A bit of symplectic geometry

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1 Symplectic vector spaces

Let V be a vector space (over $\mathbb{R}$). A symplectic structure on V is a bilinear, skewsymmetric, non-degenerate bilinear form ω on V.

1.1 Structure results

Let ω be a symplectic structure on V. Given any subset $A \subset V$, we set

$$A^{\perp_\omega} = \{v \in V \mid \forall a \in A: \ \omega(a, v) = 0\}$$

Definition 1.1 *A linear subspace $W \subset V$ is said to be*

(i) isotropic, if $W \subset W^{\perp_\omega}$;

(ii) lagrangian, if $W = W^{\perp_\omega}$ and

(iii) coisotropic, if $W \supset W^{\perp_\omega}$

Lemma 1.2 *(i) Any maximal isotropic subspace of V is lagrangian.*

(ii) Let $L \subset V$ be lagrangian. Then any linear complement L_1 to L in V is lagrangian and the map

$$L_1 \ni l_1 \to \omega(\cdot, l_1) \in L^*$$

is bijective. In particular the dimension of V is even

(iii) Given a decomposition $V = L + L_1$ as above, there are linear bases $e_1, \dots e_n$ of L and $f_1, \dots f_n$ of L_1 such that in the basis $e_1, \dots e_n, f_1, \dots f_n$ the bilinear form ω is given by

$$\omega(v, w) = w^t \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} v$$

Proof. Let $W \subset V$ be maximal isotropic, and let W_1 be its linear complement. Then the map

$$W_1 \ni l_1 \stackrel{I_\omega}{\to} \omega(\cdot, l_1) \in W^*$$

is injective by maximality of W and surjective by non-degeneracy of ω. Hence if we choose a linear bases $e_1, \dots e_n$ of W and the dual basis $w_1, \dots w_n$ of W^*, $e_1, \dots e_n, I_\omega^{-1}(w_1), \dots, I_\omega^{-1}(w_n)$ forms a linear basis of V in which

$$\omega = \omega_{st} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

and all the claims of the lemma follow.

Let $\mathrm{Sp}(V, \omega)$ denote the subgroup og $\mathrm{Gl}(V)$ of ω preserving transformations. We will write

$$Sp(2n, \mathbb{R}) = Sp(\mathbb{R}^{2n}, \ \omega_{st}).$$

Then

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n, \mathbb{R}) \Longleftrightarrow (A^tC)^t = (A^tC), \ (B^tD)^t = B^tD, \ A^tD - C^tB = 1.$$

Note that

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in Sp(2n, \mathbb{R}) \Longleftrightarrow A + iB \in U(n),$$

where $U(n)$ is the group of unitary n×n (complex) matrices. We will see later that $U(n)$ is in fact a maximal compact subgroup of $Sp(2n, \mathbb{R})$, hence $\pi_0(Sp(2n, \mathbb{R})) = \pi_0(U(n))$ and $\pi_1(Sp(2n, \mathbb{R})) = \pi_1(U(n))$. Hence the following is useful.

Lemma 1.3 *$U(n)$ is connected and $\pi_1(U(n)) = \mathbb{Z}$.*

Proof. Let U(n) acts naturally on the unit sphere S^{2n-1} in $\mathbb{C}^n$, with the stabiliser of (1,0,...,0) equal to U(n-1). This gives a fibration

$$\begin{array}{ccc} U(n-1) & \rightarrow & U(n) \\ & & \downarrow \\ & & S^{2n-1} \end{array}$$

The long exact homotopy sequence of this fibration and an iduction argument provide the proof. Note that

$$U(n) = \mathbb{T} \times SU(n)$$

and the simply connected covering group of U(n) is $\mathbb{R} \times SU(n)$.

1.2 Complex structures

A complex structure on a real vector space V is a linear transformation J such that $J^2 = -1$. Note that, given such a J, V becomes a complex vector space $V_{\mathbb{C}}$with the action of scalars given by

$$(a + ib)v := av + bJ(v).$$

Suppose moreover, that V is a symplectic vector space, with the symplectic structure given by ω.

