On the BKS Pairing for Kähler Quantizations of the Cotangent Bundle of a Lie Group

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Abstract

A natural one-parameter family of Kähler quantizations of the cotangent bundle T^*K of a compact Lie group K, taking into account the half-form correction, was studied in [FMMN]. In the present paper, it is shown that the associated Blattner-Kostant-Sternberg (BKS) pairing map is unitary and coincides with the parallel transport of the quantum connection introduced in our previous work, from the point of view of [AdPW]. The BKS pairing map is a composition of (unitary) coherent state transforms of K, introduced in [Ha1]. In the limit when one of the Kähler polarizations degenerates to the vertical real polarization, our results reproduce the unitarity up to scaling of the appropriate BKS pairing map, established by Hall.

Contents

1 Introduction

Let K be a compact, connected Lie group and let T^*K be its cotangent bundle. We start by recalling some aspects of [FMMN] where, in connection with work of Hall in [Ha3], the geometric quantization of T^*K was studied using a natural one-parameter family of Kähler structures. These Kähler structures are induced on T^*K via the following natural identifications of T^*K with the complexified group $K_{\mathbb{C}}$. Consider, for any real parameter $s > 0$, the diffeomorphisms

$$\begin{array}{rcl} \psi_s : T^*K & \to & K_{\mathbb{C}} \\ (x, Y) & \mapsto & \psi_s(x, Y) = xe^{isY} \ . \end{array} \tag{1.1}$$

Here, $x \in K$, $Y \in \mathfrak{K} \equiv \mathrm{Lie}(K)$, and we identify T^*K with $K \times \mathfrak{K}^*$ using left invariant forms and then with $K \times \mathfrak{K}$ by means of a fixed Ad-invariant inner product on $\mathfrak{K}$. The diffeomorphisms ψ_s endow T^*K with a family of complex structures J_s and it is easy to check that, together with the canonical symplectic structure ω on T^*K, the pair (ω, J_s) defines a Kähler structure on T^*K for every $s \in \mathbb{R}_+$. This family includes the Kähler structure on T^*K considered by Hall in [Ha3].

In this paper, we consider the Blattner-Kostant-Sternberg (BKS) pairing between two different Kähler quantizations of T^*K. To describe the results, let us consider the framework used in [FMMN]. Let L denote the trivial complex line bundle on T^*K, with trivial Hermitian structure (its sections are therefore identified with C^∞ functions on T^*K). Following the geometric quantization program with half-form correction, let us introduce the half-form bundle δ_s, which is a square root of the (trivial) J_s-canonical bundle κ_s over T^*K. Choosing canonical trivializing J_s-holomorphic sections Ω_s of $\kappa_s = \delta_s^2$ and $\sqrt{\Omega_s}$ of δ_s (we refer to the next section for precise formulas) one introduces a natural Hermitian structure on $L \otimes \delta_s$ so that, for a smooth section σ_s of the form

$$\sigma_s = f\sqrt{\Omega_s}, \tag{1.2}$$

with $f \in C^\infty(T^*K)$, one has

$$|\sigma_s|^2 := |f|^2 |\Omega_s|, \tag{1.3}$$

where $|\Omega_s|$ is defined by $\overline{\Omega}_s \wedge \Omega_s = |\Omega_s|^2 b\epsilon$, $b = (2i)^n(-1)^{n(n-1)/2}$ and $\epsilon = \frac{1}{n!}\omega^n$ is the Liouville measure on T^*K.

The prequantum Hilbert space $\mathcal{H}_s^{\mathrm{prQ}}$, depending on s, is then the (norm completion of the) space of C^∞ sections σ_s of $L \otimes \delta_s$ which are square-integrable with respect to the Hermitian structure (1.3), that is

$$\langle \sigma_s, \sigma_s \rangle^{\mathrm{prQ}} := \int_{T^*K} |\sigma_s|^2 \epsilon < \infty. \tag{1.4}$$

Proceeding with the quantization program, the quantum Hilbert space $\mathcal{H}_s^{\mathrm{Q}}$ is defined to be the subspace of $\mathcal{H}_s^{\mathrm{prQ}}$ consisting of polarized (J_s-holomorphic) sections of $L \otimes \delta_s$. This is naturally a sub-Hilbert space of $\mathcal{H}_s^{\mathrm{prQ}}$.

Both families of Hilbert spaces can be collected to form the Hilbert prequantum bundle $\mathcal{H}^{\mathrm{prQ}} \to \mathbb{R}_+$ and the quantum bundle $\mathcal{H}^{\mathrm{Q}} \to \mathbb{R}_+$, which is naturally a sub-bundle of $\mathcal{H}^{\mathrm{prQ}}$.

In the spirit of [AdPW], we consider a natural connection δ^{prQ} on $\mathcal{H}^{\mathrm{prQ}}$ and use it to induce, by orthogonal projection, an Hermitian connection δ^{Q} on $\mathcal{H}^{\mathrm{Q}}$. Note that, from (1.2-1.4), $\mathcal{H}^{\mathrm{prQ}}$ has a natural global (inner product preserving) trivializing morphism defined by

$$\begin{array}{rcl} L^2(T^*K, \epsilon) \times \mathbb{R}_+ & \to & \mathcal{H}^{\mathrm{prQ}} \\ (f, s) & \mapsto & \dfrac{f}{\sqrt{|\Omega_s|}}\sqrt{\Omega_s}. \end{array}$$

The prequantum connection δ^{prQ} on $\mathcal{H}^{\mathrm{prQ}}$ is defined to be the connection induced via this map from the trivial bundle $L^2(T^*K, \epsilon) \times \mathbb{R}_+ \to \mathbb{R}_+$ equipped with the trivial connection. Then, the quantum connection δ^{Q} is defined to be simply the orthogonal projection of δ^{prQ} to the quantum bundle. This construction is a natural generalization, to the case with the half-form correction and with a natural trivializing section of the square root of the canonical bundle, of the framework considered in [AdPW]. One of the advantages of this approach consists in the fact that it ensures automatically that the quantum connection is Hermitian.

