frac{1}{2}\pi)$	2 1	$\partial/\partial\rho - 2\sqrt{2}\partial/\partial\theta$ $\partial/\partial\theta$	Unstable node

In order to investigate its stability, we use Liapunov’s theorem [15]. Consider the Liapunov function

$$\mathcal{L}(x, y) = 2(y^2 + \frac{1}{32}x^4 + \frac{1}{4}\mu^2x^2). \tag{37}$$

Since  $\mathcal{L}(x, y)$  obeys the conditions

$$\begin{aligned} \mathcal{L}(0, 0) &= 0 & \mathcal{L}((x, y) \neq (0, 0)) &> 0 \\ \frac{d\mathcal{L}}{d\zeta} &< 0 & \text{in } R^2/\{(0, 0)\} \end{aligned} \tag{38}$$

we conclude that  $F$  is a stable focus and all trajectories in the  $x, y$  plane asymptotically approach this point. Furthermore, trajectories in the neighbourhood of  $F$  are clockwise-directed spirals.

Regarding the critical point at infinity  $S_1$ , the linear approximation of the system (36) around this point is sufficient and we conclude that  $S_1$  is a saddle point (see table 1). As for  $N_1$ , the situation is similar after we perform a change of time variable ( $\zeta \rightarrow \hat{\zeta}$ ) such that  $d\hat{\zeta}/d\zeta = (\rho - 1)$  and we conclude that  $N_1$  is an improper unstable node (see table 1). We shall also use the fact that the trajectories in the vicinity of  $N_1$  are tangential to the eigenvector which corresponds to the lowest eigenvalue—see table 1 and figure 1.

In order to obtain a more detailed picture of phase space we divide it into four distinct regions and find approximate analytic expressions for the trajectories in each of these regions, namely

$$\begin{aligned}
 \text{(a)} \quad x &= \frac{2}{\mu} \frac{1}{\tau + C_1} \cos \tau & y &= \frac{1}{\tau + C_1} \sin \tau & |x| \ll 1; |y| \ll 1 \\
 \text{(b)} \quad x + y &= 2C_2 x^2 & |x| \simeq |y| &\gg 1 \\
 \text{(c)} \quad y &= x + C_3 & |x| \gg |y| &\gg 1 \\
 \text{(d)} \quad y &= C_4 x^2 & |y| \gg x^2; |x|, |y| &\gg 1
 \end{aligned} \tag{39}$$

where the  $C_i$  ( $i = 1, \dots, 4$ ) are arbitrary constants. In case (a), i.e. near the origin, we see that the shape of the trajectories depends on the value of  $\mu^2$ . In fact, for small values of  $\mu^2$  the trajectories wind around  $F$  with an oblate shape whereas for large values of  $\mu^2$  they have a prolate shape. It is indeed in this region that the behaviour of the system is most sensitive to the presence of the mass term.

We now turn our attention to the analysis of the effective equation of state for the system near the critical points. In the vicinity of  $S_1$ ,  $\theta \simeq \frac{3}{4}\pi$ ; introducing this approximation into the dynamical system, we get the following solutions

$$a = a_0 t^{1/2} \quad \chi_0 = -\frac{2a_0}{(2k^2)^{1/4}} \quad (t \rightarrow 0). \tag{40}$$

Equation (29) then implies  $\rho \simeq 3p$ , indicating that the effective equation of state near  $S_1$  is approximately that of a radiation fluid. In fact this is true everywhere except in the vicinity of  $F$ . This is to be expected since, were it not for the mass term, the equation of state would be  $\rho = 3p$  throughout all regions of phase space and, as we have already mentioned, it is near the origin that the mass term most affects the trajectories.

The effective equation of state near the focus can be analysed using a procedure similar to that for  $S_1$ , which leads us to conclude that

$$a \sim t^{2/3} \quad \chi_0 \sim t^{-1/3} \sin \frac{t}{\sqrt{k}} \quad (t \rightarrow +\infty) \tag{41}$$

implying that  $p \simeq 0$ , corresponding to the equation of state for dust matter.

