Strong cosmic censorship in spherical symmetry

José Natário
(Instituto Superior Técnico, Lisbon)

CENTRA, December 2014
Outline

- Strong cosmic censorship
- (Weak cosmic censorship)
- What do you mean “inextendible”? 
- Einstein-Maxwell-scalar field equations in spherical symmetry
- Christodoulou’s results
- Dafermos’ results
- Our results
Strong cosmic censorship

singularity

$r = 0$

initial data
• The appearence of a visible singularity destroys determinism: the singularity can radiate gravitationally or otherwise (mathematically it is a singular boundary).

• Obvious conjecture: the singularities that form are never locally naked, that is, visible by some observer.

• Well known to be false: counter-examples are the dust cloud solutions of Christodoulou, or the Reissner-Nordström solution.
$r = 0$  \quad \mathcal{I}^{+}$
• Initial surface $t = 0$: 
• However, the blueshift effect should make the Cauchy horizon unstable:

\[ r = r_- \]

\[ r = r_- \]

\[ \mathcal{I}^+ \]

\[ \mathcal{I}^+ \]
• Corrected conjecture: For generic solutions of the Einstein field equations with reasonable matter models, the singularities that form are never locally naked.

• PDE version: For generic asymptotically flat or compact initial data the future maximal globally hyperbolic development is inextendible.
(Weak cosmic censorship)

- For generic asymptotically flat solutions of the Einstein field equations with reasonable matter models, the singularities that form are never **naked**, that is, visible by observers **at infinity**.

- PDE version: For generic asymptotically flat initial data the future maximal globally hyperbolic development possesses a **complete future null infinity** $\mathcal{I}^+$.

- Strong and weak CCCs are logically independent (examples are Reissner-Nordström and **thunderbolts**).
$r = 0 \quad i^+$
$r = 0$
$r = 0$
What do you mean “inextendible”?

- Inextendible as a Lorentzian manifold or inextendible as a solution?

- Strongest possible form: no $C^0$ extensions (that is, extensions with continuous metric).

- What you get if curvature blows up: no $C^2$ extensions.
• What you really want: inextendible as a solution, in the class of functions you are considering (not necessarily $C^2$: shocks, impulsive gravitational waves, ...).

• If you want to prevent any conceivable extension as solution: no $C^0$ extension with Christoffel symbols in $L^2_{loc}$ (Christodoulou).

\[ 0 = \int_M Ric \cdot \varphi = \int_M (\partial \Gamma + \Gamma \Gamma) \cdot \varphi = \int_M (-\Gamma \cdot \partial \varphi + \Gamma \Gamma \cdot \varphi) \]
Einstein-Maxwell-scalar field equations in spherical symmetry

- **Birkhoff’s theorem**: there are no gravitational degrees of freedom in spherical symmetry.

- Simplest **hyperbolic** matter model: massless scalar field.

- Simplest spherically symmetric solution containing a Cauchy horizon: Reissner-Nordström (**electromagnetic field**).
The equations for a gravitating massless scalar field $\phi$ in a sourceless electromagnetic field $F$ with a cosmological constant $\Lambda$ are

\[ R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = 2T_{\mu\nu} \]

\[ T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \partial_\alpha \phi \partial^\alpha \phi g_{\mu\nu} + F_{\mu\alpha} F^\alpha_{\nu} - \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} g_{\mu\nu} \]

\[ \Box \phi = 0 \]

\[ dF = d^* F = 0 \]

(using units for which $c = 4\pi G = \varepsilon_0 = 1$)
If we impose spherical symmetry then the metric and the fields become

\[ ds^2 = -\Omega^2(u, v) du \, dv + r^2(u, v) (d\theta^2 + \sin^2 \theta \, d\varphi^2) \]
\[ \phi = \phi(u, v) \]
\[ F = -E(u, v) \frac{\Omega^2(u, v)}{2} du \wedge dv \]

