Plan of the lectures & motivation

1. CS theory : basic ingredients & non-perturbative computations.
2. 1/N expansion in CS. 3. CS as an open topological string
4. CS as a closed topological string & the conifold transition
5. More general geometries

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1. CS theory

M 3-manifold,

A U(N) or SU(N) connection

(other groups G possible)

$$S = \frac{k}{4\pi} \int_M \mathrm{Tr}\,(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

* k integer, $M = S^3$ most of the time
* first remark: this action is topological

$\rightarrow$ $\boxed{\text{TFT}}$

* warning: this is just at the classical level
  it may happen that at the quantum level
  the topological symmetry is anomalous
  will see an example later

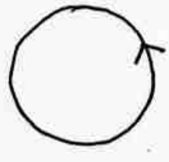
(gauge-invariant)
operators: we want to maintain metric independence
so things like $\mathrm{Tr}\,(F^2)$ are not "allowed".

we will ... then use WILSON LOOP operators

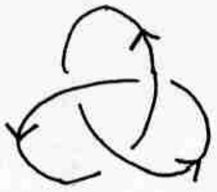
$W_R^K(A)$ labeled by
R irrep of G
K oriented path in M

notice that classification of oriented paths in $M = S^3$ is highly nontrivial ! this is the subject of KNOT THEORY

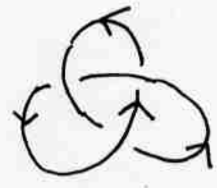
Examples:



unknot
or trivial knot



trefoil knot



another trefoil ... the mirror of the above

and much more! check for example a knot table

what we do now is to consider the holonomy of the connection A along K, called U_K

$$U_K = P\exp \oint_K A \qquad \text{path-ordered exponential}$$

U_K is not gauge invariant, but $\mathrm{Tr}_R U_K$ is. This is our gauge-invariant, metric independent operator

So we got action + operators.

Go to the quantum theory:

$$Z(M) = \int DA \; e^{iS(A)}$$

partition function
$\rightarrow$ topological invariant of M

$$\langle W_{R_1}^{K_1} \cdots W_{R_L}^{K_L} \rangle = W_{R_1 \ldots R_L}(\mathcal{L}) = \frac{1}{Z(M)} \int DA \, W_{R_1}^{K_1} \cdots W_{R_L}^{K_L} \, e^{iS(A)}$$

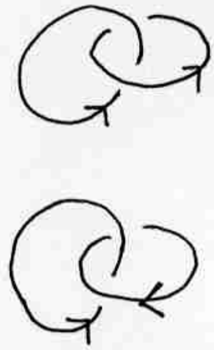
I denoted by $\mathcal{L}$ the set of knots $K_1, \dots, K_L$

this is in general a <u>link</u> of L components

it can be trivial :

trivial link

but it can be "linked"



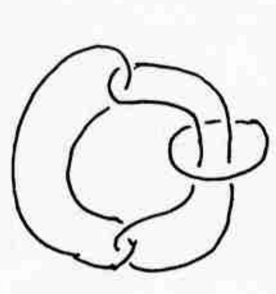
+ 1

Hopf links

−1

So we also have the problem of classifying links with a given number of components. Notice that out of trivial knots we can get nontrivial links, as shown above.

Another example: the Borromean link ! (from the Borromeo family, Italy)



How do we distinguish the above Hopf links ? The simplest link invariant is the linking number

$$lk\,(K_1, K_2) = \frac{1}{2} \sum_{\substack{\text{crossings} \\ p}} \epsilon(p)$$

notice the topological invariance:

back to the quantum theory:

* perturbation theory:

  - find the "classical solutions to the EQNs of motion

    in CS these are $F=0$ flat connections on M

    S^3 $\quad A=0$ trivial flat connections

    in general maps $\pi_1(M) \to G$

    so for example

    $S^3/\mathbb{Z}_p = L(p,1)$ lens space

    has $\mathbb{Z}_p \to U(N)$ nontrivial flat connect

  - expand around a flat connection and compute Feynman diagrams.

(c) labels flat connection $\quad Z^{(c)}(M)$ a series in $1/k$

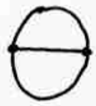
$$Z = \sum_{(c)} Z^{(c)}(M) \quad \text{series in } 1/k$$

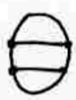
$$Z^{(c)}(M) = Z^{(c)}_{1\text{-loop}}(M)\, e^{\sum_{\ell=1}^{\infty} S_\ell^{(c)} x^\ell}$$

$x = \frac{2\pi i}{k+N}$ effective parameter

$\ell = \#$ number of vertices$/2$ $= \#$ loops -1

$\sum_{\ell=1}^{\infty} S_\ell^{(c)} x^\ell$ sum over connected diagrams (since they "vacuum bubbles" are exponentiated

$\ell = 1$ 

$\ell = 2$  ...

$S_\ell^{(c)}$ = Feynman integral $\times$ group theory factor

$$Z^{(c) = \text{trivial}}_{1\text{-loop}} \propto \frac{1}{\text{vol}(G)}$$

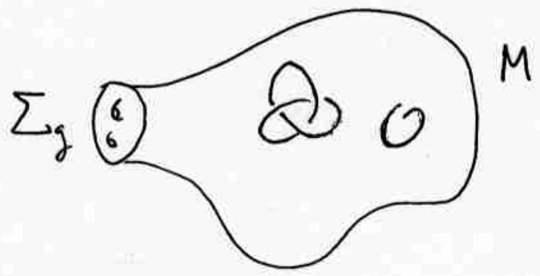
This is a very interesting approach, but we want to do things in a "non-perturbative" way.

warning: in CS theory there are no, strictly speaking, nonperturbative effects like instantons, so by non-pert we mean essentially a way to sum up the full perturbative series

canonical quantization:

a general fact in QFT

take M a manifold whose boundary is $\partial M = \Sigma$



in our case

Σ_g = Riemann surface

wavefunctionals

$$\Psi_{M,\mathcal{O}}(A) = \int_{A|_\Sigma = A} DA\, e^{iS(A)}\, \mathcal{O} = \langle A | \Psi_{M,\mathcal{O}} \rangle$$

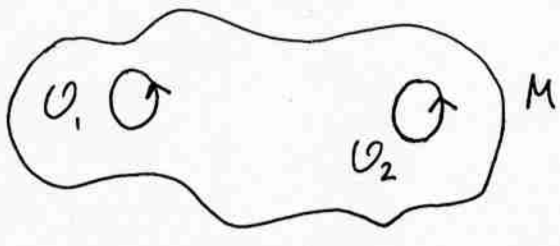
$\mathcal{O}$ local or nonlocal, in our case these are Wilson loop

these wavefunctions live in a Hilbert space $\underline{\underline{\mathcal{H}(\Sigma_g)}}$

There is also a beautiful relation between inner products and path integrals:

$Z(M, \mathcal{O}_1 \mathcal{O}_2) =$

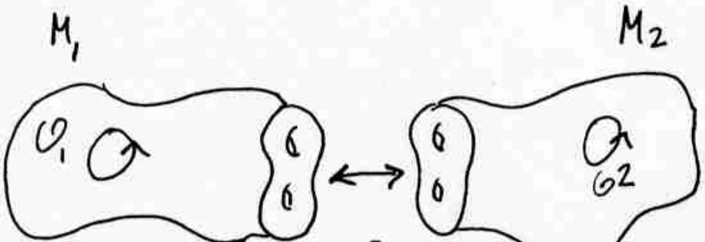
$$\int DA\, e^{iS(A)}\, \mathcal{O}_1 \mathcal{O}_2$$



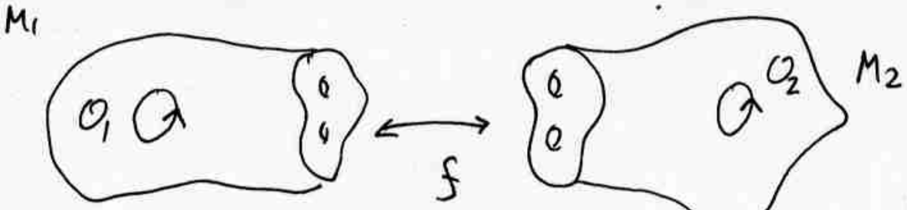
$\downarrow$ cut it

$\partial M_2 = \Sigma_g$

$\partial M_1 = \overline{\Sigma_g}$ (with opposite orientation



$f : \Sigma_g \to \Sigma_g$ "gluing instructions



$$\langle \Psi_{M_1,\mathcal{O}_1}| \qquad U_f \qquad |\Psi_{M_2,\mathcal{O}_2}\rangle \in \mathcal{H}(\Sigma_g)$$

in $\mathcal{H}^*(\Sigma_g)$

$$U_f : \mathcal{H}(\Sigma_g) \to \mathcal{H}(\Sigma_g)$$

so $$\boxed{Z(M,\mathcal{O}_1\mathcal{O}_2) = \langle \Psi_{M_1,\mathcal{O}_1}|U_f|\Psi_{M_2,\mathcal{O}_2}\rangle}$$

Great result of Witten: determining $\mathcal{H}(\Sigma_g)$ for CS theory!

<u>fact</u>: $\mathcal{H}(\Sigma_g)$ = space of conformal blocks of WZW on Σ_g with gauge group G at level k.

if you don't know this, don't worry!

being more concrete:

$g=0$ $\quad \mathcal{H}(S^2)$ = one single element

$g=1$ $\quad \mathcal{H}(T^2)$ = integrable reps of affine Lie algebra $\hat{G}$ at level k.

labeled by reps R of G together with some constraints

Example: for $SU(2)$ at level k,

we have $j = 0, 1, \ldots, k$ — this truncates the admissible reps

we will not worry about this since we are going to work with CS $U(N)$ or $SU(N)$ with k, N "very large".

