Cosmic censorship in spherical symmetry

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Braga, November 2011

Outline

- Solving the spherically symmetric wave equation
- Einstein equations in spherical symmetry
- Schwarzschild solution
- Weak Cosmic Censorship
- Reissner-Nordström solution
- Strong Cosmic Censorship

Solving the spherically symmetric wave equation

- The wave equation in 3 dimensions

$$-\frac{\partial^2\phi}{\partial t^2}+\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0$$

is the "Laplace equation"

$$g^{\mu\nu}\partial_\mu\partial_\nu\phi=0$$

for the Minkowski metric

$$ds^2=-dt^2+dx^2+dy^2+dz^2.$$

- In spherical coordinates

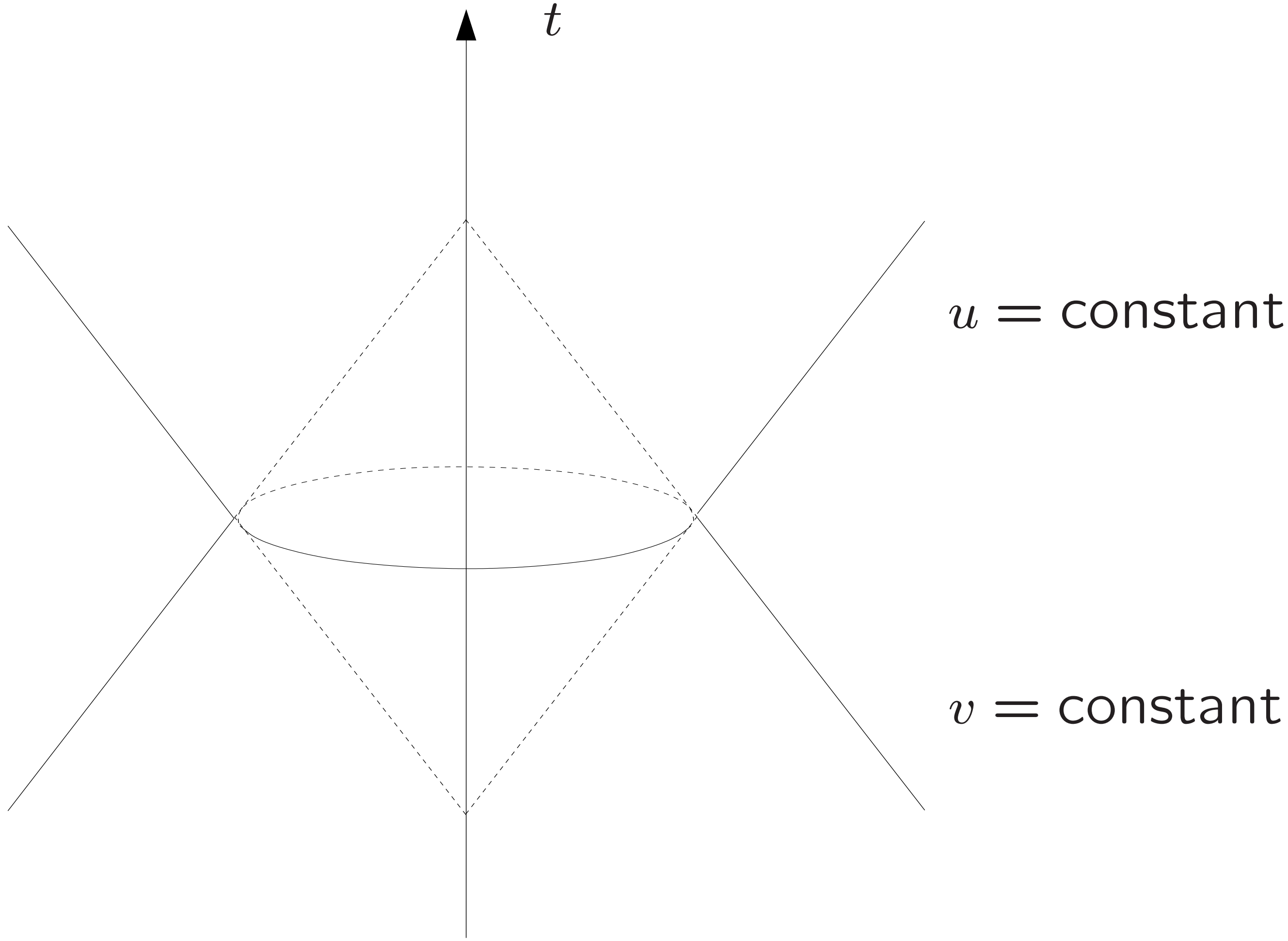
$$ds^2 = -dt^2 + dr^2 + r^2 d\Theta^2.$$

