The Hidden Geometry

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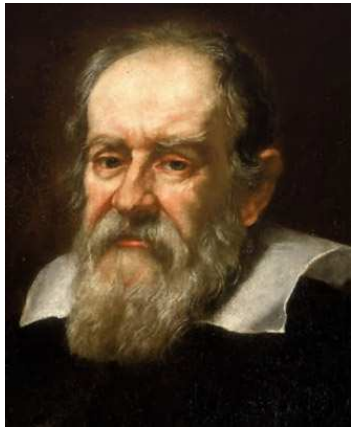
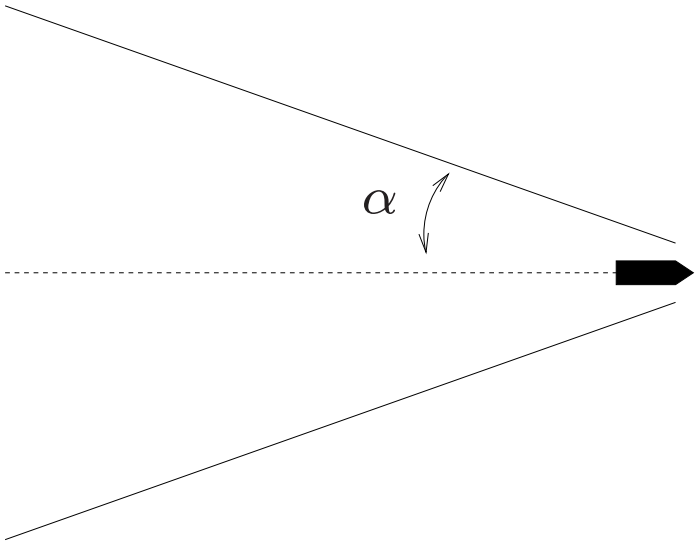
Galileu Galilei (1564–1642)

"Philosophy is written in this grand book – I mean the universe – which stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures, without which it is humanly impossible to understand a single word of it."



Manneken Pis (Brussels).

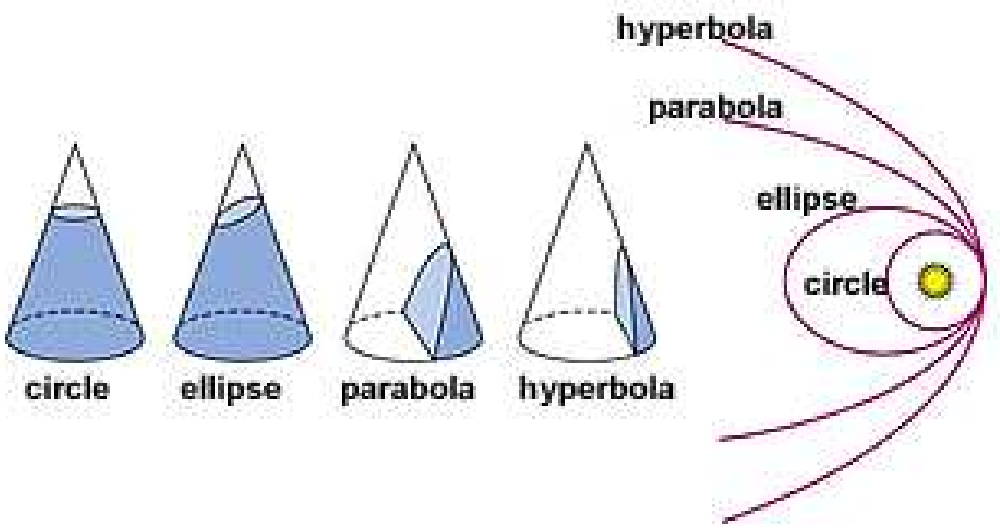
Kelvin's Theorem (1824–1907)



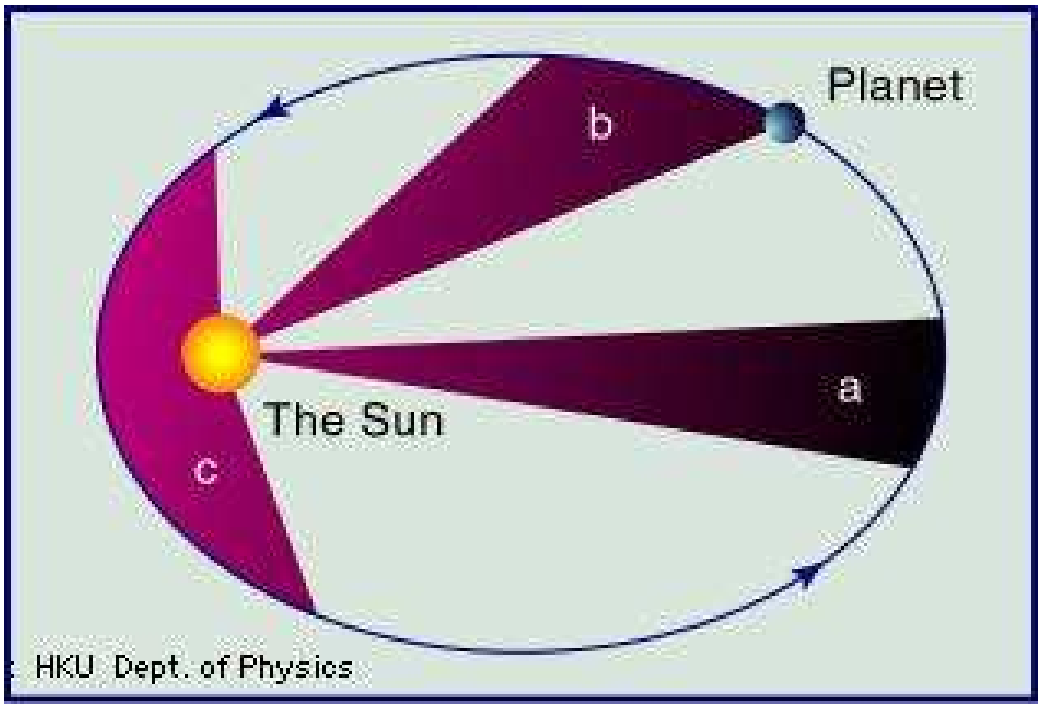
$\sin\alpha = \frac{1}{3}$ (so $\alpha \simeq 19,5^{\circ}$).

Kepler Laws (1571–1630)

1. The orbits of the heavenly bodies are conic sections with the Sun at one of the foci.

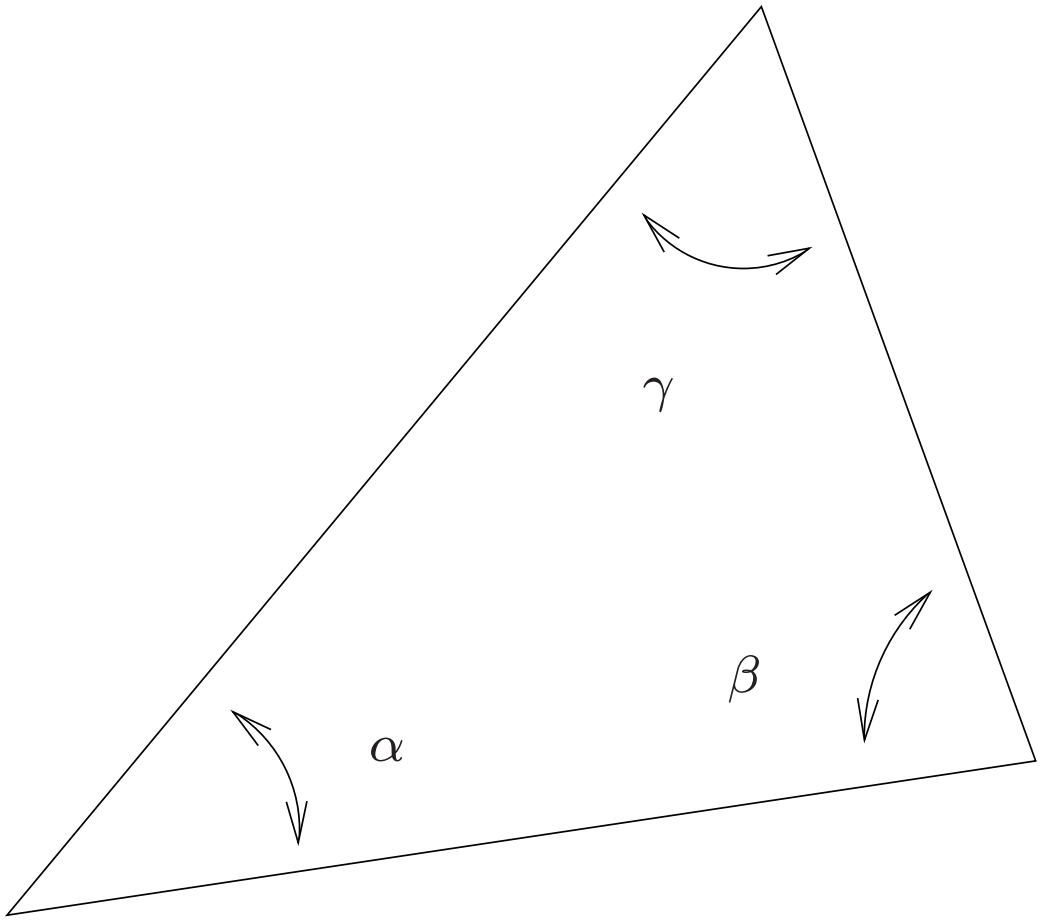


2. The line joining the Sun to the body sweeps equal areas in equal times.



Euclidean geometry ($4^{th} - 3^{rd}$ century b.C.)