In the present paper, we consider the Blattner-Kostant-Sternberg (BKS) pairing between two different fibers of $\mathcal{H}^{\mathrm{Q}}$, corresponding to two different quantizations of T^*K (see, for instance [Wo].) This is the restriction of a Hermitian pairing between the corresponding prequantum Hilbert spaces, which we call the prequantum BKS pairing, and which is defined by

$$\langle \sigma_s, \sigma'_{s'} \rangle^{\mathrm{BKS}} = \int_{T^*K} \bar{f} f' \sqrt{\frac{\bar{\Omega}_s \wedge \Omega_{s'}}{b\epsilon}}\epsilon, \tag{1.5}$$

for $\sigma_s = f\sqrt{\Omega_s} \in \mathcal{H}_s^{\mathrm{prQ}}$ and $\sigma'_{s'} = f'\sqrt{\Omega_{s'}} \in \mathcal{H}_{s'}^{\mathrm{prQ}}$. This prequantum BKS pairing defines a map

$$B_{ss'}^{\mathrm{prQ}} : \mathcal{H}_{s'}^{\mathrm{prQ}} \to \mathcal{H}_s^{\mathrm{prQ}}, \tag{1.6}$$

whose infinitesimal version induces a, also natural, connection δ on $\mathcal{H}^{\mathrm{prQ}}$. We prove in theorem 1 that δ coincides with the connection δ^{prQ}. Therefore, the quantum connection δ^{Q} will also be induced from the BKS pairing on $\mathcal{H}^{\mathrm{Q}}$. This result shows that the approach of the infinitesimal BKS pairing coincides with that of [AdPW] considered in [FMMN]. We note, however, that the prequantum BKS pairing map does not coincide with the parallel transport associated to δ^{prQ} (see theorem 2.)

In [Ha3], Hall considered two polarizations for the quantization of T^*K. Fixing the same inner product as Hall on $\mathfrak{K}$, one of these polarizations, the Kähler one, corresponds to the case of $s = 1$, and the other, the vertical real polarization, corresponds, in our framework, to letting s go to 0. He proved that the BKS pairing map between the Hilbert spaces associated to these two quantizations coincides (up to scale) with the Segal-Bargmann coherent state transform (CST) for K introduced in [Ha1], provided that one takes into account the half-form correction. In theorem 5, in the third section of this paper, we extend this result, showing that the (quantum) BKS pairing map

$$B_{ss'}^{\mathrm{Q}} : \mathcal{H}_{s'}^{\mathrm{Q}} \to \mathcal{H}_s^{\mathrm{Q}} \tag{1.7}$$

obtained by restriction of (1.6) to the quantum bundle is, in fact, unitary.

Moreover, and unlike the prequantum case, it coincides with the parallel transport of $\delta^{\mathcal{Q}}$ and corresponds to the CST.

Therefore, we have obtained an example of quantization where the BKS pairing map, between half-form corrected Hilbert spaces corresponding to two different Kähler quantizations, is an isometric isomorphism and also confirm in a different way the result of [FMMN] that the parallel transport of the quantum connection coincides with the one given by the CST. We stress that we prove unitarity of the quantum BKS pairing map and not just unitarity up to scale. Therefore, for the family of polarizations that we consider, cotangent bundles of compact Lie groups provide new examples in which the half-form correction leads to the unitarity of the BKS pairing map [Wo]. Note, however, that our results agree with [Ha3] in that the BKS pairing map between the vertically polarized Hilbert space and a Kähler polarized one is only unitary up to scale.

For other related works on the dependence of the quantization on the choice of polarization, the unitarity of the BKS pairing map and the relation between vertical and Kähler polarizations, see also [FY, Hi, KW, Ra, Th].

2 The prequantum BKS pairing

Let K be a compact, connected Lie group of real dimension n, and let us fix once and for all an Ad-invariant inner product $(\cdot\,,\cdot)$ on its Lie algebra $\mathfrak{K}$. Let $\{X_i\}_{i=1}^n$ be left-invariant vector fields forming an orthonormal basis for $\mathfrak{K}$. The cotangent bundle $T^*K \cong K \times \mathfrak{K}$ is naturally a symplectic manifold with a canonical symplectic 2-form defined by $\omega = -\mathrm{d}\theta$, where

$$\theta = \sum_{i=1}^{n} y^i w^i. \tag{2.1}$$

Here, $(y^1, \dots, y^n)$ are the global coordinates on $\mathfrak{K}$ corresponding to the basis $\{X_i\}_{i=1}^n$, and $\{w^i\}_{i=1}^n$ is the basis of left-invariant 1-forms on K dual to $\{X_i\}_{i=1}^n$, pulled-back to T^*K by the canonical projection. We let ϵ denote the Liouville volume form on T^*K, given by

$$\epsilon = \frac{1}{n!}\omega^n \ . \tag{2.2}$$

For each $s > 0$, let $\psi_s : T^*K \to K_{\mathbb{C}}$ denote the diffeomorphism defined in (1.1) by $\psi_s(x, Y) = xe^{isY}$, which induces on T^*K a Kähler structure (ω, J_s).

2.1 The prequantum bundle $\mathcal{H}^{\mathrm{prQ}}$

We start by recalling from [FMMN], some of the formulas that will be needed later on.

