1 Chern character of a finitely generated projective module

Let A be an algebra over k. For a finitely generated projective module P over A we construct a class in $HC_0^{per}(A)$ as follows.

Since P is finitely generated projective, it is of the form

$$A^N e$$

for some idempotent e in $M_N(A)$.

$$ch(e) = e + \sum_{n=1}^{\infty} \frac{(-1)^n (2n)!}{n!} (e - \frac{1}{2}) \otimes e^{\otimes 2n} \tag{1.1}$$

Theorem 1.1

$$(B + b)ch(e) = 0$$

Proof. Direct computation

Applying the map

$$tr : CC_*^{per}(M_N(A)) \to CC_*^{per}(A)$$

we get the cycle in periodic cyclic homology of A called the **Chern character** of e and also denoted by $ch(e)$.

Theorem 1.2 *The homology class of $ch(e)$ depends only on the isomorphism class of the projective A-module $A^N e$.*

Proof. The proof will be given later.

2 Group homology and periodic cyclic cohomology

Let Γ denote a discrete group. Our goal in this section is to construct a morphism of complexes

$$\chi : C_*(\Gamma, \mathbb{Z}) \to CC_*^{per}(k[\Gamma]). \tag{2.2}$$

More generally, let us consider a group Γ together with a homomorphism

$$\epsilon : \Gamma \to \mathbb{Z}/2\mathbb{Z}.$$

We will identify $\mathbb{Z}/2\mathbb{Z}$ with the multiplicative group { 1,-1} and let $\mathbb{Z}_\epsilon$ denote the Γ-module $\mathbb{Z}$ with the action given by

$$\gamma m = \epsilon(\gamma) m.$$

Below we will construct a morphism of complexes

$$\chi : C_*(\Gamma, \mathbb{Z}_\epsilon) \to CC_*^{per}(k[\Gamma]), \tag{2.3}$$

where the right hand side represents the periodic cyclic complex of the $\mathbb{Z}/2\mathbb{Z}$-graded algebra:

$$k[\Gamma] = k[\Gamma_0] \oplus k[\Gamma_1]; \quad \Gamma_i = \{\gamma \in \Gamma | \ \epsilon(\gamma) = (-1)^i\}.$$

Construction of χ

Step 1. **Denis map**

This is a morphism of complexes

$$\begin{array}{l} D: \ C_*(\Gamma, \mathbb{Z}_\epsilon) \to C_*(k[\Gamma]) \\ D(g_1, \ldots, g_n) = \prod_{i \geq 1} \epsilon(g_{2i}) \ g_n^{-1} \ldots g_1^{-1} \otimes g_1 \otimes \ldots \otimes g_n \end{array} \tag{2.4}$$

It is easy to see that D intertwines the Hochschild differential in the R.H.S. with the differential

$$(g_1, \ldots, g_n) \ \mapsto \ \epsilon(g_1)((g_2, \ldots, g_n) + \sum_{i=1}^{n-1} (g_1, \ldots, g_i g_{i+1} \ldots, g_n) + (-1)^n (g_1, \ldots, g_{n-1}). \tag{2.5}$$

Step 2.

Let $R_*(\Gamma)$ denote the standard bar resolution of the trivial Γ-module $\mathbb{Z}$:

$$\begin{array}{l} R_n(\Gamma) = \mathbb{Z}\Gamma^{n+1} \\ d_i(g_0, \ldots, g_n) = (g_0, \ldots, g_i g_{i+1}, \ldots, g_n), \ i < n; \\ d_n(g_0, \ldots, g_n) = (g_0, \ldots, g_{n-1}); \ \partial = \sum_{i=0}^n (-1)^i d_i; \\ s_i(g_0, \ldots, g_n) = d_i(g_0, \ldots, g_i, 1, g_{i+1}, \ldots, g_n). \end{array} \tag{2.6}$$

The Denis map induces a morphism of complexes

$$\mathbb{Z}_\epsilon \otimes R_*(\Gamma) \ \to \ C_*(k[\Gamma]) \tag{2.7}$$

We define

$$t(g_0, \ldots, g_n) = (g_0 \ldots g_n, g_n^{-1} \ldots g_1^{-1}, g_1 \ldots, g_{n-1}) \tag{2.8}$$

It is not difficult to check that this together with the d_i's from above endowes $R_*(\Gamma)$ with the structure of a cyclic object in the category of Γ-modules and that the Denis map is in fact a morphism of cyclic objects.