- Two distinct lines intersect at most once.
- There are lines which do not intersect (parallel).
- The internal angles of a triangle add up to 180^o.

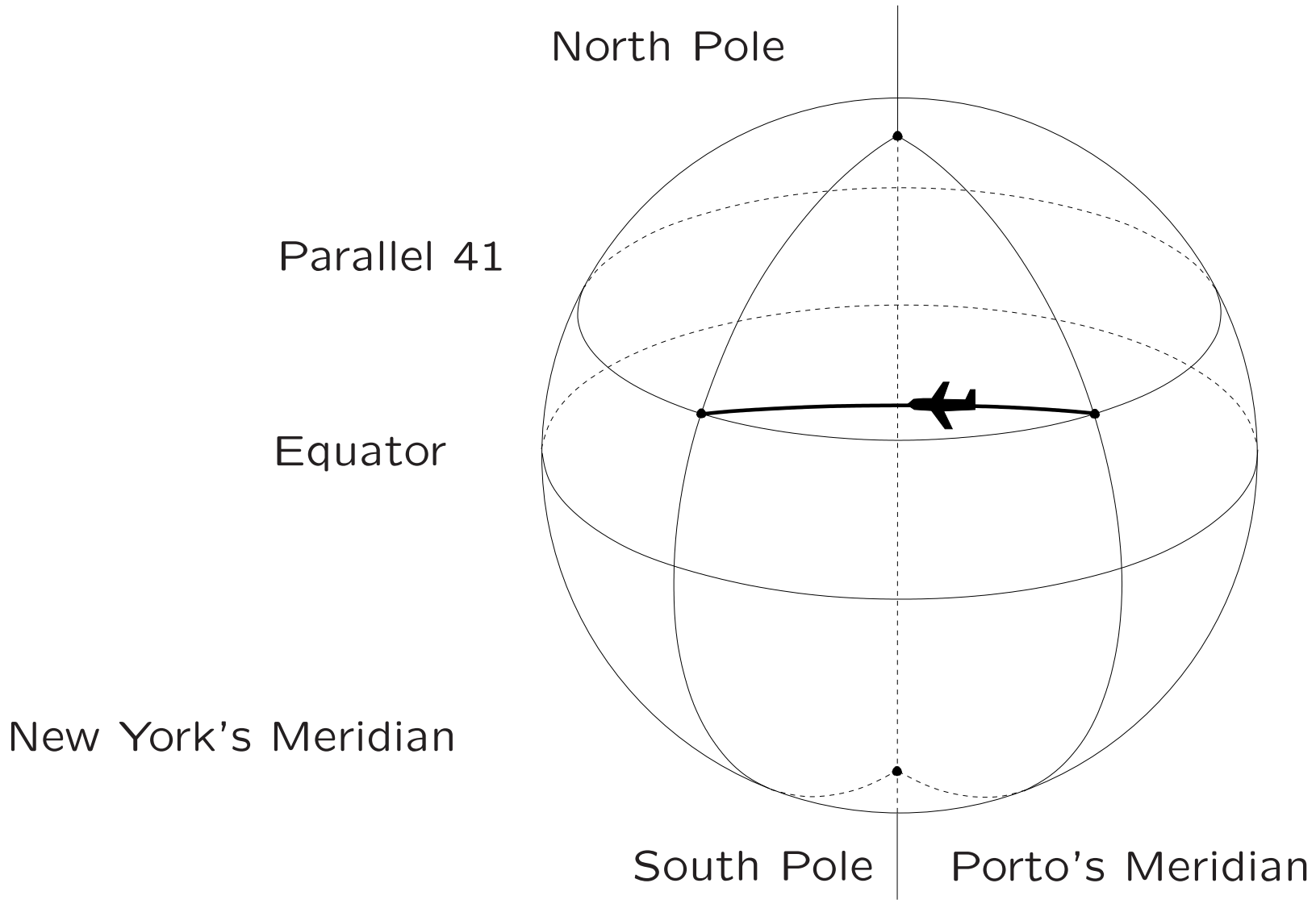


$$\alpha + \beta + \gamma = 180^{\circ}$$

Riemannian geometry (1826–1866)

It is the geometry of curved surfaces (spaces). Instead of lines we have geodesics (curves of minimum length).

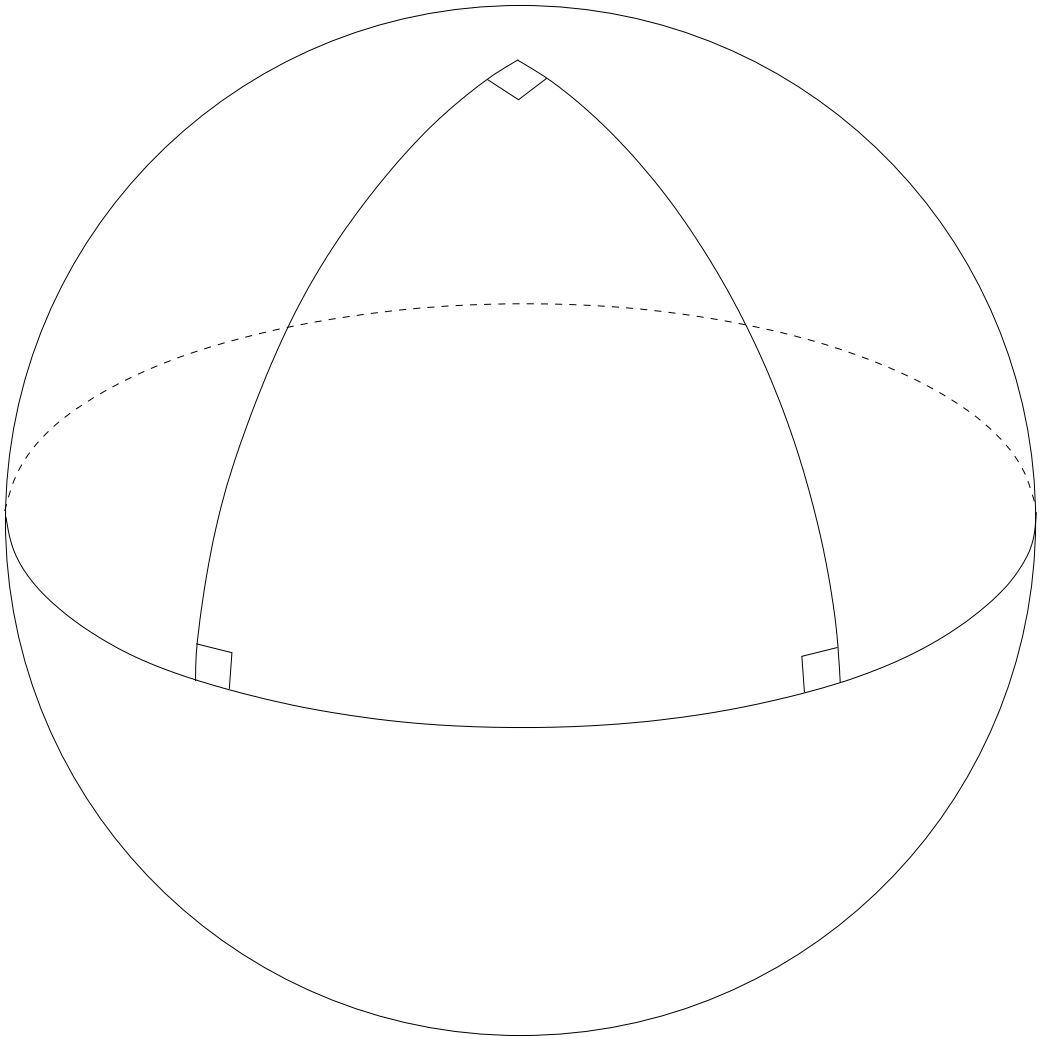
In the sphere, for instance, the geodesics are great circles, as the equator or the meridians. That is why to go from Porto to New York the plane does not fly westwards.


North Pole
Parallel 41
Equator
New York's Meridian
South Pole
Porto's Meridian

Sphere's geometry

- Two distinct geodesics intersect at exactly two points (there are no parallel geodesics).

- The internal angles of a triangle add up to more than 180^o.