Let $\tilde{X}_j$, $j = 1, ..., n$, be the vector fields on T^*K generating the right action of K lifted to T^*K and given by

$$\psi_{s*}\tilde{X}_j = X_{j,\mathbb{C}},$$

where $X_{j,\mathbb{C}}$ denotes the extension of X_j from a left-invariant vector field on $K \subset K_{\mathbb{C}}$ to the corresponding left-invariant vector field on $K_{\mathbb{C}}$. Let $\{\tilde{w}^j\}$ be the basis of left invariant 1-forms defined by $\tilde{w}^j(\tilde{X}_k) = \delta^j_k$ and $\tilde{w}^j(J_s\tilde{X}_k) = 0$, for $j, k = 1, ..., n$. For every $s \in \mathbb{R}_+$, a frame of left invariant J_s-holomorphic 1-forms is then

$$\left\{\tilde{\eta}^j_s = \tilde{w}^j - iJ_s\tilde{w}^j\right\}_{j=1}^n ,$$

where $(J_s w)(X) = w(J_s X)$, for a vector field X and a 1-form w on T^*K. Consider now the J_s-canonical bundle $\kappa_s \to T^*K$ whose sections are J_s-holomorphic n-forms with natural Hermitian structure defined as follows. For a J_s-holomorphic n-form α_s, let $|\alpha_s|$ be the unique non-negative C^∞ function on T^*K such that $\overline{\alpha}_s \wedge \alpha_s = |\alpha_s|^2 b\epsilon$, where $b = (2i)^n(-1)^{n(n-1)/2}$. Following [Ha3] we write

$$|\alpha_s|^2 = \frac{\overline{\alpha}_s \wedge \alpha_s}{b\epsilon}. \tag{2.3}$$

Given the inner product on $\mathfrak{K}$, a canonical trivializing J_s-holomorphic section of κ_s is given by

$$\Omega_s := \tilde{\eta}^1_s \wedge \cdots \wedge \tilde{\eta}^n_s$$

and has norm [FMMN]

$$|\Omega_s|^2 = s^n \eta^2(sY), \tag{2.4}$$

where $\eta(Y)$ is the Ad_K-invariant function, defined for Y in a Cartan subalgebra by the following product over a set R^+ of positive roots of $\mathfrak{K}$,

$$\eta(Y) = \prod_{\alpha \in R^+} \frac{\sinh \alpha(Y)}{\alpha(Y)}. \tag{2.5}$$

The following proposition generalizes (2.4)

Proposition 1. *Let $s, s' > 0$. We have*

$$\bar{\Omega}_s \wedge \Omega_{s'} = \left(\frac{s+s'}{2}\right)^n \eta^2 \left(\frac{s+s'}{2} Y\right) b\epsilon = \left|\Omega_{\frac{s+s'}{2}}\right|^2 b\epsilon. \tag{2.6}$$

Proof. The result follows by direct computation. From [Ha2] and the definition of ψ_s, we can write $D\psi_s : TT^*K \to TK_{\mathbb{C}}$ as

$$D\psi_s(x, Y) = \begin{bmatrix} \cos \mathrm{ad} sY & \frac{1-\cos \mathrm{ad} sY}{\mathrm{ad} Y} \\ -\sin \mathrm{ad} sY & \frac{\sin \mathrm{ad} sY}{\mathrm{ad} Y} \end{bmatrix}, \tag{2.7}$$

using the (x, Y) coordinate basis on T^*K and the basis $\{X_{j,\mathbb{C}}, JX_{j,\mathbb{C}}\}_{j=1,\dots,n}$ on $K_{\mathbb{C}}$, where J is the complex structure on $K_{\mathbb{C}}$. From the explicit expressions for the forms $\tilde{\eta}_s^j$ we then find, for $s, s' > 0$,

$$\bar{\Omega}_s \wedge \Omega_{s'} = \bar{\tilde{\eta}}_s^1 \wedge \cdots \wedge \bar{\tilde{\eta}}_s^n \wedge \tilde{\eta}_{s'}^1 \wedge \cdots \wedge \tilde{\eta}_{s'}^n = \det \begin{bmatrix} \bar{M}_s & \bar{N}_s \\ M_{s'} & N_{s'} \end{bmatrix} (-1)^{\frac{n(n-1)}{2}} \epsilon, \tag{2.8}$$

where the endomorphisms M_s and N_s are defined by

$$M_s = e^{-i \mathrm{ad} sY}, \quad N_s = \frac{1 - e^{-i \mathrm{ad} sY}}{\mathrm{ad} Y}.$$

From the left invariance of the forms, this determinant can be evaluated for Y in the Cartan subalgebra which, after taking care of the contribution from the null space of adY, yields

$$\bar{\Omega}_s \wedge \Omega_{s'} = \left(\frac{s+s'}{2}\right)^n \prod_{\alpha \in R} \frac{(e^{s'\langle\alpha, Y\rangle} - e^{-s\langle\alpha, Y\rangle})}{(s+s')\langle\alpha, Y\rangle} b\epsilon,$$

where the product runs over the set R of all roots of $\mathfrak{K}$. The result then follows from definition (2.5). □

As in the introduction, let δ_s be the J_s-holomorphic bundle of half-forms on T^*K, with trivializing section whose square is Ω_s. Following [Ha3], we will denote this section by $\sqrt{\Omega_s}$. As above, let L denote the trivial complex line bundle on T^*K and let the prequantum Hilbert space $\mathcal{H}_s^{\mathrm{prQ}}$ be the norm completion of the space of C^∞-sections of $L \otimes \delta_s$ which are integrable with respect to the Hermitian structure $\langle\cdot,\cdot\rangle^{\mathrm{prQ}}$ defined in (1.4). We will write it in the form

$$\langle\sigma_s, \sigma_s'\rangle^{\mathrm{prQ}} = \int_{T^*K} \bar{f} f' |\Omega_s| \epsilon = \int_{T^*K} \bar{f} f' \sqrt{\frac{\bar{\Omega}_s \wedge \Omega_s}{b\epsilon}} \epsilon, \tag{2.9}$$

for two smooth sections of $L \otimes \delta_s$ written as $\sigma_s = f\sqrt{\Omega_s}, \sigma'_s = f'\sqrt{\Omega_s}$, with $f, f' \in C^\infty(T^*K)$.

