Physical Properties of Quantum Field Theory Measures

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Abstract

Well known methods of measure theory on infinite dimensional spaces are used to study physical properties of measures relevant to quantum field theory. The difference of typical configurations of free massive scalar field theories with different masses is studied. We apply the same methods to study the Ashtekar-Lewandowski (AL) measure on spaces of connections. In particular we prove that the diffeomorphism group acts ergodically, with respect to the AL measure, on the Ashtekar-Isham space of quantum connections modulo gauge transformations. We also prove that a typical, with respect to the AL measure, quantum connection restricted to a (piecewise analytic) curve leads to a parallel transport discontinuous at every point of the curve.

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I Introduction

Path integrals play an important role in modern quantum field theory. The application in this context of methods of the mathematical theory of measures on infinite dimensional spaces is due to constructive quantum field theorists[1-5]. With the help of these methods important physical results have been obtained, especially concerning two and three dimensional theories. Recently analogous methods have been applied within the framework of Ashtekar non-perturbative quantum gravity to give a rigorous meaning to the connection representation[6-10], solve the diffeomorphism constraint[11] and define the Hamiltonian constraint[12-18]. These works all crucially depend on the use of (generalized) Wilson loop variables which have been considered for the first time, within a Hamiltonian formulation of gauge theories, by Gambini, Trias and collaborators[19,20,21] and were rediscovered for canonical quantum gravity by Rovelli and Smolin[22]. In fact, instead of working in the connection representation for which Wilson loops are just convenient functions, one can use a so-called loop representation (see Ref. 23 and references therein) by means of which a rich arsenal of (formal) results was obtained which complement those obtained in the connection representation. For previous works on measures on spaces of connections see e.g. Refs. 24 and 25.

The present paper has two main goals. The first consists in studying physical properties of the support of the path integral measure of free massive scalar fields. We use a system with a countable number of simple random variables which probe the typical scalar fields over cubes, with volume L^{d+1}, placed far away in (Euclidean) space-time. With these probes we are able to study the difference between the supports of two free scalar field theories with different masses. The results that we obtain provide a characterization of the supports which is physically more transparent than those obtained previously[26-30].

Our second goal consists in providing the proof of analogous results for the Ashtekar-Lewandowski (AL) measure on the space $\overline{\mathcal{A}/\mathcal{G}}$ of quantum connections modulo gauge transformations[6,7]. First we prove that the group of diffeomorphisms acts ergodically, with respect to the AL measure, on $\overline{\mathcal{A}/\mathcal{G}}$. Second we show that the AL measure is supported on connections which, restricted to a curve, lead to parallel transports discontinuous at every point of the curve.

Quantum scalar field theories have been intensively studied both from the mathematical and the physical point of view. Divergences in Schwinger functions (which, in measure theoretical terminology, are the moments of the measure) are directly related with the fact that the relevant measures are supported not on the space of nice smooth scalar field configurations, that enter the classical action, but rather on spaces of distributions. This gives a strong motivation for more detailed studies of the support of relevant measures. In physical terms this corresponds to finding "typical (quantum) scalar field configurations", the set of which has measure one.

The present paper is organized as follows. In section II we recall some results from the theory of measures on infinite dimensional spaces. Namely Bochner-Minlos theorems are stated and the concept of ergodicity of (semi-)group actions is introduced. In section III we study properties of simple measures: the countable product of identical one-dimensional Gaussian measures and the white noise measure. In the first example we illustrate the disjointness of the support of different measures, which are invariant under a fixed ergodic action of the same group. For the white noise measure we choose random variables which probe the support and which will be also used in section IV to study the support of massive scalar field theories. In section V we obtain properties of the support of the AL measure that complement previously obtained results[8] and prove that the diffeomorphism group acts ergodically on the space of connections modulo gauge transformations. In section VI we present our conclusions.

II Review of Results From Measure Theory

II.1 Bochner-Minlos Theorems

In the characterization of typical configurations of measures on functional spaces the so called Bochner-Minlos theorems play a very important role. These theorems are infinite dimensional generalizations of the Bochner theorem for probability measures on $\mathbb{R}^N$. Let us, for the convenience of the reader, recall the latter result. Consider any (Borel) probability measure μ on $\mathbb{R}^N$, i.e. a finite measure, normalized so that $\mu(\mathbb{R}^N) = 1$. The generating functional χ_μ of this measure is its Fourier transform, given by the following function on $\mathbb{R}^N$ ($\cong (\mathbb{R}^N)'$, the prime denotes the topological dual, see below)

$$\chi_\mu(\lambda) = \int_{\mathbb{R}^N} d\mu(x) e^{i(\lambda,x)} \ , \tag{1}$$

where $(\lambda, x) = \sum_{j=1}^N \lambda^j x_j$. Generating functionals of measures satisfy the following three basic conditions,

(*i*) Normalization: $\chi(0) = 1$;

(*ii*) Continuity: χ is continuous on $\mathbb{R}^N$;

(*iii*) Positivity: $\sum_{k,l=1}^m c_k \overline{c_l} \chi(\lambda_k - \lambda_l) \geq 0$, for all $m \in \mathbb{N}$, $c_1, ..., c_m \in \mathbb{C}$ and $\lambda_1, ..., \lambda_m \in \mathbb{R}^N$.

The last condition comes from the fact that $||f||_\mu \geq 0$, for $f(x) = \sum_k^m c_k e^{i(\lambda_k, x)}$, where $||\cdot||_\mu$ denotes the $L^2(\mathbb{R}^N, d\mu)$ norm. The finite dimensional Bochner theorem states that the converse is also true. Namely, for any function χ on $\mathbb{R}^N$ satisfying (*i*), (*ii*) and (*iii*) there exists a unique probability measure on $\mathbb{R}^N$ such that χ is its generating functional.

Both in statistical mechanics and in quantum field theory one is interested in the so-called correlators, or in probabilistic terminology, the moments of the measure μ,

$$< (x_{i_1})^{p_1} ... \ (x_{i_k})^{p_k} >:= \int_{\mathbb{R}^N} d\mu(x)(x_{i_1})^{p_1} ... \ (x_{i_k})^{p_k} \ . \tag{2}$$

For the correlators of any order to exist the measure μ must have a rapid decay at $x-$infinity, in order to compensate the polynomial growth in (2) (examples are Gaussian measures and measures with compact support). In the λ-space the latter condition turns out to be equivalent to χ being infinitely differentiable (C^∞). The correlators are then just equal to partial derivatives of χ at the origin, multiplied by an appropriate power of $-i$.

Let us now turn to the infinite dimensional case. The role of the space of λ's will be played by $\mathcal{S}(\mathbb{R}^{d+1})$, the Schwarz space of C^∞-functions on (Euclideanized) space-time with fast decay at infinity. So we have the indices $\lambda(i) := \lambda^i$ replaced by $f(x)$. The space $\mathcal{S}(\mathbb{R}^{d+1})$ has a standard (nuclear) topology. Its elements are functions with regularity properties both for small and for large distances. The physically interesting measures will "live" on spaces dual to $\mathcal{S}(\mathbb{R}^{d+1})$. Consider the space $\mathcal{S}^{'}(\mathbb{R}^{d+1})$ of all continuous linear functionals on $\mathcal{S}(\mathbb{R}^{d+1})$ (i.e. the topological dual of $\mathcal{S}(\mathbb{R}^{d+1})$). This is the so called space of tempered distributions, which includes delta functions and their derivatives, as well as functions which grow polynomially at infinity. We will consider also the even bigger space $\mathcal{S}^a(\mathbb{R}^{d+1})$ of all linear (not necessarily continuous) functionals on $\mathcal{S}(\mathbb{R}^{d+1})$. Then the simplest generalization of the Bochner theorem states that a function $\chi(f)$ on $\mathcal{S}(\mathbb{R}^{d+1})$ satisfies the following conditions,

(i') Normalization: $\chi(0) = 1$;

(ii') Continuity: χ is continuous on any finite dimensional subspace of $\mathcal{S}(\mathbb{R}^{d+1})$;

(iii') Positivity: $\sum_{k,l=1}^m c_k \overline{c_l} \chi(f_k - f_l) \geq 0$, for all $m \in \mathbb{N}$, $c_1, ..., c_m \in \mathbb{C}$ and $f_1, ..., f_m \in \mathcal{S}(\mathbb{R}^{d+1})$,

if and only if it is the Fourier transform of a probability measure μ on $\mathcal{S}^a(\mathbb{R}^{d+1})$, i.e.

$$\chi(f) = \int_{\mathcal{S}^a(\mathbb{R}^{d+1})} d\mu(\phi) e^{i\phi(f)} \quad . \tag{3}$$