In particular the electromagnetic field completely decouples:

\[ {}^*F = E(u, v) r^2(u, v) \sin \theta \, d\theta \wedge d\varphi \Rightarrow E(u, v) = \frac{e}{r^2(u, v)} \]
• The total charge $4\pi e$ is topological: initial surface $t = 0$ in Reissner-Nordström is
• Introducing the renormalized Hawking mass $\varpi$ through

\[
1 - \frac{2\varpi}{r} + \frac{e^2}{r^2} - \frac{\Lambda}{3}r^2 = -\frac{4\partial_u r \partial_v r}{\Omega^2} = (\text{grad } r)^2
\]

the Einstein-Maxwell-scalar field equations become

\[
\partial_u \partial_v \phi = -\frac{\partial_u r}{r} \frac{\partial_v \phi}{r} - \frac{\partial_v r}{r} \frac{\partial_u \phi}{r}
\]

\[
\partial_u \partial_v r = \partial_u r \partial_v r \frac{\frac{2\varpi}{r^2} - \frac{2e^2}{r^3} - \frac{2\Lambda}{3}r}{1 - \frac{2\varpi}{r} + \frac{e^2}{r^2} - \frac{\Lambda}{3}r^2}
\]

\[
\partial_u \varpi = \left(1 - \frac{2\varpi}{r} + \frac{e^2}{r^2} - \frac{\Lambda}{3}r^2\right) \frac{(r \partial_u \phi)^2}{2\partial_u r}
\]

\[
\partial_v \varpi = \left(1 - \frac{2\varpi}{r} + \frac{e^2}{r^2} - \frac{\Lambda}{3}r^2\right) \frac{(r \partial_v \phi)^2}{2\partial_v r}
\]
- Characteristic initial data:
Christodoulou’s results

• Christodoulou (1999): Strong cosmic censorship is true for the Einstein-scalar field system \((e = \Lambda = 0)\) in the \(C^0\) formulation.

• Generic: dispersive or black hole.

• Non-generic: light cone singularity (possibly naked), collapsed light cone singularity, black hole with light cone singularity, black hole with collapsed light cone singularity.
\[ r > 0 \quad \text{i+ or i\text{naked}} \]

\[ r = 0 \quad \text{i+} \]
$r > 0$ 

$\mathcal{I}^+$ 

$r = 0$ 

$\mathcal{I}^+$
Dafermos' results

\[ r = 0 \]

\[ I^+ \]
Characteristic initial data:
Poisson and Israel (1989) gave a nonlinear heuristic analysis suggesting that for $\Lambda = 0$ the Cauchy horizon of generic solutions ($\phi \neq 0$) still has $r \sim r_-$, but $\omega \to +\infty$ (mass inflation).

Brady, Moss and Myers (1998) performed a linear analysis suggesting that mass inflation might not occur for $\Lambda > 0$ near extremality (but the curvature still blows up at the Cauchy horizon).
• Dafermos (2005) proved the following two results for the spherically symmetric Einstein-Maxwell-scalar field system.

1. If $|\partial_v \phi| \lesssim v^{-1}$ along the event horizon then $r$ can be extended to a nonvanishing continuous function on the Cauchy horizon, and so the metric can be extended as a $C^0$ metric.

2. If $v^{-3} \lesssim |\partial_v \phi| \lesssim v^{-1}$ along the event horizon then the Hawking mass blows up identically along the Cauchy horizon, and so the metric is inextendible as a $C^1$ metric.

• The first hypothesis (*Price’s law*) was subsequently proved to occur by Dafermos and Rodnianski (2005).
\[ r \sim r_-, \mathcal{W} \rightarrow \infty \]

\[ v^{-3} \lesssim |\partial_v \phi| \lesssim v^{-1} \]
Our results

• We (J. Costa, P. Girão, J. Natário, J. Silva) consider the simpler case where $\phi = 0$ on the event horizon (so that it is exactly Reissner-Nordström), for any $\Lambda$.

• Assume $\phi(u, 0) \sim u^{s+1}$ for the normalization $r(u, 0) = r_+ - u$.