In practical terms

$$\mathcal{H}(\mathbb{T}^2) = \{ |R\rangle \}_{\substack{R \text{ irrep of} \\ U(N)}} \quad \text{(plus some constraint)}$$

$\boxed{R=0 \text{ included}}$

R: Young tableau, highest weight Λ

orthonormal $\langle R' | R \rangle = \delta_{RR'}$

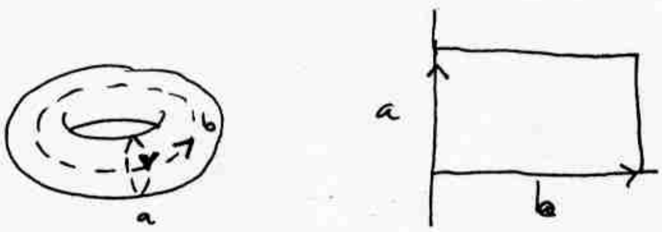
another ingredient we will need are a special class of gluing functions $f: \mathbb{T}^2 \longrightarrow \mathbb{T}^2$

given by $SL(2,\mathbb{Z})$ transformations

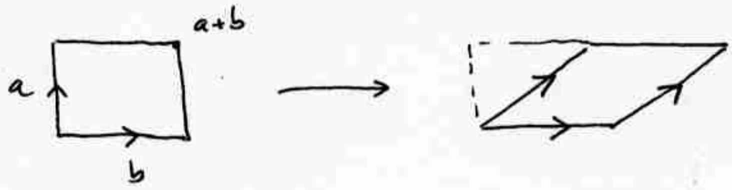
$SL(2,\mathbb{Z})$ generated by $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix}$ $\qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

where we regard them as matrices acting on the one-cycles of a torus



$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ b \end{pmatrix}$

T: 

S: 

we know how these homeomorphisms "lift" to $\mathcal{H}(T^2)$:

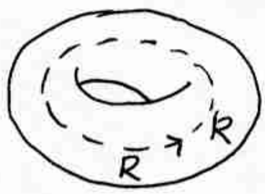
$$\underset{\text{"}T\text{"}}{U_T} |R\rangle = e^{2\pi i (h_R - c/24)} |R\rangle$$

$$\underset{\text{"}S\text{"}}{U_S} |R\rangle = \sum_{R'} S_{R'R} |R'\rangle$$

What is $|R\rangle$ from the point of view of wavefunctions that we used before?

$$|Z_{M,\mathcal{O}}\rangle = |R\rangle \quad \text{for some } M, \ \partial M = T^2 \text{ and some } \mathcal{O}$$

<u>answer</u>: take a <u>solid torus</u>

$M =$ 

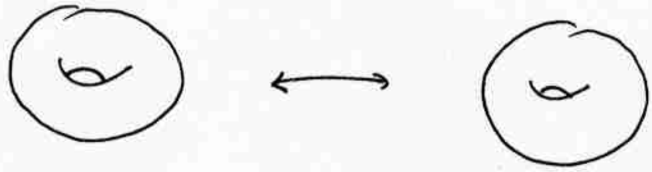
$= D \times S^1$

only one noncontractible cycle

$\mathcal{O} = \mathrm{Tr}_R U_R$ $\qquad |R\rangle = |Z_{D\times S^1, \mathrm{Tr}_R U_R}\rangle$

no insertion  $\rightarrow |0\rangle$ "vacuum" applications of all this:

1) consider two solid tori and glue them with $f = 1$ identity (with no insertions)



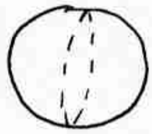
$\langle 0|$ $\qquad\qquad$ $|0\rangle$ $=$ $\langle 0|0\rangle$

$S^1 \times$

$\times\ S^1$

disc $\qquad$ disc

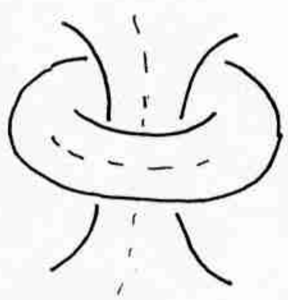
$= \quad S^1 \times$



$= \quad S^1 \times S^2$

$$\Rightarrow \quad Z(S^1 \times S^2) = \langle 0|0\rangle = 1$$

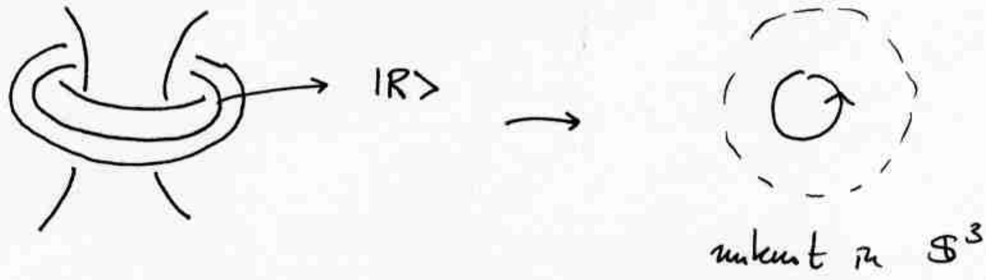
2) same but $f = S$ $\qquad$ $\langle 0|S|0\rangle = S_{00}$



$\rightarrow$ glueing them gives S^3

the complement of a solid torus on S^3 is another solid torus

3)



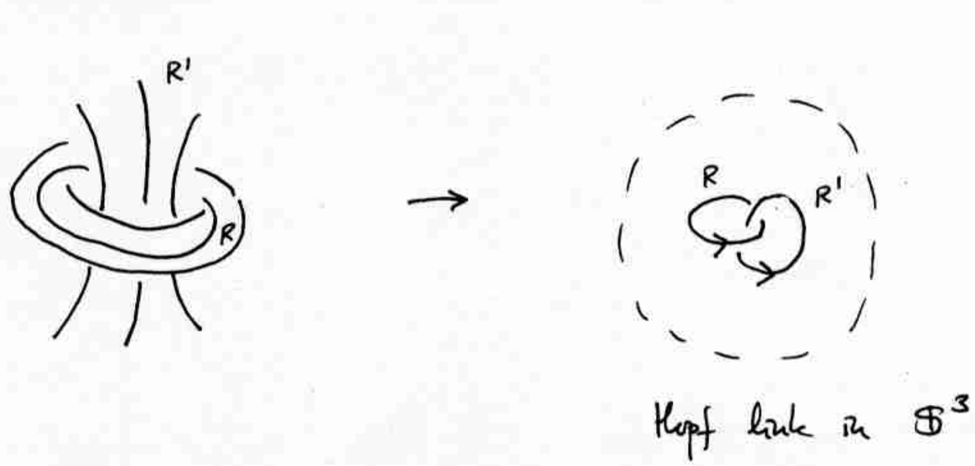
$$\langle 0 | S | R \rangle = S_{0R}$$

$$W_R(\text{unknot}) = \frac{S_{0R}}{S_{00}} = \text{quantum dimension of } R$$

$$q = e^{\frac{2\pi i}{k+N}} \qquad \lambda = q^N$$

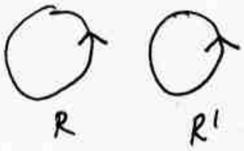
$$R = \square \qquad \frac{S_{0R}}{S_{00}} = \frac{\lambda^{\frac{1}{2}} - \lambda^{-\frac{1}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$$

4)



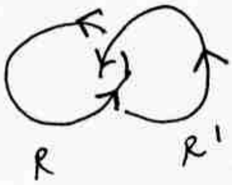
$$\langle R | S | R' \rangle = S_{RR'}$$

show that the theory distinguishes things:



$$\frac{S_{0R}}{S_{00}}\,\frac{S_{0R'}}{S_{00}} \xrightarrow{R=R'=\square} \left(\frac{\lambda^{1/2}-\lambda^{-1/2}}{q^{\frac{1}{2}}-q^{\frac{1}{2}}}\right)^2$$

$$\langle W_R^{R_1}\; W_{R'}^{R_2}\rangle = \langle W_R^{R_1}\rangle\langle W_{R'}^{R_2}\rangle$$



$$\frac{S_{RR'}}{S_{00}} \xrightarrow[R=R'=\square]{} \left(\frac{\lambda^{1/2}-\lambda^{-1/2}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}\right)^2 + \lambda^{-1}-1$$

$$W_{RR'}(\mathrm{Hopf}^{-1}) = \frac{S_{RR'}}{S_{00}}$$

$$W_{RR'}(\mathrm{Hopf}^{+1}) = \frac{S^{-1}_{RR'}}{S_{00}}$$

are also rational functions in $q^{\pm 1/2}, \lambda^{\pm 1/2}$

$$\mathrm{Tr}_R U_{R^{-1}} = \mathrm{Tr}_{\bar{R}} U_R \quad \text{and} \quad S_{\bar{R}R'} = S^{-1}_{RR'}$$

$\bar{R}$ complex conjugate

in general: $W_{R_1 \ldots R_L}(\mathcal{L})$ is
for $G = SU(2)$ $R_1 = \cdots R_L = \square$: Jones
$= SU(N)$ " : HOMFLY

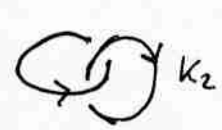
So far we have ignored an important subtlety – the framing of the theory!