The topology of convergence on finite dimensional subspaces of $\mathcal{S}(\mathbb{R}^{d+1})$ is unnaturally strong. Demanding in *(ii')* continuity of χ with respect to the much weaker standard nuclear topology on $\mathcal{S}(\mathbb{R}^{d+1})$ yields a measure supported on the topological dual $\mathcal{S}^{'}(\mathbb{R}^{d+1})$ of $\mathcal{S}(\mathbb{R}^{d+1})$[31]. This is the first version of the Bochner-Minlos theorem. Further refinement can be achieved if χ is continuous with respect to a even weaker topology induced by an inner product. We present a special version of this result, suitable for the purposes of the present work;

for different, more general formulations see Refs. 3 and 32. Let P be a linear continuous operator from $\mathcal{S}(\mathbb{R}^{d+1})$ onto $\mathcal{S}(\mathbb{R}^{d+1})$, with continuous inverse. Suppose further that P is positive when viewed as an operator on $L^2(\mathbb{R}^{d+1}, d^{d+1}x)$ and that the bilinear form

$$< f_1, f_2 >_{P^{1/2}} := (P^{1/2} f_1, P^{1/2} f_2) \ , \quad f_1, f_2 \in \mathcal{S}(\mathbb{R}^{d+1}) \tag{4}$$

defines an inner product on $\mathcal{S}(\mathbb{R}^{d+1})$, where (,) denotes the $L^2(\mathbb{R}^{d+1}, d^{d+1}x)$ inner product. Let χ satisfy (*i'*), (*iii'*) and be continuous with respect to the norm associated with the inner product $< \, , \, >_{P^{1/2}}$. Natural examples are provided by Gaussian measures μ_C with covariance C and Fourier transform

$$\chi_C(f) = e^{-\frac{1}{2}(f, Cf)} \ , \tag{5}$$

in which case one can take the positive operator P to be the covariance C itself. A particular case is the path integral measure for free massive scalar fields with mass m, which is the Gaussian measure with covariance

$$C_m = (-\Delta + m^2)^{-1} \ , \tag{6}$$

where Δ denotes the Laplacian on $\mathbb{R}^{d+1}$. In the general (not necessarily Gaussian) case let $\mathcal{H}_{P^{1/2}}$ ($\mathcal{H}_{P^{-1/2}}$) denote the completion of $\mathcal{S}(\mathbb{R}^{d+1})$ with respect to the inner product $< \, , \, >_{P^{1/2}}$ ($< \, , \, >_{P^{-1/2}}$). Then the measure on $\mathcal{S}'(\mathbb{R}^{d+1})$ corresponding to χ is actually supported on a proper subset of $\mathcal{S}'(\mathbb{R}^{d+1})$ given by an extension of $\mathcal{H}_{P^{-1/2}}$ defined by a Hilbert-Schmidt operator on $\mathcal{H}_{P^{1/2}}$. We see that in the scalar field case $\mathcal{H}_{C_m^{-1/2}}$ is the space of finite action configurations and therefore typical quantum configurations live in a bigger space. In order to define the extension mentioned above, recall that an operator H on a Hilbert space is said to be Hilbert-Schmidt if given an (arbitrary) orthonormal basis $\{e_k\}$ one has

$$\sum_{k=1}^{\infty} < He_k, He_k > \ < \ \infty \ .$$

Given such a Hilbert-Schmidt operator H on $\mathcal{H}_{P^{1/2}}$, which we require to be invertible, self-adjoint and such that $H(\mathcal{S}(\mathbb{R}^{d+1})) \subset \mathcal{S}(\mathbb{R}^{d+1})$, define the new inner product $< \, , \, >_{P^{-1/2}H}$ on $\mathcal{S}(\mathbb{R}^{d+1})$ by

$$< f_1, f_2 >_{P^{-1/2}H} := (P^{-1/2} H f_1, P^{-1/2} H f_2) \ , \quad f_1, f_2 \in \mathcal{S}(\mathbb{R}^{d+1}). \tag{7}$$

Consider $\mathcal{H}_{P^{-1/2}H}$, the completion of $\mathcal{S}(\mathbb{R}^{d+1})$ with respect to the inner product $< \, , \, >_{P^{-1/2}H}$, and identify its elements with linear functionals on $\mathcal{S}(\mathbb{R}^{d+1})$ through the $L^2(\mathbb{R}^{d+1}, d^{d+1}x)$ inner product. Under the above conditions, the (second version of the) Bochner-Minlos Theorem states that

(Bochner-Minlos) A generating functional χ, continuous with respect to the inner product $<\ ,\ >_{P^{1/2}}$, is the Fourier transform of a unique measure supported on $\mathcal{H}_{P^{-1/2}H}$, for every Hilbert-Schimdt operator H on $\mathcal{H}_{P^{1/2}}$ such that $\mathcal{S}(\mathbb{R}^{d+1}) \subset \mathrm{Ran}H$, $H^{-1}(\mathcal{S}(\mathbb{R}^{d+1}))$ is dense and $H^{-1} : \mathcal{S}(\mathbb{R}^{d+1}) \to \mathcal{H}_{<\ ,\ >_{P^{1/2}}}$ is continuous.

In section III.1 we will also use an obvious adaptation of this result to the space of infinite sequences $\mathbb{R}^{\mathbb{N}}$.

A common feature of the two versions of the Bochner-Minlos theorem is that they give the support as a linear subspace of the original measure space. Nonlinear properties of the support have to be obtained in a different way. In particular one can show (see section III.2) that the white noise measures with

$$\chi_{\sigma_1}(f) \ := \ e^{-\frac{\sigma_1}{2}(f,f)} \tag{8}$$

and

$$\chi_{\sigma_2}(f) \ := \ e^{-\frac{\sigma_2}{2}(f,f)} \tag{9}$$

have disjoint supports for $\sigma_1, \sigma_2 > 0$ and $\sigma_1 \neq \sigma_2$, while Bochner-Minlos theorem would give the same results in both cases.

II.2 Ergodic Actions

We review here some concepts and results from ergodic theory[32,33]. Let φ denote an action of the group G on the space M, endowed with a probability measure μ, by measure preserving transformations $\varphi_g : M \to M\ ,\ g \in G$, i.e. $\varphi_{g*}\mu = \mu\ ,\quad \forall g \in G$ or, equivalently, for every measurable set $A \subset M$ and for every $g \in G$, the measure of A equals the measure of the pre-image of A by φ_g. The action φ is said to be ergodic if all G-invariant sets have either measure zero or one. The fact that φ is measure preserving implies that the (right) linear representation U of G on $L^2(M, d\mu)$ induced by φ

$$(U_g\psi)(x) := \psi(\varphi_g x) \tag{10}$$

is unitary. The action φ on M is ergodic if and only if the only U_G-invariant vectors on $L^2(M, d\mu)$ are the (almost everywhere) constant functions. This follows easily from the fact that the linear space spanned by characteristic functions of measurable sets (equal to one on the set and zero outside) is dense in $L^2(M, d\mu)$. The above also applies for a discrete semi-group generated by a single (not necessarily invertible) measure preserving transformation T, in which case the role of the group G is played by the additive semi-group $\mathbb{N}$, T being identified with φ_1. Notice that in the latter case the linear representation on $L^2(M, d\mu)$ may fail to be unitary, although isometry still holds. For actions of $\mathbb{R}$ and $\mathbb{N}$, respectively, the following properties are equivalent to ergodicity[33]

$$\lim_{\tau\to\infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} dt \psi(\varphi_t x_0) = \int_M d\mu(x)\psi(x) \tag{11a}$$

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} \psi(\varphi_n x_0) = \int_M d\mu(x)\psi(x) \ , \tag{11b}$$

where $\varphi_n := \varphi_1^n$ and the equalities hold for μ-almost every x_0 and for all $\psi \in L^1(M, d\mu)$. One important consequence of (11) is that if μ_1 and μ_2 are two different measures and a given action φ is ergodic with respect to both μ_1 and μ_2 then these measures must have disjoint supports (points x_0 for which (11) holds). Recall also that the action of $\mathbb{R}$ and $\mathbb{N}$, respectively, is called mixing if for every $\psi_1, \psi_2 \in L^2(M, d\mu)$ we have

$$\lim_{t\to\infty} < \psi_1, U_t\psi_2 > = < \psi_1, 1 >< 1, \psi_2 > , \tag{12a}$$

$$\lim_{n\to\infty} < \psi_1, U_n\psi_2 > = < \psi_1, 1 >< 1, \psi_2 > \ . \tag{12b}$$

It follows from (12) that every U-invariant $L^2(M, d\mu)$-function is constant almost everywhere and therefore every mixing action is ergodic (see Ref. 33 for details). If M is a linear space then (11) gives a non-linear characterization of the support. Indeed if x_1 and x_2 are typical configurations, in the sense that (11) holds for them, then $x_1 + x_2$ and λx_1 for $\lambda \neq 1$, are in general not typical configurations. The nonlinearity of supports is best illustrated by the action of $\mathbb{N}$ on the space of infinite sequences endowed with a Gaussian measure that we recall in the next subsection.