Defining the retarded time u and the advanced time v as

$$u = t - r, \qquad v = t + r,$$

we can write the Minkowski metric as

$$ds^2 = -du dv + r^2 d\Theta^2.$$


t
u = constant
v = constant

- The wave equation

$$\partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \phi \right) = 0$$

for a spherically symmetric function

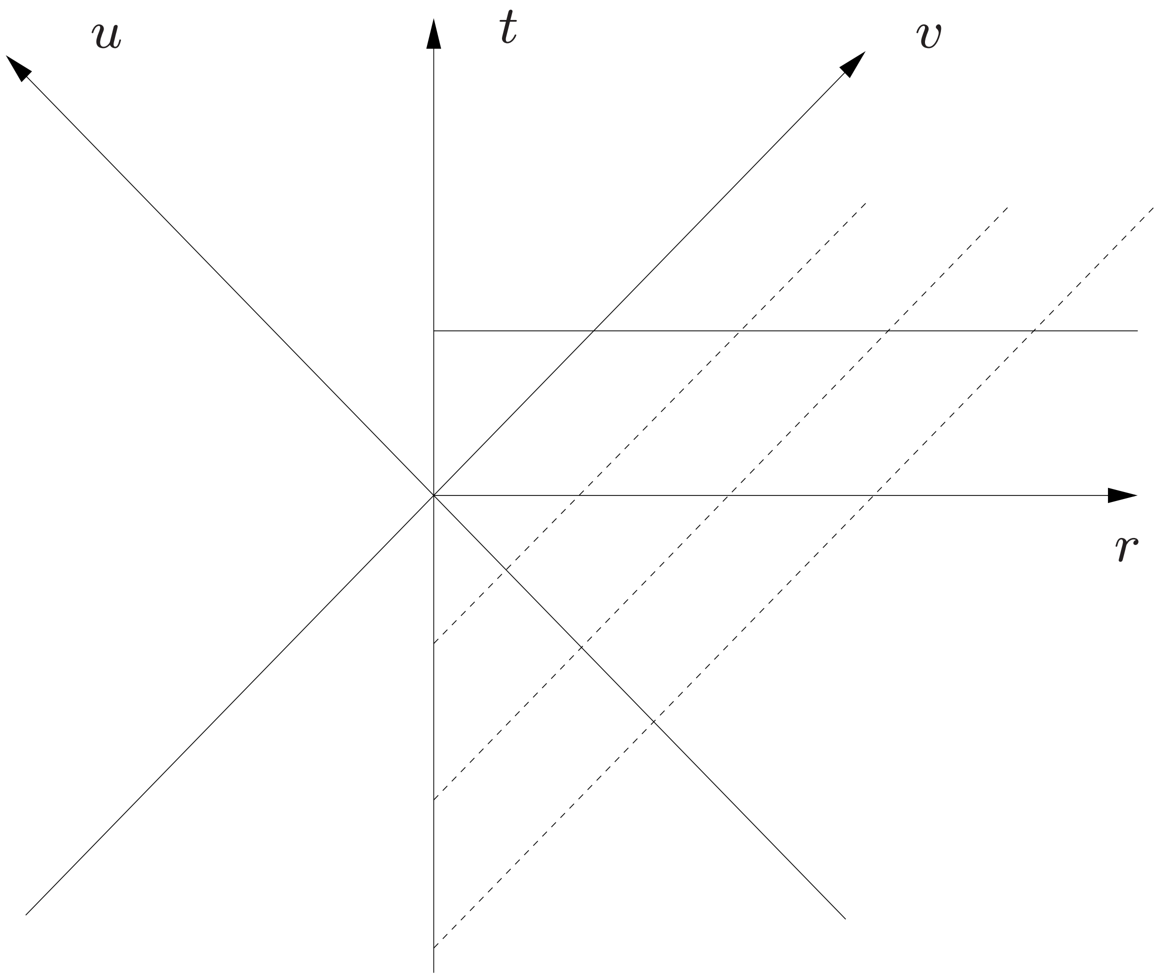
$$\phi = \phi(t, r) = \phi(u, v)$$

then becomes

$$\partial_u \partial_v (r\phi) = 0,$$

and so

$$r\phi = F(u) + G(v).$$



u
t
v
r

- Setting $\tilde{u} = \tanh u$ and $\tilde{v} = \tanh v$ we can write the Minkowski metric as

$$ds^2 = -\Omega^2 d\tilde{u} d\tilde{v} + r^2 d\Theta^2.$$

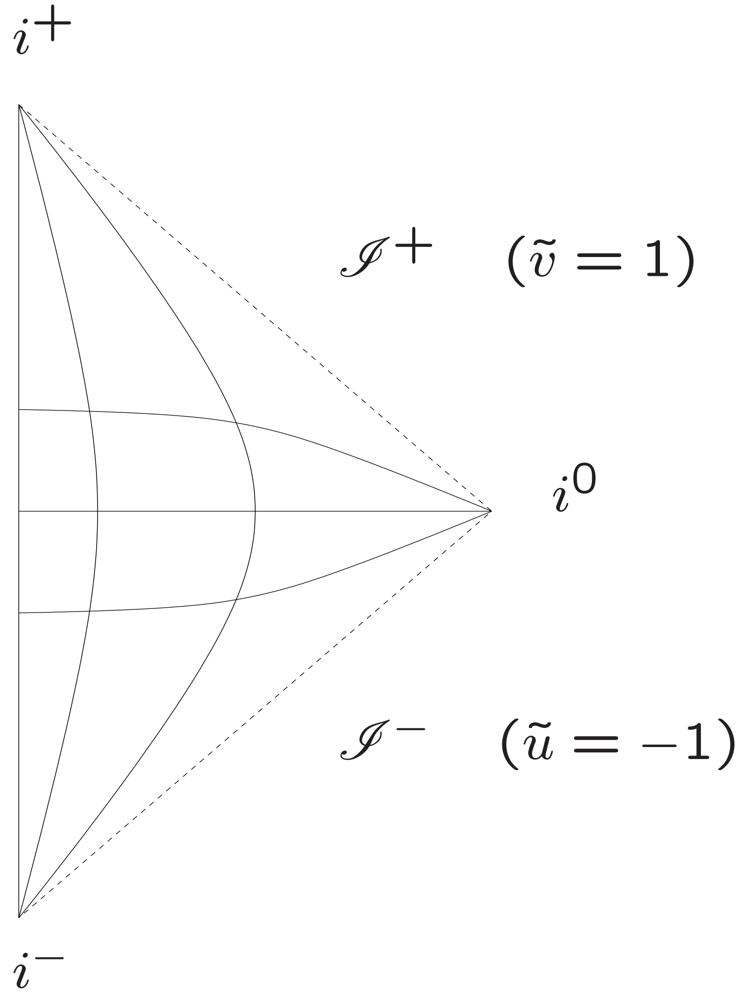
Wave equation remains

$$\partial_{\tilde{u}} \partial_{\tilde{v}} (r\phi) = 0.$$

The domain

$$D = \{(\tilde{u}, \tilde{v}) \in \mathbb{R}^3 \mid -1 < \tilde{u}, \tilde{v} < 1; \tilde{u} \leq \tilde{v}\}$$

is called the Penrose diagram for the Minkowski spacetime.


i^+
$\mathscr{I}^+$ $(\tilde{v} = 1)$
i^0
$\mathscr{I}^-$ $(\tilde{u} = -1)$
i^-

Einstein equations in spherical symmetry

- General relativity: space-time is a (curved) 4-dimensional Lorentzian manifold (M, g); matter is represented by fields $\phi : M \to \mathbb{R}$ satisfying (hyperbolic) evolution equations; the metric satisfies the Einstein field equations

$$Ric(g) - \frac{1}{2}\mathrm{tr}(Ric(g))g + \Lambda g = 8\pi T(\phi, g).$$

- Spherical symmetry: in this case the metric is

$$ds^2 = -\Omega^2(u,v)dudv + r^2(u,v)d\Theta^2$$

- Example: massless scalar field with zero cosmological constant. Introducing the Hawking mass

$$m = \frac{r}{2}(1 - g(dr, dr)) = \frac{r}{2}(1 + 4\Omega^{-2}\partial_u r \partial_v r)$$

we have

$$\begin{aligned}
&\partial_u \partial_v (r\phi) = (\partial_u \partial_v r)\phi; \\
&\partial_u (\Omega^{-2}\partial_u r) = -4\pi r \Omega^{-2}(\partial_u \phi)^2; \\
&\partial_v (\Omega^{-2}\partial_v r) = -4\pi r \Omega^{-2}(\partial_v \phi)^2; \\
&\partial_u m = -8\pi r^2 \Omega^{-2}(\partial_u \phi)^2 \partial_v r; \\
&\partial_v m = -8\pi r^2 \Omega^{-2}(\partial_v \phi)^2 \partial_u r.
\end{aligned}$$