To understand what is framing, it is useful to look at $U(1)$ CS, which is a Gaussian theory:

$$S = \frac{k}{4\pi} \int_M A \wedge dA$$

since $A \wedge A \wedge A = A_i A_j A_k \epsilon^{ijk} = 0$ if there are no gauge indices.

$$\left\langle e^{\oint_{K_1} A} \, e^{\oint_{K_2} A} \right\rangle_{\substack{\text{Abelian}\\ CS}}$$



$$= \sum_{m=0}^{\infty} \left\langle \frac{\left(\oint_{K_1} A\right)^m}{m!} \frac{\left(\oint_{K_2} A\right)^n}{n!} \right\rangle =$$

$$= e^{\frac{1}{2}\langle \oint_{K_1} A \oint_{K_1} A \rangle + \frac{1}{2}\langle \oint_{K_2} A \oint_{K_2} A \rangle + \langle \oint_{K_1} A \oint_{K_2} A \rangle}$$

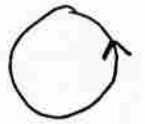
$$\langle \oint_K A \oint_K A \rangle \propto \phi(K) = \frac{1}{4\pi} \oint_{K_1} dx^\mu \oint_{K_1} dy^\nu \, \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3} \quad (*)$$

$$\langle \oint_{K_1} A \oint_{K_2} A \rangle \propto \frac{1}{4\pi} \oint_{K_1} dx^\mu \oint_{K_2} dy^\nu \, \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3} \quad (**)$$

$(**)$ <u>is</u> a topological invariant of the link made out of K_1, K_2 : it is the <u>linking number</u> of K_1, K_2, $lk(K_1, K_2)$

but $(*)$ is <u>not</u> a topological invariant : it is the "self-linking" of K_1, also called the writhe or cotorsion of K_1 :

has cotorsion -1

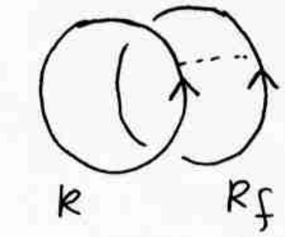


has cotorsion 0

so if we deform the knot we <u>change</u> the writhe

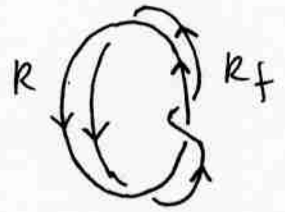
This is the kind of "anomaly" we were mentioning before, and in principle ruins the topological invariance of the theory!

How do we cure that?



no twist

instead of doing the integral twice around R, take a "copy" of R which runs parallel to R but can "twist" p times around it, R_f



– one twist

now, instead of having $\phi(R)$ we will have $lk(R, R_f) = p$

The knot R together with its "copy" R_f is called a FRAMED knot. The framing is specified by an integer p which gives the number of twists.

MAIN POINT: we don't have a topological theory of knots, but we <u>do</u> have a " " of <u>framed</u> knots!

another way of understanding this from QFT point of view is that the composite operator $\left(\oint_R A\right)^2$ is ambiguous, and by doing the framing we are choosing a definition of this operator through some kind of point-splitting.

What saves the day is that the change of vev's under change of framing is perfectly under control.

The precise rule in the <u>nonabelian</u> theory is:

$$\boxed{\begin{aligned} W_R(K) &= \langle W_R^K(A) \rangle \\ &\longrightarrow e^{2\pi i p h_R}\, W_R(K) \end{aligned}} \qquad (*)$$

p change of framing, $h_R = \dfrac{C_R}{2(k+N)}$

notice: conformal weight of a primary R in WZW

$C_R = \mathrm{Tr}_R(T_a T_a)$ quadratic Casimir

<u>NOTE:</u> this is easy to understand in perturbation theory, since contractions now involve $A_\mu = \sum_a A_\mu^a T_a$, T_a basis of Lie algebra

$$\overbrace{A_\mu^a A_\nu^b} \sim \delta^{ab}$$

and $\mathrm{Tr}_R(A_\mu A_\nu) \rightarrow \mathrm{Tr}_R(T_a T_a)$

For knots in S^3, there is something called the CANONICAL or STANDARD framing (where we declare the self-linking to be zero), and any other framing is related to this one by the rule $(*)$

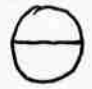
2. $1/N$ expansion & the string – gauge theory correspondence

back to the perturbative expansion of CS in $U(N)$

$$F(M) = \log Z(M) = \log Z_{1\text{-loop}} + \sum_{\ell=1}^{\infty} S_\ell \cdot x^\ell$$

S_ℓ depends on N

$S_\ell = S_\ell(N)$ due to group factors

$\ell=1$  $\rightarrow N(N^2-1)$

$\ell=2$  $\rightarrow N^2(N^2-1)$ and so on

note: each usual Feynman diagram has different powers of N associated to it

We want to <u>isolate the N-dependence</u> in a more precise way. To do that, we introduce the DOUBLE LINE notation:

A_{ij} has two matrix indices (remember adjoint $U(N) = N \times \bar{N}$)

 $\rightarrow$ 

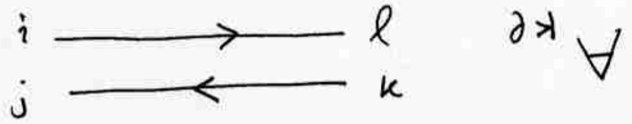
what is the index structure of the propagator?

look at the kinetic term

$x = \frac{2\pi i}{k+N}$ $\qquad \frac{1}{x} \mathrm{Tr}(A \wedge dA) \rightarrow A_{ij} \wedge dA_{ji}$

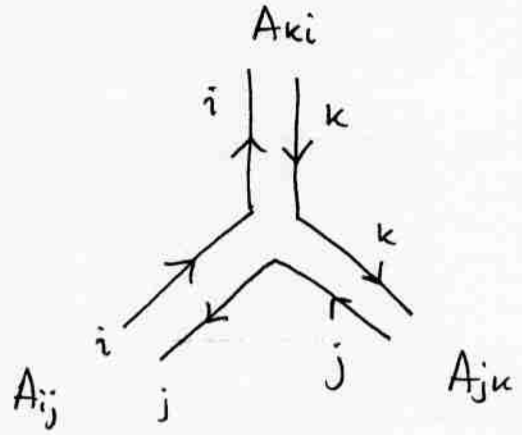
$$= A_{ij} \wedge dA_{k\ell}\ \delta_{i\ell}\, \delta_{jk}$$

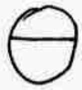
i.e. $\langle A_{ij} A_{kl} \rangle \propto X\, \delta_{il}\, \delta_{jk}$

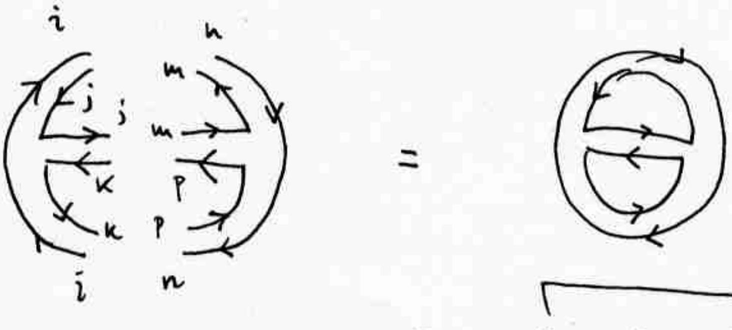
A_{ij} 

Cubic interaction:

$$\frac{1}{X}\, \mathrm{Tr}\,(A \wedge A \wedge A) = \frac{1}{X}\, A_{ij} \wedge A_{jk} \wedge A_{ki}$$



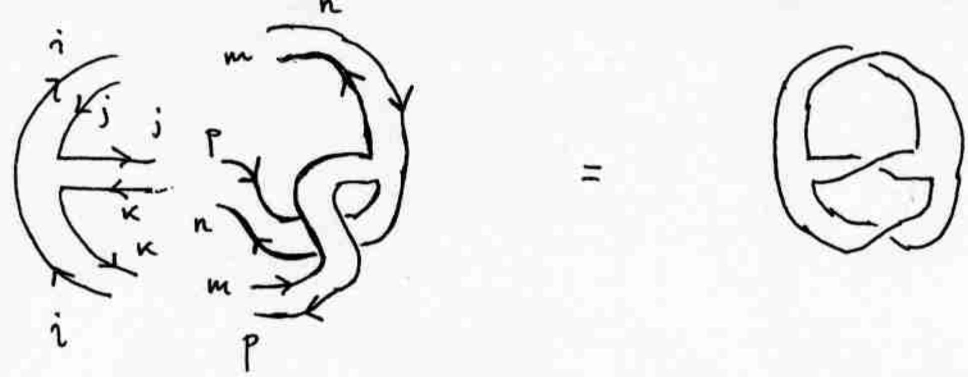
revisit 



$$\sum_{ijkmnp} A_{ij}\, A_{jk}\, A_{ki}\, A_{mn}\, A_{np}\, A_{pm}$$

$$\propto \sum_{ijkmnp} \delta_{in}\, \delta_{jm}\, \delta_{jm}\, \delta_{kp}\, \delta_{kp}\, \delta_{in} = N^3$$

3 free indices = 3 closed loops in the diagram



$$\sum_{ijkmnp} A_{ij} A_{jk} A_{ki} A_{mn} A_{np} A_{pm}$$

$$\propto \sum_{ijkmnp} \delta_{in}\,\delta_{jm}\,\delta_{jp}\,\delta_{kn}\,\delta_{km}\,\delta_{ip} \;=\; N$$

one free index = one single closed loop in the diagram

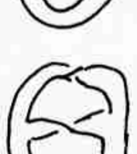
therefore, given a diagram with V vertices, E propagators and h closed loops, we have a factor

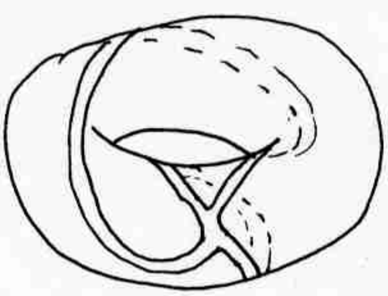
$$\chi^{E-V}\,N^{h}$$

we now look at the "fat graphs" as Riemann surfaces, and with holes we have

$$2g-2 = E-V-h$$

 $E=3,\ V=2,\ h=3 \rightarrow g=0$ a sphere with three holes

 $E=3,\ V=2,\ h=1 \rightarrow g=1$ a torus with one hole



$g=0$ are called PLANAR diagrams

$g \geq 1$ are called NON-PLANAR diagrams

$$x^{E-V} N^h = x^{2g-2+h} N^h$$

Therefore, the part of the free energy which comes from vacuum bubbles can be written as:

$$\text{vacuum bubbles} = \sum_{g=0}^{\infty} \sum_{h=1}^{\infty} \underbrace{F_{g,h}} \, x^{2g-2+h} N^h \qquad (*)$$

numerical coefficients (they involve Feynman integrals, symmetry factors, etc.)