The smooth Hilbert bundle structure on $\mathcal{H}^{\mathrm{prQ}}$ and the prequantum connection δ^{prQ} are chosen to be the ones compatible with the global trivializing map

$$\begin{aligned} L^2(T^*K, \epsilon) \times \mathbb{R}_+ &\rightarrow \mathcal{H}^{\mathrm{prQ}} \\ (f, s) &\mapsto \frac{f}{\sqrt{|\Omega_s|}}\sqrt{\Omega_s}. \end{aligned} \tag{2.10}$$

Note that the section of δ_s given by $\frac{\sqrt{\Omega_s}}{\sqrt{|\Omega_s|}}$ has unit norm, and its use would simplify some of the formulas. However, since this section is not holomorphic, we use $\sqrt{\Omega_s}$ as trivializing section of δ_s, which will be more suited for describing the polarized sections in section 3.

2.2 The prequantum BKS pairing and its associated connection on $\mathcal{H}^{\mathrm{prQ}}$

The general procedure, as described for instance in [Wo], for defining a BKS pairing, suggests, in our setting, the definition of a Hermitian pairing which we call the prequantum BKS pairing, as follows.

Definition 1. Let $\sigma_s = f\sqrt{\Omega_s} \in \mathcal{H}^{\mathrm{prQ}}_s$ and $\sigma'_{s'} = f'\sqrt{\Omega_{s'}} \in \mathcal{H}^{\mathrm{prQ}}_{s'}$. Their BKS pairing is defined by

$$\langle \sigma_s, \sigma'_{s'} \rangle^{\mathrm{BKS}} = \int_{T^*K} \bar{f} f' \sqrt{\frac{\bar{\Omega}_s \wedge \Omega_{s'}}{b\epsilon}}\epsilon. \tag{2.11}$$

Note that the integral above exists for $\sigma_s, \sigma'_{s'}$ satisfying the conditions for sections of the prequantum bundle in (1.4), which in this case read $f\sqrt{|\Omega_s|}, f'\sqrt{|\Omega_{s'}|} \in L^2(T^*K, \epsilon)$. This can be readily checked by using Proposition 1 to write (2.11) as

$$\langle \sigma_s, \sigma'_{s'} \rangle^{\mathrm{BKS}} = \int_{T^*K} \bar{f} f' \left| \Omega_{\frac{s+s'}{2}} \right| \epsilon, \tag{2.12}$$

and using the definition of $\eta(Y)$ in (2.5) to prove that the smooth function

$$\phi(s, s', Y) := \frac{\left|\Omega_{\frac{s+s'}{2}}\right|^2}{|\Omega_s||\Omega_{s'}|} = \left(\frac{s+s'}{2\sqrt{ss'}}\right)^n \frac{\eta^2\left(\frac{s+s'}{2}Y\right)}{\eta(sY)\eta(s'Y)} \tag{2.13}$$

is real, positive and bounded, for fixed s, s'.

We remark that, in the case when s' and s coincide, the prequantum BKS pairing is equal to the Hermitian structure $\langle\cdot,\cdot\rangle^{\mathrm{prQ}}$ on $\mathcal{H}^{\mathrm{prQ}}_s$. Also, we note that the pairing (2.11) is nondegenerate.

As mentioned above, δ^{prQ} was defined as the Hermitian connection for which sections $\mathcal{H}^{\mathrm{prQ}}$ of the form $\sigma_s = \frac{f}{\sqrt{|\Omega_s|}}\sqrt{\Omega_s}$, with $f \in L^2(T^*K, \epsilon)$, are parallel. It is also natural to consider a connection, δ, on $\mathcal{H}^{\mathrm{prQ}}$, induced from the infinitesimal prequantum BKS pairing by the formula

$$\langle \sigma_s, \delta_{\frac{\partial}{\partial s}} \sigma'_s\rangle^{\mathrm{prQ}} = \left.\frac{\partial}{\partial s'}\right|_{s'=s} \langle \sigma_s, \sigma'_{s'}\rangle^{\mathrm{BKS}}. \tag{2.14}$$

Let us prove that these two connections are the same.

Theorem 1. *The connections δ^{prQ} and δ on $\mathcal{H}^{\mathrm{prQ}}$ are equal.*

Proof. Let σ be a smooth section of $\mathcal{H}^{\mathrm{prQ}}$, such that $\sigma_s = \frac{f}{\sqrt{|\Omega_s|}}\sqrt{\Omega_s} \in \mathcal{H}^{\mathrm{prQ}}_s$, for $f \in L^2(T^*K, \epsilon)$, $s > 0$. Since sections of this type can be used to form a global moving frame for $\mathcal{H}^{\mathrm{prQ}}$, to prove the theorem, it suffices to show that σ is parallel with respect to the connection δ. Let $\tau_s = g\sqrt{\Omega_s} \in \mathcal{H}^{\mathrm{prQ}}_s$. We have, from (2.12) and (2.13),

$$\langle \tau_s, \delta_{\frac{\partial}{\partial s}} \sigma_s\rangle^{\mathrm{prQ}} = \left.\frac{\partial}{\partial s'}\right|_{s'=s} \langle \tau_s, \sigma_{s'}\rangle^{\mathrm{BKS}} = \int_{T^*K} \bar{g} f |\Omega_s|^{\frac{1}{2}} \left(\frac{\partial\sqrt{\phi}}{\partial s'}\right)_{s'=s} \epsilon.$$

From a straightforward computation using (2.13) we obtain

$$\left.\frac{\partial \phi(s, s', Y)}{\partial s'}\right|_{s'=s} = 0,$$

which implies that $\delta_{\frac{\partial}{\partial s}} \sigma_s = 0$, as required. $\square$

In the next section, we will show that the restriction of the prequantum BKS pairing to $\mathcal{H}^{\mathrm{Q}}$ yields the (unitary) parallel transport associated to δ^{Q}. This, however, does not hold for the prequantum bundle. In fact, let us define the prequantum BKS pairing map $B^{\mathrm{prQ}}_{ss'} : \mathcal{H}^{\mathrm{prQ}}_{s'} \to \mathcal{H}^{\mathrm{prQ}}_s$ by

$$\langle \sigma_s, \sigma'_{s'}\rangle^{\mathrm{BKS}} = \langle \sigma_s, B^{\mathrm{prQ}}_{ss'} \sigma'_{s'}\rangle^{\mathrm{prQ}}, \tag{2.15}$$

Theorem 2. *The prequantum BKS pairing map (2.15), $B^{\mathrm{prQ}}_{ss'} : \mathcal{H}^{\mathrm{prQ}}_{s'} \to \mathcal{H}^{\mathrm{prQ}}_{s}$ is given by*

$$B^{\mathrm{prQ}}_{ss'}(f'\sqrt{\Omega_{s'}}) = f'\frac{\left|\Omega_{\frac{s+s'}{2}}\right|}{|\Omega_s|}\sqrt{\Omega_s},$$

for $f'\sqrt{\Omega_{s'}} \in \mathcal{H}^{\mathrm{prQ}}_{s'}$.