III Support Properties of Simple Measures

III.1 Countable Product of Gaussian Measures

We will consider here the simplest case of a Gaussian measure in an infinite dimensional space. We will see however that many aspects of Gaussian measures on functional spaces can be rephrased in this simple context.

Let $M = \mathbb{R}^{\mathbb{N}}$, the set of all real sequences (maps from $\mathbb{N}$ to $\mathbb{R}$)

$$x = \{x_1, x_2, \ldots\}$$

and consider on this space the measure given by the infinite product of identical Gaussian measures on $\mathbb{R}$, of mean zero and variance ρ

$$d\mu_\rho(x) = \prod_{n=1}^{\infty} e^{-\frac{x_n^2}{2\rho}} \frac{dx_n}{\sqrt{2\pi\rho}} \ . \tag{13}$$

As we saw above, an equivalent way of defining μ_ρ is by giving its Fourier Transform. Let

$$< y, z >_{\sqrt{\rho}} := \rho\,(y, z) \ , \quad y, z \in \mathcal{S} \ , \tag{14}$$

where $(y,z)=\sum_{n=1}^{\infty} y_n z_n$ and $\mathcal{S}$ is the space of rapidly decreasing sequences

$$\mathcal{S} := \left\{ y \in \mathbb{R}^{\mathbb{N}} \ : \quad \sum_{n=1}^{\infty} n^k y_n^2 < \infty \ , \ \forall k > 0 \right\} .$$

Then

$$\chi_\rho(y) := e^{-\frac{1}{2}<y,y>_{\sqrt{\rho}}} = \int e^{i(y,x)} d\mu_\rho(x) \ , \tag{15}$$

where $y \in \mathcal{S}$, $x \in \mathbb{R}^{\mathbb{N}}$.

Consider now the ergodic (in fact mixing) action φ of $\mathbb{N}$ generated by $T \equiv \varphi_1$,

$$(\varphi_1(x))_n := x_{n+1} \ , \quad x \in \mathbb{R}^{\mathbb{N}} \tag{16}$$

which, as we will see, is a discrete analogue of the action of $\mathbb{R}$ by translations on the quantum configuration space of a free scalar field theory. The transformation φ_1 is clearly measurable and measure preserving, since all the measures in the product (13) are equal. It is also not difficult to see that φ is mixing. In fact, invoking linearity and continuity, one only needs to show that (12b) is verified for a set of L^2-functions whose span is dense. The functions of the form $\exp(i(y,x))$, $y \in \mathcal{S}$ form such a set, and one has for them

$$\begin{aligned} \lim_{n\to\infty} < e^{i\,(y,x)}, U_n e^{i\,(z,x)} > \ &= \ e^{-\frac{1}{2}\left(<z,z>_{\sqrt{\rho}} + <y,y>_{\sqrt{\rho}}\right)} \\ &= \ < e^{i\,(y,x)}, 1 >< 1, e^{i\,(z,x)} > \ , \quad \forall y, z \in \mathcal{S} \ , \end{aligned}$$

where, in the first and last terms, $< \ , \ >$ denotes the $L^2(\mathbb{R}^{\mathbb{N}}, d\mu_\rho)$ inner product and U denotes the isometric representation associated with φ. Thus, φ is an ergodic action with respect to μ_ρ, for any ρ. Of course, $\rho \neq \rho'$ implies $\mu_\rho \neq \mu_{\rho'}$ and one concludes that μ_ρ and $\mu_{\rho'}$ must be mutually singular (i.e. have disjoint supports). In fact, taking in (11b) $\psi(x) = \exp(i(y,x))$ for both μ_ρ and $\mu_{\rho'}$ leads to a contradiction, unless a x_0 satisfying (11b) for both μ_ρ and $\mu_{\rho'}$ cannot be found.

The aim now is to find properties of typical configurations which allow to distinguish the supports of μ_ρ and $\mu_{\rho'}$. Unfortunatelly the Bochner-Minlos theorem cannot help, due to the fact that the inner products (14), that define the measures μ_ρ and $\mu_{\rho'}$ via (15), are proportional to each other and therefore the corresponding extensions in the Bochner-Minlos theorem are equal: $\mathcal{H}_{\rho^{-1/2}H} = \mathcal{H}_{\rho'^{-1/2}H}$ $(= \mathcal{H}_H)$ for any ρ, ρ'.

Let us now find a better characterization of the support, for which the mutual singularity of μ_ρ and $\mu_{\rho'}$ becomes explicit. In order to achieve this we use a slight modification of an argument given in Ref. 1, that provides convenient sets, both of measure zero and one.

Proposition 1 *Given a sequence* $\{\Delta_j\}$, $\Delta_j > 1$ *the* μ_ρ*-measure of the set*

$$Z_\rho(\{\Delta_j\}) := \{x \ : \quad \exists N_x \in \mathbb{N} \quad \text{s.t.} \quad |x_n| < \sqrt{2\rho \ln \Delta_n} \ , \ \text{for } n \geq N_x\} \tag{17}$$

is one (zero) if $\sum 1/\left(\Delta_j \sqrt{\ln \Delta_j}\right)$ *converges (diverges).*

This can be proven as follows. For fixed integer N and positive sequence $\{\Lambda_j\}$ define sets $Z_N(\{\Lambda_j\})$ by

$$Z_N(\{\Lambda_j\}) := \{x \ : \quad |x_n| < \Lambda_n \ , \ \text{for } n \geq N\} \tag{18}$$

The μ_ρ-measure of each of these sets is

$$\mu_\rho(Z_N(\{\Lambda_j\})) \quad = \quad \prod_{n=N}^{\infty} \mathrm{Erf}\left(\frac{\Lambda_n}{\sqrt{2\rho}}\right) \ , \tag{19}$$

where $\mathrm{Erf}(x) = 1/\sqrt{\pi} \int_{-x}^{x} e^{-\xi^2} d\xi$ is the error function. The sequence of sets $Z_N(\{\Lambda_j\})$ is an increasing sequence for fixed $\{\Lambda_j\}$, and the set $Z_\rho(\{\Delta_j\})$ defined in (17) is just their infinite union, for $\Delta_j = \exp\bigl(\Lambda_j^2/2\rho\bigr)$:

$$Z_\rho(\{\Delta_j\}) = \bigcup_{N \in \mathbb{N}} Z_N(\{\Lambda_j\}) \ . \tag{20}$$

From σ-additivity one gets

$$\mu_\rho(Z_\rho(\{\Delta_j\})) = \lim_{N\to\infty} \mu_\rho(Z_N(\{\Lambda_j\})) \tag{21}$$

and therefore

$$\mu_\rho(Z_\rho(\{\Delta_j\})) = \exp\left(\lim_{N\to\infty} \sum_{n=N}^{\infty} \ln(\mathrm{Erf}(\sqrt{\ln \Delta_n}))\right) \ . \tag{22}$$

Notice that the exponent is the limit of the remainder of order N of a series. Since only divergent sequences $\{\Delta_j\}$ may lead to a non-zero measure, one can use the asymptotic expression for $\mathrm{Erf}(x)$, which gives

$$\mu_\rho(Z_\rho(\{\Delta_j\})) = \exp\left(-\lim_{N\to\infty} \sum_{n=N}^{\infty} \frac{1}{\Delta_n \sqrt{\ln \Delta_n}}\right) \ . \tag{23}$$

Depending on the sequence $\{\Delta_j\}$, only two cases are possible: either the series exists, or it diverges to plus infinity, since $\Delta_j > 1 \, , \forall j$. In the first case the limit of the remainder is zero, and so the measure of $Z_\rho(\{\Delta_j\})$ will be one. If the sum diverges so does the remainder of any order and therefore the measure of $Z_\rho(\{\Delta_j\})$ is zero. □

Let us now discuss the meaning of this result. To begin with, it is easy to present disjoint sets A, A' s.t $\mu_\rho(A) = 1$, $\mu_{\rho'}(A) = 0$ and $\mu_\rho(A') = 0$, $\mu_{\rho'}(A') = 1$, for $\rho \neq \rho'$. Without loss of generality, take $\rho = a\rho'$, $a > 1$. The set $Z_\rho(\{n\})$ has μ_ρ-measure zero, since $\sum 1/(n\sqrt{\ln n})$ diverges. But $Z_\rho(\{n\}) = Z_{\rho'}(\{n^a\})$ (see (17)) and $Z_{\rho'}(\{n^a\})$ has $\mu_{\rho'}$-measure one, since $\sum 1/(n^a\sqrt{\ln n^a})$ converges, for $a > 1$. On the other hand the sets $Z_\rho(\{n^{1+\epsilon}\})$ are such that

$$\mu_\rho(Z_\rho(\{n^{1+\epsilon}\})) = \mu_{\rho'}(Z_\rho(\{n^{1+\epsilon}\})) = 1 \ , \quad \forall \epsilon > 0$$

and since $Z_\rho(\{n\}) \subset Z_\rho(\{n^{1+\epsilon}\}),\ \forall \epsilon > 0$, the difference sets

$$A_\rho^\epsilon := Z_\rho(\{n^{1+\epsilon}\}) \backslash Z_\rho(\{n\}) \tag{24}$$

are such that $\mu_\rho(A_\rho^\epsilon) = 1$ and $\mu_{\rho'}(A_\rho^\epsilon) = 0\,,\ \forall \epsilon > 0$.