This structure is of course typical of any field theory with gauge group $U(N)$ (i.e., from matrix models to QCD)

Notice that in the resulting series we have now two parameters (x, N). It is also useful to introduce the so-called 't Hooft parameter:

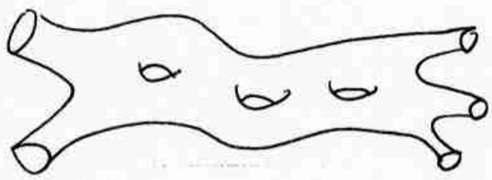
$$t = xN$$

and in terms of this we have

$$\sum_{g=0}^{\infty} \sum_{h=1}^{\infty} F_{g,h}\, x^{2g-2}\, t^h \qquad (**)$$

The expansion (*) looks very much like an OPEN STRING theory expansion with $U(N)$ Chan-Paton factors,

where
$$x \longrightarrow g_s, \quad \underline{\underline{\text{open}}} \text{ string coupling constant}$$



$$\longrightarrow g_s^{2g-2+h}\, N^h$$

<u>temptation</u>: interpret $F_{g,h}$ as an amplitude of an open string theory on a Riemann surface $\Sigma_{g,h}$ (say, the partition function)

How do we compute $F_{g,h}$?

In CS theory on S^3 there is an obvious way: since we know $Z(S^3)$ exactly, we can just take the log and expand!

I will give some indications on this:

$$Z = \frac{1}{(k+N)^{N/2}} \prod_{\alpha>0} 2\sin\left(\frac{\pi(\alpha\cdot\rho)}{k+N}\right)$$

α positive roots $= e_i - e_j$

ρ Weyl vector $= \frac{1}{2}\sum_k (N-2k+1)\, e_k$

$$= \frac{1}{(k+N)^{N/2}} \prod_{j=1}^{N-1} \left(2\sin\left(\frac{\pi j}{k+N}\right)\right)^{N-j}$$

$$F(S^3) = -\frac{N}{2}\log(N+k) + \sum_{j=1}^{N-1} \log\left[2\sin\frac{\pi j}{k+N}\right]$$

$$= F^{1\text{-loop}} + F^{\substack{\text{vacuum} \\ \text{bubbles}}}$$

$$\boxed{\text{use } \sin \pi z = \pi z \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)}$$

$$F^{1\text{-loop}} = -\frac{N^2}{2}\log(k+N) + \frac{1}{2}N(N-1)\log 2\pi + \sum_{j=1}^{N-1}(N-j)\log j$$

$$= \log \frac{(2\pi g_s)^{\frac{1}{2}N^2}}{\mathrm{vol}(U(N))} \qquad g_s = x = \frac{2\pi}{k+N}$$

$$= \log Z_{1\text{-loop}}$$

$$F^{\text{bubbles}} = \sum_{j=1}^{N-1}(N-j) \cdot \sum_{n=1}^{\infty} \log\left(1 - \frac{j^2 g_s^2}{4\pi^2 n^2}\right)$$

$$= \sum_{g=0}^{\infty} \sum_{h=2}^{\infty} g_s^{2g-2+h} N^h F_{g,h}$$

$$F_{0,h} = \frac{B_{h-2}}{(h-2)h!} \qquad h \geq 4$$

$$F_{1,h} = \frac{1}{12} \frac{B_h}{h \cdot h!}$$

$$F_{g,h} = 2 \frac{3(2g-2+h)}{(2\pi)^{2g-2+h}} \binom{2g-3+h}{h} \frac{B_{2g}}{2g(2g-2)} \qquad g \geq 2$$

From open strings to closed strings:

we can rewrite $(**)$ as:

$$\sum_{g=0}^{\infty} \sum_{h=1}^{\infty} F_{g,h} t^h x^{2g-2} = \sum_{g=0}^{\infty} F_g(t) x^{2g-2}$$

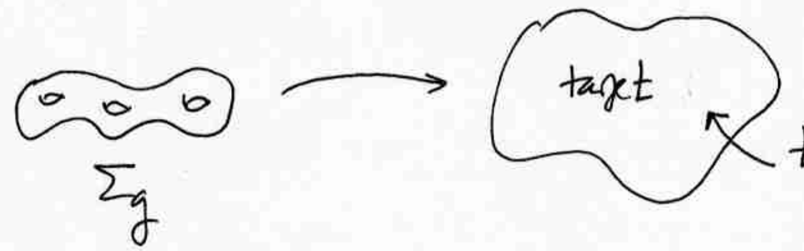
$$F_g(t) = \sum_{h=1}^{\infty} F_{g,h} t^h \qquad \text{a function of } t$$

now this looks like a CLOSED string expansion where

$x = g_s$ closed string coupling constant

$F_g(t)$ amplitude for a closed string at genus g

t should have, in this context, the interpretation as a modulus of the target string geometry



geometric modulus of the target

notice that in this context we sum over all open string diagrams with a fixed genus and all possible # of holes to get $F_g(t)$:

$g=0$:  t^2 +  t^3 +  $t^4 + \dots$

=  sphere $\boxed{\text{planar}}$ piece

$g=1$  t +  $t^2 + \dots$

=  torus and so on

so by summing over the holes we fill up or close the holes.

$F_0(t)$ comes from planar diagrams

$x = t/N$

notice that if we write $\displaystyle\sum_{g=0}^{\infty} x^{2g-2} F_g(t)$

$$= \sum_{g=0}^{\infty} N^{2-2g} \left[t^{2g-2} F_g(t) \right]$$

we have that at $\underline{\text{fixed}}$ t, large N,

$F_0(t)$ is the "leading" piece (goes like N^2)

$F_g(t)$ are subleading (go like $\mathcal{O}(N^0)$)

3. CS theory as an open string theory

Witten hep-th/9207094

We want now to give an interpretation of CS theory (realization) as an open string theory.

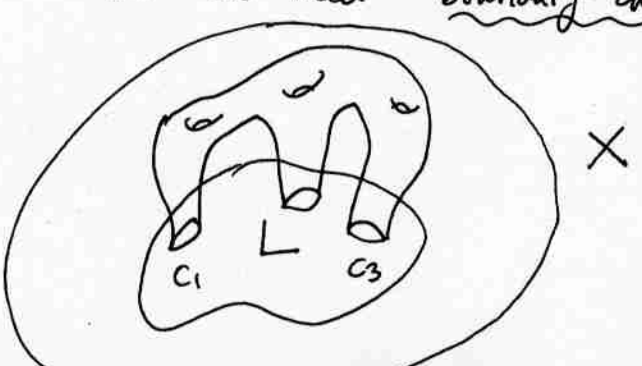
Since CS is a TFT, it seems reasonable to look for an open topological string theory. The answer (Witten, 1992) is that $\underbrace{\text{CS}}_{\text{on } M}$ can be realized as a topological string theory on T^*M.

★ open topological strings

remind that a closed topological string is a theory of holomorphic maps $f: \Sigma_g \to X$ X CY 3-fold

open: $f: \Sigma_{g,h} \to X$

but we need $\underline{\text{boundary conditions}}$: $\begin{cases} \bigcup_i C_i = \partial\Sigma \\ f(C_i) \subset \mathcal{L} \end{cases}$



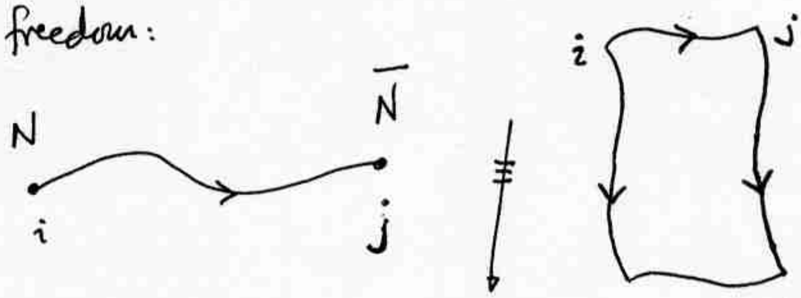
$\mathcal{L}$ $\underline{\text{Lagrangian}}$ submanifold of X

$J|_{\mathcal{L}} = 0$

NOTE: This is like in the analysis of b.c. for $N=2$ SCFT (Ooguri, Oz, Yin), but since we do not worry about spacetime SUSY we just get Lagrangian.