Recall that the (B,b) periodic double complex of a cyclic object is the following:

$$\begin{array}{l} CC_*^{per}(R_*) = \prod_{i \cong n (mod 2)} R_n / \sum_{i \geq 0} \mathrm{Im} s_i; \\ b = \sum_{i=0}^n d_i; \ B = s \sum_{i=0}^n (-1)^{ni} t^i, \end{array} \tag{2.9}$$

where $s = t s_n$. This gives us the map of complexes:

$$D: \ CC_*^{per}(\mathbb{Z}_\epsilon \otimes_{\mathbb{Z}\Gamma} R_*(\Gamma)) \ \to \ CC_*^{per}(k[\Gamma]). \tag{2.10}$$

Note that $B|_{R_0(\Gamma)} = 0$, in fact

$$B(g_0) = t s_0(g_0) = t(g_0 \otimes 1) = g_0 \otimes 1 = s_0 g_0 = 0.$$

Remark 2.1 *The map D above is a split monomorphism of complexes. Its image is the direct summand of $CC^{per}_*(k[\Gamma])$ given by the linear span of*

$$\{g_0 \otimes \ldots \otimes g_n | g_0 \ldots g_n = 1\}.$$

Step 3. **construction** of χ

Our map χ will be the composition of D above (2.10) with a morphism:

$$\phi : C_*(\Gamma, \mathbb{Z}_\epsilon) \to CC^{per}_*(\mathbb{Z}_\epsilon \otimes_{\mathbb{Z}\Gamma} R_*(\Gamma)) \tag{2.11}$$

constructed by recursion as follows. It is sufficient to construct a Γ equivariant map from the standard resolution $R_*(\Gamma)$ to $CC^{per}_* R_*(\Gamma))$. This has to be of the form

$$\phi = \nVdash + \sum_{i \geq 1} \phi_i \; ; \phi_i : R_* \to R_{*+2i}$$

where the equation $(B + b)\phi = \phi b$ gives the recursive system

$$B = \phi_1 b - b\phi_1; \;\; B\phi_i = \phi_{i+1} b - \phi_{i+1} b. \tag{2.12}$$

Now suppose that ϕ_i have been constructed for all $i \leq m$ so that the above equations are satisfied. Then

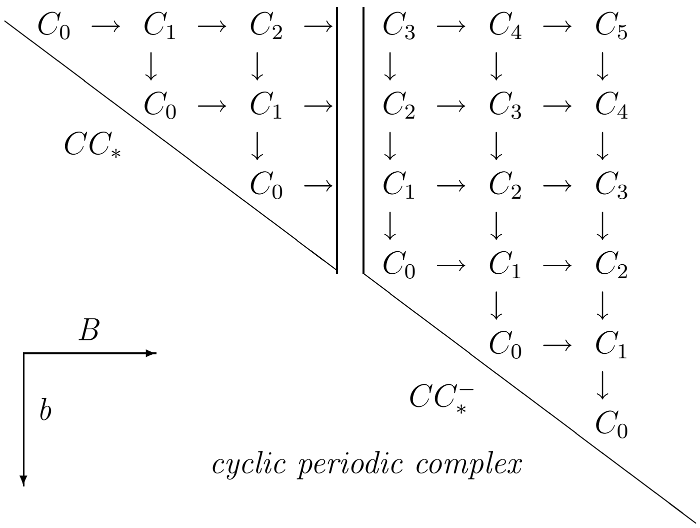
$$[b, B\phi_m] = -B[b, \phi_m] = BB\phi_{m-1} = 0,$$

hence $B\phi_m$ is an endomorphism of R_* of degree 2m+1 which commutes with b, therefore, since R_* is a projective resolution of $\mathbb{Z}$, ϕ_{m+1} exists (and can be explicitly constructed).