Definition 1.4 *J is compatible with ω if*

$$\forall\ v, w \in V,\ \omega(Jv, Jw) = \omega(v, w).$$

J is positive, if

$$\forall v \neq 0 \in V,\ \omega(Jv, v) > 0.$$

Lemma 1.5 *The set of complex structures on $\mathbb{R}^{2n}$ compatible with ω_{st} is $Sp(2n, \mathbb{R})/U(n)$. There exists a unique positive complex structure.*

Proof. To begin with, there exists a positive complex structure. In fact set, for $(\mathbb{R}^{2n}, \omega_{st})$,

$$Je_i = f_i;\ Jf_i = -e_i$$

where $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ is the standard basis of $\mathbb{R}^{2n}$ in which ω_{st} has the form $\left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$. It is immediate to check that this gives a compatible positive complex structure.

In fact, identifying $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ by

$$V \ni \sum_i x_i e_i + \xi_i f_i \to \{z_i = x_i + i\xi_i\}_{i=1,\dots n} \in \mathbb{C}^n$$

we get $\omega_{st}(z, w) = Im < z \mid w >$ and J equal to the usuall coordinatewise complex conjugation on $\mathbb{C}^n$.

A positive compatible complex structure is unique, since it defines on $V_{\mathbb{C}}$ a positive definite scalar product

$$< v, w >_J = \omega(Jv, w) + i\omega(v, w).$$

But then $Im <, >$ is uniqely determined by ω and in turn determines uniquely $Re <, >$, and hence, by nondegeneracy of ω, gives J.

Now, the symplectic transformations that commute with the positive J are the same as the unitary transformations of the associated complex Hilbert space

$$V_{\mathbb{C}_J}, < \cdot, \cdot >_J$$

and the rest of the lemma follows.

Let Λ denote the grassmanian of lagrangian subspaces of $(\mathbb{R}^{2n}, \omega_{st})$.

Lemma 1.6 *$\Lambda \simeq U(n)/O(n)$ and $\pi_1(\Lambda) = \mathbb{Z}$. The simply connected covering space $\tilde{\Lambda}$ of Λ is*

$$\{(u, s) \subset U(n) \times \mathbb{R} \mid (detu)^2 = \exp(2\pi i x)\}/O(n)$$

Proof. We will use the standard complex structure on $\mathbb{R}^{2n}$. Given any lagrangian subspace $W \subset \mathbb{R}^{2n}$, $<, >$ is real-valued on W (since $Im(<, >) = \omega$). If we choose an orthonormal basis $(e_1, \dots, e_n)$ of W and an orthogonal transformation

$$O : \mathbb{R}^n \ni (x_1, \dots, x_n) \to \sum_i x_i e_i$$

then

$$U : z \to O(Re(z)) + iO(Im(z))$$

is a unitary transformation which maps $\mathbb{R}^n$ onto W. Since the stabiliser of $\mathbb{R}^n$ is O(n), the equality

$$\Lambda \simeq U(n)/O(n)$$

follows. To prove the rest, one uses the homotopy exact sequence of the fibration

$$\begin{array}{ccc} O(n) & \to & U(n) \\ & & \downarrow \\ & & \Lambda \end{array}$$

and standard fact about homotopy groups of $O(n)$.

Let us describe a Čech one-cocycle determining the covering $\tilde{\Lambda} \to \Lambda$. For $I \subset \{1, \dots, n\}$, let $x_1 = (x^k | k \in I)$, $x_2 = (x^k | k \in \overline{I})$, $\xi_1 = (\xi^k | k \in I)$, $\xi_2 = (\xi^k | k \in \overline{I})$ where $\overline{I}$ is the complement of I. Let L_I be the Lagrangian subspace $\{x_2 = 0, \xi_1 = 0\}$. Let U_I be the open subset of those L for which the projection onto L_I along $L_{\overline{I}}$ is an isomorphism. In other words, U_I consists of those L which are defined by equations

$$\xi_1 = Ax_1 + B\xi_2 \tag{1}$$

$$x_2 = -B^t x_1 - C\xi_2 \tag{2}$$

where A, C are self-adjoint matrices and t means transposition.