These are essentially equivalent to the nonlinear wave equations

$$\partial_u \partial_v (r\phi) = -\frac{m\Omega^2}{2r^2}\phi;$$
$$\partial_u \partial_v r = -\frac{m\Omega^2}{2r^2};$$
$$\partial_u \partial_v m = -16\pi r \Omega^{-2}\Big(2\pi r^2 (\partial_u \phi)^2 (\partial_v \phi)^2 + \partial_u r \partial_v r \partial_u \phi \partial_v \phi\Big)$$

with

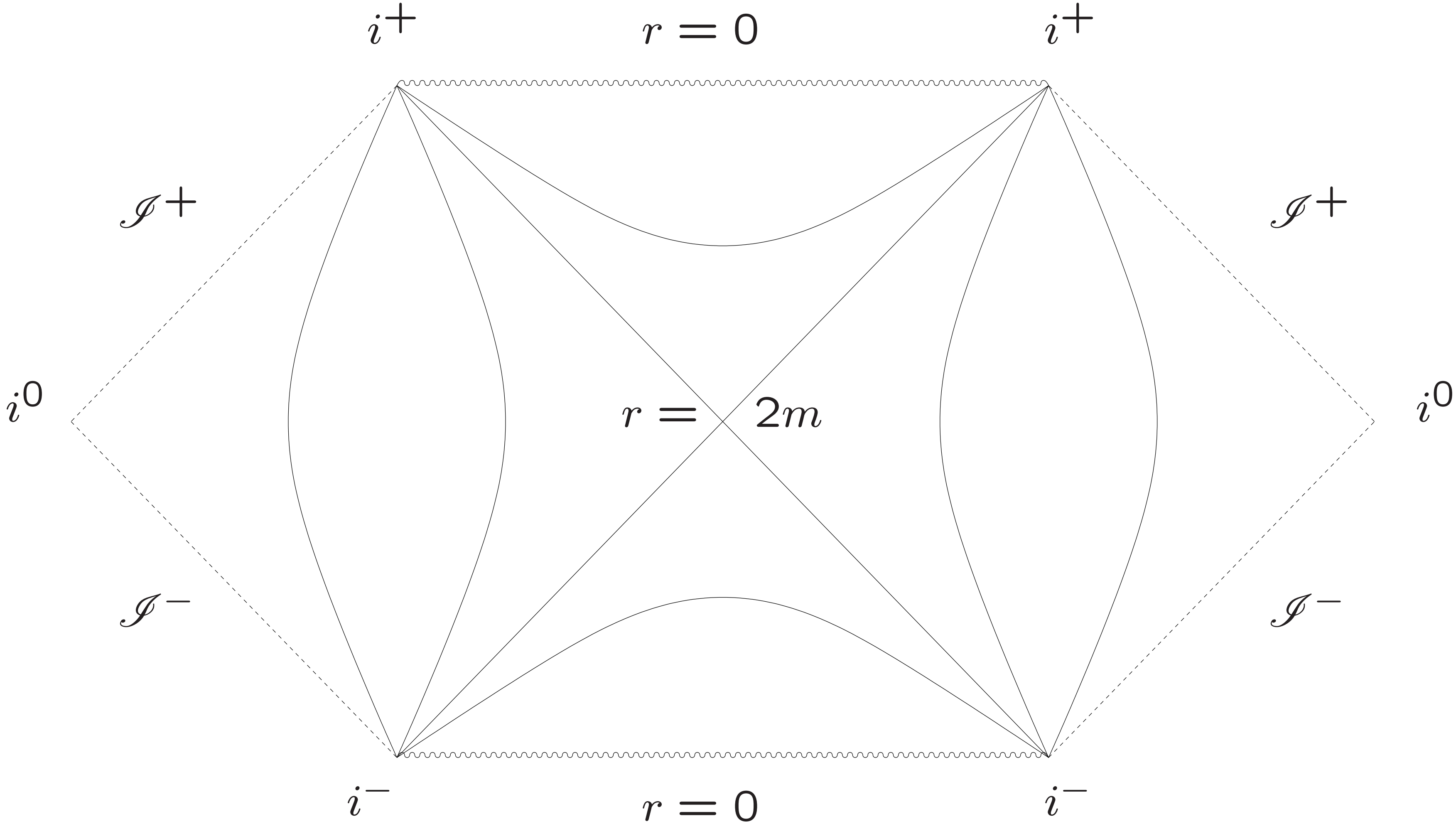
$$\Omega^2 = -\frac{4\partial_u r \partial_v r}{1 - \frac{2m}{r}}.$$

Schwarzschild solution

- Vacuum, zero cosmological constant.

- The metric is written
$$ds^2 = -\left(1 - \frac{2m}{r}\right) dudv + r^2 d\Theta^2$$
with m constant.

- Extending this through $r = 2m$ (event horizon) we obtain the full Penrose diagram.

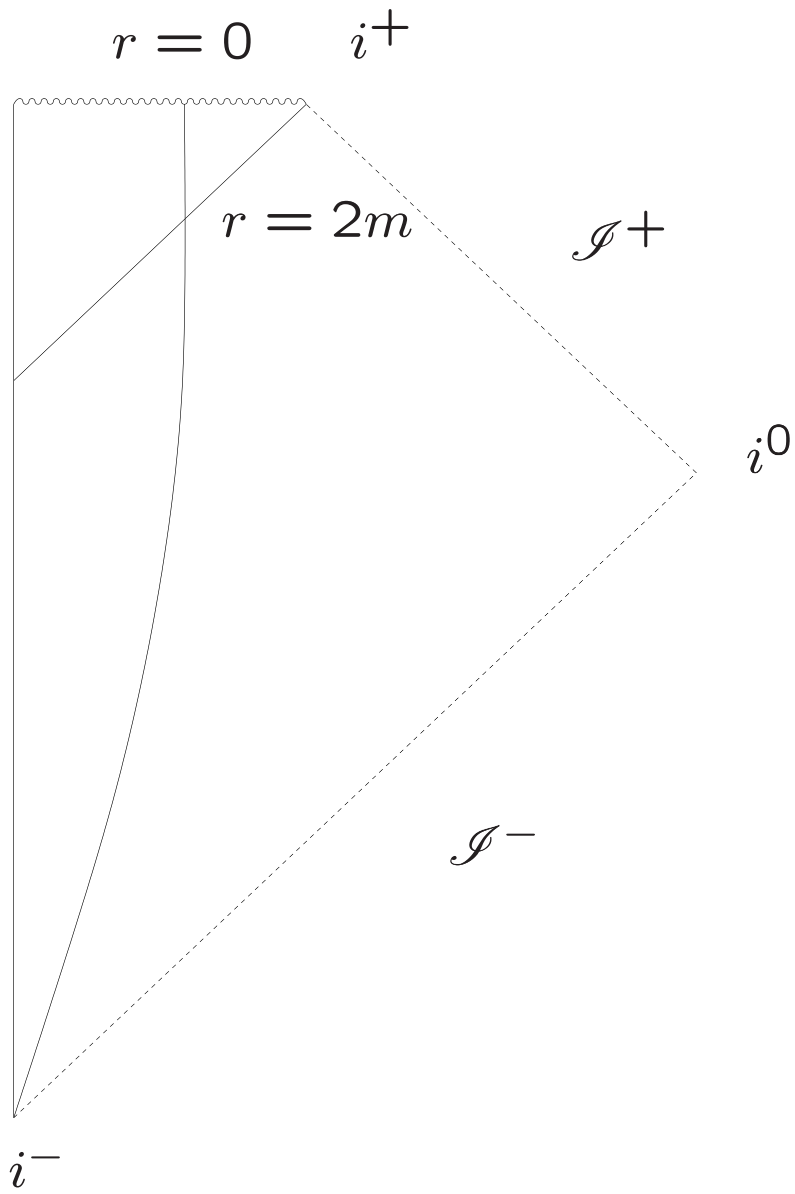


i^+
$r = 0$
i^+
$\mathscr{I}^+$
$\mathscr{I}^+$
i^0
$r = 2m$
i^0
$\mathscr{I}^-$
$\mathscr{I}^-$
i^-
$r = 0$
i^-

- $r = 0$ is a singularity, where the equations break down (and the curvature explodes).

