1. We define the **symplectic gradient** of a smooth function $F \in C^\infty(\mathbb{R}^2)$ to be the vector field $X_F \in \mathfrak{X}(\mathbb{R}^2)$ given in the usual Cartesian coordinates $(x, y)$ by

$$X_F = \frac{\partial F}{\partial y} \frac{\partial}{\partial x} - \frac{\partial F}{\partial x} \frac{\partial}{\partial y}.$$

Show that:

(a) If $a \in \mathbb{R}$ is a regular value of $F$ then $X_F$ is tangent to the submanifold $F^{-1}(a)$.

(b) If $F$ does not have critical points and

$$\lim_{(x,y)\to\infty} F(x,y) = +\infty$$

then $X_F$ is complete.

(c) $[X_F, X_G] = X_{\{F,G\}}$, where $\{F,G\} \in C^\infty(\mathbb{R}^2)$ is the function

$$\{F,G\} = X_F \cdot G = \frac{\partial F}{\partial y} \frac{\partial G}{\partial x} - \frac{\partial F}{\partial x} \frac{\partial G}{\partial y}.$$

(d) If either $F$ or $G$ have a critical point and $X_F, X_G$ are complete then their flows commute if and only if $\{F,G\} = 0$.

(e) $\omega(X_F, Y) = dF(Y)$ for any $Y \in \mathfrak{X}(\mathbb{R}^2)$, where $\omega = dx \wedge dy$ is the standard volume form.

(f) If $X_F$ is complete with flow $\psi_t : \mathbb{R}^2 \to \mathbb{R}^2$ and $U \subset \mathbb{R}^2$ is a bounded open set such that $\overline{U}$ is a manifold with boundary then

$$\int_U \omega = \int_{\psi_t(U)} \omega.$$

(*Hint:* Use the Stokes Theorem on the three-dimensional manifold with boundary $M = \{(s, x, y) \mid s \in [0, t] \text{ and } (x, y) \in \psi_s(\overline{U})\}$).

(g) Check the result in (f) for the function $F(x, y) = x^2 + y^2$ by computing the flow of $X_F$ explicitly.

(4/20) 2. A smooth manifold $M$ is called **contractible** if the identity map is smoothly homotopic to a constant map. Show that if $M$ is compact and orientable (with $\dim M > 0$) then $M$ is not contractible.
1. Let \( f : \mathbb{R} \to \mathbb{R} \) be a diffeomorphism, and consider the embedding \( \varphi : \mathbb{R}^2 \to \mathbb{R}^3 \) given by \( \varphi(s, t) = (s \cos(t), s \sin(t), f(t)) \).

Show that:

(a) The metric induced on \( \mathbb{R}^2 \) by this embedding and the Euclidean metric of \( \mathbb{R}^3 \) is
\[
g = ds \otimes ds + R^2(s, t) \, dt \otimes dt,
\]
where
\[
R(s, t) = \sqrt{s^2 + (f'(t))^2}.
\]

(b) The nonvanishing connection forms associated to the orthonormal coframe
\[
\omega^s = ds, \quad \omega^t = R \, dt
\]
are
\[
\omega^s_t = -\omega^t_s = \frac{\partial R}{\partial s} \, dt.
\]

(c) The Gauss curvature is
\[
K = -\frac{1}{R} \frac{\partial^2 R}{\partial s^2} = -\frac{(f'(t))^2}{(s^2 + (f'(t))^2)^2}.
\]

(d) The curve \( s = 0 \) and the curves of constant \( t \) are (images of) geodesics.

(e) \( \mathbb{R}^2 \) with this induced metric is complete.

2. Prove the ultraparallel theorem: given two non-intersecting geodesics \( \gamma_1, \gamma_2 \) of the hyperbolic plane ("parallel geodesics") there exists at most a third geodesic \( \gamma_3 \) (up to reparameterization) intersecting \( \gamma_1 \) and \( \gamma_2 \) orthogonally. Show by means of examples that \( \gamma_3 \) may or may not exist (that is, \( \gamma_1 \) and \( \gamma_2 \) may be ultraparallel or asymptotic).

3. Let \( N \subset \mathbb{R}^3 \) be a 2-dimensional submanifold formed by straight lines through the origin (excluding the origin). Prove that \( N \) with the metric induced by the Euclidean metric of \( \mathbb{R}^3 \) is flat, that is, has zero Gauss curvature.

\[
\text{Cartan equations: } \quad \left\{ \begin{array}{l}
d\omega^i = \sum_{j=1}^n \omega^j \wedge \omega^i_j \\
\Omega^i_j = d\omega^i_j - \sum_{k=1}^n \omega^i_k \wedge \omega^j_k
\end{array} \right.
\]