Remark 2.2 *Note that in fact we have constructed a morphism of complexes*

$$\chi :\to CC^-_*(k[\Gamma]), \tag{2.13}$$

where the negative cyclic homology is defined as the kernel of the projection of the periodic cyclic complex to its left half, the cyclic homology:



cyclic periodic complex

3 Chern character in higher algebraic K theory

Let A be a $\mathbb{Z}/2\mathbb{Z}$-graded algebra over k and P_0 and P_1 be free modules over A of rank n and m respectively. Set

$$M_{n|m}(A) = End_A(P_0 \oplus P_1). \tag{3.14}$$

We will view $P = P_0 \oplus P_1$ as a $\mathbb{Z}/2\mathbb{Z}$-graded A-module. In particular $M_{n|m}(A)$ becomes a $\mathbb{Z}/2\mathbb{Z}$-graded algebra.

Definition 3.1 *$GL_{m,n}(A)$ is the group of homogeneous invertible elements of $M_{n|m}(A)$ equipped with the homomorphism*

$$\epsilon : GL_{m,n}(A) \to \mathbb{Z}/2\mathbb{Z};\ \ \epsilon(g) = (-1)^{deg\ g}.$$

Now the composition

$$C_*(GL_{m,n}(A)_h) \xrightarrow{\chi} CC_*^{per}(k[GL_{m,n}(A)_h]) \xrightarrow{id} CC_*^{per}(M_{n|m}(A)) \xrightarrow{tr_s} CC_*^{per}(A) \tag{3.15}$$

gives the **chern character** map:

$$ch : C_*(GL_{m,n}(A)_h, \mathbb{Z}_\epsilon) \to CC_*^{per}(A). \tag{3.16}$$

In particular, for a k-algebra A, if we set

$$GL(A) = \lim_{\to} GL_n(A)$$

we get

$$ch : K_i(A) \to H_I(GL(A)) \to HC_i^{per}(A), \tag{3.17}$$

the chern character in higher algebraic K-theory. Note that this factors in fact through the negative cyclic homology:

$$ch : K_i(A) \to HC_i^-(A) \to HC_i^{per}(A) \tag{3.18}$$

3.1 Regulators for nilpotent algebras

Recall that if A is an algebra without unit, we denote by $\tilde{A}$ the unitalisation of A and define

$$\begin{array}{lcl} K_i(A) & = & ker(K_i(\tilde{A}) \to K_i(k)) \\ HC_i^{per}(A) & = & ker(HC_i^{per}(\tilde{A}) \to HC_i^{per}(k)). \end{array} \tag{3.19}$$

Let A be a nilpotent algebra without unit. As we shall see, in this case the periodic cyclic homology is zero and therefore

$$HC_i^-(A) \xrightarrow{\sim} HC_{i-1}(A),$$

hence we get the **regulators**

$$\tau_i : K_i(A) \mapsto HC_{i-1}(A). \tag{3.20}$$

In fact in the case when char(k)=0, these are isomorphisms [**?**].

3.2 Explicit formulas for χ and ch

The morphism χ above was defined by an explicit recursion procedure, which we will make explicit for the case of one-cochains on the group Γ.