- $r = 2m$ is the event horizon. It is the boundary of the black hole region, which contains the singularity and is not visible from the outer region.

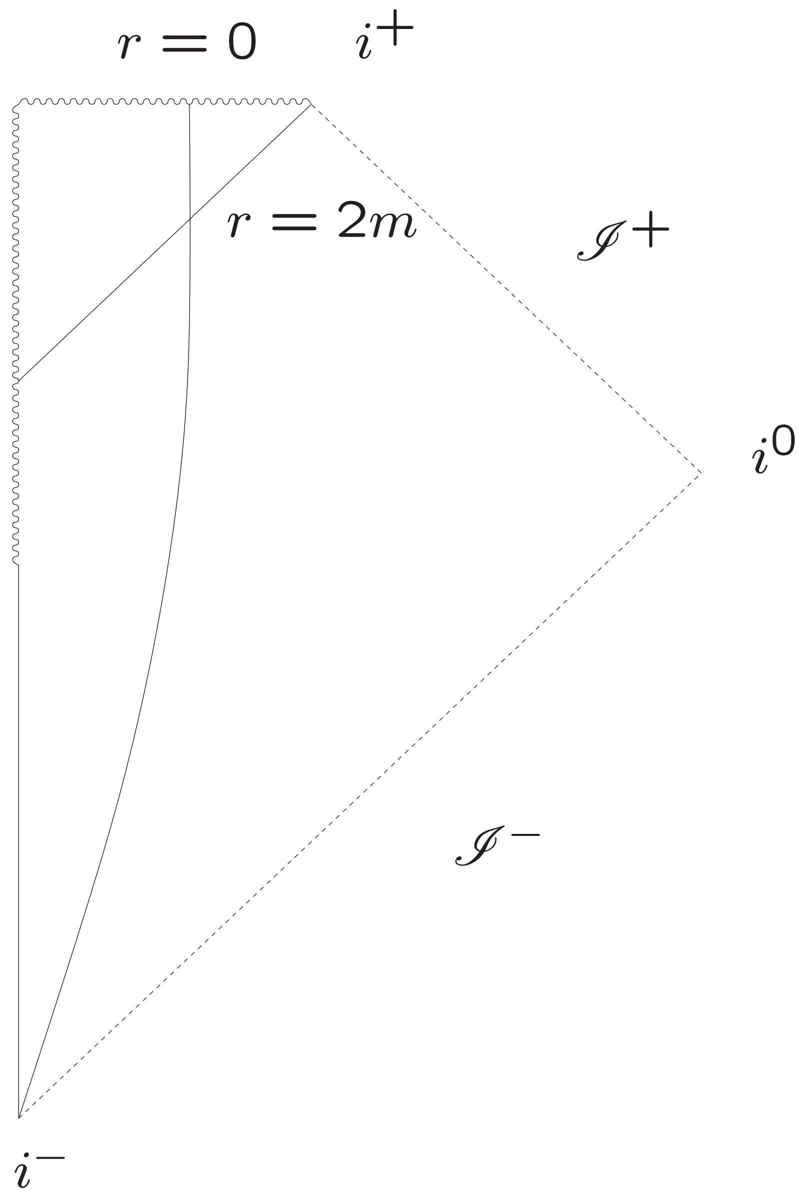
- Physically one expects a black hole to form from the collapse of, say, a homogeneous ball of dust (Oppenheimer-Snyder model).



$r = 0$
i^+
$r = 2m$
$\mathscr{I}^+$
i^0
$\mathscr{I}^-$
i^-

- Collapse of non-homogeneous balls of dust (Tolman-Bondi models) lead to naked singularities (that is, singularities which are visible from the outer region).

- Matter model (dust) leads to singularities even in flat Minkowski spacetime.