In addition, we find that the time of transition between the regimes described by the different equations of state depends on the value of  $m^2$  such that the transition occurs earlier for more massive vector fields. Before the transition the energy density and pressure evolve as expected in a radiation-dominated universe and, as the system approaches the focus  $F$ , the universe becomes matter dominated and the pressure starts to oscillate around zero (see figure 2) with ever decreasing amplitude and quickly vanishes.

### 5. Conclusions

We have shown that the Einstein–Yang–Mills theory, with gauge group  $SO_1(3)$ , broken by the presence of a mass term for the gauge fields in such a way that

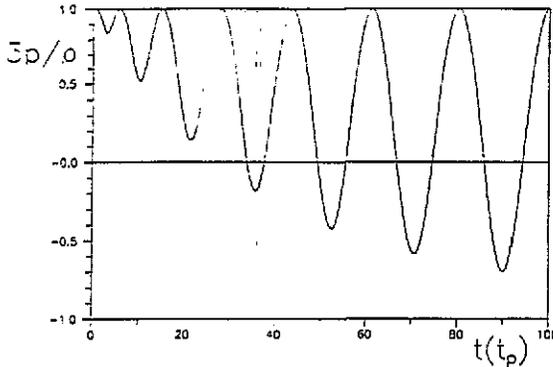


Figure 2.  $3p/\rho$  as a function of time showing the change of the equation of state of the universe from radiation dominated to matter dominated.

only a global  $SO_1(3)$  symmetry remains, admits homogeneous and isotropic flat FRW solutions. To prove this, we have constructed an  $E^3$ -symmetric ansatz for the (massive) vector fields, parametrized in terms of an arbitrary function of time  $\chi_0(t)$ .

We have studied the equations of motion of this theory using the methods of the theory of dynamical systems and found that the system has just one stable critical point, the origin, which is a focus. We have checked that the behaviour of the system is basically the same as for the unbroken phase except near the origin; in this region, the change is such that the effective equation of state of the system becomes that of a matter-dominated universe.

## Appendix

Let us consider a spacetime of the form

$$M^d = R \times G/H \quad (\text{A.1})$$

where  $\{t_o\} \times G/H$  are spatial hypersurfaces,  $G$  is the isometry group and  $t \in R$  is a time-like coordinate. There are two classes of models for which vector fields can be compatible with this geometry, in the sense that solutions of the coupled Einstein-vector field equations can be found in a simple way: non-Abelian gauge theories and theories with a global internal symmetry. In the former case, since the theory is invariant under an internal local symmetry, the total EMT is also gauge invariant, which implies that, for the total EMT to be  $G$ -invariant, the fields  $A_\mu^a$  need only to be  $G$ -invariant up to a gauge transformation. For a gauge group  $K$  containing an  $H$  subgroup such fields often exist, which has allowed the study of FRW cosmologies in the presence of non-Abelian gauge sectors [3, 7, 8]. The latter case has, however, to our knowledge, not yet been studied in the literature. In section 2, we analysed a particular example, with  $G/H = E^3/SO(3)$  and  $K = SO_1(3)$ . We now study the general case.

Consider configurations  $A_\mu^a(x)$  for which the action of  $g \in G$  can be compensated by an internal global  $K$ -rotation

$$g^* (A_\mu^a(x) dx^\mu) = \Psi(\lambda(g^{-1}))_b^a A_\mu^b(x) dx^\mu \quad (\text{A.2})$$

where  $\lambda: G \rightarrow K$  is an homomorphism and  $\Psi$  is the representation of  $K$  in the internal space. Clearly, configurations satisfying (A.2) will generate  $G$ -invariant (spatially homogeneous and isotropic in the FRW case) EMTs. Notice that condition (A.2) is less restrictive than that of  $G$ -invariance (corresponding in (A.2) to  $\lambda(g) = e, \forall g \in G$ ) but still much more restrictive than that for gauge theories since the compensating internal rotation,  $\Psi(\lambda(g))$ , cannot depend on the points  $x \in R \times G/H$ . Nevertheless, we will show that, for a class of homogeneous spaces  $G/H$  which includes the spatial hypersurfaces of closed and flat FRW models, field configurations satisfying (A.2) can be found.

