Mathematical Relativity

Homework 13

Due on May 25

1. (a) Check that the Klein-Gordon equation in Minkowski spacetime, \( \Box \phi - m^2 \phi = 0 \), can be derived from the Lagrangian density
\[
L = -\frac{1}{2} \left( \partial_{\mu} \phi \partial^{\mu} \phi + m^2 \phi^2 \right).
\]

(b) Show that the corresponding Hamiltonian is
\[
H = \int_{\mathbb{R}^n} \frac{1}{2} \left( (\partial_0 \phi)^2 + \ldots + (\partial_n \phi)^2 + m^2 \phi^2 \right) \, dx^1 \wedge \ldots \wedge dx^n.
\]

(c) Starting with the Einstein-Hilbert-Klein-Gordon action
\[
S = \int_{M} \left[ R - 8\pi \left( g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2 \right) \right] \, dV_{n+1}
\]

obtain the energy-momentum tensor for \( \phi \):
\[
T_{\mu \nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu \nu} \left( \partial_\alpha \phi \partial^\alpha \phi + m^2 \phi^2 \right).
\]

(d) Check that \( T_{00} \) coincides with the Hamiltonian density \( H \) (i.e. the integrand in the expression for the Hamiltonian \( H \)).

2. Let \( \gamma \) be the spherically symmetric Riemannian metric defined in \( \mathbb{R}^3 \) by
\[
\gamma = \frac{d r^2}{1 - \frac{2m(r)}{r}} + r^2 \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right),
\]
where \( m \) is a smooth function whose derivative has compact support.

(a) Check that in Cartesian coordinates we have
\[
\gamma_{ij} = \delta_{ij} + \frac{2m(r)}{r^2} \frac{r \, x_i \, x_j}{1 - \frac{2m(r)}{r}}.
\]

(b) Show that if the limit
\[
M = \lim_{r \to \infty} m(r)
\]
exists then \( \gamma \) is asymptotically flat with ADM mass \( M \) (which in particular coincides with the Komar mass when appropriate).
(c) Check that $\gamma$ has scalar curvature

$$R = \frac{4}{r^2} \frac{dm}{dr},$$

and use this to prove the Riemannian Positive Mass Theorem for $\gamma$.

(d) Show that $r = r_0$ is a minimal surface if and only if $m(r_0) = \frac{r_0}{2}$ (in which case $r$ is a well-defined coordinate only for $r > r_0$), and use this to prove the Riemannian Penrose Conjecture for $\gamma$. 