It does not coincide with the parallel transport of the connection δ^{prQ}.

Proof. Let us write $B^{\mathrm{prQ}}_{ss'}(f'\sqrt{\Omega_{s'}}) = g\sqrt{\Omega_s}$ for some vector $g\sqrt{\Omega_s} \in \mathcal{H}^{\mathrm{prQ}}_s$. Then, using (2.12) we have for any $f\sqrt{\Omega_s} \in \mathcal{H}^{\mathrm{prQ}}_s$, that

$$\int_{T^*K} \bar{f}f'\left|\Omega_{\frac{s+s'}{2}}\right|\epsilon = \langle f\sqrt{\Omega_s}, B^{\mathrm{prQ}}_{ss'}(f'\sqrt{\Omega_{s'}})\rangle^{\mathrm{prQ}} = \int_{T^*K} \bar{f}g|\Omega_s|\epsilon.$$

Since $f \in L^2(T^*K, |\Omega_s|^{\frac{1}{2}})$ is arbitrary, we obtain $g = f'\frac{\left|\Omega_{\frac{s+s'}{2}}\right|}{|\Omega_s|}$, as wanted.

On the other hand, the parallel transport $P_{ss'} : \mathcal{H}^{\mathrm{prQ}}_{s'} \to \mathcal{H}^{\mathrm{prQ}}_s$ of δ^{prQ} is determined by (2.10) giving

$$P_{ss'}(f'\sqrt{\Omega_{s'}}) = \frac{|\Omega_{s'}|^{\frac{1}{2}}}{|\Omega_s|^{\frac{1}{2}}}f'\sqrt{\Omega_s}.$$

Clearly, this is not equal to $B^{\mathrm{prQ}}_{ss'}$ since ϕ is not identically equal to 1. ☐

3 Unitarity of the quantum BKS pairing and the CST

Consider the quantum Hilbert bundle $\mathcal{H}^{\mathrm{Q}} \to \mathbb{R}_+$, whose fiber over each $s > 0$ consists of J_s-polarized sections of $L \otimes \delta_s$. Recall from [FMMN] that $\mathcal{H}^{\mathrm{Q}}_s$ is given by

$$\left\{\sigma_s = Fe^{-\frac{s|Y|^2}{2\hbar_0}}\sqrt{\Omega_s},\ F \text{ is } J_s\text{-holomorphic and } ||\sigma_s||^{\mathrm{Q}}_s < \infty\right\}, \qquad (3.1)$$

where the norm refers to the Hermitian structure on $\mathcal{H}^{\mathrm{Q}}$ as a sub-bundle of $\mathcal{H}^{\mathrm{prQ}}$. The pairing (2.11) between fibers of $\mathcal{H}^{\mathrm{prQ}}$ restricted to polarized sections defines the BKS pairing bewteen fibers of $\mathcal{H}^{\mathrm{Q}}$, for which we use the same notation. As in (2.14), the BKS pairing induces a connection, δ^{BKS}, on $\mathcal{H}^{\mathrm{Q}}$ defined by the same formula

$$\langle \sigma_s, \delta^{\mathrm{BKS}}_{\frac{\partial}{\partial s}} \sigma'_s \rangle^{\mathrm{Q}} = \frac{\partial}{\partial s'}\bigg|_{s'=s} \langle \sigma_s, \sigma'_{s'} \rangle^{\mathrm{BKS}}. \tag{3.2}$$

for $\sigma_s \in \mathcal{H}^{\mathrm{Q}}_s$ and $\sigma'_{s'} \in \mathcal{H}^{\mathrm{Q}}_{s'}$. Recall that, along the lines of [AdPW], the quantum bundle is equipped with a quantum connection δ^{Q}, obtained from the orthogonal projection $P : \mathcal{H}^{\mathrm{prQ}} \to \mathcal{H}^{\mathrm{Q}}$. More precisely, we have $\delta^{\mathrm{Q}} = P \circ \delta^{\mathrm{prQ}}$, which implies immediately

Theorem 3. *The connections δ^{BKS} and δ^{Q} on $\mathcal{H}^{\mathrm{Q}}$ coincide.*

Proof. This is an obvious corollary of theorem 1, since the pairing on $\mathcal{H}^{\mathrm{Q}}$ is obtained by restriction from $\mathcal{H}^{\mathrm{prQ}}$. □

The BKS pairing formalism in this case provides results consistent with the approach of [AdPW]. We will establish below the fact that the BKS pairing map for $\mathcal{H}^{\mathrm{Q}}$ is unitary and coincides with the parallel transport of δ^{Q}.

3.1 The quantum BKS pairing and the K-averaged heat kernel

In [FMMN], we showed that the parallel transport associated to δ^{Q} corresponds to Hall's CST, so that, in fact, δ^{Q}-parallel sections of $\mathcal{H}^{\mathrm{Q}}$ satisfy a heat equation on $K_{\mathbb{C}}$. This provides a better understanding of some of the results of [Ha3].

