1. (a) Show that if the mass distribution of the reference configuration of a rigid body is symmetric with respect to reflections in a plane containing the origin then there exists a principal axis orthogonal to the reflection plane.

(b) Determine the principal axes and the corresponding principal moments of inertia of:

   i. a homogeneous rectangular parallelepiped with mass $M$, sides $2a$, $2b$, $2c \in \mathbb{R}^+$ and centered at the origin;

   ii. a homogeneous (solid) ellipsoid with mass $M$, semiaxes $a, b, c \in \mathbb{R}^+$ and centered at the origin. (Hint: Use the coordinate change $(x, y, z) = (au, bv, cw)$).

2. (Precession of the equinoxes) Due to its rotation, the Earth is not a perfect sphere, but an oblate ellipsoid; therefore its moments of inertia are not quite equal, satisfying approximately

$$I_1 = I_2 \neq I_3;$$

$$\frac{I_3 - I_1}{I_1} \simeq \frac{1}{306}.$$ 

As a consequence, the combined gravitational attraction of the Moon and the Sun disturbs the Earth’s rotation motion. This perturbation can be approximately modelled by the potential energy $U : SO(3) \to \mathbb{R}$ given in the Euler angles $(\theta, \varphi, \psi)$ by

$$U = -\frac{\Omega^2}{2} (I_3 - I_1) \cos^2 \theta,$$

where $\frac{2\pi}{\Omega} \simeq 168$ days.

(a) Write the equations of motion and determine the equilibrium points.

(b) Show that there exist solutions such that $\theta, \dot{\varphi}$ and $\dot{\psi}$ are constant, which in the limit $|\dot{\varphi}| \ll |\dot{\psi}|$ (as is the case with the Earth) satisfy

$$\dot{\varphi} \simeq -\frac{\Omega^2 (I_3 - I_1) \cos \theta}{I_3 \dot{\psi}}.$$

Given that for the Earth $\theta \simeq 23^\circ$, determine the approximate value of the period of $\varphi(t)$. 