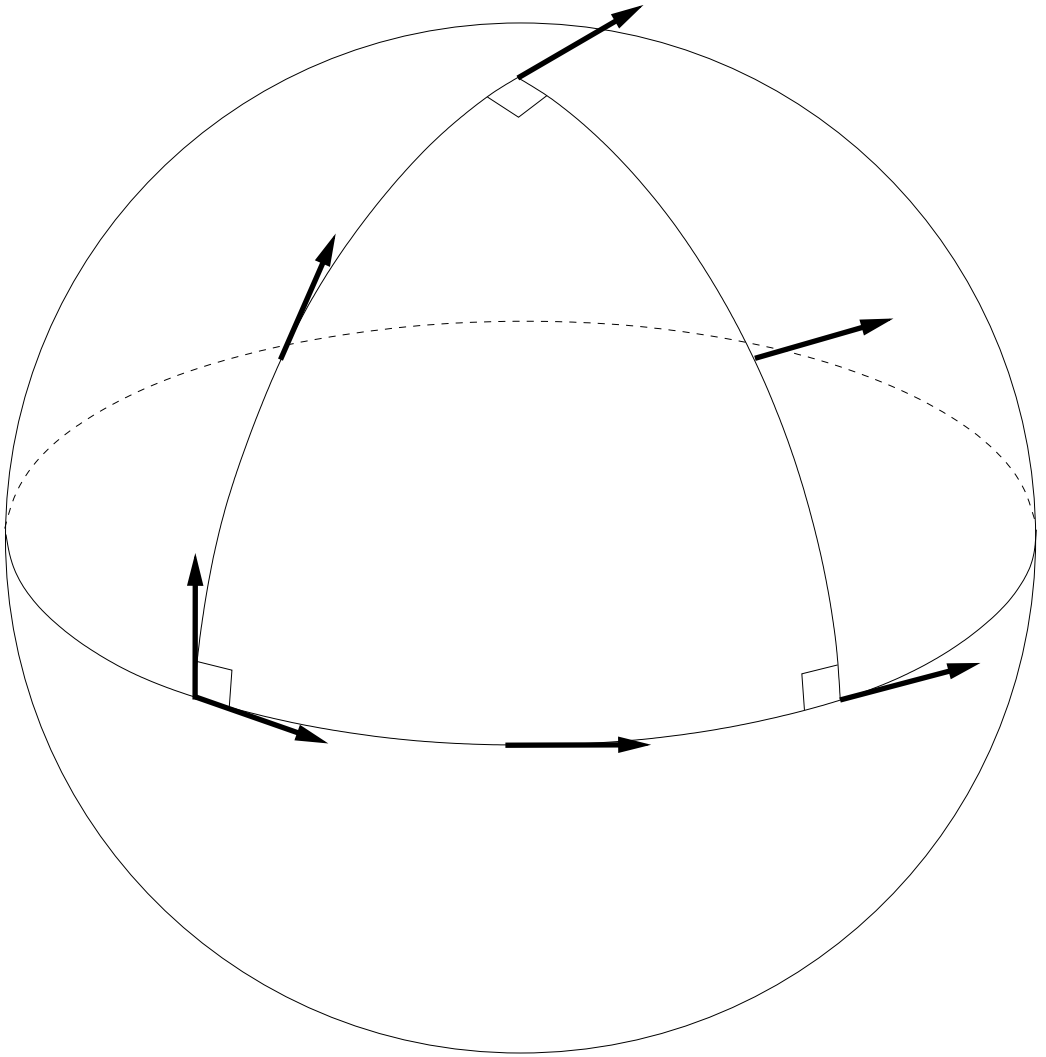


- Average curvature $= \dfrac{\text{Excess angle}}{\text{Triangle's area}} = \dfrac{\frac{\pi}{2}}{\frac{4\pi R^2}{8}} = \dfrac{1}{R^2}$.

- In the sphere all triangles have the same average curvature (constant curvature surface). In general, the curvature of a surface at a point is the limit of the average curvature of increasingly smaller triangles.

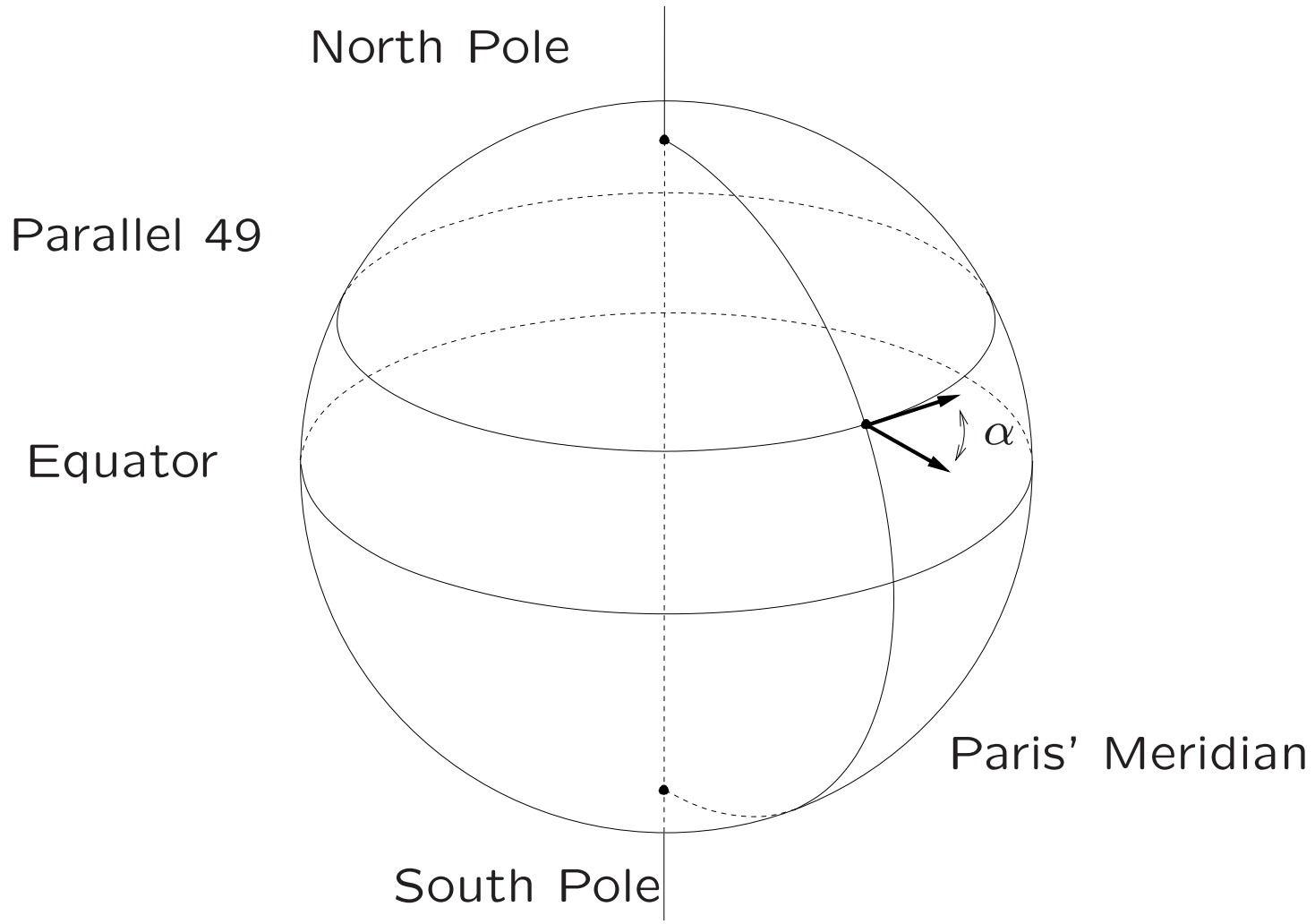
Parallel transport

A tangent vector which is parallel-transported along a closed curve returns to the initial point rotated by an angle equal to the excess.



That is what happens with the Foucault pendulum:

Foucault pendulum at the Panthéon in Paris.


