Strong cosmic censorship with a cosmological constant I

José Natário
(Instituto Superior Técnico, Lisbon)

Braga, December 2013
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Einstein-Maxwell-scalar field equations in spherical symmetry

• The equations for a gravitating massless scalar field in a sourceless electromagnetic field with a cosmological constant are

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 2 T_{\mu\nu} \]

\[ T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \partial_\alpha \phi \partial^\alpha \phi g_{\mu\nu} + F_{\mu\alpha} F^{\alpha}_\nu - \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} g_{\mu\nu} \]

\[ \Box \phi = 0 \]

\[ dF = d^* F = 0 \]

(using units for which \( c = 4\pi G = \varepsilon_0 = 1 \))
• If we assume spherical symmetry then the metric and the fields become

\[ ds^2 = -\Omega^2(u, v) du \, dv + r^2(u, v) \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right) \]

\[ \phi = \phi(u, v) \]

\[ F = -E(u, v) \frac{\Omega^2(u, v)}{2} du \wedge dv \]

• In particular the electromagnetic field completely decouples:

\[ *F = E(u, v) r^2(u, v) \sin \theta \, d\theta \wedge d\varphi \quad \Rightarrow \quad E(u, v) = \frac{e}{r^2(u, v)} \]
- The total charge $4\pi e$ is topological:
Introducing the renormalized Hawking mass \( \varpi \) through

\[
1 - \frac{2\varpi}{r} + \frac{e^2}{r^2} - \frac{\Lambda}{3}r^2 = -\frac{4\partial_u r \partial_v r}{\Omega^2}
\]

the Einstein-Maxwell-scalar field equations become

\[
\partial_u \partial_v \phi = -\frac{\partial_u r \partial_v \phi}{r} - \frac{\partial_v r \partial_u \phi}{r}
\]

\[
\partial_u \partial_v r = \partial_u r \partial_v r \left( \frac{2\varpi}{r^2} - \frac{2e^2}{r^3} - \frac{2\Lambda}{3}r \right)
\]

\[
\partial_u \varpi = \left( 1 - \frac{2\varpi}{r} + \frac{e^2}{r^2} - \frac{\Lambda}{3}r^2 \right) \frac{(r \partial_u \phi)^2}{2\partial_u r}
\]

\[
\partial_v \varpi = \left( 1 - \frac{2\varpi}{r} + \frac{e^2}{r^2} - \frac{\Lambda}{3}r^2 \right) \frac{(r \partial_v \phi)^2}{2\partial_v r}
\]
• Characteristic initial data:

\[ \varphi(0, v), r(0, v) \]

\[ \varphi(u, 0), r(u, 0) \]

\[ \varpi(0, 0) \]
Reissner-Nordström-(anti-)de Sitter solution

- Vacuum: $\phi = 0 \Rightarrow \omega = \omega_0$.

- The metric is written

$$ds^2 = \frac{4\partial_u r \partial_v r}{1 - \frac{2\omega_0}{r} + \frac{e^2}{r^2} - \frac{\Lambda}{3} r^2} du \, dv + r^2 \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right)$$

with $m$ and $e$ constant.

- Zeros $r = r_-$ (Cauchy horizon), $r = r_+$ (event horizon) and $r = r_c$ (cosmological horizon) (for $\Lambda > 0$ only) must be null: $\partial_u r = 0$ or $\partial_v r = 0$. 
• $r = 0$ is a singularity, where the equations break down (and the curvature explodes).

• $r = r_+$ is the event horizon. It is the boundary of the black hole region, which contains the singularity and is not visible from the outer region.

• $r = r_-$ is the Cauchy horizon. Observers crossing this horizon can see the singularity. However, the blueshift effect should make this horizon unstable.
\[ r = r_- \]

\[ r = r_- \]
Strong cosmic censorship

- For **generic solutions** of the Einstein field equations with **reasonable matter models** no observer can see the singularity.

- **PDE version:** For **generic initial data** the **maximal globally hyperbolic development** is inextendible.
Mass inflation: heuristics

• Poisson and Israel (1989) gave a nonlinear heuristic analysis suggesting that for $\Lambda = 0$ the Cauchy horizon of generic solutions ($\phi \neq 0$) still has $r \to r_-$, but $\varpi \to +\infty$ (mass inflation).

• Brady, Moss and Myers (1998) performed a linear analysis suggesting that mass inflation might not occur for $\Lambda > 0$ near extremality (but the curvature still blows up at the Cauchy horizon).

• The whole question hinges on the decay properties of $\phi$ along the event horizon.
Mass inflation: rigorous results

• Dafermos (2005) proved the following two results for the spherically symmetric Einstein-Maxwell-scalar field system.

1. If $|\partial_v \phi| \lesssim v^{-1}$ along the event horizon then $r$ can be extended to a nonvanishing continuous function on the Cauchy horizon, and so the metric can be extended as a $C^0$ metric.

2. If $v^{-3} \lesssim |\partial_v \phi| \lesssim v^{-1}$ along the event horizon then the Hawking mass blows up identically along the Cauchy horizon, and so the metric is inextendible as a $C^1$ metric.

• The first hypothesis (Price’s law) was subsequently proved to occur by Dafermos and Rodnianski (2005).
\[ r^{-3} \wedge \frac{\partial \phi}{\partial \phi} \lesssim r^{-1} \]
• In a previous paper, Dafermos had assumed $\phi = 0$ on the event horizon and could only prove mass inflation away from extremality.

• The decay properties of $\phi$ along the event horizon appear to be crucial to mass inflation.

• For $\Lambda > 0$ one expects an exponential decay in Price’s law, so it is likely that there is no mass inflation near extremality.