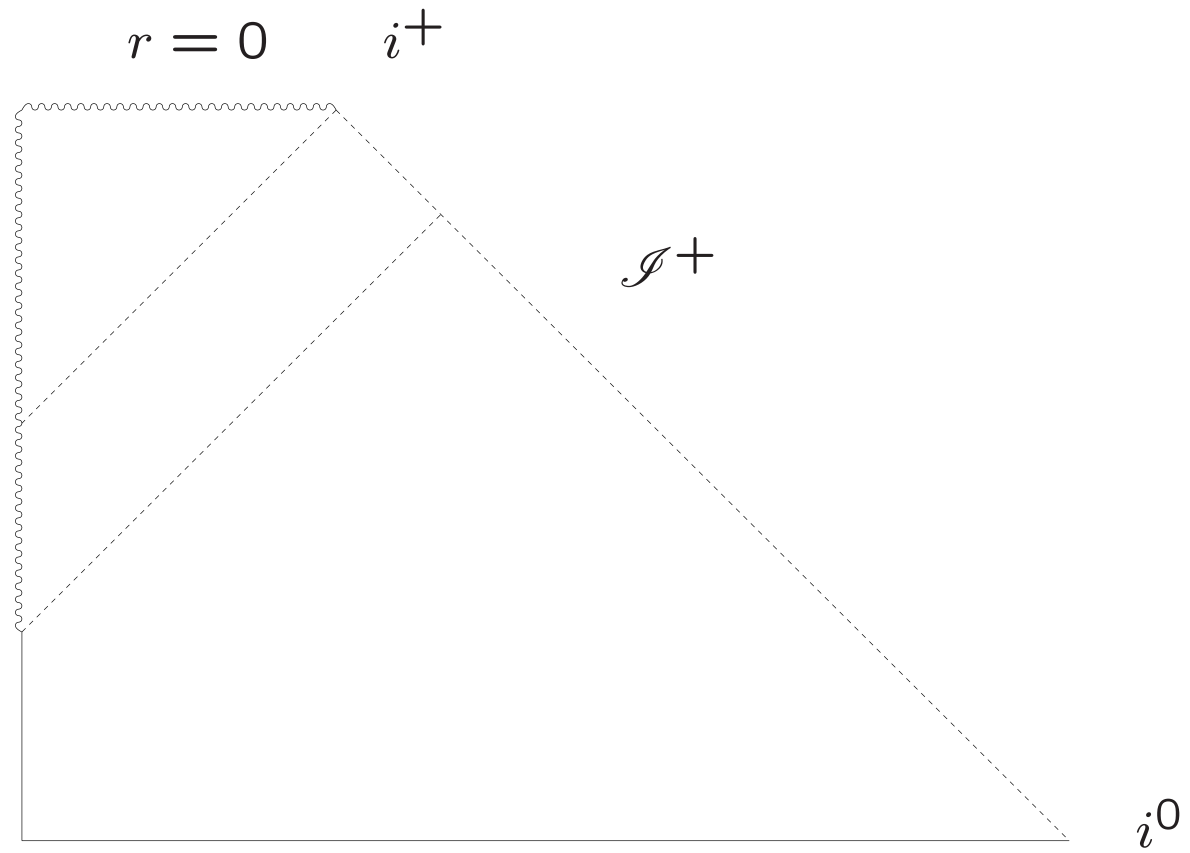


$r = 0$
i^+
$r = 2m$
$\mathscr{I}^+$
i^0
$\mathscr{I}^-$
i^-

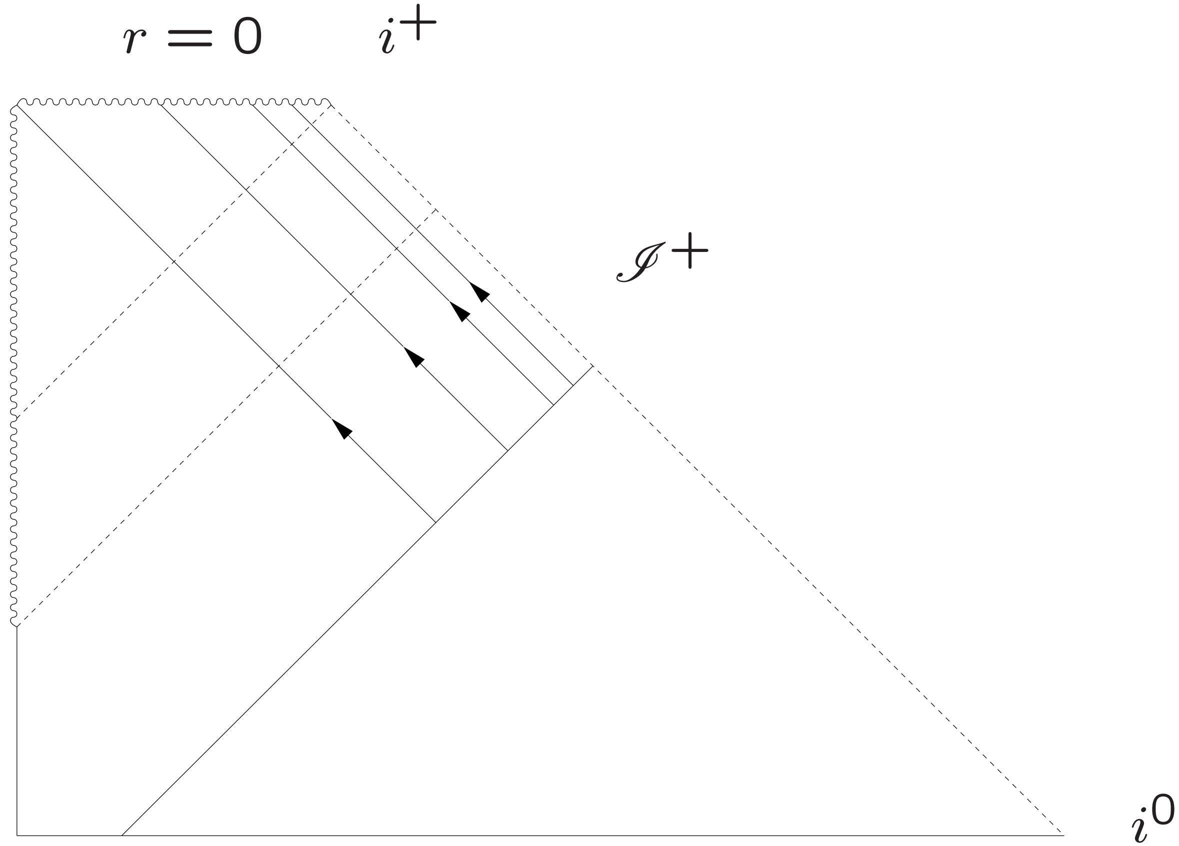
Weak Cosmic Censorship

- For generic solutions of the Einstein field equations with reasonable matter models observers at infinity cannot see the singularity.

- PDE version: For generic initial data the maximal globally hyperbolic development has a complete $\mathscr{I}^+$.



