Riemannian Geometry

Homework 7

Due on November 3

1. Consider the manifolds

$$S^3 = \left\{(x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + w^2 = 2\right\};$$
$$T^2 = \left\{(x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 = z^2 + w^2 = 1\right\}.$$

The submanifold $T^2 \subset S^3$ splits S^3 into two connected components. Let M be one of these components and let ω be the 3-form

$$\omega = zdx \wedge dy \wedge dw - xdy \wedge dz \wedge dw.$$

Compute the two possible values of $\int_M \omega$ using:

(a) The definition of integral of a 3-form;

(b) The Stokes Theorem.

2. (a) Two smooth maps $f_0, f_1 : M \to N$ are said to be **smoothly homotopic** if there exists a differentiable map $H : \mathbb{R} \times M \to N$ such that $H(0, p) = f_0(p)$ and $H(1, p) = f_1(p)$. If M is a compact oriented manifold of dimension n and ω is a closed n-form on N, show that

$$\int_M f_0^* \omega = \int_M f_1^* \omega.$$

(b) Let $X \in \mathfrak{X}(S^n)$ be a vector field with no zeros. Show that

$$H(t, p) = \cos(\pi t)p + \sin(\pi t)\frac{X_p}{\|X_p\|}$$

is a smooth homotopy between the identity map and the antipodal map, where we make use of the identification

$$X_p \in T_p S^n \subset T_p \mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}.$$

(c) Using the Stokes Theorem, show that

$$\int_{S^n} \omega > 0,$$

where

$$\omega = \sum_{i=1}^{n+1} (-1)^{i+1} x^i dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^{n+1}$$

and $S^n = \partial\{x \in \mathbb{R}^{n+1} : \|x\| \leq 1\}$ has the orientation induced by the standard orientation of $\mathbb{R}^{n+1}$.

(d) Show that if n is even then X cannot exist. What about when n is odd?

3. **(Optional)** Let $i_1 : \mathfrak{X}(\mathbb{R}^3) \to \Omega^1(\mathbb{R}^3)$ and $i_2 : \mathfrak{X}(\mathbb{R}^3) \to \Omega^2(\mathbb{R}^3)$ be the isomorphisms defined in Homework 6 and let $X \in \mathfrak{X}(\mathbb{R}^3)$ be a vector field. Check that:

   (a) If $C \subset \mathbb{R}^3$ is a one-dimensional submanifold of $\mathbb{R}^3$ parameterized by $g : I \subset \mathbb{R} \to \mathbb{R}^3$ then for the orientation defined by this parameterization

   $$\int_C i_1(X) = \int_C \langle X, dg \rangle := \int_I \left\langle X(g(t)), \frac{dg}{dt}(t) \right\rangle dt$$

   (where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product).

   (b) If $S \subset \mathbb{R}^3$ is a two-dimensional submanifold of $\mathbb{R}^3$ parameterized by $g : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ then for the orientation defined by this parameterization

   $$\int_S i_2(X) = \int_S \langle X, n \rangle := \int_U \left\langle X(g(u,v)), \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right\rangle du dv.$$

   (c) If $S \subset \mathbb{R}^3$ is a two-dimensional submanifold with boundary of $\mathbb{R}^3$ then the Stokes Theorem implies

   $$\int_S \langle \nabla \times X, n \rangle = \int_{\partial S} \langle X, dg \rangle$$

   where n is related to the orientation of ∂S by the right hand rule.

   (d) If $D \subset \mathbb{R}^3$ is a three-dimensional submanifold with boundary of $\mathbb{R}^3$ then the Stokes Theorem implies

   $$\int_D \nabla \cdot X = \int_{\partial D} \langle X, n \rangle,$$

   where n is the the outward pointing unit normal.

4. **(Optional)**

   (a) Let M be an n-dimensional compact orientable manifold with boundary $\partial M \neq \varnothing$. Show that there exists no smooth map $f : M \to \partial M$ satisfying $f|_{\partial M} = \mathrm{id}$.

   (b) Prove the **Brouwer Fixed Point Theorem**: Any smooth map $g : B \to B$ of the closed ball $B := \{x \in \mathbb{R}^n : |x| \leq 1\}$ to itself has a fixed point, that is, a point $p \in B$ such that $g(p) = p$.

5. **(Optional)** Let $f : S^n \to S^n$ be the antipodal map. Recall that the n-dimensional projective space is the differential manifold $\mathbb{R}P^n = S^n/\mathbb{Z}_2$, where the group $\mathbb{Z}_2 = \{1, -1\}$ acts on S^n through $1 \cdot x = x$ and $(-1) \cdot x = f(x)$. Let $\pi : S^n \to \mathbb{R}P^n$ be the natural projection.

   (a) Prove that $\omega \in \Omega^k(S^n)$ is of the form $\omega = \pi^*\theta$ for some $\theta \in \Omega^k(\mathbb{R}P^n)$ if and only if $f^*\omega = \omega$.

   (b) Show that $\mathbb{R}P^n$ is orientable if and only if n is odd.