Análise Matemática IV

2º semestre de 2000/2001

Exercício-teste 5

Determine a série de Laurent convergente em $\mathbb{C}\setminus\{1\}$ da função

$$f(z) = z\exp\left(\frac{z}{z-1}\right).$$

Aproveite o resultado para calcular

$$\oint_{|z-1|=1} z\exp\left(\frac{z}{z-1}\right)dz,$$

onde a circunferência $|z-1|=1$ é percorrida uma vez no sentido positivo.

Resolução

Temos

$$\begin{aligned}
z\exp\left(\frac{z}{z-1}\right) &= z\exp\left(1+\frac{1}{z-1}\right)\\
&= ((z-1)+1)\exp(1)\exp\left(\frac{1}{z-1}\right)\\
&= ((z-1)+1)\,e\sum_{n=0}^{+\infty}\frac{1}{n!}\left(\frac{1}{z-1}\right)^n\\
&= (z-1)\sum_{n=0}^{+\infty}\frac{e}{n!}(z-1)^{-n}+\sum_{n=0}^{+\infty}\frac{e}{n!}(z-1)^{-n}\\
&= \sum_{n=-1}^{+\infty}\frac{e}{(n+1)!}(z-1)^{-n}+\sum_{n=0}^{+\infty}\frac{e}{n!}(z-1)^{-n}\\
&= e(z-1)+\sum_{n=0}^{+\infty}\left(\frac{1}{(n+1)!}+\frac{1}{n!}\right)e(z-1)^{-n}\\
&= e(z-1)+\sum_{n=0}^{+\infty}\frac{1}{(n+1)!}(1+n+1)\,e(z-1)^{-n}\\
&= e(z-1)+\sum_{n=0}^{+\infty}\frac{n+2}{(n+1)!}e(z-1)^{-n}
\end{aligned}$$

A série de Laurent pretendida é $\sum\limits_{n=0}^{+\infty}\frac{n+1}{n!}e(z-1)^{-n+1}$.

Pelo teorema dos resíduos obtemos:

$$\begin{aligned}
\oint_{|z-1|=1} z\exp\left(\frac{z}{z-1}\right)dz &= 2\pi i\operatorname*{Res}_{z=1}\left(z\exp\left(\frac{z}{z-1}\right)\right)\\
&= 2\pi i\left[\frac{n+2}{(n+1)!}e\right]_{n=1}\\
&= 3\pi e\,i.
\end{aligned}$$