DIFFERENTIAL GEOMETRY

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Contents

Preface

These are lecture notes for the courses "Differentiable Manifolds I" and "Differentiable Manifolds II", that I am lecturing at UIUC. This course is usually taken by graduate students in Mathematics in their first or second year of studies. The background for this course is a basic knowledge of analysis, algebra and topology.

My main aim in writing up these lectures notes is to offer a written version of the lectures. This should give a chance to students to concentrate more on the class, without worrying about taking notes. It offers also a guide for what material was covered in class. These notes do not replace the recommended texts for this course, quite the contrary: I hope they will be a stimulus for the students to consult those works. In fact, some of these notes follow the material in theses texts.

These notes are organized into sections, each of these should correspond to lectures of approximately to 1 hour and 30 minutes of classroom time. However, some sections do include more material than others, which correspond to different rhythms in class. The exercises at the end of each section are a very important part of the course, since one learns a good deal about mathematics by solving exercises. Moreover, sometimes the exercises contain results that were mentioned in class, but not proved, and which are used in later sections. The students should also keep in mind that the exercises are not homogeneous: this is in line with the fact that in mathematics when one faces for the first time a problem, one usually does not know if it has an easy solution, a hard solution or if it is an open problem.

These notes are a modified version of similar lectures notes in Portuguese that I have used at IST-Lisbon. For the Portuguese version I have profited from comments from Ana Rita Pires, Georgios Kydonakis, Miguel Negrão, Miguel Olmos, Ricardo Inglês, Ricardo Joel, José Natário and Roger Picken. A special thanks goes to my colleague from IST Silvia Anjos, who has pointed out many typos and mistakes, and has suggested several corrections. I continue to update these notes and I will be grateful for any corrections and suggestions for improvement that are sent to me.

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Part 1. Basic Concepts

The notion of a *smooth manifold* of dimension d makes precise the concept of a space which locally looks like the usual euclidean space $\mathbb{R}^d$. Hence, it generalizes the usual notions of curve (locally looks like $\mathbb{R}^1$) and surface (locally looks like $\mathbb{R}^2$). This course consists of a precise study of this fundamental concept of Mathematics and some of the constructions associated with it. We will see that many constructions familiar in *infinitesimal analysis* (i.e., calculus) extend from euclidean space to smooth manifolds. On the other hand, the *global analysis* of manifolds requires new techniques and methods, and often elementary questions lead to open problems.

In this first part of the lectures we will introduce the most basic concepts of Differential Geometry, starting with the precise notion of a smooth manifold. The main concepts and ideas to keep in mind from these first part are:

- Section 0: A *manifold as a subset of Euclidean space*, and the various *categories of manifolds*: topological, smooth and analytic manifolds.
- Section 1: The *abstract notion of smooth manifold* (our objects) and *smooth map* (our morphisms).
- Section 2: A technique of gluing called *partitions of unity.*
- Section 3: *Manifolds with boundary* and *smooth maps between manifolds with boundary.*
- Section 4: *Tangent vector*, *tangent space* (our infinitesimal objects).
- Section 5: The *differential of a smooth map* (our infinitesimal morphisms).
- Section 6: Important classes of smooth maps: *immersions*, *submersions* and *local difeomorphisms*. *Submanifolds* (our sub-objects).
- Section 7: *Embedded sub manifolds* and the *Whitney Embedding Theorem*, showing that any smooth manifold can be embedded in some Euclidean space $\mathbb{R}^n$.
- Section 8 *Foliations*, which are certain partitions of a manifold into submanifolds, a very useful generalization of the notion of manifold.
- Section 9: *Quotients of manifolds*, i.e., smooth manifolds obtained from other smooth manifolds by taking equivalence relations.

0. Manifolds as subsets of Euclidean space

Recall that the *Euclidean space of dimension* n is:

$$\mathbb{R}^n := \left\{(x^1, \dots, x^n) : x^1, \dots, x^n \in \mathbb{R}\right\}$$

We will also denote by $x^i : \mathbb{R}^n \to \mathbb{R}$ the i-th coordinate function in $\mathbb{R}^n$. If $U \subset \mathbb{R}^n$ is an open set, a map $f : U \to \mathbb{R}^m$ is called a **smooth map** if all its partials derivatives of every order:

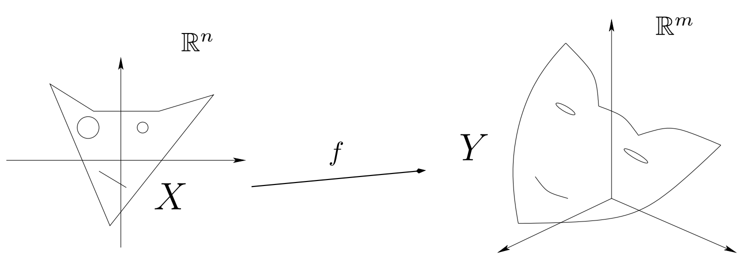
$$\frac{\partial^r f^j}{\partial x^{i_1} \cdots \partial x^{i_r}}(x),$$

exist and are continuous functions in U. More generally, given any subset $X \subset \mathbb{R}^n$ and a map $f : X \to \mathbb{R}^m$, where X is not necessarily an open set, we say that f is a **smooth map** if for each $x \in X$ there is an open neighborhood $U \subset \mathbb{R}^n$ and a smooth map $F : U \to \mathbb{R}^m$ such that $f|_{X \cap U} = F|_{X \cap U}$.