In [Ha3], Hall has also shown that, up to a constant, the CST corresponds to the BKS pairing map between the vertical real polarized and the Kähler polarized quantum Hilbert spaces. In this section, we further relate the BKS pairing on the quantum bundle $\mathcal{H}^{\mathrm{Q}}$ to the unitary parallel transport of the quantum connection δ^{Q} and to the CST.

Let now Δ be the invariant Laplacian on K and $\mathcal{C}$ denote the analytic continuation from K to $K_{\mathbb{C}}$. We recall from [Ha1], that the CST is a unitary isomorphism of Hilbert spaces defined by

$$\begin{array}{rcl} C_\hbar : L^2(K, dx) & \to & \mathcal{H}L^2(K_{\mathbb{C}}, d\nu_\hbar) \\ f & \mapsto & C_\hbar(f) = \mathcal{C} \circ e^{\frac{\hbar}{2}\Delta} f, \end{array}$$

where dx is the Haar measure on K, $d\nu_\hbar(g) = \nu_\hbar(g)dg$ is the K-averaged heat kernel measure of [Ha1], and dg is the Haar measure on $K_\mathbb{C}$.

Recall from [Ha2] the explicit form of the K-averaged heat kernel measure for $\hbar = s\hbar_0$

$$\nu_\hbar(g) = (a_s s^{n/2}\eta(Y))^{-1}e^{-\frac{|Y|^2}{\hbar}}, \tag{3.3}$$

where

$$a_s = (\pi\hbar_0)^{n/2}e^{|\rho|^2\hbar_0 s},$$

and ρ is the Weyl vector given by half the sum of the positive roots of $\mathfrak{K}$. Using the relation of parallel sections of $\mathcal{H}^{\mathrm{Q}}$ with holomorphic functions on $K_\mathbb{C}$ we will now obtain an explicit formula for the BKS pairing as an integral over $K_\mathbb{C}$. As in the definition of the quantum Hilbert space, let

$$\sigma_s = Fe^{-\frac{s|Y|^2}{2\hbar_0}}\sqrt{\Omega_s} \in \mathcal{H}^{\mathrm{Q}}_s \text{ and } \sigma'_{s'} = F'e^{-\frac{s'|Y|^2}{2\hbar_0}}\sqrt{\Omega_{s'}} \in \mathcal{H}^{\mathrm{Q}}_{s'}, s, s' > 0, \tag{3.4}$$

where $F = \hat{F}\circ\psi_s$ and $F' = \hat{F}'\circ\psi_{s'}$ are, respectively, J_s-holomorphic and $J_{s'}$-holomorphic functions on T^*K, with $\hat{F}, \hat{F}'$ homolorphic functions on $K_\mathbb{C}$.

Proposition 2. *The BKS pairing on $\mathcal{H}^{\mathrm{Q}}$ for $\sigma_s, \sigma'_{s'}$ as above is given by*

$$\langle\sigma_s, \sigma'_{s'}\rangle^{\mathrm{BKS}} = a_{\frac{s+s'}{2}}\int_{K_\mathbb{C}} \overline{\hat{F}(gZ)}\hat{F}'(gZ^{-1})\, d\nu_{\hbar''}(g), \tag{3.5}$$

where $g = xe^{iY}$, $Z = e^{i\frac{s-s'}{s+s'}Y}$ and $\hbar'' = \frac{s+s'}{2}\hbar_0$.

Proof. From definition 1 and proposition 1 we have

$$\langle\sigma_s, \sigma'_{s'}\rangle^{\mathrm{BKS}} = \int_{T^*K}\overline{\hat{F}(xe^{isY})}\hat{F}'(xe^{is'Y})e^{-\frac{(s+s')}{2\hbar_0}|Y|^2}\left(\frac{s+s'}{2}\right)^{\frac{n}{2}}\eta\left(\frac{s+s'}{2}Y\right)\epsilon. \tag{3.6}$$

We recall from [Ha2] and [FMMN] that $\psi_s^* dg = |\Omega_s|^2\epsilon = s^n\eta^2(sY)\epsilon$. The formula then follows from a change of variables of integration, from (x, Y) to $g = xe^{iY'}$ where $Y' = \frac{s+s'}{2}Y$, and where at the end we renamed Y' as Y again. □

3.2 Unitarity of the quantum BKS pairing

We now recall that in [FMMN] a relation was given between parallel sections of the quantum bundle and the coherent state transform for K. This will now enable us to find a relation between the BKS pairing and the CST.

Let then $\sigma_s = Fe^{-\frac{s|Y|^2}{2\hbar_0}}\sqrt{\Omega_s} \in \mathcal{H}_s^{\mathrm{Q}}$ be of the form given in (3.4), with $F = \hat{F} \circ \psi_s$ and $\hat{F}$ holomorphic on $K_{\mathbb{C}}$. It follows from [FMMN] that $\hat{F} \in \mathcal{H}L^2(K_{\mathbb{C}}, d\nu_\hbar)$ with $\hbar = s\hbar_0$.

Therefore, for σ_s and $\sigma'_{s'}$, $s, s' > 0$, as in (3.4) we conclude that, due to the fact that the coherent state transform is an isomorphism, there are functions $f, f' \in L^2(K, dx)$ such that $\hat{F} = C_\hbar f$ and $\hat{F}' = C_{\hbar'} f'$, where $\hbar = s\hbar_0$ and $\hbar' = s'\hbar_0$.

Formula (3.6), for $\hat{F}' = C_{\hbar'} f'$ with fixed $f' \in L^2(K, dx)$, for fixed s and with $\hbar' = s'\hbar_0$, implies that the BKS pairing between polarized sections has a well defined $s' \to 0$ limit. Moreover, in this limit, setting $s = 1$ and taking care of the different normalizations, we obtain the expression of [Ha3] (see Theorem 2.6) for the pairing between a vertically polarized section and a Kähler polarized one. (In our definition of the pairing there is an extra factor of $2^{-n/2}$ as compared to [Ha3].) This can also be checked directly from the explicit formula for the BKS pairing in the next theorem.