North Pole
Parallel 49
Equator
α
Paris' Meridian
South Pole

At latitude λ, Foucault's pendulum rotates

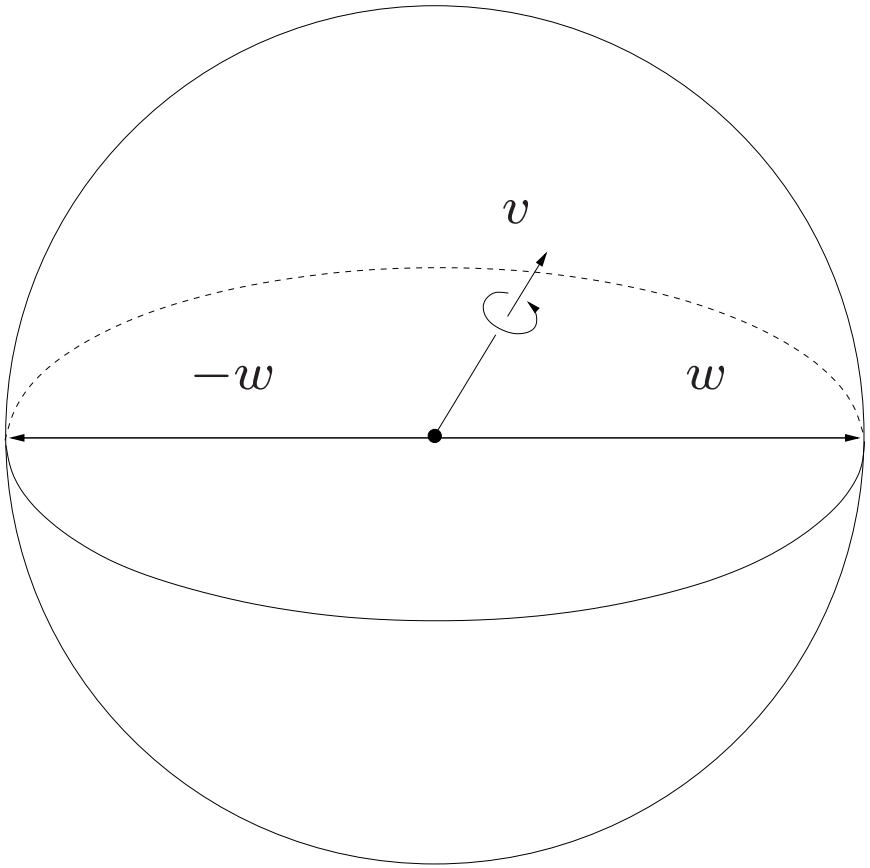
$$\alpha = 2\pi - \overbrace{\underbrace{2\pi R(R - R\,\mathrm{sen}\,\lambda)}_{\text{area}}\ \underbrace{\frac{1}{R^2}}_{\text{curvature}}}^{\text{excess}} = 2\pi\,\mathrm{sen}\,\lambda$$

radians per day.

Rigid body

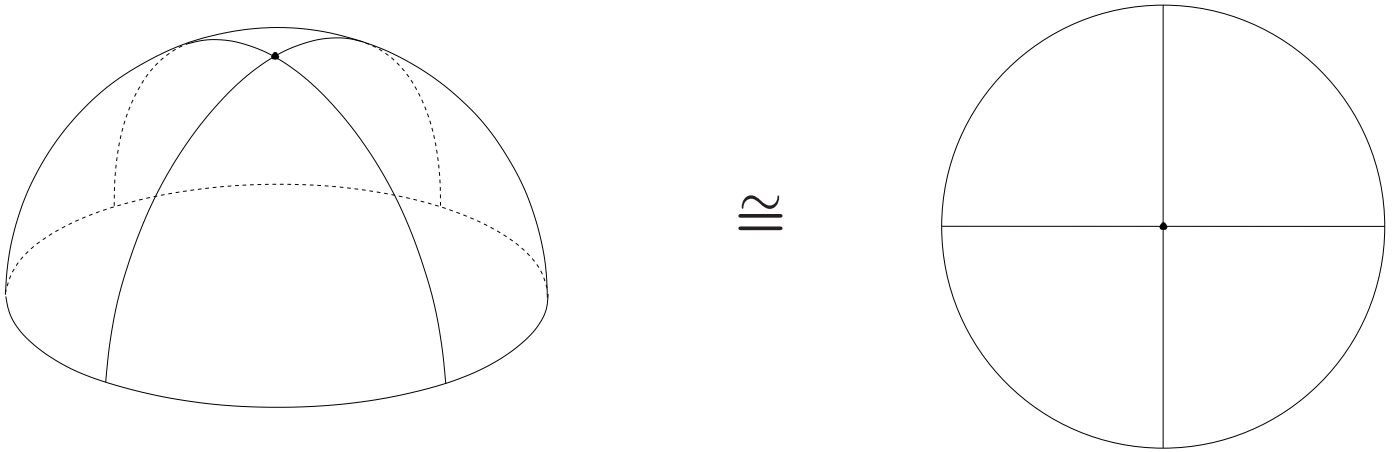
Asteroid 433 Eros.

Any position of a rigid body is given by an element of the rotation group $SO(3)$ applied to a reference position. So the motion of a rigid body corresponds to a curve on $SO(3)$, which is a geodesic for a certain geometry (determined by the kinetic energy).

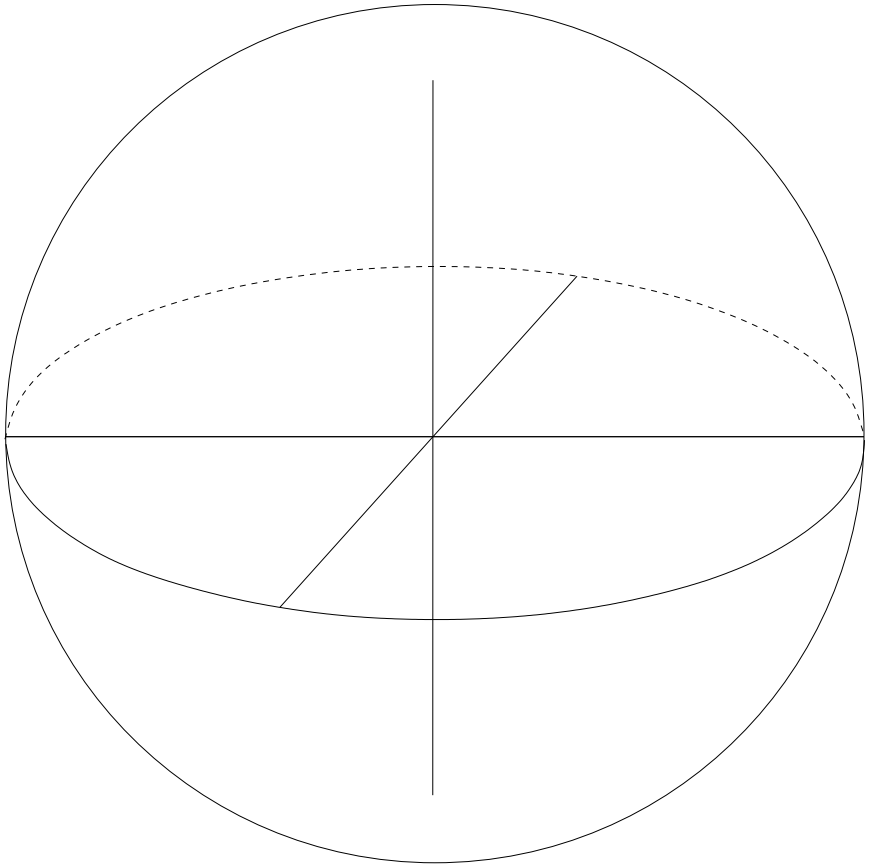


SO(3)

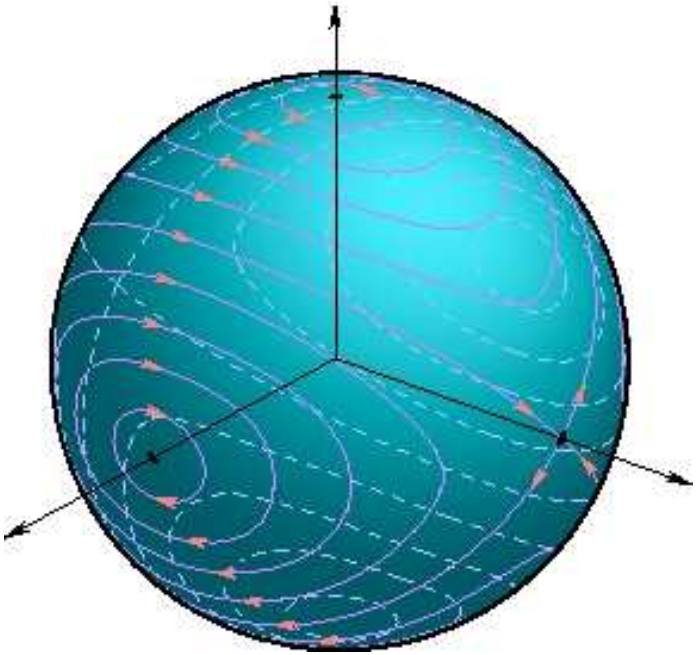
- v represents the positive rotation by an angle $\|v\|$ around v.
- If $\|w\| = \pi$ then w and $-w$ represent the same rotation.
- So $SO(3) \cong \dfrac{S^3}{\{\pm 1\}} \cong \mathbb{R}P^3$.