Notice that the "square-root-of-logarithm" nature of the support A_ρ^ϵ of μ_ρ does not mean that the typical sequence x approaches $\sqrt{2\rho \ln n}$, as $n \to \infty$. To clarify this point let us appeal to the Bochner-Minlos Theorem (see section II.1). Take $\mathcal{H}_{P^{1/2}}$ to be ℓ^2, the completion of $\mathcal{S}$ with respect to the inner product $(\ ,\ )$:

$$\ell^2 = \left\{ y \in \mathbb{R}^{\mathbb{N}} \ :\ \sum_{n=1}^{\infty} y_n^2 \ <\ \infty \right\}.$$

Consider a vector $a = \{a_1, a_2, \ldots\} \in \ell^2$ and a Hilbert-Schmidt operator H_a defined by

$$(H_a(x))_n := a_n x_n\ ,\quad x \in \ell^2\,. \tag{25}$$

Then the Bochner-Minlos Theorem leads to the conclusion that a typical sequence x in the support of the measure must satisfy

$$\sum_{n=1}^{\infty} a_n^2 x_n^2 < \infty\ , \tag{26}$$

which is certainly not true for a sequence behaving asymptotically like $\sqrt{2\rho \ln n}$, if we choose appropriately $a \in \ell^2$. However, the Bochner-Minlos theorem does not forbid the appearance of a subsequence behaving asymptotically even worse than $\sqrt{2\rho \ln n}$. Therefore proposition 1 means that in a typical sequence $\{x_n\}$ no subsequence $\{x_{n_k}\}$ can be found such that, $\forall n_k,\ |x_{n_k}| > \sqrt{2(1+\epsilon)\rho \ln n_k}$, for any arbitrary but fixed ϵ greater than zero, and that one is certainly found if ϵ is taken to be zero. But this subsequence is rather sparse, as demanded by Bochner-Minlos Theorem; from a stochastic point of view the occurrence of values $|x_n|$ greater than $\sqrt{2\rho \ln n}$ is a rare event. The typical sequence in the support is one that is generated with a probability distribution given by the measure. The measure in this case is just a product of identical Gaussian measures in $\mathbb{R}$, so the typical sequence is one obtained by throwing a "Gaussian dice" an infinite number of times.

Notice that a typical $\mu_{\rho'}$ sequence can be obtained from a μ_ρ typical sequence simply by multiplying by $\sqrt{\rho'/\rho}$. This follows from the fact that the map $x \mapsto \sqrt{\rho'/\rho}\, x$ is a isomorphism of measure spaces $(\mathbb{R}^{\mathbb{N}}, \mu_\rho) \to (\mathbb{R}^{\mathbb{N}}, \mu_{\rho'})$.

III.2 The White Noise Measure

We consider now the so called "white noise" measure, which in some sense is the continuous analogue of the previous case[4]. Again, we will look for convenient sets of measure one, in the sense given in section III.1. This will be achieved by a proper choice of random variables, i.e. measurable functions, which will reduce the present case to the previous discrete one.

As mentioned in section II.1 the d+1-dimensional white noise is the Gaussian measure μ_σ with Fourier transform $\chi_\sigma(f) = \exp(-\sigma/2(f,f))$. Notice that here σ has dimensions of inverse mass squared.

The Euclidean group $\mathcal{E}$ acts on $\mathcal{S}(\mathbb{R}^{d+1})$, $\mathcal{S}^{'}(\mathbb{R}^{d+1})$ and (unitarily) on $L^2(\mathcal{S}^{'}(\mathbb{R}^{d+1}), d\mu_\sigma)$ respectively by

$$\begin{array}{rcl} (\tilde{\varphi}_g f)\,(x) & = & f(g^{-1}x)\,, \\ (\varphi_g \phi)\,(f) & = & \phi(\tilde{\varphi}_{g^{-1}} f)\,, \\ (U_g \psi)\,(\phi) & = & \psi\,(\varphi_g \phi)\ , \end{array} \tag{27}$$

where $g \in \mathcal{E}$, $g\,x$ denotes the standard action of $\mathcal{E}$ on $\mathbb{R}^{d+1}$ by translations, rotations and reflections, $f \in \mathcal{S}(\mathbb{R}^{d+1})$, $\phi \in \mathcal{S}^{'}(\mathbb{R}^{d+1})$ and $\psi \in L^2(\mathcal{S}^{'}(\mathbb{R}^{d+1}), d\mu_\sigma)$.

It is easy to see that a subgroup of translations in a fixed direction, say the time direction, is mixing. One just has to consider the set with dense span of L^2-functions of the form $\exp(i\,\phi(f))$, $f \in \mathcal{S}(\mathbb{R}^{d+1})$ and use the Riemann-Lebesgue Lemma to prove that

$$\lim_{t\to\infty} \int f(x_0+t,\ldots,x_d)\,g(x_0,\ldots,x_d)\,d^{d+1}x = 0\ ,\ \forall f,g \in \mathcal{S}(\mathbb{R}^{d+1})\,. \tag{28}$$

This implies that the measures μ_σ and $\mu_{\sigma'}$ for $\sigma \neq \sigma'$ have disjoint supports, even though the Bochner-Minlos theorem gives us for the support in both cases an extension of $L^2(\mathbb{R}^{d+1}, d^{d+1}x)$ through an Hilbert-Schmidt operator (see section III.1). Since the choice of a complete (,)-orthonormal system $\{f_n\}$ gives us a isomorphism of measure spaces

$$\begin{array}{rcl} (\mathcal{S}^{'}(\mathbb{R}^{d+1}), \mu_\sigma) & \to & (\mathbb{R}^{\mathbb{N}}, \mu_\rho)_{|\rho=\sigma} \\ \phi & \mapsto & \{\phi(f_n)\}\ , \end{array} \tag{29}$$

for every such basis one can find sets of the type of those found in the previous subsection and which put in evidence the mutual singularity of μ_σ and $\mu_{\sigma'}$. However, for the convenience of our analysis of free massive scalar fields in the next section, let us study the x-behavior of typical white noise configurations ϕ. Since

$$\delta_x\ :\ \phi \mapsto \phi(x) \tag{30}$$

is not a good random variable, we fix in $\mathbb{R}^{d+1}$ a family of non-intersecting cubic boxes, $\{B_j\}_{j=1}^{\infty}$, with sides of length L. Then the mean value of ϕ over B_j is a well defined random variable

$$F_{B_j}\ :\ \phi \mapsto F_{B_j}(\phi) \equiv \phi(f_j) = \frac{1}{L^{d+1}} \int_{B_j} \phi(x) d^{d+1}x\ , \tag{31}$$

where f_j denotes the characteristic function of the set B_j divided by the volume L^{d+1}, and the map

$$\begin{array}{rcl} \mathcal{S}'(\mathbb{R}^{d+1}) & \to & \mathbb{R}^{\mathbb{N}} \\ \phi & \mapsto & \{\phi(f_j)\} \end{array} \tag{32}$$

defines (by push-forward) a measure on $\mathbb{R}^{\mathbb{N}}$ of the form (13) with $\rho = \sigma/L^{d+1}$. This can be seen from the fact that

$$\int_{\mathcal{S}'(\mathbb{R}^{d+1})} d\mu_\sigma \exp\Big(i\sum_{j=1}^{\infty} y_j \phi(f_j)\Big) = \exp\Big(-\frac{\sigma}{L^{d+1}} \frac{\sum_{j=1}^{\infty} y_j^2}{2}\Big) \ . \tag{33}$$

We then conclude from section III.1 that the sets

$$\begin{array}{rcl} W_\sigma^\epsilon := \big\{\phi & \in & \mathcal{S}'(\mathbb{R}^{d+1}) \,:\, \exists N_\phi \in \mathbb{N} \quad \text{s.t.} \\ |\phi(f_n)| & < & \sqrt{2(1+\epsilon)(\sigma/L^{d+1})\ln n}, \quad \text{for } n \geq N_\phi \big\} \end{array} \tag{34}$$

have μ_σ-measure one for every positive ϵ, and that for $\sigma' > \sigma$ an $\epsilon(\sigma') > 0$ can be found such that $\mu_{\sigma'}(W_\sigma^{\epsilon(\sigma')}) = 0$. This shows that the supremum of the mean value of ϕ over N boxes with volume L^{d+1} goes like $\sqrt{2(1+\epsilon)(\sigma/L^{d+1})\ln N}$. A white noise with a bigger variance $\sigma' > \sigma$ has the latter behavior on larger boxes with volume $L'^{d+1} = \frac{\sigma'}{\sigma} L^{d+1}$.

