1. Consider the manifolds

\[ S^3 = \{ (x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + w^2 = 2 \} ; \]

\[ T^2 = \{ (x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 = z^2 + w^2 = 1 \} . \]

The submanifold \( T^2 \subset S^3 \) splits \( S^3 \) into two connected components. Let \( M \) be one of these components and let \( \omega \) be the 3-form

\[ \omega = zdx \wedge dy \wedge dw - xdy \wedge dz \wedge dw. \]

Compute the two possible values of \( \int_M \omega \) using:

(a) The definition of integral of a 3-form;

(b) The Stokes Theorem.

2. (a) Two smooth maps \( f_0, f_1 : M \to N \) are said to be smoothly homotopic if there exists a differentiable map \( H : \mathbb{R} \times M \to N \) such that \( H(0, p) = f_0(p) \) and \( H(1, p) = f_1(p) \).

If \( M \) is a compact oriented manifold of dimension \( n \) and \( \omega \) is a closed \( n \)-form on \( N \), show that

\[ \int_M f_0^* \omega = \int_M f_1^* \omega. \]

(b) Let \( X \in \mathfrak{X}(S^n) \) be a vector field with no zeros. Show that

\[ H(t, p) = \cos(\pi t)p + \sin(\pi t)\frac{X_p}{\|X_p\|} \]

is a smooth homotopy between the identity map and the antipodal map, where we make use of the identification

\[ X_p \in T_p S^n \subset T_p \mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}. \]

(c) Using the Stokes Theorem, show that

\[ \int_{S^n} \omega > 0, \]

where

\[ \omega = \sum_{i=1}^{n+1} (\prod_{i=1}^{n+1} x_i dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^{n+1}) \]

and \( S^n = \partial \{ x \in \mathbb{R}^{n+1} : \| x \| \leq 1 \} \) has the orientation induced by the standard orientation of \( \mathbb{R}^{n+1} \).
(d) Show that if $n$ is even then $X$ cannot exist. What about when $n$ is odd?

3. (Optional) Let $i_1 : \mathcal{X}(\mathbb{R}^3) \to \Omega^1(\mathbb{R}^3)$ and $i_2 : \mathcal{X}(\mathbb{R}^3) \to \Omega^2(\mathbb{R}^3)$ be the isomorphisms defined in Homework 6 and let $X \in \mathcal{X}(\mathbb{R}^3)$ be a vector field. Check that:

(a) If $C \subset \mathbb{R}^3$ is a one-dimensional submanifold of $\mathbb{R}^3$ parameterized by $g : I \subset \mathbb{R} \to \mathbb{R}^3$ then for the orientation defined by this parameterization

$$\int_C i_1(X) = \int_C \langle X, dg \rangle := \int_I \langle X(g(t)), \frac{dg}{dt}(t) \rangle \, dt$$

(where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product).

(b) If $S \subset \mathbb{R}^3$ is a two-dimensional submanifold of $\mathbb{R}^3$ parameterized by $g : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ then for the orientation defined by this parameterization

$$\int_S i_2(X) = \int_S \langle X, n \rangle := \int_U \langle X(g(u, v)), \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \rangle \, dudv.$$ 

(c) If $S \subset \mathbb{R}^3$ is a two-dimensional submanifold with boundary of $\mathbb{R}^3$ then the Stokes Theorem implies

$$\int_S \langle \nabla \times X, n \rangle = \int_{\partial S} \langle X, dg \rangle$$

where $n$ is related to the orientation of $\partial S$ by the right hand rule.

(d) If $D \subset \mathbb{R}^3$ is a three-dimensional submanifold with boundary of $\mathbb{R}^3$ then the Stokes Theorem implies

$$\int_D \nabla \cdot X = \int_{\partial D} \langle X, n \rangle,$$

where $n$ is the the outward pointing unit normal.

4. (Optional)

(a) Let $M$ be an $n$-dimensional compact orientable manifold with boundary $\partial M \neq \emptyset$. Show that there exists no smooth map $f : M \to \partial M$ satisfying $f|_{\partial M} = \text{id}$.

(b) Prove the **Brouwer Fixed Point Theorem**: Any smooth map $g : B \to B$ of the closed ball $B := \{x \in \mathbb{R}^n : |x| \leq 1\}$ to itself has a fixed point, that is, a point $p \in B$ such that $g(p) = p$.

5. (Optional) Let $f : S^n \to S^n$ be the antipodal map. Recall that the $n$-dimensional projective space is the differential manifold $\mathbb{R}P^n = S^n/\mathbb{Z}_2$, where the group $\mathbb{Z}_2 = \{1, -1\}$ acts on $S^n$ through $1 \cdot x = x$ and $(-1) \cdot x = f(x)$. Let $\pi : S^n \to \mathbb{R}P^n$ be the natural projection.

(a) Prove that $\omega \in \Omega^k(S^n)$ is of the form $\omega = \pi^*\theta$ for some $\theta \in \Omega^k(\mathbb{R}P^n)$ if and only if $f^*\omega = \omega$.

(b) Show that $\mathbb{R}P^n$ is orientable if and only if $n$ is odd.