We also want to add Chan-Paton degrees of freedom to the open strings, so that we get in the spectrum nonabelian degrees of freedom:

$i, j = 1, \ldots, N$



as they evolve in time

notice the similarity to fatgraphs!

remind: in the usual open string, this gives nonabelian gauge fields

$$\alpha^{\mu}_{-1}|0\rangle \longleftrightarrow A^{\mu}$$

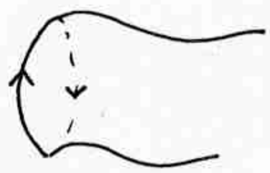
$$\alpha^{\mu}_{-1}|i;j\rangle \longleftrightarrow A^{\mu}_{ij} \qquad U(N) \text{ gauge field}$$

Having open strings ending on a Lagrangian submanifold $\mathcal{L}$ with $U(N)$ Chan-Paton degrees of freedom is what is usually called having open strings in the presence of N D-branes wrapping $\mathcal{L} \subset X$. The D-branes are called topological in the context of topological strings, since we do not require them to be spacetime SUSY and they are just Lagrangian.

One thing we will need: coupling of Chan-Paton to background fields

A $\overset{U(N)}{\text{gauge}}$ connection on $\mathcal{L}$

$$\rightarrow \quad e^{-L_{\text{top string}}} \prod_{i=1}^{h} \mathrm{Tr}\, P\exp \oint_{C_i} f^*(A)$$



C_i

★ T^*M

M 3-manifold, T^*M

symplectic $J = \sum_{a=1}^{3} dp_a \wedge dq_a$ — q_a local coords M, p_a " cotangent

$\downarrow$

find a complex structure s.t. J is a Kähler form

it is in fact a CY

NON-COMPACT one!

see this more clearly with our main interest case: T^*S^3

which is nothing but

$$\sum_{\mu=1}^{4} \eta_\mu^2 = a \qquad \eta_\mu = x_\mu + i v_\mu$$

$$\swarrow \qquad \searrow$$

$$\sum_{\mu=1}^{4} (x_\mu^2 - v_\mu^2) = a \qquad \sum_{\mu=1}^{4} x_\mu v_\mu = 0$$

x_μ S^3 sphere coords

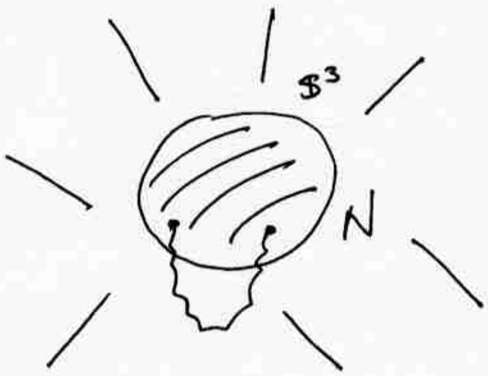
v_μ bundle coordinates

so T^*S^3 = DEFORMED CONIFOLD

Finally: $M \subset T^*M$ is obviously Lagrangian since $J|_M = 0$

Therefore we can consider open topological strings with $U(N)$ Chan-Paton factors on T^*M and b.c. specified by M

equivalently, N top D-branes wrapping M inside T^*M



We want to argue that the theory of open strings here reduces to CS theory. Equivalently, we want to argue that the analog of BI Lagrangian for N D-branes on M reduces to CS theory on M.

<u>Shortcut</u>: use open string field theory for the usual bosonic string ! Why? Because the algebraic structure of bosonic strings and topological strings is <u>quite</u> similar

The basic ingredient is the existence of a BRST operator Q_{BRST} with $Q^2_{BRST} = 0$ and

$$T_{\mu\nu} = \{Q_{BRST}, b_{\mu\nu}\}$$

Witten showed for the bosonic string that the spacetime action is just: $$S = \frac{1}{g_s} \mathrm{Tr} \int \frac{1}{2} \Psi * Q_{BRST} \Psi + \frac{1}{3} \Psi * \Psi * \Psi$$

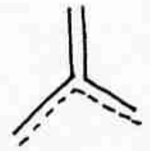
which describes the spacetime dynamics of N D-25 branes.

Here,

Ψ = string field $= T\,|p\rangle + A_\mu \alpha^\mu_{-1}\,|p\rangle + \cdots$

= functional of spacetime position of midpoint of the string + all oscillator modes

$*$ star product



$\int$:  integration of string functionals

Since open topological strings have the same structure, one should work out the meaning of Ψ, $*$, $\int$ for them, and then the spacetime action should be the same as above.

First question: what is the string field for open top strings on T^*M?

it turns out that here strings reduce to point particles!

stretched strings decouple from the spectrum in the same way that in computing $\mathrm{Tr}\,(-1)^F e^{-\beta H}$ only states with $H \sim 0$ are not suppressed as $\beta \to \infty$

so we have

M

and oscillator modes (which are the responsibles for strechting the string) shouldn't be included

$\Rightarrow$ downsizing of Ψ! Ψ is just a functional of a point in M

more precisely, $\Psi = A_\mu^{ij}\, dx^\mu$ ij Chan-Paton

a $U(N)$ connection

<u>NOTE</u> for cognoscendi: g.n. $\Psi = 1$ = degree of the form and this is the usual restriction on g.n. for string fields.

$\star \to \wedge$ wedge product (since Witten's star product reduces to the usual product in the limit of point-partic

$$\int \to \int_M$$

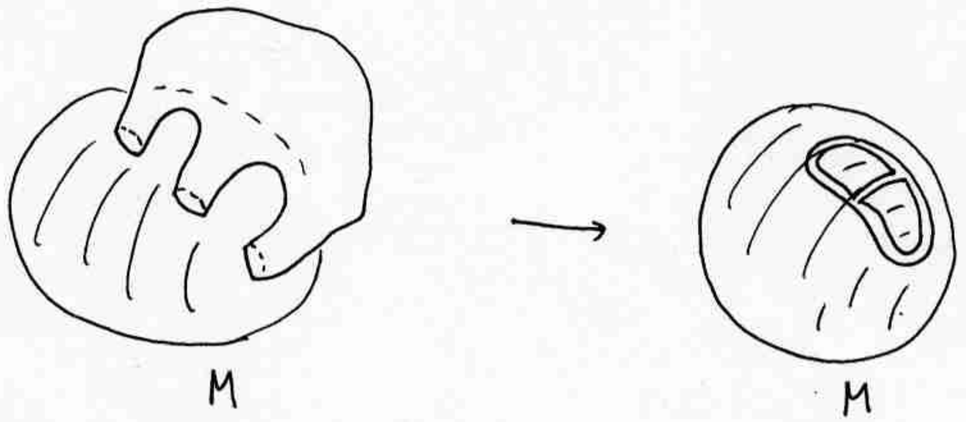
$$S \to \frac{1}{2g_s}\int_M \left(A\wedge dA + \frac{2}{3}\, A\wedge A\wedge A\right)$$

$$g_s = \frac{2\pi}{k+N}$$

But you must think I'm crazy, since after all a theory of open topological strings is a theory of worldsheet instantons

What are the worldsheet instantons here?

It turns out that, in the limit of point-particles, the worldsheet instantons reduce to fatgraphs of CS theory!



this is a known degeneration limit of open string theory...

4. CS as a closed string theory and the conifold transition

We want to know if there is a closed string description in the sense we explained before, i.e.

$$F = \sum_{g=0}^{\infty} \sum_{h=1}^{\infty} F_{g,h}\, x^{2g-2+h} N^h$$

$$= \sum_{g=0}^{\infty} F_g(t)\, x^{2g-2}$$

$$x = \frac{2\pi}{k+N} = g_s$$

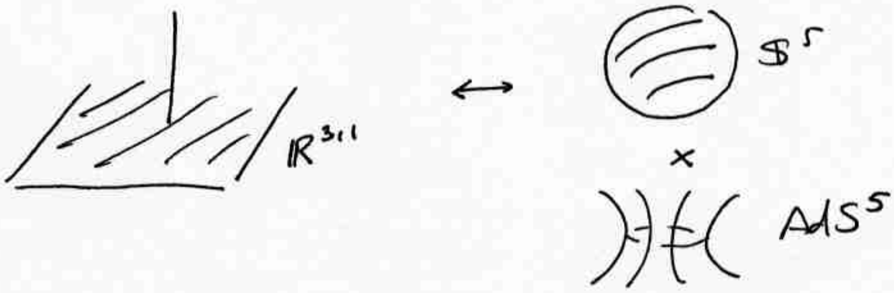
$$F_g(t) = \sum_{h=1}^{\infty} F_{g,h}\, t^h$$

$$t = Nx$$

for $\underline{M = S^3}$

hint: AdS/CFT $\quad$ IIB/$\mathbb{R}^{9,1}$ $=$ IIB/$AdS_5 \times S^5$

N D-branes on $\mathbb{R}^{3,1}$ $\qquad$ <u>no</u> D-branes



trade the N-branes for their effect on the geometry

So we should expect

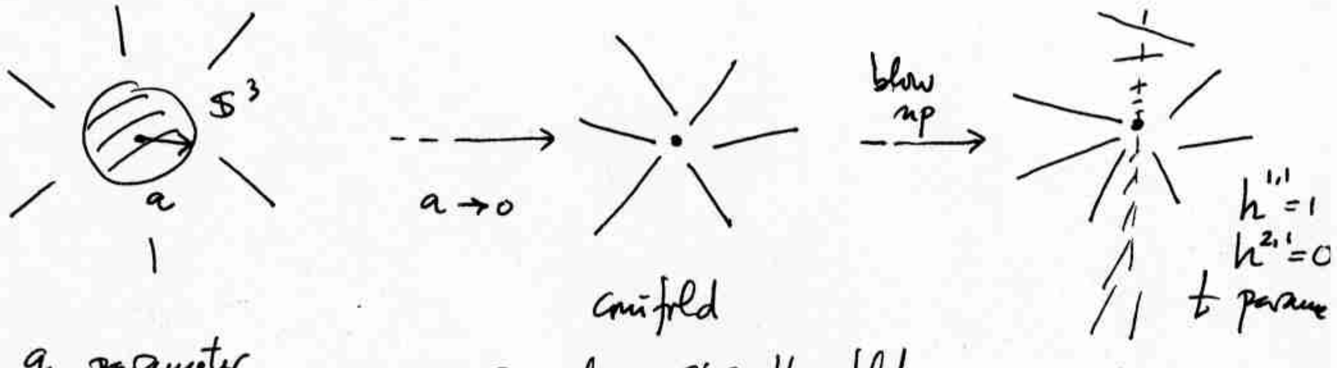
$T^* S^3$ $\longleftrightarrow$ X a different background geometry with <u>no</u> D-bra[nes]

N D-branes on S^3

$\Longrightarrow$ <u>GEOMETRIC TRANSITION</u> of the background geometry

but for T^*S^3 = deformed conifold there is a well-known transition of the geometry: the <u>CONIFOLD</u> transition.