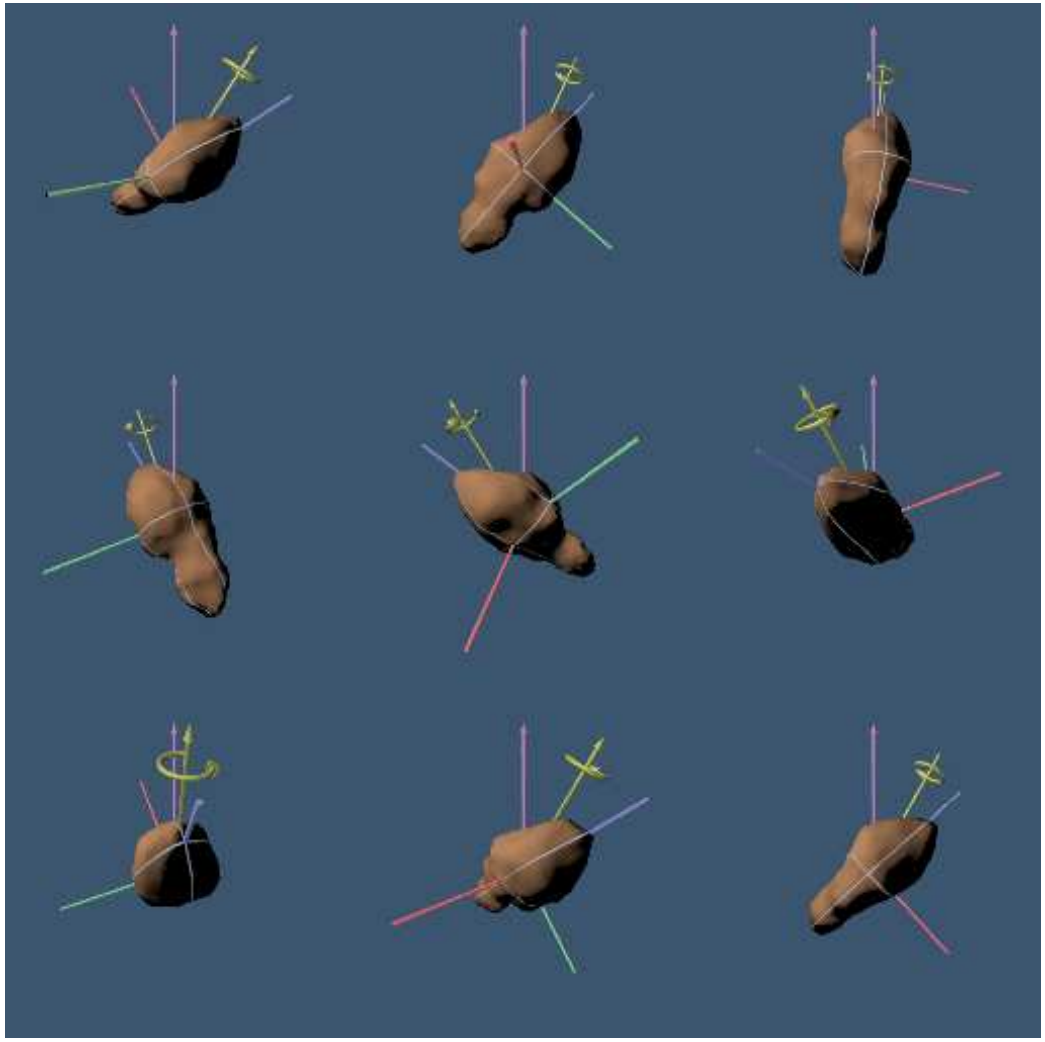
If the rigid body is sufficiently symmetric then the geodesics are exactly those of the round sphere S^3:



The general case is more complicated:



Trajectories of the angular momentum vector as seen by the rigid body.

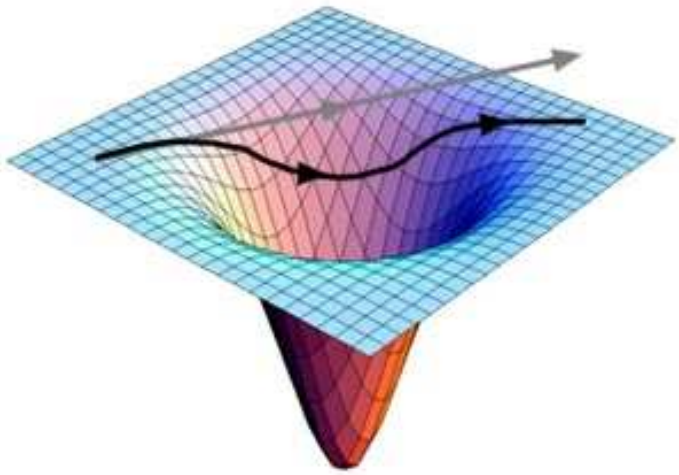


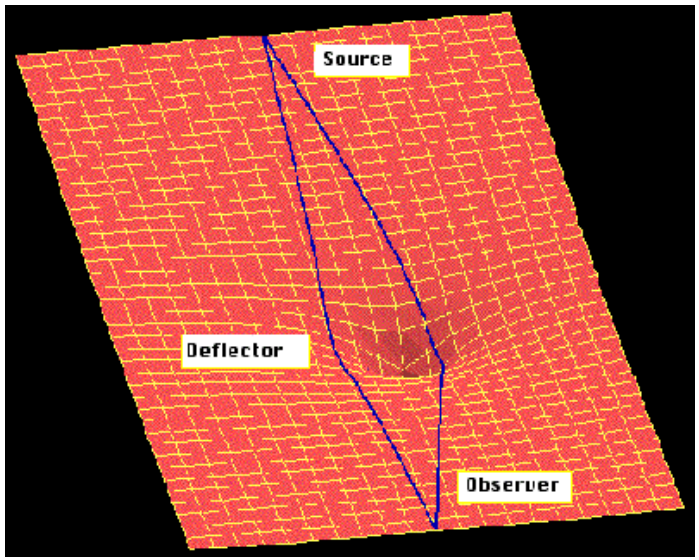
Asteroid 4179 Toutatis.

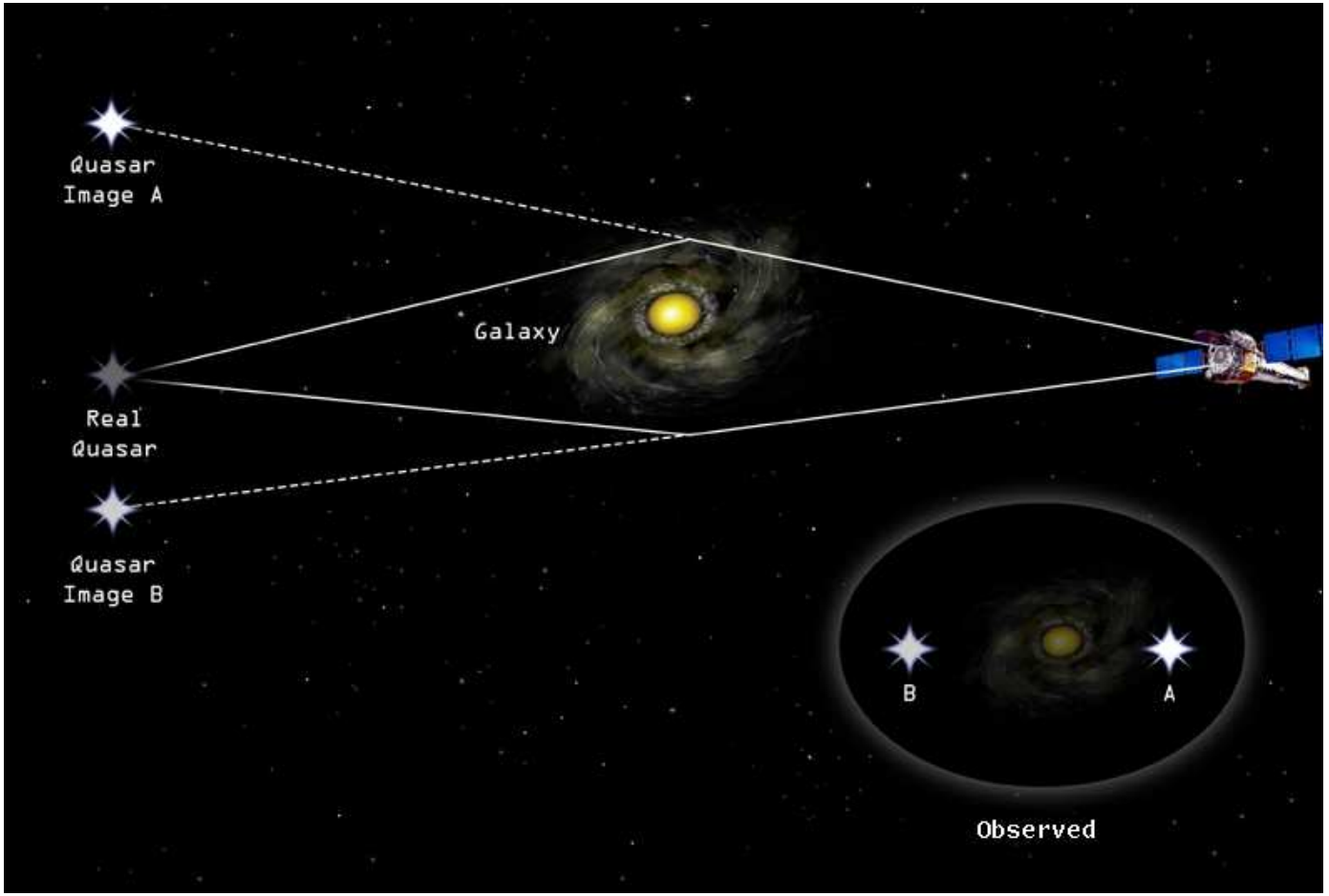
Einstein (1879–1955)