IV Quantum Scalar Field Theories

IV.1 Constructive Quantum Scalar Field Theories

Here we recall briefly some aspects of constructive quantum field theory that will be relevant for the next subsection[2]. A quantum scalar field theory on $d+1$-dimensional (flat Euclideanized) space-time is a measure μ on $\mathcal{S}'(\mathbb{R}^{d+1})$ with Fourier transform χ (generating functional or $\chi(f) = Z(-if)/Z(0)$ in theoretical physics terminology) satisfying the Osterwalder-Schrader (OS) axioms. We will be interested in the axioms which state the Euclidean invariance of the measure (OS2) and ergodicity of the action of the time translation subgroup (OS4), i.e. for $g = T_t :\ T_t(t', \mathbf{x}) = (t'+t, \mathbf{x})$,

$$\lim_{\tau\to\infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} dt \psi(\varphi_{T_t}\phi_0) =_{a.e.} \int_{\mathcal{S}'(\mathbb{R}^{d+1})} \psi(\phi) d\mu(\phi) \ . \tag{35}$$

The action of the Euclidean group on $\mathcal{S}(\mathbb{R}^{d+1})$, $\mathcal{S}'(\mathbb{R}^{d+1})$ and $L^2(\mathcal{S}'(\mathbb{R}^{d+1}), d\mu)$ is defined as in section III.2. So OS2 states that $\varphi_{g*}\mu = \mu$ for all elements g of the Euclidean group $\mathcal{E}$. Notice that OS2+OS4 imply that ergodicity under the subgroup of time translations is equivalent to ergodicity under the full Euclidean group. The vacua of the theory correspond to Euclidean invariant vectors on $L^2(\mathcal{S}'(\mathbb{R}^{d+1}), d\mu)$ and the axioms OS2 and OS4 imply that the vacuum is unique and given by the constant function. Examples of measures satisfying the OS axioms are the Gaussian measures μ_{C_m} corresponding to free massive quantum scalar field theories (see eqs. (5,6)).

IV.2 Support of Free Scalar Field Measures

Since the Euclidean group $\mathcal{E}$ acts ergodically on $(\mathcal{S}'(\mathbb{R}^{d+1}), \mu_{C_m})$ we conclude that the measures μ_{C_m} and $\mu_{C_{m'}}$ (see eq. (6)) with $m \neq m'$ must have disjoint supports. Like in section III.2 we will characterize the difference of supports in terms of the mean value of ϕ over a region with volume L^{d+1}. Before going into the details of the calculations notice that the map

$$\begin{aligned} (\mathcal{S}'(\mathbb{R}^{d+1}), \mu_\sigma) &\to (\mathcal{S}'(\mathbb{R}^{d+1}), \mu_{C_m}) \\ \phi &\mapsto \left[\sigma(-\Delta + m^2)\right]^{-1/2} \phi \end{aligned} \tag{36}$$

is an isomorphism of measure spaces which maps typical white noise configurations to typical μ_{C_m}-configurations. Heuristically this means that for big distances ($\Delta x >> \frac{1}{m}$) or small momenta the correlation imposed by the kinetic term in the action is lost and the typical configurations approach those of white noise with $\sigma = \frac{1}{m^2}$. Let us now obtain a formal derivation of this fact as far as the x-space behavior of typical configurations of free massive scalar fields is concerned.

Consider the same random variables F_{B_j} as in section III.2 but, in order to eliminate the correlation, the cubic boxes B_j will be chosen centered in the points $x^j = (x_0^j, ..., x_d^j) = (\frac{j^2}{m}, 0, ..., 0)$ and with sides parallel to the coordinate axes. The push-forward of μ_{C_m} with respect to the map

$$\begin{aligned} \mathcal{S}'(\mathbb{R}^{d+1}) &\to \mathbb{R}^{\mathbb{N}} \\ \phi &\mapsto \{\phi(f_j)\} \end{aligned} \tag{37}$$

is a Gaussian measure μ_{M_m} in $\mathbb{R}^{\mathbb{N}}$ with covariance matrix M_m given by

$$\begin{aligned} (M_m)_{jl} &= C_m(f_j, f_l) = \\ &= \left(\frac{2}{\pi}\right)^{d+1} \frac{1}{L^{2(d+1)}} \int_{\mathbb{R}^{d+1}} d^{d+1}p \, e^{i\frac{p_0}{m}(j^2-l^2)} \frac{1}{p^2+m^2} \prod_{k=0}^{d} \frac{\sin^2(p_k L/2)}{p_k^2} . \end{aligned} \tag{38}$$

Let us denote the (constant) value of the diagonal elements of M_m by C_m^L, i.e.

$$\begin{aligned} C_m^L &:= (M_m)_{ii} = \\ &= \left(\frac{2}{\pi}\right)^{d+1} \frac{1}{L^{2(d+1)}} \int_{\mathbb{R}^{d+1}} d^{d+1}p \, \frac{1}{p^2+m^2} \prod_{k=0}^{d} \frac{\sin^2(p_k L/2)}{p_k^2} . \end{aligned} \tag{39}$$

Proposition 2 *The set*

$$\begin{aligned} Y^{\epsilon}_{(m)} \quad &:= \quad \{\phi \in \mathcal{S}'(\mathbb{R}^{d+1}) : \exists N_\phi \in \mathbb{N} \ \text{ s.t.} \\ |\phi(f_n)| \quad &< \sqrt{2(1+\epsilon) C^L_m \ln n} \ , \ n \geq N_\phi\} \end{aligned} \tag{40}$$

has μ_{C_m}-measure one for any $\epsilon > 0$.

Like in Section III.2, we will show that the μ_{M_m}-measure of the image of $Y^{\epsilon}_{(m)}$ in $\mathbb{R}^{\mathbb{N}}$ is one. To prove this we will relate the measure μ_{M_m} with a diagonal Gaussian measure of the form (13). Let $\mu_{C^L_m}$ be the Gaussian measure in $\mathbb{R}^{\mathbb{N}}$ with diagonal covariance matrix $C^L_m \delta_{ij}$.

Lemma 1 *The measures μ_{M_m} and $\mu_{C^L_m}$ are mutually absolutely continuous, i.e. have the same zero measure sets.*

To prove the lemma we will rely on Theorem I.23 in pag. 41 of Ref. 1 (see also Theorem 10.1 in pag. 160 of Ref. 32), which gives necessary and sufficient conditions for two covariances to give rise to mutually absolutely continuous Gaussian measures. In our case, since the covariance of $\mu_{C^L_m}$ is proportional to the identity, it is sufficient to show that (i) the operator $T := M_m - C^L_m \mathbf{1}$ is Hilbert-Schmidt and (ii) the operator M_m is bounded, positive with bounded inverse in ℓ^2. Let us first prove that T is Hilbert-Schmidt. The matrix elements of T are $T_{ii} = 0$ and $T_{jl} = (M_m)_{jl}$, for $j \neq l$. One can see from (38) that the off-diagonal elements of $(M_m)_{jl}$ are the values at the points $j^2 - l^2$ of the Fourier transform of a real function f. Explicitly,

$$(M_m)_{jl} = \int_{\mathbb{R}} d\nu_0 \, e^{i\nu_0 (j^2 - l^2)} f(\nu_0) \ , \tag{41}$$

where

$$\begin{aligned} f(\nu_0) \quad := \quad & \left(\frac{2}{\pi}\right)^{d+1} \frac{1}{m^{d+3} L^{2(d+1)}} \frac{\sin^2(mL\nu_0/2)}{\nu_0^2} \ \cdot \\ \cdot \quad & \int_{\mathbb{R}^d} d^d\nu \, \frac{1}{1 + \nu_0^2 + \sum_1^d \nu_k^2} \prod_1^d \frac{\sin^2(mL\nu_k/2)}{\nu_k^2} \ . \end{aligned} \tag{42}$$