$r = 0$
i^+
$\mathscr{I}^+$
i^0

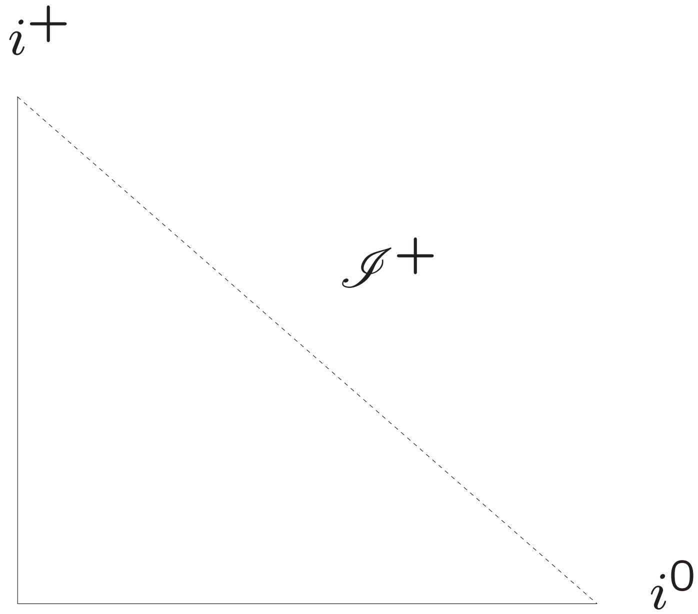


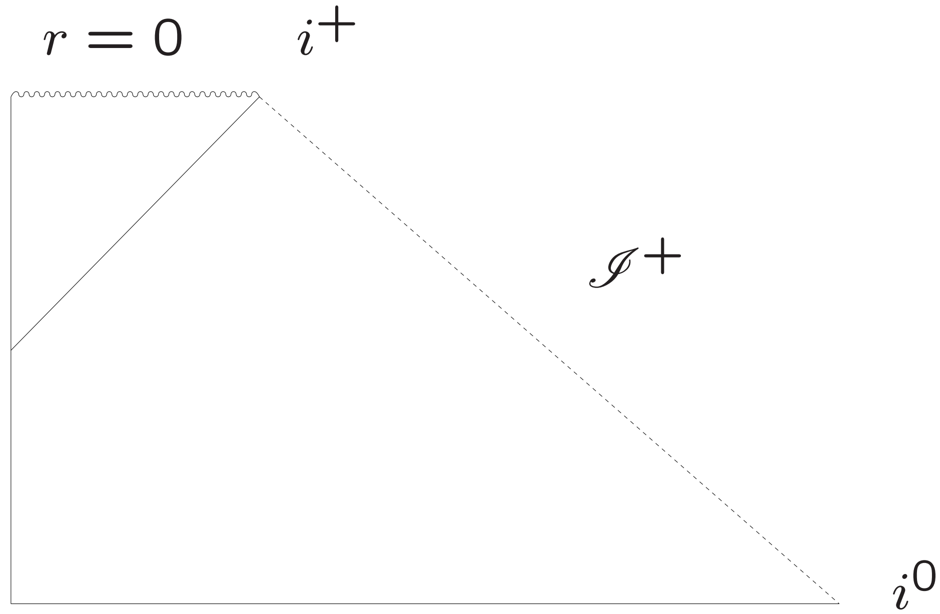
$r = 0$
i^+
$\mathscr{I}^+$
i^0

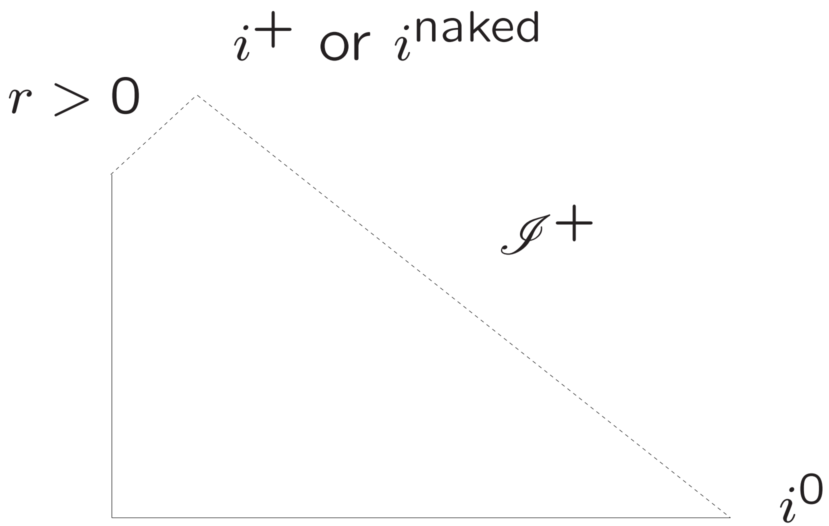
- Christodoulou (1999): Weak cosmic censorship is true for the Einstein-scalar field system.

- Generic: dispersive or black hole.

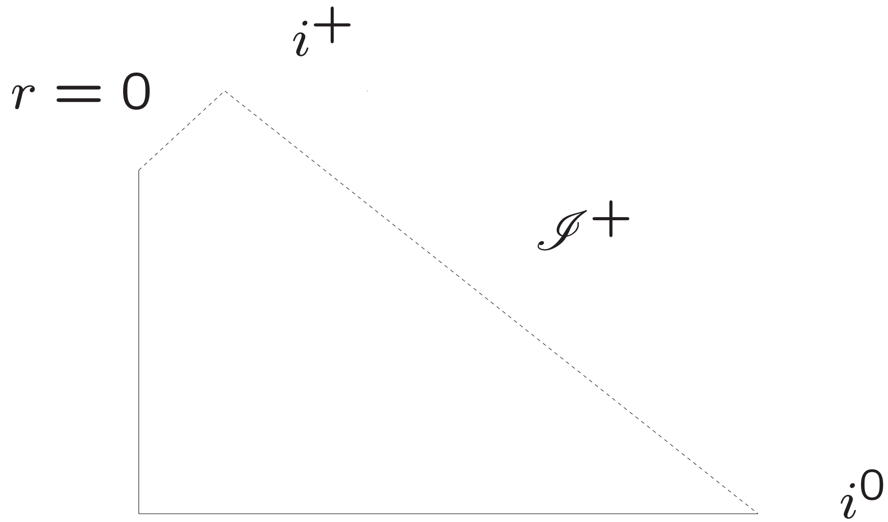
- Non-generic: light cone singularity (possibly naked), collapsed light cone singularity, black hole with light cone singularity, black hole with collapsed light cone singularity.