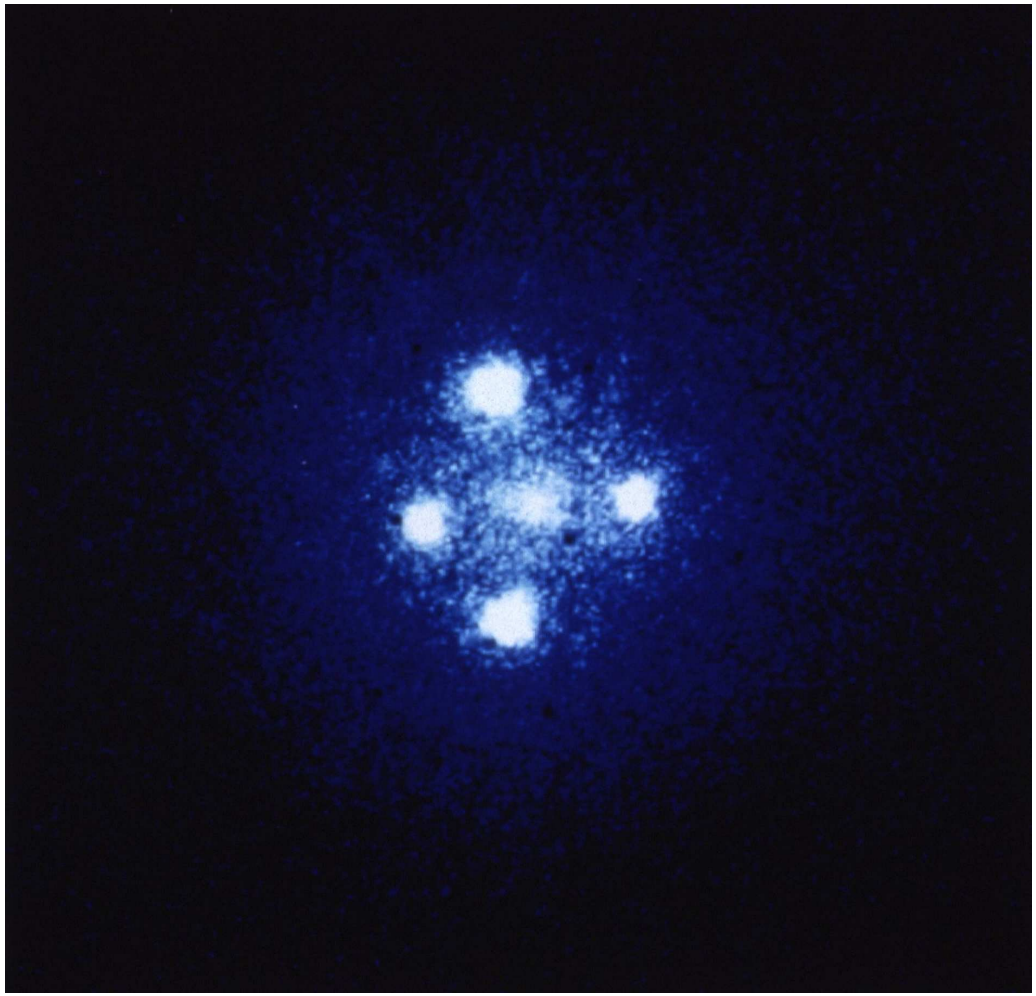
Space(-time) is curved and light rays follow geodesics.

One consequence is the gravitational lens effect, which originates multiple images of astronomical objects.





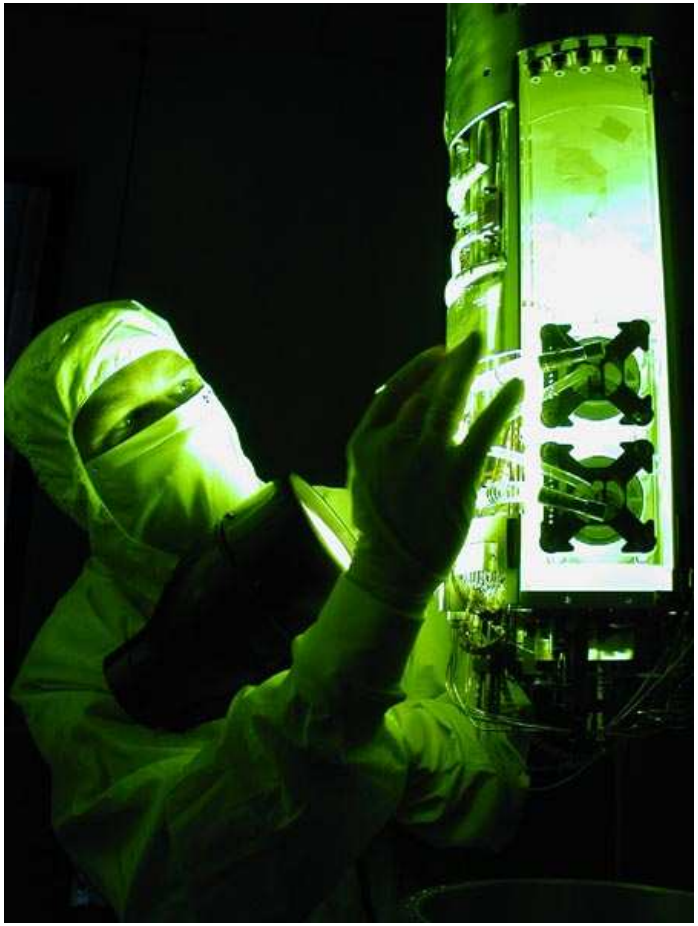

Quasar
Image A
Galaxy
Real
Quasar
Quasar
Image B
B
A
Observed



Einstein Cross.

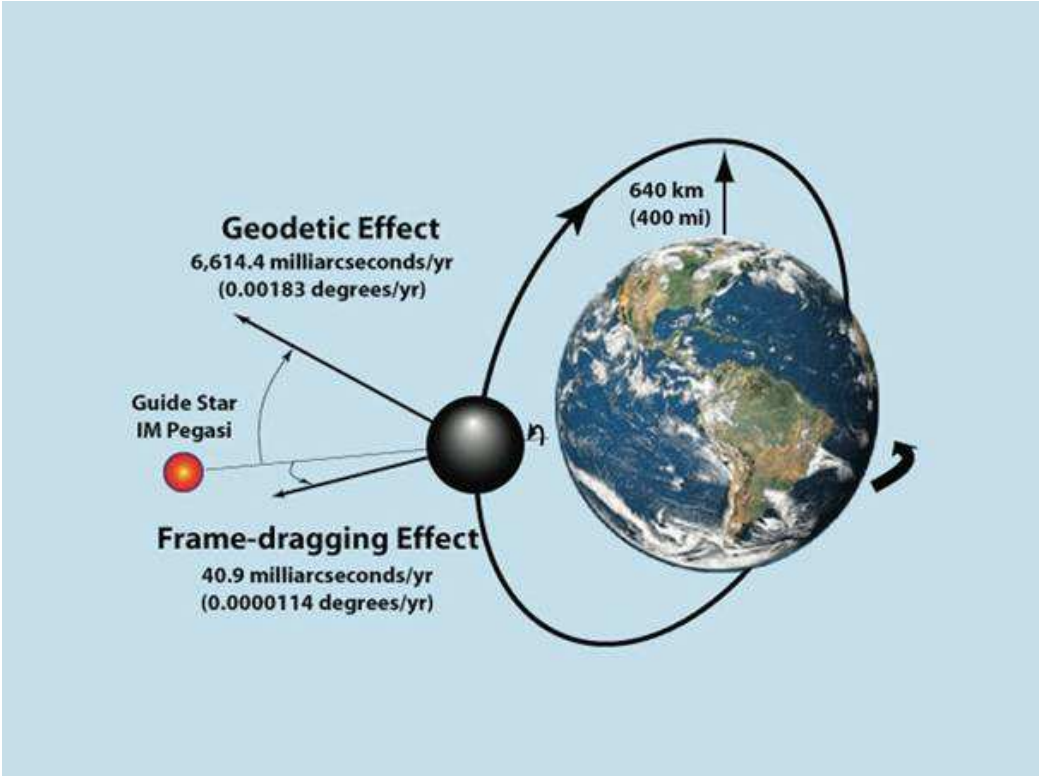
Gravity Probe B (launched in 2004)

The most spherical spheres in the world...