Since both f and its derivative f' are L^1, one gets for the Fourier transform $\tilde{f}$ of f:

$$|\tilde{f}(t)| = \frac{|\widetilde{f'}(t)|}{|t|} \ , \tag{43}$$

with $\widetilde{f'}$ continuous, bounded and approaching zero at infinity. Therefore exists $A > 0$ such that

$$|(M_m)_{jl}|^2 \leq \frac{A}{(j^2 - l^2)^2} \ , \quad \text{for } j \neq l \ , \tag{44}$$

and therefore

$$\sum_{j,\,l} |T_{jl}|^2 \leq A \sum_{j \neq l} \frac{1}{(j^2 - l^2)^2} < \infty \ , \tag{45}$$

thus proving that T is Hilbert-Schmidt. Let us now prove (ii). The operator M_m is a positive operator on ℓ^2 since it is given by the restriction of the positive covariance C_m on $L^2(\mathbb{R}^{d+1})$ to the linearly independent system $\{f_j\}_{j\in\mathbb{N}}$. Positivity of M_m and the fact that $M_m = C_m^L \mathbf{1} + T$, T being compact (in fact Hilbert-Schmidt), implies that M_m is bounded, has a trivial kernel and therefore is invertible with bounded inverse (see e.g. Theorem 4.25 and open mapping theorem in Ref. 34). Proposition 2 now follows, given the characterization of the support of $\mu_{C_m^L}$ one gets from section III.1. □

Using the fact that

$$\frac{1}{\alpha\pi} \frac{\sin^2(\alpha p)}{p^2} \tag{46}$$

tends to $\delta(p)$ when $\alpha \to \infty$, one sees from (39) that

$$\lim_{L\to\infty} L^{d+1} C_m^L = \frac{1}{m^2} \ . \tag{47}$$

Thus, comparing (40) with (34) we see that, in accordance with the discussion above, when averaged over widely separated large boxes ($L >> 1/m$), the typical free field distribution approaches white noise with $\sigma = 1/m^2$.

The explicitness of the mutual singularity of μ_{C_m} and $\mu_{C_{m'}}$, with $m \neq m'$ now follows easily from (24), (40) and the fact that C_m^L is a monotonous (decreasing) function of m.

V Properties of the Ashtekar-Lewandowski Measure on Spaces of Connections

V.1 Ergodic Action of the Group of Diffeomorphisms

The diffeomorphism-invariant Ashtekar-Lewandowski measure μ_{AL} on the space of connections modulo gauge transformations $\mathcal{A}/\mathcal{G}$ over the manifold Σ plays an important role in rigorous attempts to find a quantization of canonical general relativity. In the present section we study properties of this measure. We show that $Diff_0(\Sigma)$, the connected component of the group of diffeomorphisms of (the connected analytic manifold) Σ, acts ergodically on $\overline{\mathcal{A}/\mathcal{G}}$. In the next subsection we also obtain results concerning the properties of the support of μ_{AL}.

Let us recall the definition of μ_{AL}. We denote by a hoop $[\alpha]$ in Σ the equivalence class of (piecewise analytic) loops $\{\widetilde{\alpha}\}$ based on $x_0 \in \Sigma$ such that

$$U_{\widetilde{\alpha}}(A) = U_\alpha(A) \ , \forall A \in \mathcal{A} \ , \tag{48}$$

where $U_\alpha(A)$ denotes the holonomy corresponding to the loop α, the connection A and a chosen point in the fiber over x_0 of the (fixed) principal G-bundle over Σ and $\mathcal{A}$ is the space of all connections on this bundle. The set of all hoops forms a group $\mathcal{HG}$, called the hoop group[7]. We note that for all $G = SU(N)$ with $N \geq 2$ the group $\mathcal{HG}$ does not depend on N nor on the principal bundle[7]. Throughout the present section we will assume G to be a compact connected Lie group. It is well known that a connection A defines through $U_.(A)$ a homomorphism from $\mathcal{HG}$ to the gauge group G: $[\alpha] \mapsto U_\alpha(A)$. In fact, the space $\mathcal{A}/\mathcal{G}$ is in a natural bijection with the space of all appropriately smooth homomorphisms of this type, modulo conjugation at the base point[35,36,37]. On the other hand, and as expected from the example of scalar fields, the measure μ_{AL} lives in a space bigger than the classical space $\mathcal{A}/\mathcal{G}$. This is the space of *all*, not necessarily smooth or even continuous, homomorphisms from $\mathcal{HG}$ to G modulo conjugation, denoted by $\overline{\mathcal{A}/\mathcal{G}}$ [6,7]. The space $\mathcal{A}/\mathcal{G}$ of smooth classes $[A]$ was shown to be of zero measure in $\overline{\mathcal{A}/\mathcal{G}}$ [8]. In the next subsection we will deepen this result.

The measure μ_{AL} is, as in the scalar field case, completely specified by giving the result of integrating the so-called cylindrical functions, which in this case are gauge-invariant functions of a finite number of parallel transports along (analytic embedded) edges

$$f(\bar{A}) = F(\bar{A}(e_1), ..., \bar{A}(e_n)) \ , \tag{49}$$

where different edges may intersect only on the ends and $\bar{A} \in \bar{\mathcal{A}}$, the space of all connections realized as parallel transports in a natural way[9,11]. The measure μ_{AL} is then defined by

$$\int_{\overline{\mathcal{A}/\mathcal{G}}} d\mu_{AL} f(\bar{A}) = \int_{G^n} dg_1 ... dg_n F(g_1, ..., g_n) \ , \tag{50}$$

where dg is the normalized Haar measure on G.

The group $Diff_0(\Sigma)$ has a natural action on $\bar{\mathcal{A}}$ which leaves μ_{AL} invariant

$$\varphi^* \bar{A}(e) := \bar{A}(\varphi \cdot e). \tag{51}$$

As we have seen in section II.2 this action induces a unitary action of $Diff_0(\Sigma)$ on $L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_{AL})$

$$(U_\varphi f) = f(\varphi^* \bar{A}). \tag{52}$$

Consider now the so called *spin-network states*[38,39,40] $\{T_s\}$, indexed by triples $s = (\gamma, \pi, c)$, where γ is a graph, $\pi := (\pi_1, \ldots, \pi_n)$ is a labeling of the edges of γ with nontrivial irreducible representations π_i of G and $c := (c_1, \ldots, c_m)$ is a labeling of the vertices $v_1, \ldots, v_m$ of γ with *contractors* c_j, i.e. nonzero intertwining operators from the tensor product of the representations corresponding to the incoming edges at v_j to the tensor product of the representations associated with the outgoing edges. The unitary action of $Diff_0(\Sigma)$ on the spin-network states is particularly simple, being given by

$$U_\varphi T_{\gamma,\pi,c} = T_{\varphi\gamma,\varphi\pi,\varphi c} \tag{53}$$

or, in short

$$U_\varphi T_s = T_{\varphi s}\,, \tag{54}$$

where $\varphi\gamma$ is the image of the graph γ under the diffeomorphism φ and $\varphi\pi$ and φc are the corresponding representations and contractors associated with $\varphi\gamma$. A crucial property is that the contractors can be chosen in such a way that the spin-network states form an orthonormal basis on $L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_{AL})$[39,40,41]. We will assume that this has been done to prove the following

Theorem 1 *The group $Diff_0(\Sigma)$ acts ergodically on $\overline{\mathcal{A}/\mathcal{G}}$, with respect to the measure μ_{AL}.*

To prove the theorem, we will show that the only $Diff_0(\Sigma)$-invariant vector on $L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_{AL})$ is the constant function. Therefore there can be no measurable $Diff_0(\Sigma)$-invariant subsets of $\overline{\mathcal{A}/\mathcal{G}}$ with measure different from zero or one.