Proposition 3.2 *Let g be an element of a discrete group Γ. Let $\epsilon : \Gamma \to \{\pm 1\}$ be a given group homomorphism. Then*

$$\chi(g) = \sum_{n \geq 1} (n-1)!(-\epsilon(g))^n (g^{-1} \otimes g)^{\otimes n}. \tag{3.21}$$

Proof. The values of the maps ϕ_i on the element $(1, g)$ of $R_1(\Gamma)$ are given by the relations

$$B\phi_i(1, g) = (\phi_{i+1} b - b\phi_{i+1})(1, g) = -b\phi_{i+1}(1, g),$$

following from the fact that $\phi_i|_{R_0}$ are zero for $i \geq 1$. Therefore we can set

$$\phi_{i+1}(1, g) = -sB\phi_i(1, g),$$

where

$$\begin{array}{ccc} R_* & \xrightarrow{s} & R_{*+1} \\ (g_0, \ldots, g_n) & \mapsto & (1, g_0, \ldots, g_n) \end{array} \tag{3.22}$$

is the contracting homotopy for the acyclic complex R_*. The claim of the proposition now follows from the formula for the differential B in $CC_*^{per}(R_*(\Gamma))$.

4 Chern character of locally free sheaves of modules

Let X be a topological space with a sheaf of rings $\underline{A}$, and let $\underline{F}$ be a locally free sheaf of $\underline{A}$-modules of rank N. Below we construct a caracteristic cohomology class of $\underline{F}$, $ch\underline{F}$ of the even degree in the complex:

$$\check{C}^*(X, CC_*^{per}(\underline{A})),$$

where $CC_*^{per}(\underline{A})$ is the sheaf on X associated to the presheaf

$$U \mapsto CC_*^{per}(\underline{A})(U) = CC_*^{per}(\underline{A}(U)). \tag{4.23}$$

Let $\{U_\alpha\}$ be an open cover of X and

$$\Phi_\alpha : \underline{F}(U_\alpha) \xrightarrow{\sim} \underline{A}(U_\alpha)^N$$

be the associated family of local trivialisations of $\underline{F}$. Then

$$g_{\alpha\beta} = \Phi_\alpha \Phi_\beta^{-1} \in \mathrm{Gl}(\underline{A}(U_\alpha \cap U_\beta))$$

satisfy

$$g_{\alpha\beta} = 1 \ \text{ and } g_{\alpha\beta} g_{\beta\gamma} = 1 \text{ in } \mathrm{Gl}(\underline{A}(U_\alpha \cap U_\beta \cap U_\gamma)).$$

Put

$$\begin{array}{c} ch(\underline{F}, (g_{\alpha\beta}) = \sum_{p\geq 1} ch_p \ ; \ ch_p \in \check{C}^p(X, CC_p^{per}(\underline{A})), \\ (ch_p)_{\alpha_0,\ldots,\alpha_p} = (-1)^{p+1} ch(g_{\alpha_0\alpha_1}, \ldots, g_{\alpha_{p-1}\alpha_p}), \end{array} \tag{4.24}$$

where the expression on the right hand side of the above equation represents the chern character of the cochain from $C_p(\mathrm{Gl}_N(\underline{A}(U_{\alpha_0} \cap \ldots \cap U_{\alpha_p})))$.

Proposition 4.1 *$(B+b+\check{\partial}\ )(ch(\underline{F}, (g_{\alpha\beta}))=0$*

Proof.

$$\begin{array}{ll} (B+b)(ch_p)_{\alpha_0,\ldots,\alpha_p} & = (-1)^{p+1}(B+b)ch(g_{\alpha_0\alpha_1}, \ldots, g_{\alpha_{p-1}\alpha_p}) \\ & = (-1)^{p+1} ch\{(g_{\alpha_1\alpha_2}, \ldots, g_{\alpha_{p-1}\alpha_p}) \\ & + \sum_{i=1}^{p-1}(-1)^i (g_{\alpha_0\alpha_1}, \ldots, g_{\alpha_i\alpha_{i+2}}, \ldots, g_{\alpha_{p-1}\alpha_p}) \\ & +(-1)^p (g_{\alpha_0\alpha_1}, \ldots, g_{\alpha_{p-2}\alpha_{p-1}})\} \\ & = -(\check{\partial} ch_{p-1})_{\alpha_0\ldots\alpha_p}. \end{array}$$