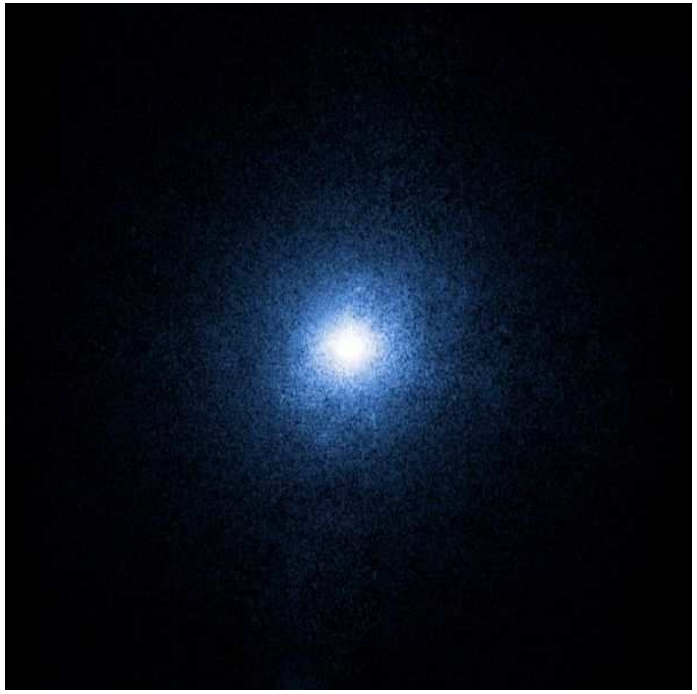


...cooled below -271^{o}C...

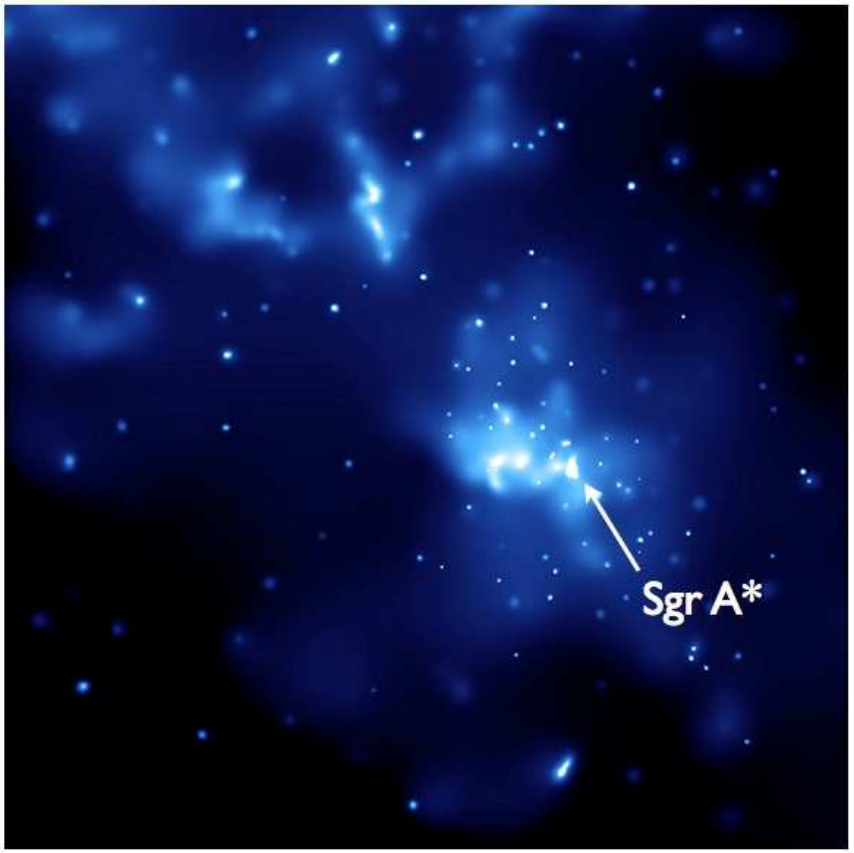
...were spun up and put in orbit.


Geodetic Effect
6,614.4 milliarcseconds/yr
(0.00183 degrees/yr)
640 km
(400 mi)
Guide Star
IM Pegasi
Frame-dragging Effect
40.9 milliarcseconds/yr
(0.0000114 degrees/yr)