Using the completeness of the spin-network states, every $\psi \in L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_{AL})$ can be represented in the form

$$\psi = \sum_s c_s T_s\,, \tag{55}$$

where no more than countably many coefficients c_s are nonzero. Since for any s and any diffeomorphism φ, $T_{\varphi s}$ belongs to the same orthonormal basis ($\varphi s \neq s \Rightarrow T_{\varphi s} \perp T_s$), we conclude that ψ in (55) is $Diff_0(\Sigma)$-invariant if and only if

$$c_{\varphi s} = c_s \quad \forall \varphi \in Diff_0(\Sigma). \tag{56}$$

A L^2-vector cannot have more than finitely many equal coefficients, and since for every nontrivial spin-network (with nontrivial graph and representations) there is an infinite (actually uncountable) number of (distinct) spin-networks in the orbit

$$Diff_0(\Sigma)\, s := \{\varphi s\,,\ \varphi \in Diff_0(\Sigma)\}\,, \tag{57}$$

we conclude that an invariant ψ in (55) is necessarily constant almost everywhere. □

From the proof it follows that if H is a subgroup of $Diff_0(\Sigma)$ s.t. the H-orbit Hs through every nontrivial spin-network s is infinite, then H acts ergodically on $\overline{\mathcal{A}/\mathcal{G}}$. So for example

Corollary 1 *If $\Sigma = \mathbb{R}^N$ or $\Sigma = T^N$ the measure space $(\overline{\mathcal{A}/\mathcal{G}}, \mu_{AL})$ admits the ergodic action of subgroups H of $Diff_0(\Sigma)$ isomorphic to $\mathbb{Z}$.*

For $\Sigma = \mathbb{R}^N$ we take the $\mathbb{Z}$-subgroup of $Diff_0(\Sigma)$ generated by φ_0:

$$\varphi_0(x^1, \ldots, x^N) = (x^1 + \omega^1, \ldots, x^N + \omega^N)\,, \tag{58}$$

where $(\omega^1, \ldots, \omega^N)$ is a fixed non-vanishing vector in $\mathbb{R}^N$ (recall that spin-networks are defined here only for finite graphs). If $\Sigma = T^N$ we take in (58) the vector $(\omega^1, \ldots, \omega^N)$ to have (at least) two irrational and incommensurable components, where $(x^1, \ldots, x^N)$ are now mod 1 coordinates of T^N. In both cases for every nontrivial spin network s, $\{\varphi_0^n s,\ n \in \mathbb{Z}\}$ contains an infinite number of distinct spin networks and so the group $\{\varphi_0^n,\ n \in \mathbb{Z}\}$ acts ergodically on $(\overline{\mathcal{A}/\mathcal{G}}, \mu_{AL})$. $\square$

V.2 Support Properties

As we mentioned in the beginning of section V.1 the space $\mathcal{A}/\mathcal{G}$ of smooth connections modulo gauge transformations is contained in a zero measure subset of $\overline{\mathcal{A}/\mathcal{G}}$, the space where the measure μ_{AL} is naturally defined[6,7,8]. The latter space is naturally identified with the space of all (not necessarily continuous) homomorphisms from the hoop group $\mathcal{HG}$ to the gauge group G modulo conjugation at the base point[9]. In the present subsection we deepen the result of Ref. 8 by showing that the parallel transport of a μ_{AL}-typical connection $\bar{A} \in \overline{\mathcal{A}}$ along an edge e leads to a nowhere continuous map

$$g(\cdot)\ :\ [0,1]\ \to\ G\ . \tag{59}$$

Indeed let $e\ :\ [0,1]\ \to\ \Sigma$ be an arbitrary edge and consider for $s \in [0,1]$ the part of the edge e_s given by

$$e_s(t) = e(st)\ ,\ t \in [0,1]\ .$$

We then have a map

$$\begin{array}{rcll} v\ :\ \overline{\mathcal{A}} & \to & G^{[0,1]} & \\ \bar{A} & \mapsto & v_{\bar{A}}\ , & \quad v_{\bar{A}}(s) := \bar{A}(e_s)\ , \end{array} \tag{60}$$

where $G^{[0,1]}$ denotes the space of all maps from $[0,1]$ to G. By choosing in $G^{[0,1]}$ the standard product space topology and as algebra of measurable sets the Borel σ-algebra the map v becomes measurable.

It is easy to see that, due to the properties of the Haar measure, the push-forward $v_*\mu_{AL}$ of μ_{AL}[42] to $G^{[0,1]}$ is a product of Haar measures, one for each point $s \in [0,1]$:

$$d\nu(g(\cdot)) = v_* d\mu_{AL} = \prod_{s\in[0,1]} dg(s)\ . \tag{61}$$

The main result of this subsection is the following

Theorem 2 *The measure μ_{AL} is supported on the set W of all connections $\bar{A}$ such that $v_{\bar{A}}$ is everywhere discontinuous as a map from $[0,1]$ to G.*

Since $W = v^{-1}(W_1)$, where

$$W_1 = \{g(\cdot) \in G^{[0,1]}\ ,\ \ \text{s.t.}\ \ g(\cdot)\ \text{is nowhere continuous}\}\ , \tag{62}$$

it is sufficient to prove that the complement of W_1,

$$W_1^c = \{g(\cdot) \in G^{[0,1]} \ : \ \exists s_0 \in [0,1] \ \text{ s.t. } \ g(\cdot) \text{ is continuous at } s_0\} \ , \tag{63}$$

is contained in a zero ν-measure subset of $G^{[0,1]}$. Consider the sets

$$\Theta_U = \{g(\cdot) \in G^{[0,1]} \ : \ \exists I \ \text{ s.t. } \ g(I) \subset U\} \ , \tag{64}$$

where U is a (measurable) subset of G with $0 < \mu_H(U) < 1$ ($\mu_H(U)$ denoting the Haar measure of U) and I is an open subset of $[0,1]$. We need the following

Lemma 2 *For every $U \subset G$ with $0 < \mu_H(U) < 1$ the set Θ_U is contained in a zero measure subset of $G^{[0,1]}$.*

To prove the lemma recall that the open balls

$$B(q, 1/m) = \{s \in [0,1] \ : \ |s - q| < 1/m\} \tag{65}$$

with rational q and integer m are a countable basis for the topology of $[0,1]$. Thus Θ_U is the countable union of the sets

$$\Theta_{U,q,m} := \{g(\cdot) \in G^{[0,1]} \ : \ g(B(q, 1/m)) \subset U\}, \quad q \in \mathbb{Q}, \ m \in \mathbb{N}\,. \tag{66}$$

It is easy to construct zero measure subsets containing $\Theta_{U,q,m}$. For this fix an infinite sequence $\{s_i\}_{i=1}^{\infty} \subset B(q, 1/m)$ of distinct points, $s_i \neq s_j$ for $i \neq j$. Then the set $Z = \{g(\cdot) \in G^{[0,1]} \ : \ g(s_i) \in U \ , \ i \in \mathbb{N}\}$ contains $\Theta_{U,q,m}$ and has zero measure:

$$\nu(Z) = \lim_{n \to \infty} (\mu_H(U))^n = 0 \ .$$

From the σ-additivity of ν we conclude that, for every subset $U \subset G$ with Haar measure less than one, the set Θ_U is contained in a zero ν-measure subset. □

We can now conclude the proof of the theorem. Let us choose a number r, $0<r<1$ and a finite open covering $\{U_i\}_{i=1}^k$ of the compact group G with $\mu_H(U_i) = r \ , \ i = 1, ..., k$. This is clearly possible since we can take a neighborhood of each point with measure r and then take a finite sub-covering. Consider $g(\cdot) \in W_1^c$. Then there exists a $s_0 \in [0,1]$ such that $g(\cdot)$ is continuous at s_0. Let i_0 be such that $g(s_0) \in U_{i_0}$. Continuity implies that there exists a neighborhood I of s_0 such that $g(I) \subset U_{i_0}$ and therefore we have $W_1^c \subset \cup_{i=1}^k \Theta_{U_i}$. □

VI Conclusions and Discussion

The knowledge of the support of measures on infinite dimensional spaces, used in quantum field theory, gives a grasp on the behavior of typical quantum field configurations associated with these measures. This may be important for a better understanding, both from the physical and mathematical points of view, of problems afflicting interacting theories like the problem of divergences.

The Bochner-Minlos theorem is very effective in capturing linear properties of the support of measures on the space $\mathcal{S}'(\mathbb{R}^{d+1})$ of quantum scalar field configurations (the support is a linear subspace of $\mathcal{S}'(\mathbb{R}^{d+1})$ spanned by configurations with a given norm finite). It allows e.g. to distinguish the support of the measure μ_{C_m}, corresponding to free scalar field theory with mass m, from that of the white noise measure. However this theorem would predict the same support for the measures μ_{C_m} and $\mu_{C_{m'}}$ with $m \neq m'$ even though these measures must have disjoint supports. The latter is due to the fact that any one parameter subgroup of translations of $\mathbb{R}^{d+1}$ acts ergodically on $\mathcal{S}'(\mathbb{R}^{d+1})$ with respect to both measures. The above two claims do not contradict and rather complement each other since the subset of a "support" which is thick, i.e. such that any measurable subset of its complement has measure zero, is also a (finer) support for the same measure. So, to distinguish between μ_{C_m} and $\mu_{C_{m'}}$, one has to find properties of the supports which complement those given by the Bochner-Minlos theorem. Support properties of μ_{C_m} were studied in Refs. 26 and 27 (see also Refs. 28, 29 and 30). Our goal in section IV.2 consisted in showing that a simple system of random variables can be used to study the difference of supports of μ_{C_m} and $\mu_{C_{m'}}$ for large distances. This simplifies part of the results of Refs. 26 and 27 and makes them physically more transparent.