Let  $\{T_i\}_{i=1}^{\dim G}$  be a basis in  $\text{Lie}(G)$ , the Lie algebra of the isometry group  $G$ , satisfying:  $[T_i, T_j] = C_{ij}^k T_k$  and  $\{X_i\}$  be the associated Killing vector fields

$$X_i(x) = \frac{d}{d\epsilon} (\exp(-\epsilon T_i)x)|_{\epsilon=0}. \tag{A.3}$$

We shall assume that the homogeneous space  $G/H$  is such that there exists in  $\text{Lie}(G)$  an invariant subalgebra (ideal) with basis  $\{T_i\}_{i=1}^{\dim G/H}$

$$[T_i, T_j] = C_{ij}^k T_k \tag{A.4}$$

and such that  $\{X_i\}_{i=1}^{\dim G/H}$  is a (local) moving frame in  $G/H$ . This is obviously the case for flat (see section 2) and closed FRW models. Consider now the moving frame in  $R \times G/H$ ,  $\{X_\mu\}_{\mu=0}^{\dim G/H} = \{\partial/\partial t, X_i\}_{i=1}^{\dim G/H}$ , and the moving coframe  $\{\omega^\mu\}_{\mu=0}^{\dim G/H} = \{dt, \omega^i; i = 1, \dots, \dim G/H\}$  dual to  $\{X_\mu\}_{\mu=0}^{\dim G/H}$ . The commutation relations define a representation of  $\text{Lie}(G)$  acting in the subalgebra of vector fields

$$C = \{X = a^\mu X_\mu, a^\mu \in R\} \tag{A.5}$$

according to

$$\mathcal{L}_{X_i} X = a'^{\mu} X_\mu \tag{A.6}$$

where

$$a'^0 = 0 \quad a'^i = C_k^i \wedge_1 a^k. \tag{A.7}$$

In the coframe  $\{\omega^\mu\}$ , we choose the fields  $A_\mu^a$  to depend only on the coordinate  $t$

$$A^a = A_\mu^a(t)\omega^\mu. \tag{A.8}$$

As the  $\omega^\mu$  are not  $G$ -invariant, the fields  $A^a$  are not  $G$ -invariant either. Infinitesimally, condition (A.2) applied to (A.8) gives

$$\mathcal{L}_{X_i} A^a = -\Psi(\lambda(T_i))_b^a A^b \tag{A.9}$$

or equivalently

$$A^a(\mathcal{L}_{X_i} X) = \Psi(\lambda(T_i))_b^a A^b(X). \tag{A.10}$$

These are linear algebraic constraints on  $A_\mu^a(t)$ , which imply that, as a mapping from  $C$  to the internal space (for fixed  $t$ )

$$A_\mu^a : \quad a^\mu \rightarrow B^a = A_\mu^a a^\mu \quad (\text{A.11})$$

$A_\mu^a$  intertwines the representation of  $\text{Lie}(G)$  in  $C$  with the representation of  $\lambda(\text{Lie}(G))$  in the internal space. From Schur's lemma, we then conclude that there exist non-trivial solutions of (A.10) if and only if there are equivalent irreducible representations in  $C$  and in the internal space. Therefore, in order to make a  $G$ -symmetric ansatz for the fields  $A_\mu^a$ , one must

(i) check whether (A.4) takes place for the given  $G$  and  $G/H$  and choose the associated coframe  $\{\omega^\mu\}$ ;

(ii) choose an homomorphism  $\lambda$  of the isometry group  $G$  to the internal symmetry group  $K$  such that there are equivalent irreducible representations in the representations of  $G$  in  $C$  (see (A.6), (A.7)) and of  $\lambda(G)$  in the internal space;

(iii) find the intertwining operators  $A_\mu^a$  between the two representations and substitute them in (A.8).

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