• Previously considered by Dafermos (2003) in the case $\Lambda = 0$.

• Results depend on $s$ and $\rho = \frac{k_-}{k_+} > 1$ (we exclude the extremal case $\rho = 1$).
\[ r \sim r_- \]
\[ \phi = 0 \]
\[ \phi \sim u^{s+1} \]
\[ s = \frac{13\rho}{9} - 1 \]

\[ s = \frac{7\rho}{9} - 1 \]

\[ s = \frac{\rho}{2} - 1 \]
no mass inflation
Kretschmann bounded
smooth extension
beyond Cauchy horizon

mass inflation or
Kretschmann unbounded

no mass inflation
Kretschmann unbounded

mass inflation
• $C^0$ extensions always exist.

• **Mass inflation**: no $C^1$ extensions (in fact no Christodoulou extensions).

• **Kretschmann unbounded**: no $C^2$ extensions.

• **No mass inflation**: Christodoulou extensions.

• **Smooth extensions**: as solutions across the Cauchy horizon, all derivatives continuous.
• So all formulations of strong cosmic censorship are violated in this particular framework.

• But we have $\phi = 0$ along the event horizon (infinitely fast decay).

• For $\Lambda > 0$ one expects an exponential decay in Price's law, so it is likely that there is no mass inflation near extremality. What about strong cosmic censorship?
\[ \partial_u r = \nu, \]
\[ \partial_v r = \lambda, \]
\[ \partial_u \lambda = -2\nu \kappa \frac{1}{r^2} \left( \frac{e^2}{r} + \frac{\Lambda}{3} r^3 - \omega \right), \]
\[ \partial_v \nu = -2\nu \kappa \frac{1}{r^2} \left( \frac{e^2}{r} + \frac{\Lambda}{3} r^3 - \omega \right), \]
\[ \partial_u \omega = \frac{1}{2} \left( 1 - \frac{2\omega}{r} + \frac{e^2}{r^2} - \frac{\Lambda}{3} r^2 \right) \left( \frac{\zeta}{\nu} \right)^2 \nu, \]
\[ \partial_v \omega = \frac{1}{2} \frac{\theta^2}{\kappa}, \]
\[ \partial_u \theta = -\frac{\zeta \lambda}{r}, \]
\[ \partial_v \zeta = -\frac{\theta \nu}{r}, \]
\[ \partial_u \kappa = \kappa \nu \frac{1}{r} \left( \frac{\zeta}{\nu} \right)^2, \]
\[ \lambda = \kappa (1 - \mu). \]
$$\kappa(u, v) = \kappa_0(v) e^{\int_0^u \frac{\zeta^2}{r^2} (u', v) \, du'},$$

$$\nu(u, v) = \nu_0(u) e^{-\int_0^v \left(2\kappa + \frac{\Lambda}{3} r^3 - \varpi\right)(u', v) \, dv'},$$

$$\lambda(u, v) = \lambda_0(v) - \int_0^u \left(2\nu\kappa \frac{1}{r^2} \left(\frac{e^2}{r} + \frac{\Lambda}{3} r^3 - \varpi\right)\right) (u', v) \, du',$$

$$\theta(u, v) = \theta_0(v) - \int_0^u \left(\frac{\zeta\lambda}{r}\right) (u', v) \, du',$$

$$\zeta(u, v) = \zeta_0(u) - \int_0^v \left(\frac{\theta\nu}{r}\right) (u, v') \, dv',$$

$$\varpi(u, v) = \varpi_0(v) e^{-\int_0^u \frac{\zeta^2}{rr'} (u', v) \, du'} + \int_0^u e^{-\int_s^u \frac{\zeta^2}{rr'} (u', v) \, du'} \left(\frac{1}{2} \left(1 + \frac{e^2}{r^2} - \frac{\Lambda}{3} r^2\right) \frac{\zeta^2}{\nu}\right) (s, v) \, ds,$$

$$r(u, v) = r_0(u) + \int_0^v \lambda(u, v') \, dv'.$$