Instituto Superior Técnico
Departamento de Matemática

Differential Geometry - homework set 2

1. Let $\mathcal{F}_1, \mathcal{F}_2$ be two foliations of a smooth manifold M.

   a) Give an appropriate definition for transversality $\mathcal{F}_1 \pitchfork \mathcal{F}_2$.

   b) If $\mathcal{F}_1 \pitchfork \mathcal{F}_2$ define $\mathcal{F}_1 \cap \mathcal{F}_2$ appropriately and compute its codimension.

2. ([RLF], Lecture 8, Problem 8) A foliation $\mathcal{F}$ of a smooth manifold M is called *simple* if M can be covered by foliated charts such that every leaf meets each of those charts at most along one plaque.

   Let $M/\mathcal{F}$ be the space of leaves of $\mathcal{F}$. Show that $\mathcal{F}$ is simple iff $M/\mathcal{F}$ has a (possibly non-Hausdorff) smooth structure such that $\pi : M \to M/\mathcal{F}$ is a submersion.

3. Consider the 3-sphere

   $$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}.$$

   The unitary group,

   $$U(1) = \{w \in \mathbb{C} : |w| = 1\} \cong S^1,$$

   acts on S^3 by

   $$e^{i\theta} \cdot (z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2).$$

   The action is free and proper so that $S^1 \setminus S^3$ is a smooth manifold and $\pi : S^3 \to S^1 \setminus S^3$ is a submersion.

   a) Show that the orbits of this action define a 1-dimensional simple foliation $\mathcal{F}$ of S^3.

   b) Show that $S^1 \setminus S^3$ is diffeomorphic to the 2-sphere S^2.
   Hint: Think about the map $p(z_1, z_2) = (2z_1\bar{z}_2, |z_1|^2 - |z_2|^2)$.

   c) Show that $\mathcal{F}$ is *not* transverse to the Reeb foliation.
   Hint: Recall that one of the leaves of the Reeb foliation is the Clifford torus.
   Note: $\pi : S^3 \to S^1 \setminus S^3 \cong S^2$ is known as the *Hopf fibration*.

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4. Let $p, q \in \mathbb{N}, q < p$, be coprime and consider the 3-sphere, $S^3 \subset \mathbb{C}^2$, as in the previous problem. Let the cyclic group $\mathbb{Z}_p$ act on S^3 by

$$e^{\frac{2\pi i}{p}} \cdot (z_1, z_2) = (e^{\frac{2\pi i}{p}} z_1, e^{\frac{2\pi q i}{p}} z_2).$$

   a) Verify that this is a properly discontinuous action and that, therefore, the *lens space* $L(p, q) = \mathbb{Z}_p \setminus S^3$ is a smooth compact 3-manifold, with $\pi_1(L(p, q)) \cong \mathbb{Z}_p$.

   b) Verify that the S^1-action on S^3 leading to the Hopf fibration commutes with the $\mathbb{Z}_p$ action. Thus, one obtains an action of S^1 on $L(p, q)$. Show that this action, however, is not free.

   c) Define a different S^1-action on $L(p, 1)$ by

$$e^{i\theta} \cdot [(z_1, z_2)]_{\mathbb{Z}_p} = [(e^{\frac{i\theta}{p}} z_1, e^{\frac{i\theta}{p}} z_2)]_{\mathbb{Z}_p}.$$

   Verify that this is a proper free action so that $S^1 \setminus L(p, 1)$ is a smooth manifold. Show that it is diffeomorphic to S^2.

Note: You could also think about considering $\mathbb{Z}_p \setminus (S^1 \setminus S^3) \cong \mathbb{Z}_p \setminus S^2$. However, the corresponding action of $\mathbb{Z}_p$ is not free; in fact, the only non-trivial group acting smoothly and freely on S^2 is $\mathbb{Z}_2$.