Now let us describe intersections $U_I \cap U_J$. Let $I = I_3 \cup I_4$, $J = I_2 \cup I_4$, where I_p, $p = 1, \dots, 4$, are disjoint and cover $\{1, \dots, n\}$. Put $x_p = (x^k | k \in I_p)$ and $\xi_p = (\xi^k | k \in I_p)$. A Lagrangian subspace L is in $U_I \cup U_J$ if and only if it can be described by equations

$$\begin{aligned} \xi_1 &= \dots \\ \xi_2 &= \dots + Ax_2 + B\xi_3 + \dots \\ x_3 &= \dots - B^t x_2 - C\xi_3 + \dots \\ x_4 &= \dots \end{aligned}$$

where the matrix $\begin{bmatrix} A & B \\ B & C \end{bmatrix}$ is nondegenerate.

Theorem 1.7 *The formula*

$$c_{IJ} = \frac{1}{2} \textit{signature} \begin{bmatrix} A & B \\ B & C \end{bmatrix}$$

defines a $\frac{1}{2}\mathbb{Z}$-valued Čech 1-cocycle for the cover $\{U_I\}$. This cocycle is cohomologous to a $\mathbb{Z}$-valued 1-cocycle whose cohomology class is the class of the universal cover of Λ.

Here by the signature of a self-adjoint nondegenerate matrix we mean the signature of the corresponding quadratic form.

There is a central extension $\tilde{Sp(n)}^N$ of $Sp(n)$ by $\mathbb{Z}/N\mathbb{Z}$ of which $\tilde{U}^N$ is a subgroup. It is defined by

$$\tilde{Sp(n)}^N = \{(g, \gamma)\}$$

where $g \in Sp(n)$ and γ is a homotopy class of a path in Λ connecting L_0 and $\gamma(L_0)$. Similarly, one constructs a central extension $\tilde{Sp(n)}$ of $Sp(n)$ by $\mathbb{Z}$.

2 Symplectic manifolds

Definition 2.1 *Let M be a smooth manifold. A symplectic structure on M is closed two-form ω such that, for all $m \in M$, ω_m is non-degenerate on $T_m M$.*

Let us introduce some terminology associated with symplectic manifold $(M, \omega$. We denote by I_ω the isomorphism

$$\Gamma(M, TM) \to \Gamma(M, T^*M)$$

given by

$$I_\omega(X)(Y) = \omega(X, Y).$$

Given a smooth function h on M, X_h will denote the vector field $I_\omega^{-1}(dh)$, called the hamiltonian vector field associated to the Hamiltonian h and, given $f, g \in C^\infty(M)$, we set the Poisson bracket of f and g to be the smootrh function on M given by

$$\{f, g\} = X_f g = -X_g f.$$

Lemma 2.2 *A flow $\phi_t^{X_h}$ of a hamiltonian vector field X_h preserves ω and, conversely, any vector field X such that its flow ϕ_t^X preserves the symplectic form is locally hamiltonian.*

Proof. The computation

$$\partial_t {\phi_X}_t^*(\omega) = {\phi_X}_t^*(\mathcal{L}_X \omega)$$

and the Cartan formula

$$\mathcal{L}_X d\iota_X + \iota_x d$$

gives:

$${\phi_X}_t^*(\omega) = \omega \Leftrightarrow d(\iota_X \omega) = 0.$$

If $X = X_h$ is hamiltonian, then $d(\iota_{X_h}\omega) = d^2 h = 0$. Conversely, if $d(\iota_X \omega) = 0$ then, locally, $\iota_X \omega$ is exact, i.e. $\iota_X \omega = df$ for some smooth function f and $X = X_f$.

Example 2.3 *Let $\pi : T^*M \to M$ be the canonical projection. We define a one-form α on T^*M by*

$$\alpha_l(v) = l(d\pi(v)), \text{ where } l \in T^*_{\pi(l)} \text{ and } d\pi(v) \in T_{\pi(l)}M.$$

Let $\omega = d\alpha$. This is a symplectic structure on T^*M. In fact, given local coordinates $(x_1, \dots, x_n)$ on M, let $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ denote the induced local coordinates on T^*M. Then

$$\alpha = \sum_i \xi_i dx_i \text{ and } \omega_{st} = \sum_i d\xi_i dx_i,$$

which is clearly non-degenerate.

