Recall that the Euler angles \((\theta, \varphi, \psi) : SO(3) \to (0, \pi) \times (0, 2\pi) \times (0, 2\pi)\) are the local coordinates defined by the parameterization

\[
S(\theta, \varphi, \psi) = \begin{pmatrix}
\cos \varphi & -\sin \varphi & 0 \\
\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
\cos \psi & -\sin \psi & 0 \\
\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Let \(S : \mathbb{R} \to SO(3)\) describe the orientation of a rigid sphere of radius \(R > 0\), mass \(M > 0\) and inertia tensor \(I = \text{diag}(I_1, I_1, I_1)\). It can be shown that the space angular velocity \(\omega = S\Omega\) is given by

\[
\omega = (\dot{\theta} \cos \varphi + \dot{\psi} \sin \theta \sin \varphi) e_1 + (\dot{\theta} \sin \varphi - \dot{\psi} \sin \theta \cos \varphi) e_2 + (\dot{\varphi} + \dot{\psi} \cos \theta) e_3.
\]

1. Consider first only the rotational motion of the sphere about its center.

(a) Show that the kinetic energy \(K : TSO(3) \to \mathbb{R}\) is

\[
K(\theta, \varphi, \psi; v^\theta, v^\varphi, v^\psi) = \frac{I_1}{2} \left( (v^\theta)^2 + (v^\varphi)^2 + (v^\psi)^2 + 2v^\varphi v^\psi \cos \theta \right).
\]

(b) Prove that the Lagrangian function \(L = K\) is hyper-regular.

(c) Determine the Hamiltonian function \(H : T^*SO(3) \to \mathbb{R}\) and show that it is completely integrable.

2. Now assume that the sphere is rolling on a horizontal plane, and let \((x, y)\) be the Cartesian coordinates of the contact point, so that the kinetic energy has an additional term

\[
\frac{1}{2} M \left( (v^x)^2 + (v^y)^2 \right).
\]

(a) Show that the constraint of rolling without slipping, that is, that the velocity of the contact point is zero, is given by the kernels of the 1-forms

\[
\alpha^1 = dx - R \sin \varphi \, d\theta + R \sin \theta \cos \varphi \, d\psi, \quad \alpha^2 = dy + R \cos \varphi \, d\theta + R \sin \theta \sin \varphi \, d\psi.
\]

(b) Prove this constraint is non-holonomic.

(c) Write the equations of motion assuming a perfect reaction force, and find all solutions satisfying \(x = vt\) and \(y = \dot{\theta} = \dot{\varphi} = 0\).
1. Recall that the Schwarzschild metric for the equatorial plane $\theta = \frac{\pi}{2}$ is given by

$$g = -\left(1 - \frac{2m}{r}\right)dt \otimes dt + \left(1 - \frac{2m}{r}\right)^{-1}dr \otimes dr + r^2d\varphi \otimes d\varphi.$$  

(a) Show that the conditions for a curve of constant $r$ to be a timelike geodesic parameterized by its proper time are

$$\begin{cases}
\ddot{t} = 0 \\
\ddot{\varphi} = 0 \\
r^2\dot{\varphi}^2 = \frac{m}{r^2}\dot{t}^2 \\
\left(1 - \frac{3m}{r}\right)\dot{t}^2 = 1
\end{cases}$$

(b) Conclude that massive particles can orbit the central mass in circular orbits for all $r > 3m$, and compute the period of these orbits as measured by:

(i) A stationary observer;

(ii) The orbiting particle.

(c) Show that there exists a circular null geodesic for $r = 3m$.

2. Recall that the upper half plane $H = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ has a Lie group structure, given by the operation

$$(a, b) \cdot (x, y) = (bx + a, by).$$

(a) Show that the formula

$$(a, b) \cdot (x, y, p_x, p_y) = \left(bx + a, by, \frac{px}{b}, \frac{py}{b}\right)$$

defines a Poisson action on $T^*H$ (with the canonical symplectic structure).

*Hint:* Show that this map preserves the canonical symplectic form.

(b) Check that the functions

$$F(x, y, p_x, p_y) = yp_x \quad \text{and} \quad G(x, y, p_x, p_y) = yp_y$$

are $H$-invariant, and use this to obtain the quotient Poisson structure on $T^*H/H$. Is this quotient Poisson manifold a symplectic manifold?