<u>brief description:</u>



a parameter $\qquad$ conifold

$h^{2,1}=1$, $h^{1,1}=0$ $\qquad$ a <u>singular</u> CY threefold $\qquad$ $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbb{P}$

$$\sum_{\mu=1}^{4} \eta_\mu^2 = 0$$

<u>resolved</u> conifold

algebraic description:

$$xy = uv + a \qquad \text{def. conifold}$$

$$xy = uv \qquad \text{conifold}$$

$$x = \lambda v, \quad u = \lambda y \qquad \lambda \text{ inhomogeneous coordinate on } \mathbb{P}^1$$

$z_1 = 0$

$z_4 = 0$

$$\lambda = \frac{z_1}{z_4}$$

$$\underbrace{\begin{matrix} (\lambda, x, u) \in \mathbb{C} \times \mathbb{C}^2 \\ \updownarrow \\ (\lambda^{-1}, v, y) \in \mathbb{C} \times \mathbb{C}^2 \end{matrix}}_{\text{gluing this gives } \mathcal{O}(-1)^2 \to \mathbb{P}^1}$$

* another algebraic description of $\mathcal{O}(-1)^2 \to \mathbb{P}^1$:

$\boxed{\text{check it with Aspinwall's lectures}}$

consider the coordinates $z_1, z_2, z_3, z_4 \in \mathbb{C}$

and the $U(1)$ action

$$z_1\ z_2\ z_3\ z_4 \longrightarrow e^{i\theta} z_1\ e^{-i\theta} z_2\ e^{-i\theta} z_3\ e^{i\theta} z_4$$

Define $$\mathcal{M} = \{ |z_1|^2 + |z_4|^2 - |z_3|^2 - |z_2|^2 = t \} / U(1)$$

$$= \mathcal{O}(-1)^2 \to \mathbb{P}^1$$

notice $z_2 = z_3 = 0$ $\quad \mathcal{M} = \{ |z_1|^2 + |z_4|^2 = t \} / U(1)$

$= \mathbb{P}^1$ with radius $t^{1/2}$

and area $\propto t$

z_2, z_3 coordinates for the fibers

$$x = z_1 z_3, \; y = x_2 x_4, \; u = x_1 x_2, \; v = x_3 x_4 \longrightarrow \lambda = x/v = u/y = x_1/x_4$$

pictorial description:

rewrite $xy = uv + a$ as follows:

* we introduce $z = uv$, so we write
$$uv = z$$
$$xy = a + z$$

* notice the $\mathbb{T}^2 = U(1)^2$ symmetry of this last equation as:

$$x, y \to x e^{i\theta_a}, \ y e^{-i\theta_a} \qquad U(1)_a \times U(1)_b$$
$$u, v \to u e^{i\theta_b}, \ v e^{-i\theta_b}$$

we have then a fibration
$$\boxed{\begin{array}{ccc} \mathbb{T}^2 \times \mathbb{R} & \longrightarrow & \mathcal{M}_{\text{deformed conifold}} \\ & & \downarrow \\ & & \mathbb{R}^3 \end{array}}$$

if we fix z, θ_a, θ_b,

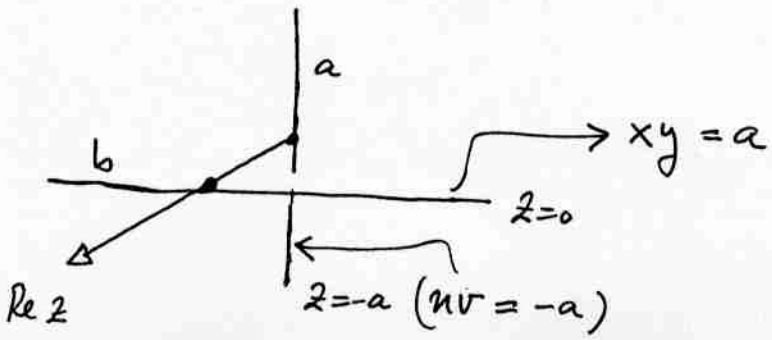
$$\left.\begin{array}{l} |u||v| = |z| \\ \theta_u + \theta_v = \theta_z \\ |x||y| = |z+a| \\ \theta_x + \theta_y = \theta_{z+a} \end{array}\right\}$$

with $\mathbb{T}^2 \times \mathbb{R}$ corresponding to $(\theta_a, \theta_b, \operatorname{Im} z)$

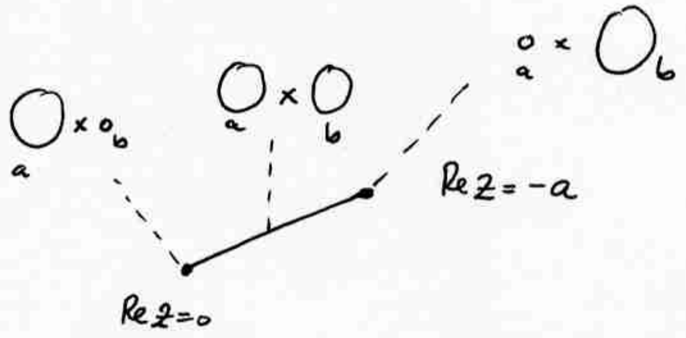
and $\mathbb{R}^3$ to $(\operatorname{Re} z, |x|, |u|)$

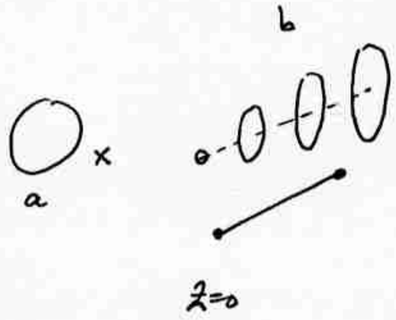
fixed points of $\mathbb{T}^2$ action:

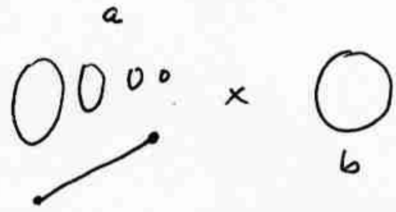
- $x = y = 0$ $U(1)_a$ degenerates at $z = -a$
- $u = v = 0$ $U(1)_b$ " at $z = 0$

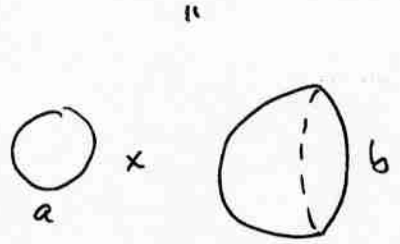


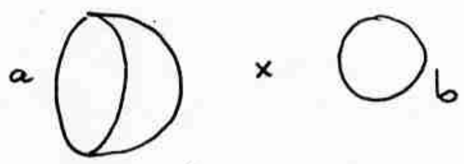
where is the 3-sphere here?









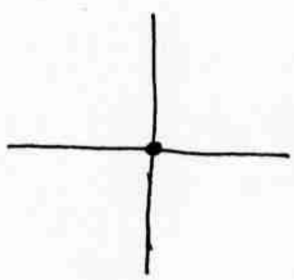


$S^1_a \times D_b \longleftrightarrow D_a \times S^1_b$

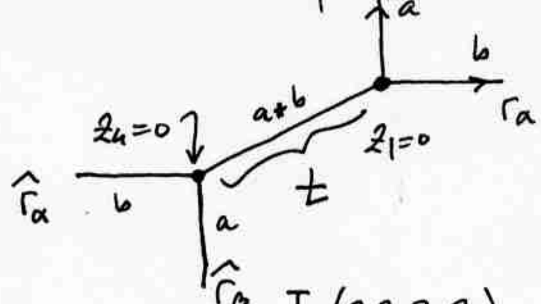
solid torus   solid torus

this is our S^3!

singular conifold



resolved conifold



$\mathrm{Im}(z_1 z_2 z_3 z_4)=0$

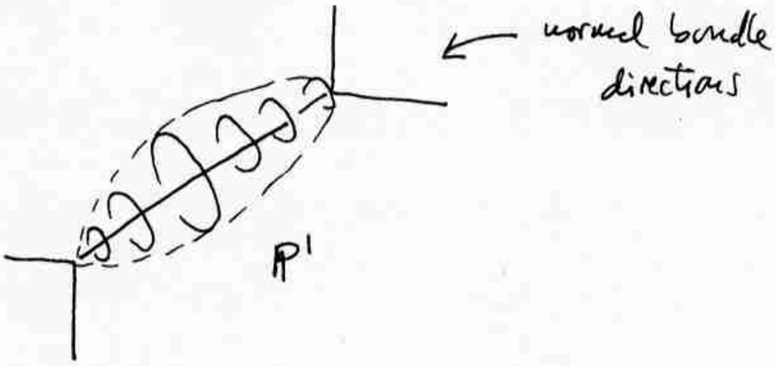
$$r_\alpha = |z_2|^2-|z_1|^2 \qquad \hat{r}_\alpha = |z_4|^2-|z_2|^2-t$$

$$r_\beta = |z_3|^2-|z_1|^2 \qquad \hat{r}_\beta = |z_4|^2-|z_3|^2-t$$

$\mathbb{T}^2 = U(1)_a \times U(1)_b$ $\qquad z_1 \to e^{i\alpha} z_1 \qquad z_3 \to e^{-i(\alpha+\beta)} z_3$

$$z_2 \to e^{i\beta} z_2$$

notice that on the diagonal there is a nontrivial S^1 cycle $(a-b)$ which degenerates at the endpoints:



<u>Summarizing</u>: there is a natural geometry, the resolved conifold, which comes from the deformed conifold through geometric transition

CONJECTURE (Gopakumar - Vafa)

CS/S^3 = open top string on T^*S^3 $\neq$ closed strings on the <u>resolved</u> conifold

in the sense of 't Hooft

<u>check</u>: we have to sum over holes

remember $F = \log Z(S^3) = F^{np} + F^p$

$$F^{np} = \log \frac{(2\pi g_s)^{\frac{1}{2}N^2}}{\mathrm{vol}(U(N))} \qquad F^p = \underbrace{\sum_{g,h} F_{g,h}\, t^h x^{2g}}_{\text{do the sum here first}}$$

$$F_g^{pert}(t) = \sum_{h=1}^{\infty} F_{g,h}\, t^h \quad \text{where} \quad F_{g,h} = \frac{2\zeta(2g-2+h)}{(2\pi)^{2g-2+h}} \frac{B_{2g}}{2g(2g-2)} \binom{2g-3+h}{h}$$

$g \geq 2$

hints for this sum:

* write $\zeta(2g-2+h) = \sum_{n=1}^{\infty} \frac{1}{n^{2g-2+h}}$ and $\underline{\underline{=0}}$ for $\underline{\underline{h \text{ odd}}}$

so that the sum for h reduces to

$$\sum_{h \text{ even}}^{\infty} \frac{1}{(2\pi n)^{2g-2+h}} \binom{2g-3+h}{h} t^h = \frac{1}{(2\pi n)^{2g-2}} \left\{ \frac{1}{\left(1-\frac{t}{2\pi n}\right)^{2g-2}} + \frac{1}{(1+t/2\pi n)^{2g-2}} \right\}$$

where we used the binomial sum $\frac{1}{(1-x)^q} = \sum_{n=0}^{\infty} \binom{q+n-1}{n}$,

so we obtain

$$F_g^{\text{pert}}(t) = \frac{(-1)^g |B_{2g} B_{2g-2}|}{2g(2g-2)(2g-2)!} +$$

$$+ \frac{B_{2g}}{2g(2g-2)} \underbrace{\sum_{n\in\mathbb{Z}}{}' \frac{1}{(t+2\pi n)^{2g-2}}}_{\text{except for } n=0} \quad \checkmark$$

$$F_g^{\text{np}}(t) = \frac{B_{2g}}{2g(2g-2)} \frac{1}{t^{2g-2}} \qquad g \geq 2$$

since $F^{\text{np}} = \log \frac{(2\pi g_s)^{\frac{1}{2}N^2}}{\text{vol}(U(N))} =$

$$= \frac{N^2}{2}\left(\log(N g_s) - \frac{3}{2}\right) - \frac{1}{2}\log N + \zeta'(-1)$$

$$+ \underbrace{\sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g-2)} N^{2-2g}}_{\sum_{g=2}^{\infty} F_g^{\text{np}}(t)\, x^{2g-2}}$$

$$F = \sum_{g=0}^{\infty} F_g(t)\, g_s^{2g-2} = \sum_{d=1}^{\infty} \frac{e^{-d t_{\text{Kähler}}}}{\left(2 \sin \frac{d g_s}{2}\right)^2}$$

$\Rightarrow$ one single Gopakumar-Vafa (BPS) invariant

$$Z(S^3; N, g_s) = \int dM\, e^{-\frac{1}{g_s}(\log M)^2}$$

$\uparrow$

* comment on CS matrix model and summing over h

$V(M) = \frac{1}{g_s}(\log M)^2$

and $\dfrac{B_{2g}}{2g(2g-2)} \displaystyle\sum_{n\in\mathbb{Z}} \dfrac{1}{(t+2\pi n)^{2g-2}} = \dfrac{|B_{2g}|}{2g(2g-2)} \mathrm{Li}_{3-2g}(e^{-it})$

$$= F_g(t) \text{ for the resolved conifold!}$$

$$-it \equiv t^{\mathbb{CP}^1}_{\text{kähler}}$$

of course, for $F_0(t) \sim \frac{t^2}{2}\left(\log t - 3/2\right) \sim V\Upsilon$ for small t

Notice that we have resummed a series expanded around $t=0$ to obtain a series around $\boxed{e^{-t} \underset{t\to\infty}{\to} 0}$ large volume limit

CS perturbation theory

large volume limit ~GW theory

$t=0$ •-- ⟶ • $t=\infty$

So we checked the conjecture at the level of partition function (free energy)

5. Generalizations

5.1. Knots in S^3

i.e. what are the implications of this for Wilson loop operators?

$\mathcal{R} \to W^{\mathcal{R}}_{\mathcal{R}}(A)$ operator

gives an open string sector in the closed theory!

in order to specify a b.c. in top strings we need a Lagrangian submanifold. So we obtain a geometric map:

$$K \longrightarrow \mathcal{L}_K \subset \mathcal{O}(-1)^2 \longrightarrow \mathbb{P}^1$$
$$\cap$$
$$S^3$$

$\boxed{\text{Ooguri-Vafa}}$

⇒ very important implications for knot invariants !!

5.2. The above is an example of string theory ⟺ gauge theory duality.

Now, on the string theory side we don't learn much — a single background $\mathcal{O}(-1)^2 \to \mathbb{P}^1$, with rather trivial invariants.

Can we understand topological strings on other CY's X using CS ?

two approaches:

1) $CS/M \longleftrightarrow$ open top string on T^*M $\longleftrightarrow$ closed strings on X_M

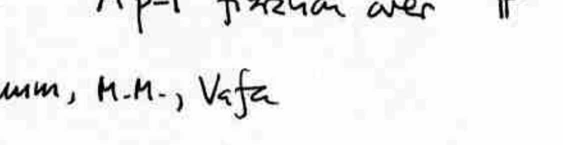
$X_{S^3} = \mathcal{O}(-1)^2 \to \mathbb{P}^1$ but also

$X_{S^3/\mathbb{Z}_p}$ = A_{p-1} fibration over $\mathbb{P}^1$ [geometric realization of $SU(p)$, $N=2$ SYM]

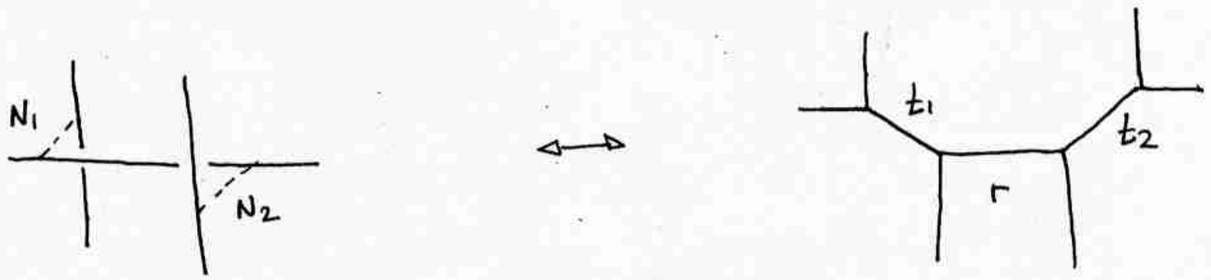
Aganagic, Klemm, M.M., Vafa

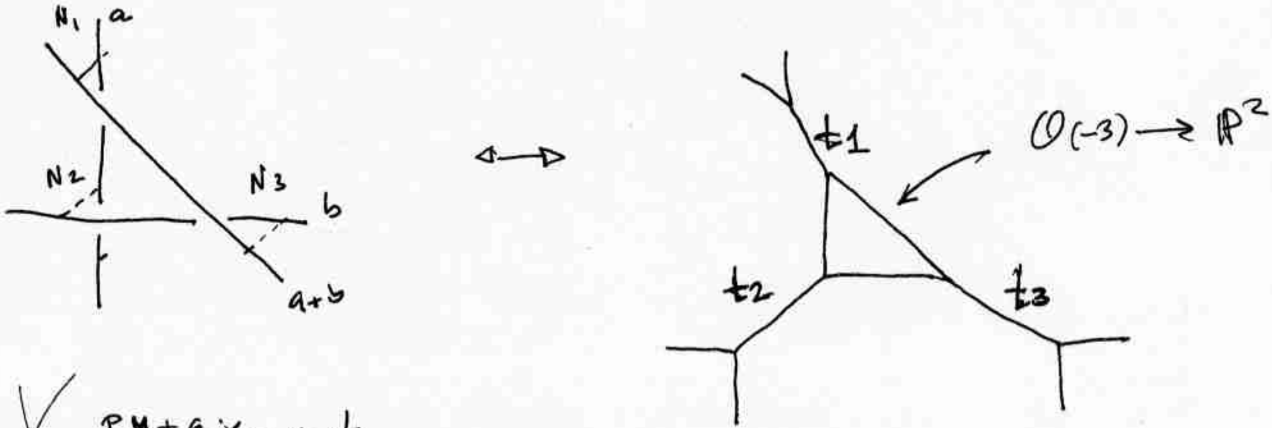
2) do large N / geometric transitions locally

remember the basic geometric statement



Now, we can use these diagrams to encode other geometries!