i^+
$\mathscr{I}^+$
i^0

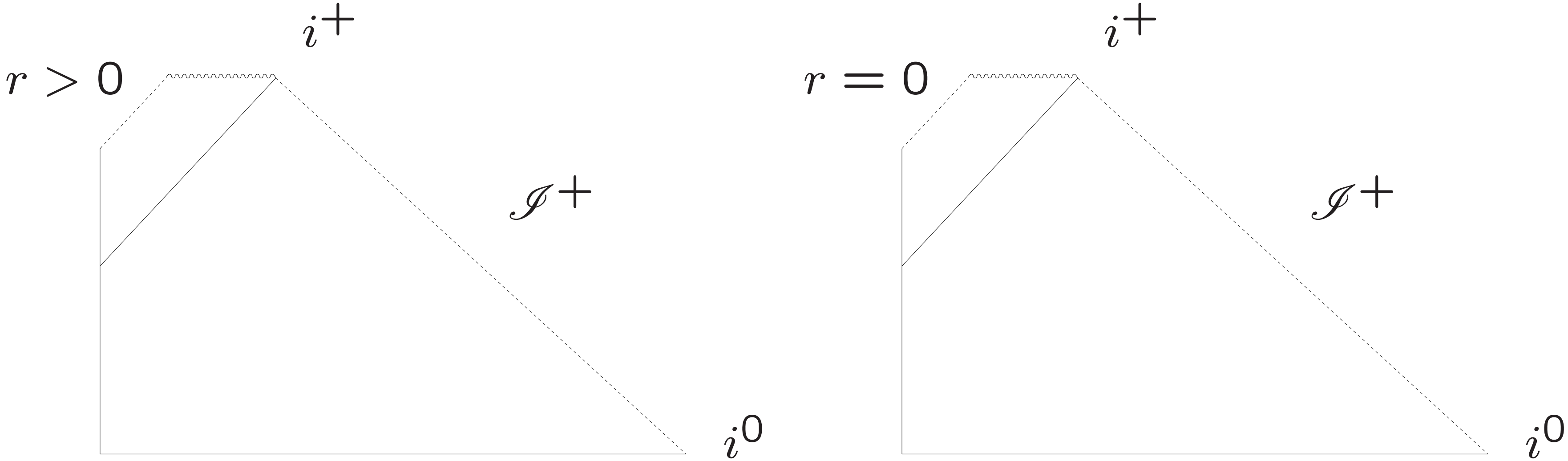

$r = 0$
i^+
$\mathscr{I}^+$
i^0



i^+ or i^{naked}
$r > 0$
$\mathscr{I}^+$
i^0



i^+
$r = 0$
$\mathscr{I}^+$
i^0


i^+
$r > 0$
$\mathscr{I}^+$
i^0
i^+
$r = 0$
$\mathscr{I}^+$
i^0

Reissner-Nordström solution

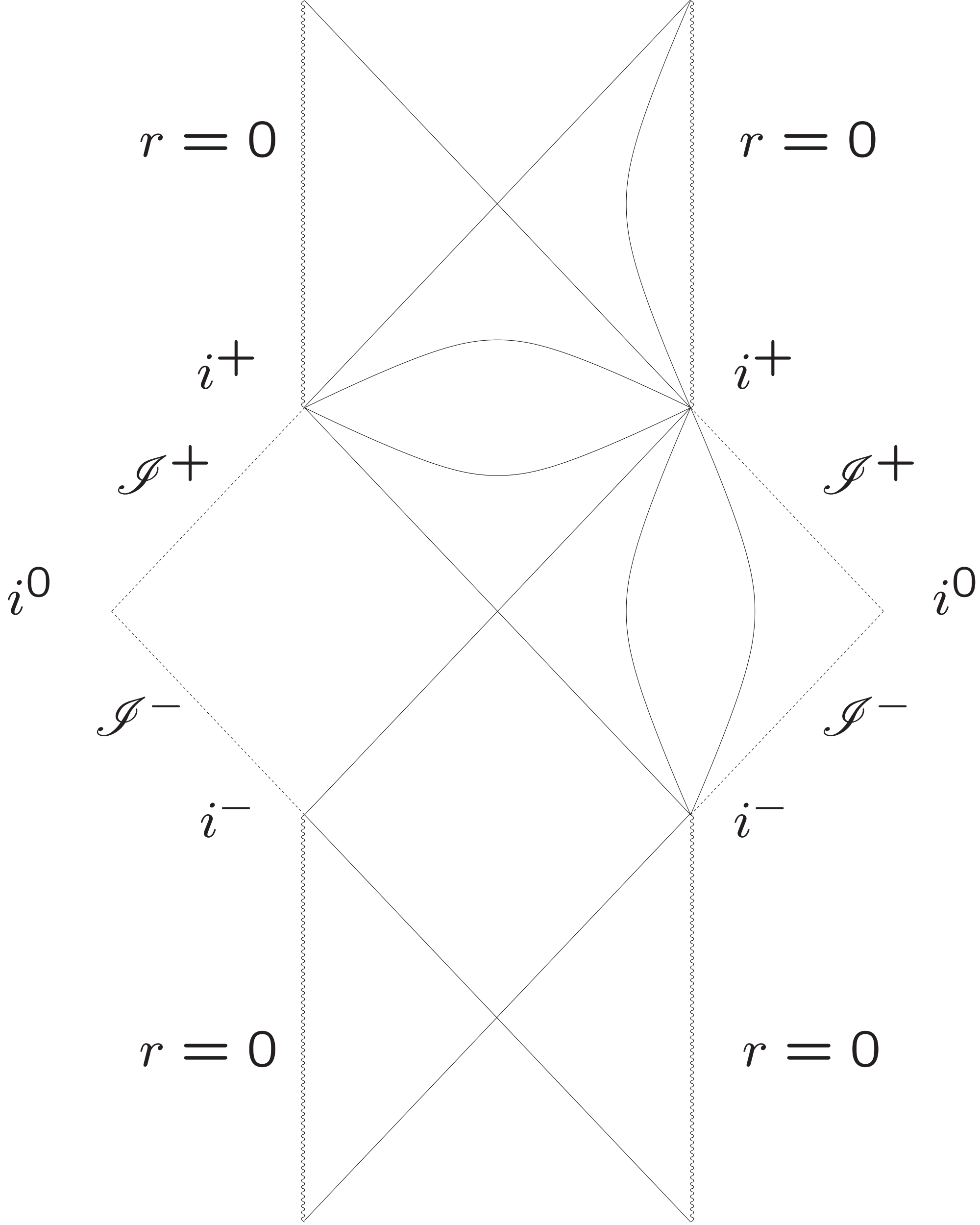
- Vacuum, nonzero electromagnetic field, zero cosmological constant.

- The metric is written

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right) dudv + r^2 d\Theta^2$$

with m and e constant.

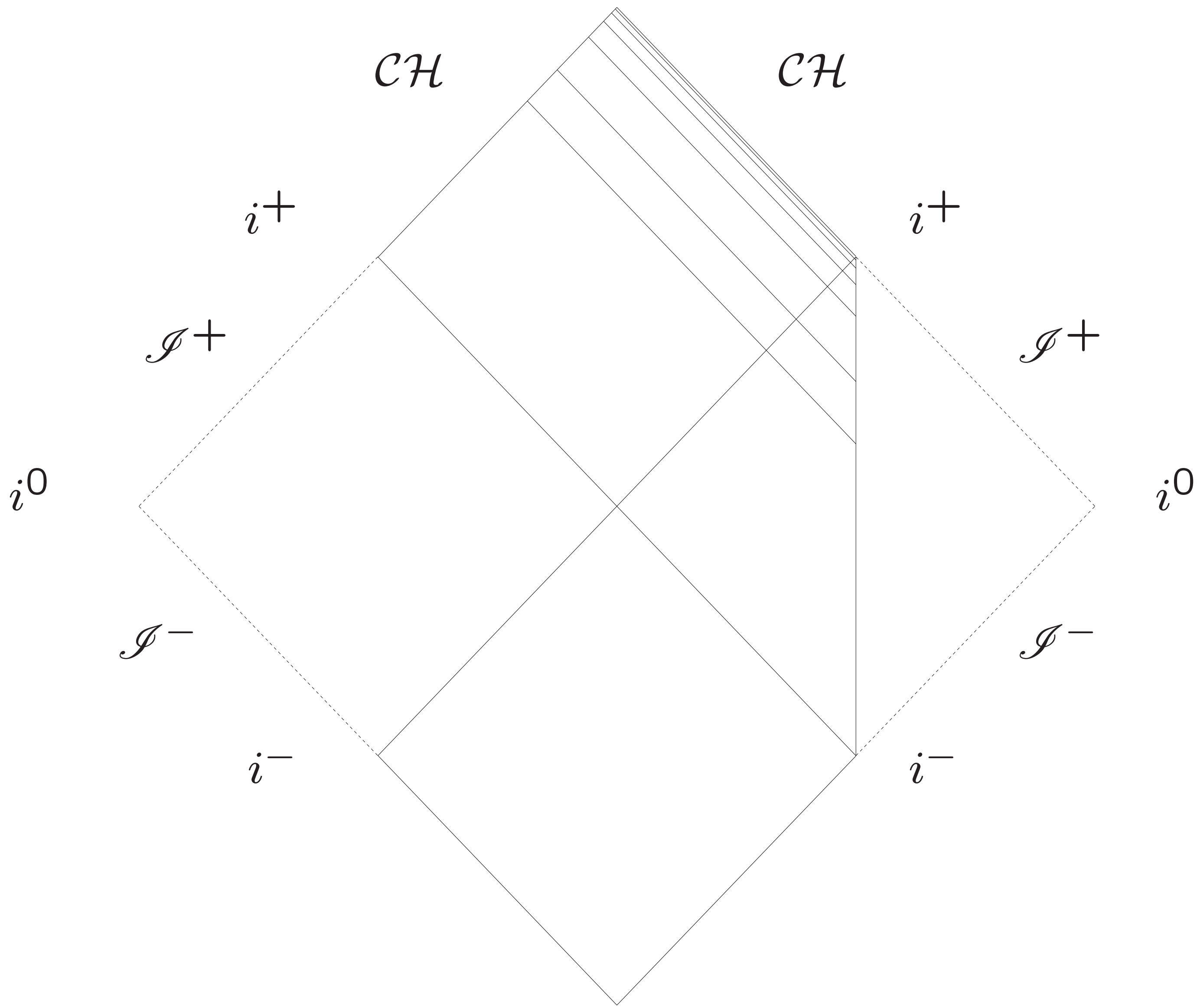
- Extending this through $r = m + \sqrt{m^2 - e^2}$ (event horizon) and $r = m - \sqrt{m^2 - e^2}$ (Cauchy horizon) we obtain the full Penrose diagram.