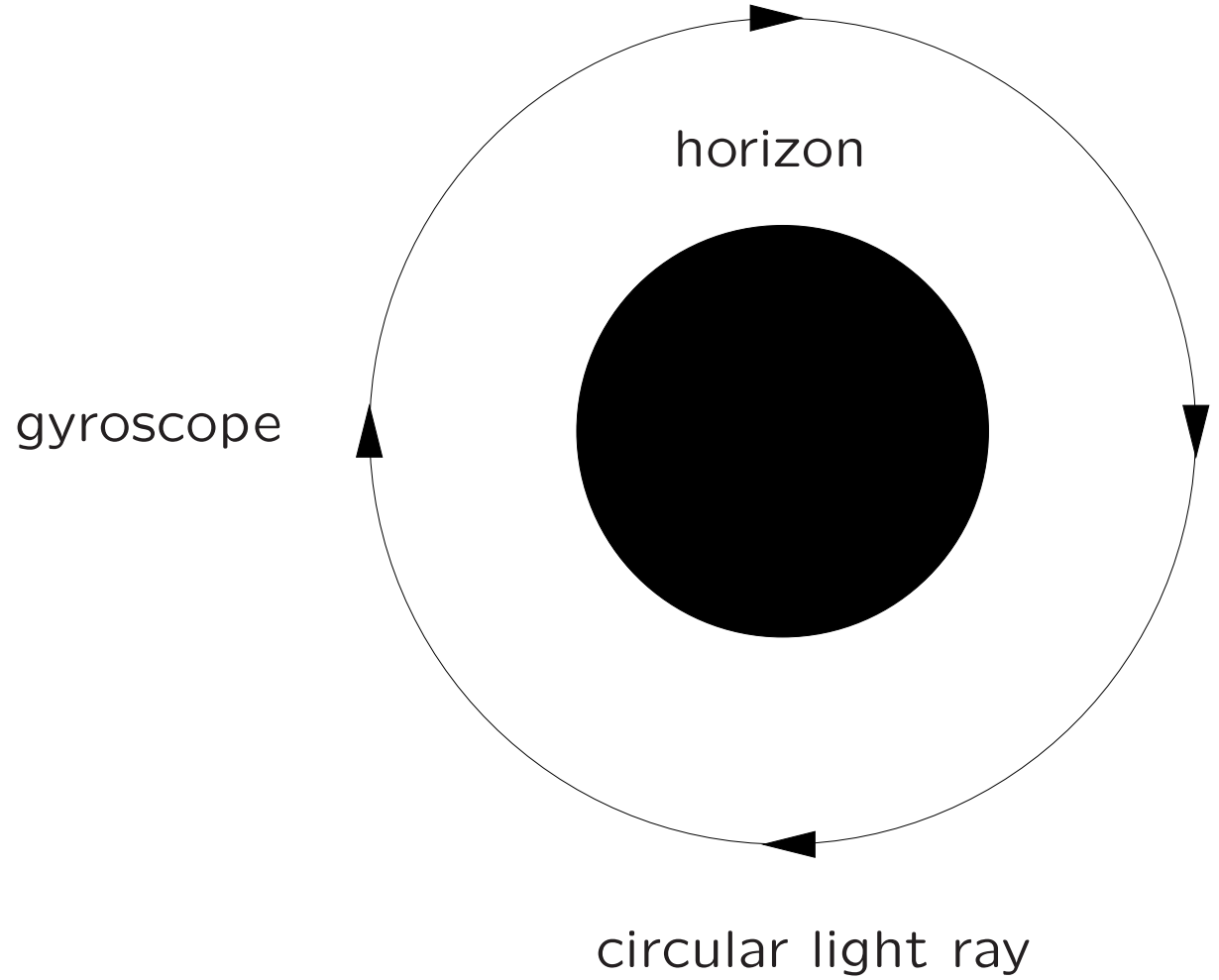
Black holes

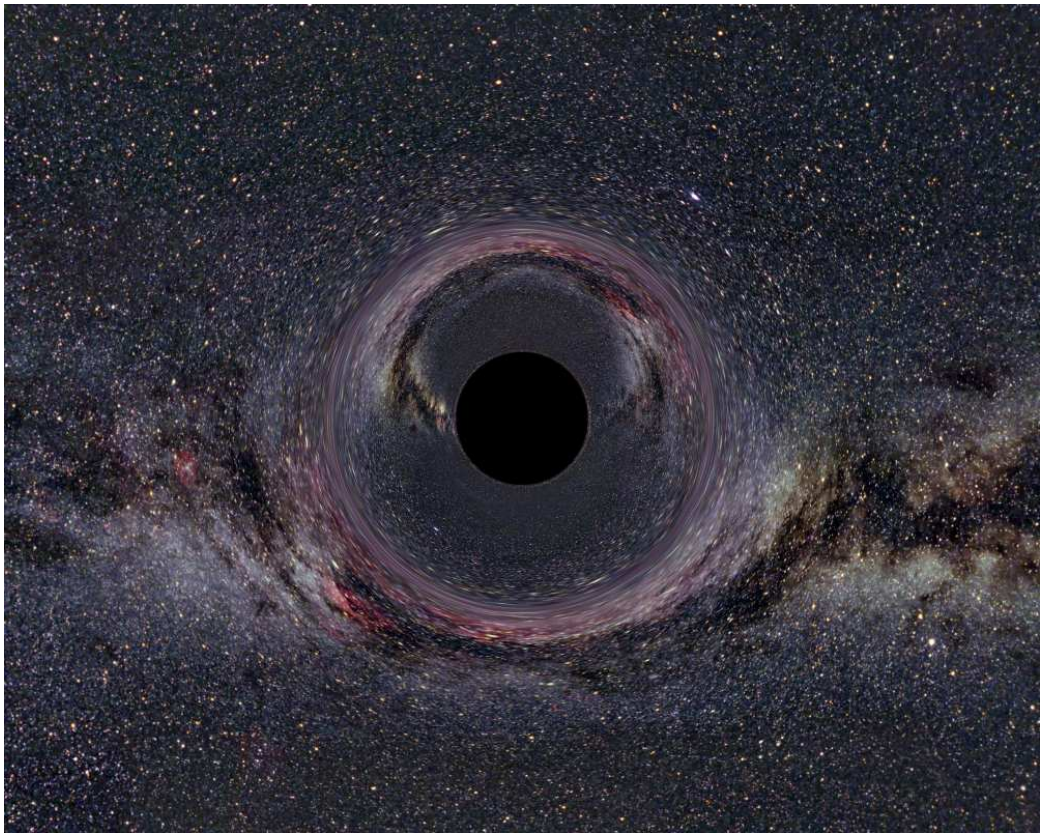


Cygnus X-1 (X-ray image by Chandra).

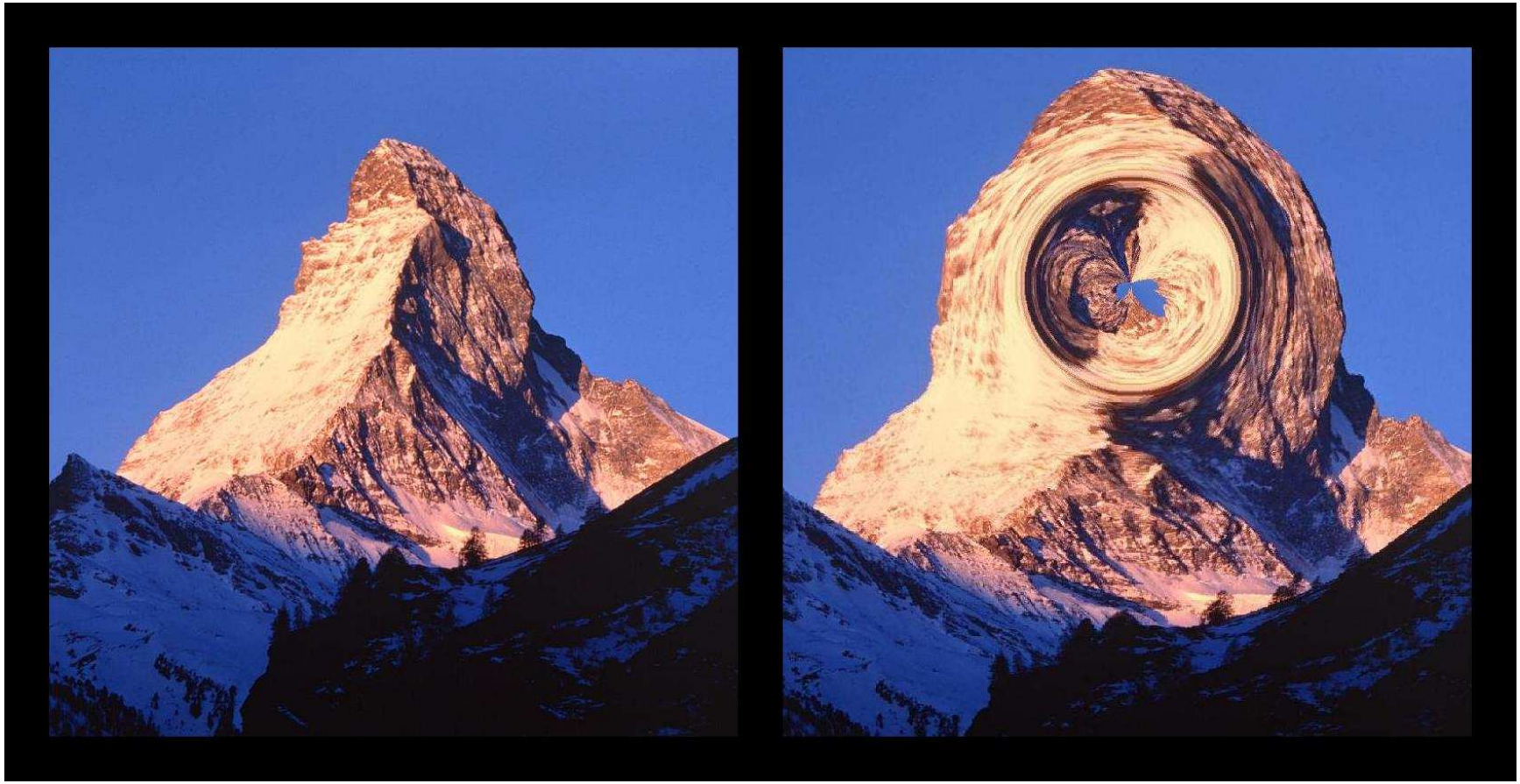


Sagittarius A* (X-ray image by Chandra).


horizon
gyroscope
circular light ray



Black hole 600 kilometers away...



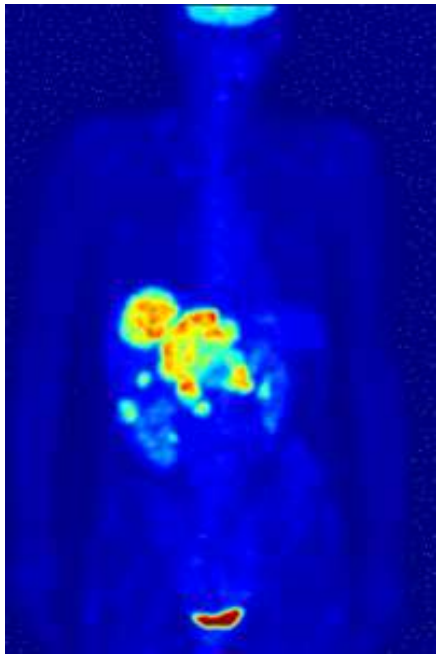
...and 600 meters away.

Gauge theories

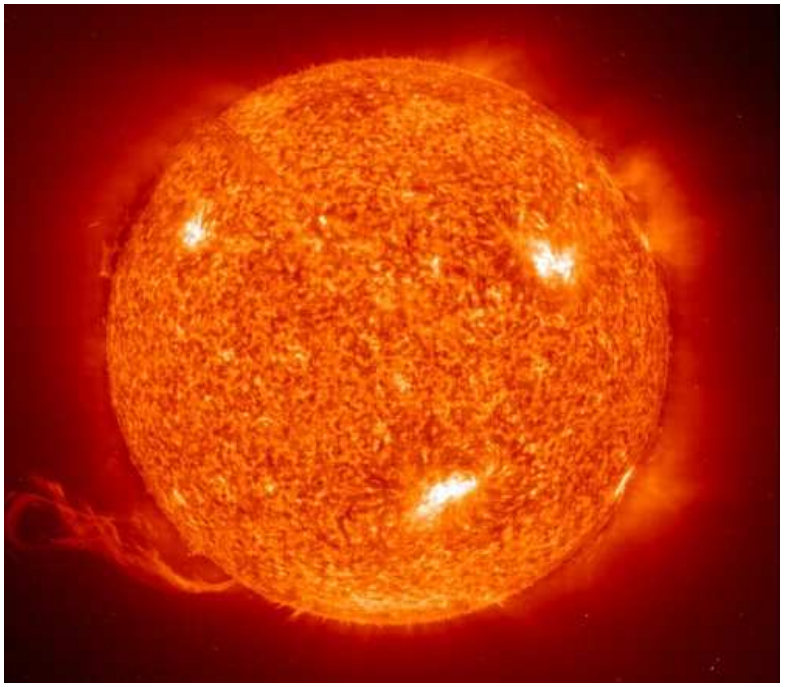
• At each point of space(-time) there exists an internal space, i.e. a complex vector space with a Hermitian inner product.

• The fundamental forces arise from the curvature of parallel transport of internal vectors.

Electromagnetic force – 1-dimensional internal space.



Weak nuclear force – 2-dimensional internal space.



Strong nuclear force – 3-dimensional internal space.

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