The use of the AL measure in attempts to construct a quantum theory of gravity[6-18,43-51] justifies the importance to study its properties. In section V we improved the results of Ref. 8. We fixed an arbitrary edge e, or equivalently a piecewise analytic curve on Σ, and considered the random variables given by $\bar{A} \mapsto \bar{A}(e_s)$, where $\bar{A}$ is a quantum connection, e_s represents the edge e up to the value $s \in [0,1]$ of the parameter and $\bar{A}(e_s)$ is the parallel transport corresponding to $\bar{A}$ and e_s. These random variables were suggested to the authors by Abhay Ashtekar and were motivated by studies of the volume operator in quantum gravity[43-53]. Varying s we obtain a (measurable) map v from the space of quantum connections $\overline{\mathcal{A}}$ to the space $G^{[0,1]}$ of maps, $g(\cdot) : [0,1] \to G$. The AL measure becomes, by push-forward, an infinite product of Haar measures. Using this we have shown that for a typical μ_{AL}-connection $\bar{A}$ the parallel transport $\bar{A}(e_s)$ is everywhere discontinuous.

We also showed that $Diff_0(\Sigma)$ acts ergodically on $\overline{\mathcal{A}/\mathcal{G}}$ with respect to the AL measure. The importance of this stems from the fact that in quantum gravity one has to solve the diffeomorphism constraint and therefore naively one would have to take functions on the quotient

$$\overline{\mathcal{A}/\mathcal{G}}/Diff_0(\Sigma)\ .$$

The ergodicity of the action of $Diff_0(\Sigma)$ on $\overline{\mathcal{A}/\mathcal{G}}$ implies that the only solution to the diffeomorphism constraint in $L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_{AL})$ is the constant function. This explains why in Ref. 11, and even though $\overline{\mathcal{A}/\mathcal{G}}$ is compact, one had to use distributional elements to solve the diffeomorphism constraint. If the action were not ergodic one would have diffeomorphism invariant measurable subsets of $\overline{\mathcal{A}/\mathcal{G}}$ (pre-images of sets in $\overline{\mathcal{A}/\mathcal{G}}/Diff_0(\Sigma)$) with μ_{AL}-measure different from zero or one. The characteristic functions of these sets would provide L^2 solutions to the diffeomorphism constraint.

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References

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[1] B. Simon, *The $P(\phi)_2$ Euclidean (Quantum) Field Theory* (Princeton University Press, Princeton, 1974).

[2] J. Glimm, A. Jaffe, *Quantum Physics* (Springer Verlag, New York, 1987).

[3] V. Rivasseau, *From Perturbative to Constructive Renormalization* (Princeton University Press, Princeton, 1991).

[4] T. Hida, H.-H. Kuo, J. Potthoff, L. Streit, *White Noise* (Kluwer Academic Publishers, 1993).

[5] S. Albeverio, J. Jost, S. Paycha, S. Scarlatti, *A Mathematical Introduction to String Theory* (Cambridge University Press, Cambridge, 1997).

[6] A. Ashtekar, C. J. Isham, Class. Quant. Grav. **9**, 1433 (1992).

[7] A. Ashtekar, J. Lewandowski, *Representation Theory of Analytic Holonomy $C^\star$ Algebras.* In: Knots and Quantum Gravity, ed. J. Baez (Oxford University Press, Oxford, 1994).

[8] D. Marolf, J. Mourão, Comm. Math. Phys. **170**, 583 (1995).

[9] A. Ashtekar, J. Lewandowski, J. Math. Phys. **36**, 2170 (1995).

[10] J. Lewandowski, Int. J. Mod. Phys. **D3**, 207 (1994).

[11] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourão, T. Thiemann, J. Math. Phys. **36**, 6456 (1995).

[12] T. Thiemann, Phys. Lett. **B380**, 257 (1996).

[13] T. Thiemann, Class. Quant. Grav. **15**, 839 (1998).

[14] T. Thiemann, Class. Quant. Grav. **15**, 875 (1998).

[15] T. Thiemann, Class. Quant. Grav. **15**, 1207 (1998).

[16] T. Thiemann, Class. Quant. Grav. **15**, 1249 (1998).

[17] T. Thiemann, Class. Quant. Grav. **15**, 1281 (1998).

[18] T. Thiemann, Class. Quant. Grav. **15**, 1463 (1998).

[19] R. Gambini, A. Trias, Phys. Rev. **D22**, 1380 (1980).

[20] C. Di Bartolo, F. Nori, R. Gambini, A. Trias, Lett. Nuov. Cim. **38**, 497 (1983).

[21] R. Gambini, A. Trias, Nucl. Phys. **B278**, 436 (1986).

[22] C. Rovelli, L. Smolin, Nucl. Phys. **B331**, 80 (1990).

[23] R. Gambini, J. Pullin, *Loops, Knots, Gauge Theories and Quantum Gravity* (Cambridge University Press, Cambridge, 1996).

[24] S. Albeverio, R. Høegh-Krohn, H. Holden, J. Funct. Anal. **78**, 154 (1988).

[25] L. Gross, Ch. King, A. Sengupta, Ann. Phys. **194**, 65 (1989).

[26] Ph. Colella, O. E. Lanford, *Sample field behavior for the free Markov random field.* In: Constructive Quantum Field Theory, The 1973 Ettore Majorana School, eds. G. Velo, A. Wightman (Springer Verlag, 1973).

[27] M. Reed, L. Rosen, Comm. Math. Phys. **36**, 123 (1974).

[28] M. Roeckner, J. Funct. Anal. **79**, 211 (1988).

[29] Z. Haba, Comm. Math. Phys. **69**, 247 (1979).

[30] S. Kusuoka, J. Fac. Sci. Univ. Tokyo **A29**, 387 (1982).

[31] The support of the measure μ on M is any (not necessarily measurable) subset $X \subset M$ such that any measurable subset Y of the complement $Y \subset X^c$ has measure zero. Then μ can be identified with the σ-additive measure μ' on X defined on sets of the form $X \cap Y$, with Y measurable on M, by $\mu'(X \cap Y) := \mu(Y)$.

[32] Y. Yamasaki, *Measures on Infinite Dimensional Spaces* (World Scientific, Singapore, 1985).

[33] Ya. G. Sinai, *Topics in Ergodic Theory* (Princeton University Press, New Jersey, 1994).

[34] W. Rudin, *Functional Analysis* (McGraw-Hill, New York, 1991).

[35] J. W. Barret, Int. J. Theor. Phys. **30**, 1171 (1991).

[36] J. Lewandowski, Class. Quant. Grav. **10**, 879 (1993).

[37] A. Caetano, R. F. Picken, Int. J. Math. **5**, 835 (1994).

[38] C. Rovelli, L. Smolin, Phys. Rev. **D52**, 5743 (1995).

[39] J. Baez, Adv. Math. **117**, 253 (1996).

[40] J. Baez, *Spin-networks in non-Perturbative Quantum Gravity.* In: The Interface of Knots and Physics, ed. L. Kauffman (AMS, Providence, 1996).

[41] T. Thiemann, J. Math. Phys. **39**, 1236 (1998).

[42] Here μ_{AL} is considered on $\overline{\mathcal{A}}$. It is defined analogously to eq. (50) with the difference that the functions of parallel transports are not necessarily gauge invariant (see Ref. 7). The push-forward of this measure to $\overline{\mathcal{A}/\mathcal{G}} = \overline{\mathcal{A}}/\overline{\mathcal{G}}$ gives the Ashtekar-Lewandowski measure on this space.

[43] C. Rovelli, L. Smolin, Nucl. Phys. **B442**, 593 (1995). Erratum: Nucl. Phys. **B456**, 753 (1995).

[44] R. Loll, Phys. Rev. Lett. **75**, 3048 (1995).

[45] R. Loll, Nucl. Phys. **B460**, 143 (1996).

[46] R. Loll, Class. Quant. Grav. **14**, 1725 (1997).

[47] R. Loll, Nucl. Phys. **B500**, 405 (1997).

[48] A. Ashtekar, J. Lewandowski, *Quantum Field Theory of Geometry.* In: Proceedings of the March '95 Boston Conference on the Foundations of Quantum Field Theory.

[49] A. Ashtekar, J. Lewandowski, Class. Quant. Grav. **14**, A55 (1997).

[50] A. Ashtekar, J. Lewandowski, Adv. Theo. Math. Phys. **1**, 388 (1997).

[51] J. Lewandowski, Class. Quant. Grav. **14**, 71 (1997).

[52] R. De Pietri, C. Rovelli, Phys. Rev. **D54**, 2664 (1996).

[53] T. Thiemann, J. Math. Phys. **39**, 3347 (1998).