$r = 0$
$r = 0$
i^+
i^+
$\mathscr{I}^+$
$\mathscr{I}^+$
i^0
i^0
$\mathscr{I}^-$
$\mathscr{I}^-$
i^-
i^-
$r = 0$
$r = 0$

- $r = 0$ is a singularity, where the equations break down (and the curvature explodes).

- $r = m + \sqrt{m^2 - e^2}$ is the event horizon. It is the boundary of the black hole region, which contains the singularity and is not visible from the outer region.

- $r = m + \sqrt{m^2 - e^2}$ is the Cauchy horizon. Observers crossing this horizon can see the singularity. However, the blueshift effect should make this horizon unstable.



$\mathcal{CH}$
$\mathcal{CH}$
i^+
i^+
$\mathscr{I}^+$
$\mathscr{I}^+$
i^0
i^0
$\mathscr{I}^-$
$\mathscr{I}^-$
i^-
i^-

Strong Cosmic Censorship

- For generic solutions of the Einstein field equations with reasonable matter models no observer can see the singularity.

- PDE version: For generic initial data the maximal globally hyperbolic development is inextendible.

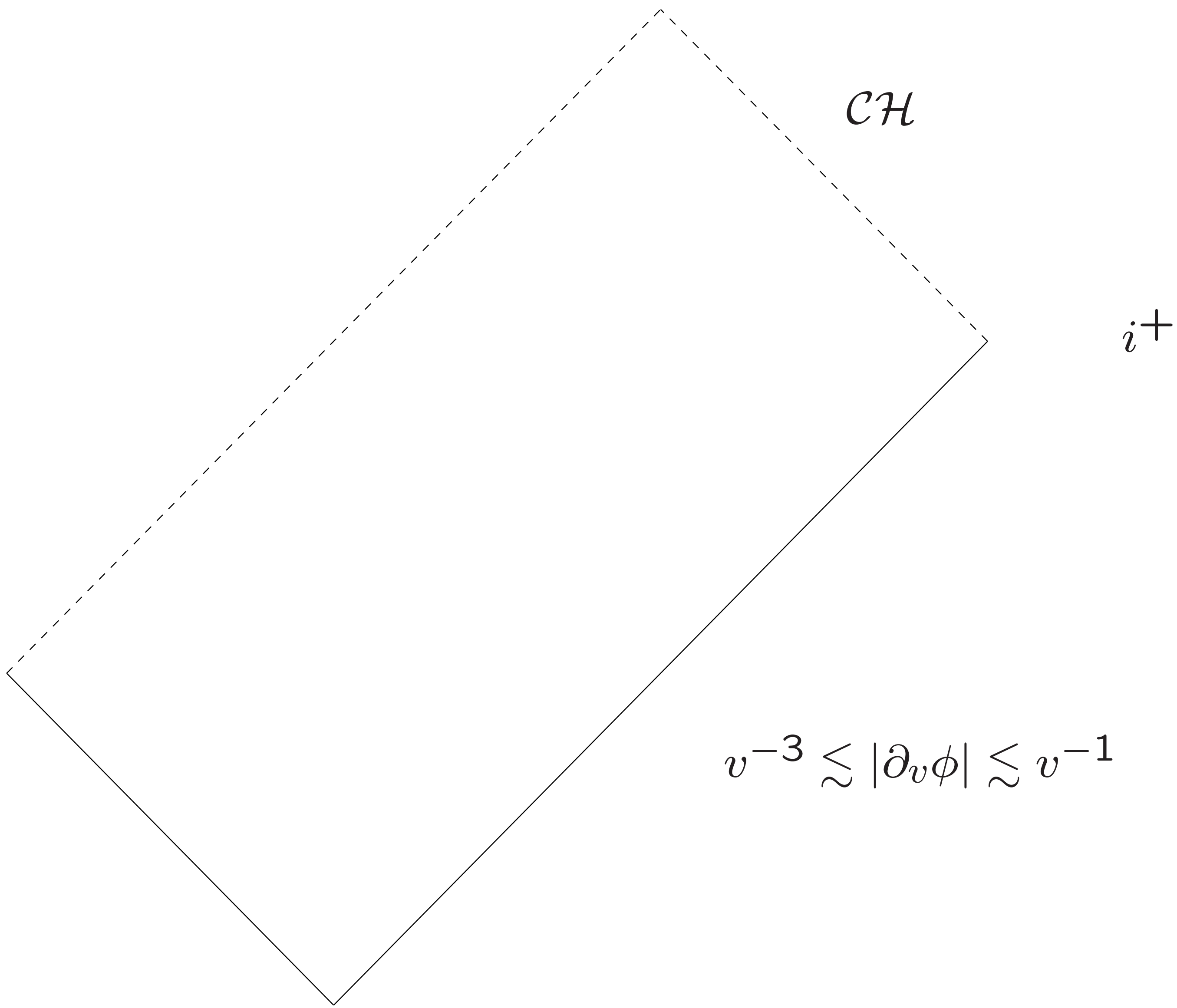
- PDE versions of weak and strong cosmic censorship conjectures are logically independent.

- Dafermos (2005) proved the following two results for the spherically symmetric Einstein-Maxwell-scalar field system.

1. If $v^{-3} \lesssim |\partial_v \phi| \lesssim v^{-1}$ along the event horizon then the Hawking mass blows up identically along the Cauchy horizon, and so the metric is inextendible as a C^1 metric.

2. If $|\partial_v \phi| \lesssim v^{-1}$ along the event horizon then r can be extended to a nonvanishing continuous function on the Cauchy horizon, and so the metric can be extended as a C^0 metric.

- The second hypothesis (Price law) was subsequently proved to occur by Dafermos and Rodnianski (2005).


$\mathcal{CH}$
i^+
$v^{-3} \lesssim |\partial_v \phi| \lesssim v^{-1}$

Bibliography

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