Theorem 4. *Let $\sigma_s \in \mathcal{H}_s^{\mathrm{Q}}$ and $\sigma'_{s'} \in \mathcal{H}_{s'}^{\mathrm{Q}}$ be as above. We have,*

$$\langle \sigma_s, \sigma'_{s'} \rangle^{\mathrm{BKS}} = a_{\frac{s+s'}{2}} \langle f, f' \rangle_{L^2(K,dx)}. \tag{3.7}$$

Before proving this theorem, we establish the following auxiliary lemma which proves an analog of equation (4) in [Ha2]. Let $\rho_\hbar$ denote the analytic continuation of the heat kernel on K to $K_{\mathbb{C}}$, as in [Ha1]. Moreover, let $* : K_{\mathbb{C}} \to K_{\mathbb{C}}$ denote the unique anti-holomorphic anti-automorphism of $K_{\mathbb{C}}$ which extends the map $x \to x^{-1}$ on K (see also [Ha4]). We then have,

Lemma 1. *The Dirac delta distribution on K, with respect to the Haar measure dx, can be written as*

$$\delta(x) = \int_{K_{\mathbb{C}}} \rho_{2\hbar}(x^{-1}g^*g)\, d\nu_\hbar(g). \tag{3.8}$$

Proof. From the Peter-Weyl theorem, to prove the lemma it suffices to show that for any matrix element $R_{i,j}$ of any irreducible representation R of K, we have

$$\int_{K_{\mathbb{C}}} \left(\int_K \rho_{2\hbar}(x^{-1}g^*g) R_{ij}(x)\ dx \right) d\nu_\hbar(g) = \delta_{ij}. \tag{3.9}$$

Recall that $\rho_\hbar = \sum_R d_R e^{-\frac{\hbar}{2}c_R}\chi_R$, where the sum extends over the irreducible representations R of K, d_R is the dimension of R, χ_R denotes its character and finally c_R is the negative of the eigenvalue of Δ corresponding to the eigenvector χ_R.

Now, recall Weyl's classical orthogonality relations for matrix elements R_{ij} of irreducible representations of K,

$$\int_K \overline{R_{ij}(x)} R'_{lk}(x) dx = \frac{\delta_{RR'}}{d_R} \delta_{jk} \delta_{il}. \tag{3.10}$$

From the unitarity of the CST of [Ha1], we then obtain the following identities,

$$e^{-\frac{\hbar}{2}(c_R + c_{R'})} \int_{K_{\mathbb{C}}} \overline{R_{ij}(g)} R'_{lk}(g) d\nu_\hbar(g) = \frac{\delta_{RR'}}{d_R} \delta_{jk} \delta_{il}, \tag{3.11}$$

where we have used the fact that $C_\hbar R_{ij} = e^{-\frac{\hbar}{2} c_R} R_{ij}$. (We are denoting by the same symbol a function on K and its analytic continuation to $K_{\mathbb{C}}$.)

Upon substituting the expression for $\rho_{2\hbar}$ in (3.9), using $\chi_R(g) = \sum_{i=1}^{d_R} R_{ii}(g)$ and the orthogonality relations (3.10), the left hand side of (3.9) becomes

$$e^{-\hbar c_R} \sum_{k=1}^{d_R} \int_{K_{\mathbb{C}}} \overline{R_{ki}(g)} R_{kj}(g) d\nu_\hbar(g).$$

Using (3.11), we obtain the lemma. $\square$

The Dirac delta distribution can also be written in a different useful form.

Lemma 2. *The Dirac delta distribution on K, with respect to the Haar measure, can be written as*

$$\delta(x_1^{-1} x_2) = \int_{K_{\mathbb{C}}} \overline{\rho_\hbar(g e^{itY} x_1^{-1})} \rho_{\hbar'}(g e^{-itY} x_2^{-1}) \, d\nu_{\hbar''}(g), \tag{3.12}$$

for $x_1, x_2 \in K$, $\hbar + \hbar' = 2\hbar''$ and any real number t.

Proof. To prove the lemma, we substitute the explicit expressions for the heat kernels in (3.12) and use the fact that, with $g = xe^{iY} \in K_{\mathbb{C}}$, we have $\overline{\chi_R(xe^{i(1+t)Y} x_1^{-1})} = \chi_R(x_1 e^{i(1+t)Y} x^{-1})$. Using again $\rho_\hbar = \sum_R d_R e^{-\frac{\hbar}{2} c_R} \chi_R$ and rewriting the characters as a sum of products of matrix elements, we express the integral as an integral on T^*K. One then finds, upon integration along K, and use of the orthogonality relations (3.10), that the t dependence in (3.12) is apparent and cancels out. Finally, from the fact that $\hbar + \hbar' = 2\hbar''$

and using $\int_K dx = 1$, one rewrites the expression as an integral on $K_{\mathbb{C}}$ so that equation (3.12) becomes

$$\delta(x_1^{-1}x_2) = \int_{K_{\mathbb{C}}} \rho_{2\hbar''}(x_2^{-1}x_1 g^* g)\, d\nu_{\hbar''}(g),$$

which, together with lemma 1, proves the lemma. $\square$

We now prove theorem 4.