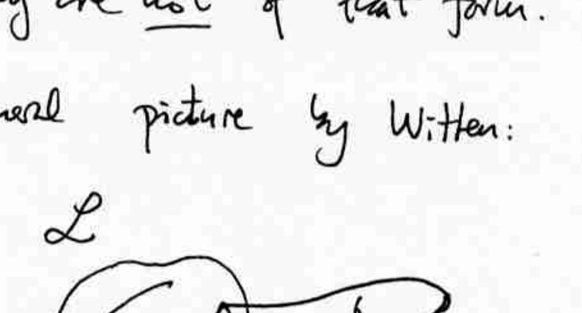
$py + qx = \text{const}$

How do we compute the open topological string on the diagrams on the left?

Notice that, although locally these manifolds contain T^*M, they are not of that form. What corrections do we expect?

General picture by Witten: there are now nontrivial open string instantons

$\mathcal{L}$



a "true" instanton

we now ~~notice~~ that the boundaries of $\Sigma_{g,h}$ are in fact knots in $\mathcal{L}$

some strings with both ends on $\mathcal{L}$ col

in general, the effective action will be if we have <u>more</u> $\mathcal{L}_i$'s we get

$$\sum_i S_{cs}^{\mathcal{L}_i}(A_i) + \sum_{\substack{\Sigma_{g,h} \\ \text{instantons}}} e^{-A_{\Sigma_{g,h}}(t)} \prod_j \mathrm{Tr}\, P\exp \oint_{R_j} A$$

A_i are $U(N_i)$ gauge fields

$\partial \Sigma_{g,h} = \cup R_j$

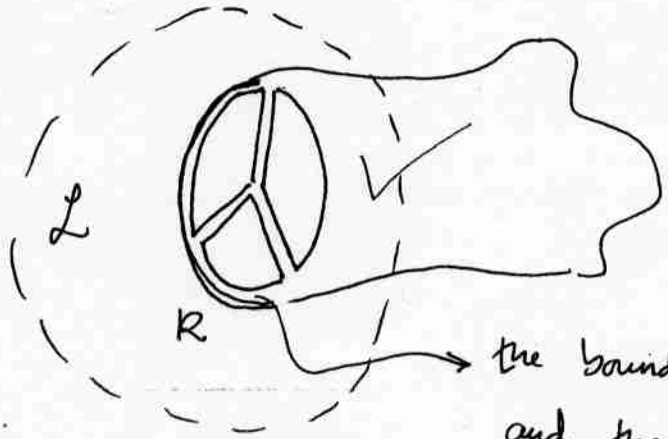
R_j ends on $\mathcal{L}_{i}$

$\boxed{\text{same}}$ coupling const

$$S = S^{\mathcal{L}}_{CS}(A) + \sum_{\substack{\Sigma_{g,h} \\ \text{instantons}}} e^{-A_{\Sigma_{g,h}}(t)} \underbrace{\prod_i \mathrm{Tr}\, P\exp \oint_{R_i} A}_{\text{coupling to background fields}}$$

$$\partial \Sigma_{g,h} = \cup R_i$$

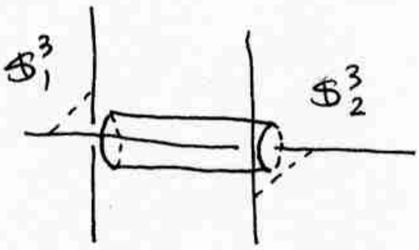
notice that in doing the path integral we insert fatgraphs inside the R_i, as it happens when computing vevs of Wilson loops in the $1/N$ double line expansion



So the resulting object is a partially degenerated instanton

the boundary of $\Sigma_{g,h}$ is "dressed" and the functional integral leads to gluon insertions

Can we make the sum over instantons more concrete in examples?



in this example we have a single annulus going from one sphere to the other! The boundaries of the annulus are unknots in S^3

$$\boxed{\text{sum over instantons} = \sum_{d=1}^{\infty} \frac{e^{-dr}}{d} \mathrm{Tr}\, U_1^{d}\, \mathrm{Tr}\, U_2^{-d}}$$

r = Kähler parameter for the tube

localization
Diaconescu-Florea-Grassi
Aganagic-M.M.-Vafa

a "primitive" annulus of area r + multicoverings

$U_{1,2}$ holonomies of $U(N_1)$, $U(N_2)$ in S^3_1, S^3_2

annuli always along the fixed point loci

Simple, important result:

$$\exp\Big(\sum_{d=1}^{\infty} \frac{e^{-dr}}{d} \operatorname{Tr} U_1^{d} \operatorname{Tr} U_2^{-d}\Big) = \sum_R e^{-\ell_R \cdot r} \operatorname{Tr}_R U_1 \operatorname{Tr}_R U_2^{-1}$$

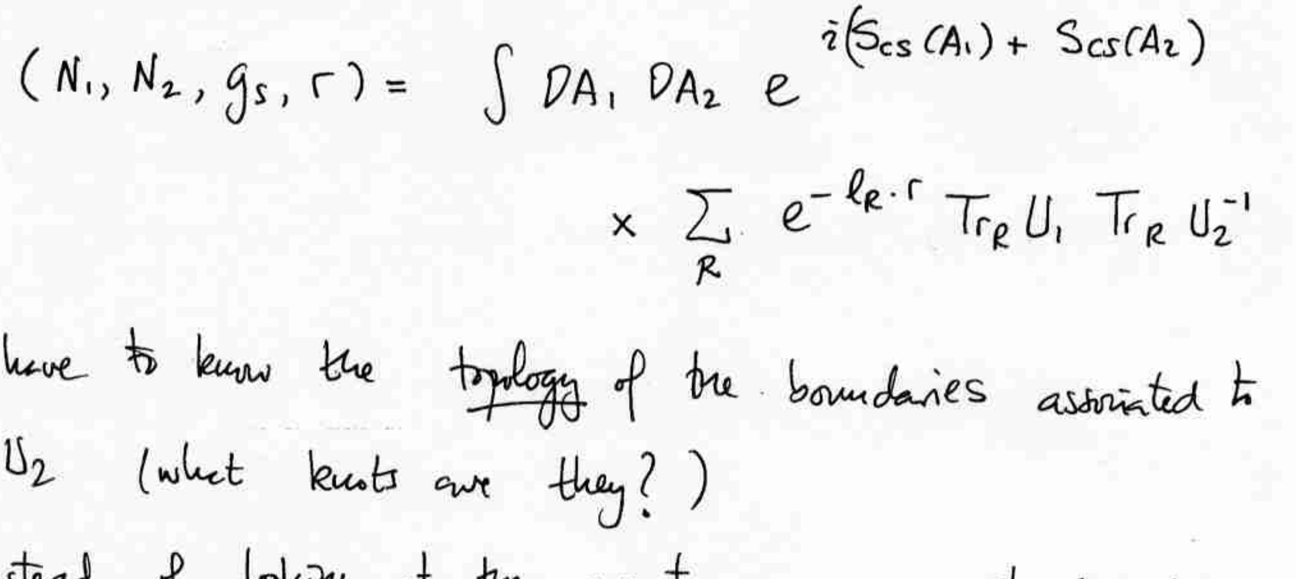
$\Rightarrow$ exercise

We have to evaluate now:

$$Z(N_1, N_2, g_s, r) = \int DA_1\, DA_2\, e^{i(S_{CS}(A_1) + S_{CS}(A_2))} \times \sum_R e^{-\ell_R \cdot r} \operatorname{Tr}_R U_1 \operatorname{Tr}_R U_2^{-1}$$

we have to know the topology of the boundaries associated to U_1, U_2 (what knots are they?)

instead of looking at the geometry, we use operator formalism and gluing



this is a state in $\mathcal{H}_1(T^2) \otimes \mathcal{H}_2^*(T^2)$

obviously $\sum_R |R\rangle_1\, e^{-\ell_R \cdot r}\, {}_2\langle R|$

Since we are inserting Wilson loops in reps R_1, R_2 on the NONCONTRACTIBLE cycle of the torus

we then evaluate:

$$Z(N_1, N_2, g_s, r) = \sum_R {}_1\langle 0| S |R\rangle_1 \, e^{-\ell_R \cdot r} \, {}_2\langle R| S |0\rangle_2$$

$$= {}_1\langle 0|S|0\rangle_1 \, {}_2\langle 0|S|0\rangle_2 \sum_R \frac{S^{(1)}_{R0}}{S_{00}} e^{-\ell_R r} \frac{S^{(2)}_{0R}}{S_{00}}$$

$$F(N_1, N_2, g_s, r) = F(S^3_1) + F(S^3_2)$$

$$+ \log \left\{ \sum_R (\dim_q R)_{U(N_1)} \, e^{-\ell_R \cdot r} (\dim_q R)_{U(N_2)} \right.$$

let's write the first terms of the above:

$$1 + \left(\frac{e^{t_1/2} - e^{-t_1/2}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \right) e^{- \quad r} \left(\frac{e^{t_2/2} - e^{-t_2/2}}{q^{1/2} - q^{-1/2}} \right)$$

$+ \ldots$

redefine $r \Rightarrow r + \frac{t_1 + t_2}{2}$

$$1 - e^{-t_1} - e^{-t_2} + e^{-t_1 - t_2}$$

so we get $\frac{(1 - e^{-t_1})(1 - e^{-t_2})}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2} e^{-r}$ at first order.

$-(4\sin^2 \frac{g_s}{2})$

so we read $n_{g=0, r, t_2, 0} = -1$, $n_{g=0, r, t_1, 0} = +1$, --- we can read GV invariants