In fact every symplectic structure looks locally as the above example.

Theorem 2.4 (Darboux) *Every symplectic manifold is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_{st})$, where dim(M)=2n.*

Proof. Since any manifold is locally diffeomorhic to an Euclidean space, we can just as well assume that we have two different symplectic structures ω_0 and ω_1 on an open contractible neighbourhood U of $0 \in \mathbb{R}^{2n}$. Since $(T_0^*(U), \omega_0)$ and $(T_0^*(U), \omega_1)$ are isomrphic as symplectic vector spaces, we can, after a linear change of coordinates, reduce to the case when ω_0 and ω_1 coincide at the origin 0.

Set

$$\omega_t = \omega_0 + t(\omega_1 - \omega_0) \; t \in [0, 1].$$

Since $\omega_1 - \omega_0$ is closed, it is exact on U and we set

$$\omega_1 - \omega_0 = d\alpha,$$

where α is chosen to vanish at the origin, and define one parameter family of vector fields on U, X_t, by

$$\iota_{X_t} \omega_t = -\alpha.$$

On $U \times [0, 1]$ we have the following:

(i) two-form μ defined by

$$\mu|_{U \times \{t\}} = \omega_t$$

and

(ii) vector field Y defined by

$$Y|_{U \times \{t\}} = X_t + \partial_s,$$

where s is the vertical coordinate.

Note that

$$\mathcal{L}_Y\mu = \iota_Y d\mu + d\iota_Y\mu = \frac{d}{ds}\omega_s + \iota_{X_s}\omega_s = 0,$$

hence μ is invariant under the flow of Y. Moreover, since $Y|_{\{0\}\times[0,1]} = \partial_t$,

$$[0,1] \ni s \to (0,s) \in U \times [0,1]$$

is a flowline of Y. From the last observation there exists an open neighbourhood $V \subset U$ of 0, so that ϕ_1^Y is defined on V and such that $\phi_1^Y(V)$ is an open neighbourhood of 0 in U. But, since μ is ϕ_t^Y-invariant, ϕ_1^Y is a symplectomorphism from (V,ω_0) to $(\phi_1^Y(V),\omega_1)$.

The following "strengthening" of Darboux theorem is very often useful.

Theorem 2.5 *Let (M,ω) be a connected symplectic manifold and P,Q be two points of M. There exists a flow ϕ_t on M such that ϕ_1 is defined in a neighbourhood of P, $\phi_1(P) = Q$ and a 1-parameter family of functions H_t on M so that the following holds*

$$\frac{d}{dt} f \circ \phi_t = \{H_t, f \circ \phi_t\}$$

Proof. Let $\gamma : [0,1] \to M$ be an embedded curve joining P and Q. Set

$$\xi_{\gamma(s)} = \iota_{\dot\gamma}\omega,$$

where $\dot\gamma$ is the vector field tangent to γ. By trivialising the bundle $TM|_\gamma$, we can choose a family of vector fields $X_1, \ldots X_{2n}$ along γ everywhere non-zero and such that $X_1 = \dot\gamma, \xi(X_2) = 1$ and $\xi(X_i) = 0$ for $i \neq 2$. We can choose a coordinate system $x_1, \ldots, x_{2n}$ in a neighbourhood of γ such that, on γ, $X_i = \partial_{x_i}$. But then on γ, $\xi = dx_2$ and the flow of the hamiltonian vector field X_{x_2}, satisfies our conditions.

A submanifold $L \subset M$ is called Lagrangian (isotropic, coisotropic) if, for every $l \in L$, $T_lL \subset T_lM$ is lagrangian (isotropic, coisotropic). The following strengthening of

Darboux theorem is due to Alan Weinstein.

Theorem 2.6 *Let (M,ω) be a symplectic manifold, and $L \subset M$ be an embedded Lagrangian submanifold. There exists an open neighbourhood U of L in M and a symplectomorphism*

$$\Phi : (U,\omega) \to (T^*L, \omega_{st})$$

such that $\Phi|_L = id_L$ and $\Phi(U)$ is an open neighbourhood of L in T^*L. Here we identified L with the zero section of T^*L.

We will not give the proof (one can adapt the above proof of Darboux theorem)