Proof. (*of theorem 4*) Let $\sigma_s, \sigma'_{s'}$ be as described in the theorem. From proposition 2, we have

$$\langle \sigma_s, \sigma'_{s'} \rangle^{\mathrm{BKS}} = a_{\frac{s+s'}{2}} \int_{K_{\mathbb{C}}} \overline{\hat{F}(gZ)}\hat{F}'(gZ^{-1})\, d\nu_{\hbar''}(g), \tag{3.13}$$

where $Z = e^{i\frac{s-s'}{s+s'}Y}$, $\hbar'' = \frac{s+s'}{2}\hbar_0$. Now, we have $\hat{F} = C_\hbar f$, so that $\hat{F}(gZ) = \int_K dx_1 \rho_\hbar(gZx_1^{-1})f(x_1)$. Analogously, $\hat{F}'(gZ^{-1}) = \int_K dx_2 \rho_{\hbar'}(gZ^{-1}x_2^{-1})f'(x_2)$, where $\hbar = s\hbar_0$ and $\hbar' = s'\hbar_0$. Substituting these expressions in (3.13), we see that to prove the theorem it is enough to show that

$$\delta(x_1^{-1}x_2) = \int_{K_{\mathbb{C}}} \overline{\rho_\hbar(gZx_1^{-1})}\rho_{\hbar'}(gZ^{-1}x_2^{-1})\, d\nu_{\hbar''}(g). \tag{3.14}$$

This follows easily from lemma 2. $\square$

In [FMMN], it was shown that the parallel transport between two fibers of $\mathcal{H}^{\mathrm{Q}}$, $\mathcal{H}^{\mathrm{Q}}_s$ and $\mathcal{H}^{\mathrm{Q}}_{s'}$, for the quantum connection δ^{Q} corresponds to $C_\hbar \circ C_{\hbar'}^{-1}$. More explicitly, we defined the CST bundle $\mathcal{H}^{\mathrm{H}} \to \mathbb{R}_+$ with $\mathcal{H}^{\mathrm{H}}_s = \mathcal{H}L^2(K_{\mathbb{C}}, d\nu_\hbar)$, where $\hbar = s\hbar_0$. The CST bundle comes equipped with a connection δ^{H} for which the parallel transport $U_{\hbar\hbar'} : \mathcal{H}^{\mathrm{H}}_{s'} \to \mathcal{H}^{\mathrm{H}}_s$ is given by the CST as $U_{\hbar\hbar'} = C_\hbar \circ C_{\hbar'}^{-1}$. Moreover, there exists a natural unitary bundle isomorphism $S : \mathcal{H}^{\mathrm{H}} \to \mathcal{H}^{\mathrm{Q}}$ such that $\delta^{\mathrm{Q}} = S \circ \delta^{\mathrm{H}} \circ S^{-1}$. At the fiber over $s > 0$ this is given by

$$S_s(\hat{F}) = \hat{F} \circ \psi_s \frac{e^{-\frac{s|Y|^2}{2\hbar_0}}}{\sqrt{a_s}} \sqrt{\Omega_s}. \tag{3.15}$$

From theorem 3, we know that δ^{Q} is also the connection induced from the BKS pairing between infinitesimally close fibers of $\mathcal{H}^{\mathrm{Q}}$. Note, however, that of course one can give many different explicit Hermitian pairings between fibers of $\mathcal{H}^{\mathrm{Q}}$, such that they all induce the same connection δ^{Q} on $\mathcal{H}^{\mathrm{Q}}$. Theorem 4 implies that, among those, the pairing map that corresponds to the CST *is* the BKS pairing map naturally defined from geometric quantization. This is the map $B^{\mathrm{Q}}_{ss'} : \mathcal{H}^{\mathrm{Q}}_{s'} \to \mathcal{H}^{\mathrm{Q}}_{s}$ obtained by restriction of the prequantum BKS map (2.15),

$$\langle \sigma_s, \sigma'_{s'} \rangle^{\mathrm{BKS}} = \langle \sigma_s, B^{\mathrm{Q}}_{ss'} \sigma'_{s'} \rangle^{\mathrm{Q}}, \tag{3.16}$$

for $\sigma_s \in \mathcal{H}^{\mathrm{Q}}_s$ and $\sigma'_{s'} \in \mathcal{H}^{\mathrm{Q}}_{s'}$. As a corollary of theorem 4 we obtain the following theorem, which adds to theorem 4 of [FMMN] the explanation of the role of the BKS pairing.

Theorem 5. *For $s, s' > 0$, the BKS pairing map $B^{\mathrm{Q}}_{ss'} : \mathcal{H}^{\mathrm{Q}}_{s'} \to \mathcal{H}^{\mathrm{Q}}_{s}$ is unitary and it is given by*

$$B^{\mathrm{Q}}_{ss'} = S_s \circ C_\hbar \circ C^{-1}_{\hbar'} \circ S^{-1}_{s'} = S_s \circ U_{\hbar\hbar'} \circ S^{-1}_{s'}. \tag{3.17}$$

This pairing coincides with the parallel trasnport of δ^{Q}.

Proof. The result follows from direct computation and from theorem 4, where we use the unitarity of $C_\hbar$ and also $a_{\frac{s+s'}{2}} = \sqrt{a_s a_{s'}}$. Since $\delta^{\mathrm{Q}} \circ S = S \circ \delta^{\mathrm{H}}$, the theorem states that the parallel transport of the quantum connection is given by the BKS pairing map which is clearly unitary. $\square$

This is in agreement with the results of [FMMN] where, however, the quantum connection was defined in a different way by orthogonal projection.

We note that the isomorphisms S_s do not have an obvious continuation to $s = 0$. From the point of view of the CST, it would be natural to define the fiber of $\mathcal{H}^{\mathrm{H}}$ over $s = 0$ as being simply $L^2(K, dx)$. However, note that the BKS pairing map between the Hilbert space of vertically polarized sections (which would correspond to taking the limit $s \to 0$) and the Hilbert space $\mathcal{H}^{\mathrm{Q}}_1$ over $s = 1$, is *not* unitary but only unitary up to scale. (Explicitly, it is unitary up to a factor of $a_{\frac{1}{2}}$.) This agrees precisely with the result of [Ha3] (again, after taking care of the overall normalization of the pairing). The reason for this breakdown of unitarity in the limit $s \to 0$ is that the isomorphisms S_s are not well defined in this limit.

We recall that sections of $\mathcal{H}^{\mathrm{Q}}$ which are horizontal with respect to δ^{Q} satisfy a heat equation on $K_{\mathbb{C}}$. The corresponding explicit form for the connection δ^{Q} can also be found directly from theorem 5, using lemma 1 from [FMMN].

We also remark that in the case when K is abelian, the formulas in the paper are valid upon setting $\rho = 0$ and $\eta \equiv 1$.

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