Consider again the spherically symmetric Lorentzian metric given by
\[ g = -(A(t, r))^2 dt^2 + (B(t, r))^2 dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2, \]
where \( A \) and \( B \) are positive smooth functions. Recall that in the orthonormal frame dual to
\[ \omega^0 = Adt, \quad \omega^r = Bdr, \quad \omega^\theta = r d\theta, \quad \omega^\varphi = r \sin \theta d\varphi, \]
the Einstein tensor has components (using the notation \( \dot{=}=\frac{\partial}{\partial t} \) and \( \'=\frac{\partial}{\partial r} \))
\[ G_{00} = 2 \frac{B'}{rB^3} + \frac{B^2 - 1}{r^2B^2} \frac{2m'}{r^2}; \]
\[ G_{0r} = G_{r0} = \frac{2B}{rAB^2}; \]
\[ G_{rr} = 2 \frac{A'}{rAB^2} - \frac{B^2 - 1}{r^2B^2}; \]
\[ G_{\theta\theta} = G_{\varphi\varphi} = \frac{A'^2}{rAB^3} + \frac{\dot{A}B - A\dot{B}}{A^2B} + \frac{A'}{rAB^2} - \frac{B'}{rB^3}, \]
where
\[ B(t, r) = \left(1 - \frac{2m(t, r)}{r}\right)^{-\frac{1}{2}}. \]

1. Assuming
\[ \bullet \quad G_{0r} = 0 \text{ (so that } B, \text{ and hence } m, \text{ do not depend on } t); \]
\[ \bullet \quad G_{00} + G_{rr} = 0 \text{ (so that } A = \frac{\alpha(t)}{r} \text{ for some positive smooth function } \alpha(t)); \]
\[ \bullet \quad \alpha(t) = 1 \text{ (which can always be achieved by rescaling } t), \]
show that
\[ G_{\theta\theta} = G_{\varphi\varphi} = \frac{1}{2} (A^2)'' + \frac{1}{r} (A^2)'. \]

2. Prove that the general spherically symmetric solution of the vacuum Einstein field equations with a cosmological constant \( \Lambda \) is the **Kottler metric**
\[ g = - \left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2\right) dt^2 + \left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2. \]

3. Obtain the Penrose diagram for the maximal extension of the Kottler solution with \( \Lambda > 0 \) and \( 0 < M < \frac{1}{3\sqrt{\Lambda}} \).
4. Consider now the spherically symmetric electromagnetic field

\[ F = E(t, r) \omega^r \wedge \omega^0. \]

Show that this field satisfies the vacuum Maxwell equations

\[ dF = d^* F = 0 \]

in and only if

\[ E(t, r) = \frac{e}{r^2}, \]

where the constant \( e \in \mathbb{R} \) is the electric charge in units where \( 4\pi \varepsilon_0 = 1 \). (Here \( * \) is the Hodge star; you will need to use \( * \omega^r \wedge \omega^0 = \omega^\theta \wedge \omega^\phi \).)

5. As we shall see, this electromagnetic field corresponds to the energy-momentum tensor

\[ T = \frac{E^2}{8\pi} \left( \omega^0 \otimes \omega^0 - \omega^r \otimes \omega^r + \omega^\theta \otimes \omega^\theta + \omega^\phi \otimes \omega^\phi \right). \]

Prove that the general spherically symmetric solution of the Einstein field equations with an electromagnetic field of this kind is the \textbf{Reissner-Nordström metric}

\[ g = -\left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \]

6. Obtain the Penrose diagram for the maximal extension of the Reissner-Nordström solution with \( M > 0 \) and \( 0 < e^2 < M^2 \).