Consider the spherically symmetric Lorentzian metric given by
\[ g = -(A(t,r))^2 dt^2 + (B(t,r))^2 dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2, \]
where \(A\) and \(B\) are positive smooth functions.

1. Use the condition of compatibility with the metric and Cartan’s first structure equations,
\[
\begin{align*}
\omega_{\mu\nu} &= -\omega_{\nu\mu} \\
d\omega^{\mu} + \omega^{\mu}_{\nu} \wedge \omega^{\nu} &= 0
\end{align*}
\]

to show that the nonvanishing connection forms for the orthonormal frame dual to
\[
\omega^0 = Adt, \quad \omega^r = Bdr, \quad \omega^\theta = r d\theta, \quad \omega^\varphi = r \sin \theta d\varphi
\]
are (using the notation \(\dot{\ } = \frac{\partial}{\partial t}\) and \(\prime = \frac{\partial}{\partial r}\))
\[
\begin{align*}
\omega^0_{\ r} &= \omega^r_{\ 0} = \frac{A'}{B} dt + \frac{\dot{B}}{A} dr; \\
\omega^\theta_{\ r} &= -\omega^r_{\ \theta} = \frac{1}{B} d\theta; \\
\omega^\varphi_{\ r} &= -\omega^r_{\ \varphi} = \frac{\sin \theta}{B} d\varphi; \\
\omega^\varphi_{\ \theta} &= -\omega^\theta_{\ \varphi} = \cos \theta d\varphi.
\end{align*}
\]

2. Use Cartan’s second structure equations
\[
\Omega^{\mu}_{\ \nu} = d\omega^{\mu}_{\ \nu} + \omega^\alpha_{\ \mu} \wedge \omega^{\alpha}_{\ \nu},
\]
to show that the curvature forms on this frame are
\[
\begin{align*}
\Omega^0_{\ r} &= \Omega^r_{\ 0} = \left( \frac{A''B - A'B'}{AB^3} + \frac{\dot{A}\dot{B} - \ddot{A}B}{A^3B} \right) \omega^r \wedge \omega^0; \\
\Omega^0_{\ \theta} &= \Omega^\theta_{\ 0} = \frac{A'}{rAB^2} \omega^\theta \wedge \omega^0 + \frac{\dot{B}}{rAB^2} \omega^\theta \wedge \omega^r; \\
\Omega^0_{\ \varphi} &= \Omega^\varphi_{\ 0} = \frac{A'}{rAB^2} \omega^\varphi \wedge \omega^0 + \frac{B}{rAB^2} \omega^\varphi \wedge \omega^r; \\
\Omega^\theta_{\ r} &= -\Omega^r_{\ \theta} = \frac{B'}{rB^3} \omega^\theta \wedge \omega^r + \frac{\dot{B}}{rAB^2} \omega^\theta \wedge \omega^0; \\
\Omega^\varphi_{\ r} &= -\Omega^r_{\ \varphi} = \frac{B'}{rB^3} \omega^\varphi \wedge \omega^r + \frac{\dot{B}}{rAB^2} \omega^\varphi \wedge \omega^0; \\
\Omega^\varphi_{\ \theta} &= -\Omega^\theta_{\ \varphi} = \frac{B^2 - 1}{r^2B^2} \omega^\varphi \wedge \omega^\theta.
\end{align*}
\]
3. Using
\[ \Omega^\mu_{\nu} = \sum_{\alpha<\beta} R_{\alpha\beta}^\mu \omega^\alpha \wedge \omega^\beta \]
determine the components \( R_{\alpha\beta}^\mu \) of the curvature tensor on this orthonormal frame, and show that the nonvanishing components of the Ricci tensor on this frame are
\[ R_{00} = \frac{A'' B - A'B'}{AB^3} + \frac{\dot{A} \dot{B} - A \ddot{B}}{A^3 B} + \frac{2A'}{rAB^2}; \]
\[ R_{0r} = R_{r0} = \frac{2\dot{B}}{rAB^2}; \]
\[ R_{rr} = \frac{A'B' - A''B}{AB^3} + \frac{A\ddot{B} - A\dddot{B}}{A^3 B} + \frac{2B'}{rB^3}; \]
\[ R_{\theta\theta} = R_{\varphi\varphi} = -\frac{A'}{rAB^2} + \frac{B'}{rB^3} + \frac{B^2 - 1}{r^2B^2}. \]

Conclude that the nonvanishing components of the Einstein tensor on this frame are
\[ G_{00} = 2\frac{B'}{rB^3} + \frac{B^2 - 1}{r^2B^2}; \]
\[ G_{0r} = G_{r0} = \frac{2\dot{B}}{rAB^2}; \]
\[ G_{rr} = 2\frac{A'}{rAB^2} - \frac{B^2 - 1}{r^2B^2}; \]
\[ G_{\theta\theta} = G_{\varphi\varphi} = \frac{A''B - A'B'}{AB^3} + \frac{\dot{A} \dot{B} - A \ddot{B}}{A^3 B} + \frac{A'}{rAB^2} - \frac{B'}{rB^3}. \]

4. Show that if we write
\[ B(t, r) = \left(1 - \frac{2m(t, r)}{r}\right)^{-\frac{1}{2}} \]
for some smooth function \( m \) then
\[ G_{00} = \frac{2m'}{r^2}. \]

Conclude that the vacuum Einstein equations \( G_{00} = G_{0r} = 0 \) are equivalent to
\[ B = \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}}, \]
where \( M \in \mathbb{R} \) is an integration constant.

5. Show that the vacuum equation \( G_{00} + G_{rr} = 0 \) is equivalent to \( A = \frac{\alpha(t)}{B} \) for some positive smooth function \( \alpha(t) \).

6. Check that if \( A \) and \( B \) are as above then the remaining vacuum equation \( G_{\theta\theta} = G_{\varphi\varphi} = 0 \) is automatically satisfied.

7. Argue that it is always possible to rescale the coordinate \( t \) so that the any metric of the given form satisfying the vacuum Einstein field equations is written
\[ g = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2. \]