A very basic property which we leave as an exercise is that:

Proposition 0.1. *Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ and $Z \subset \mathbb{R}^p$. If $f : X \to Y$ and $g : Y \to Z$ are smooth maps, then $g \circ f : X \to Z$ is also a smooth map.*

A bijection $f : X \to Y$, where $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$, with inverse map $f^{-1} : Y \to X$, such that both f and f^{-1} are smooth, is called a **diffeomorphism** and we say that X and Y are **diffeomorphic subsets**.

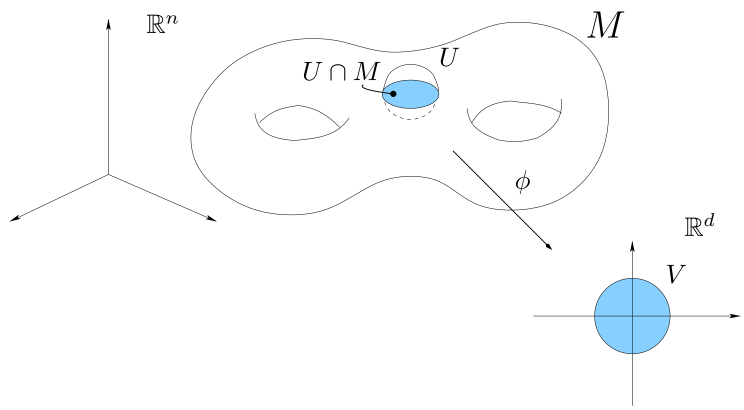


One would like to study properties of sets which are invariant under diffeomorphisms, characterize classes of sets invariant under diffeomorphisms, etc. However, in this definition, the sets X and Y are just too general, and it is hopeless to try to say anything interesting about classes of such diffeomorphic subsets. One must consider nicer subsets of Euclidean space: for example, it is desirable that the subset has at each point a tangent space and that the tangent spaces vary smoothly.

Recall that a subset $X \subset \mathbb{R}^n$ has an induced topology, called the *relative topology*, where the relative open sets are just the sets of the form $X \cap U$, where $U \subset \mathbb{R}^n$ is an open set.

Definition 0.2. *A subset $M \subset \mathbb{R}^n$ is called a **smooth manifold of dimension** d if each $p \in M$ has a neighborhood $M \cap U$ which is diffeomorphic to an open set $V \subset \mathbb{R}^d$.*

The diffeomorphism $\phi : M \cap U \to V$, in this definition, is called a **coordinate system**. The inverse map $\phi^{-1} : V \to M \cap U$, which by assumption is smooth, is called a **parameterization**.



We have the **category of smooth manifolds** where:

- the objects are smooth manifolds;
- the morphisms are smooth maps.

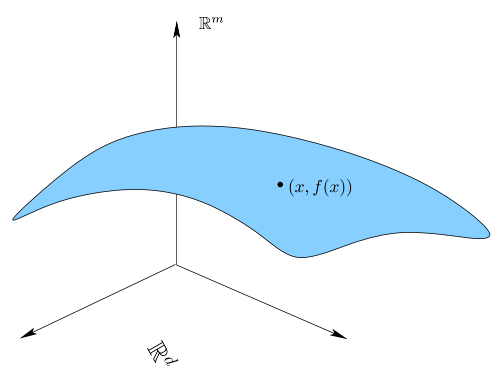
The reason they form a category is because the composition of smooth maps is a smooth map and the identity is also a smooth map.

Examples 0.3.

1. *Any open subset $U \subset \mathbb{R}^d$ is itself a smooth manifold of dimension d: the inclusion $i : U \hookrightarrow \mathbb{R}^d$ gives a global defined coordinate chart.*
2. *If $f : \mathbb{R}^d \to \mathbb{R}^m$ is any smooth map, its graph:*

$$\mathrm{Graph}(f) := \{(x, f(x)) : x \in \mathbb{R}^d\} \subset \mathbb{R}^{d+m}$$

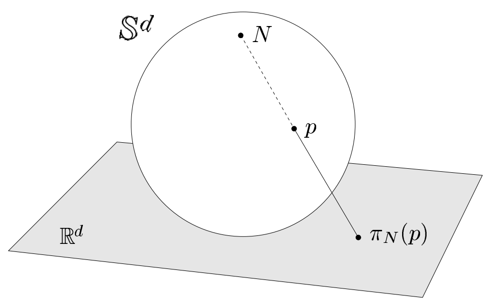
is a smooth manifold of dimension d: the map $x \mapsto (x, f(x))$ is a diffeomorphism $\mathbb{R}^d \to \mathrm{Graph}(M)$, so gives a global parametrization of $\mathrm{Graph}(f)$.



3. The ***unit*** d***-sphere*** *is the subset of* $\mathbb{R}^{d+1}$ *formed by all vectors of length 1:*

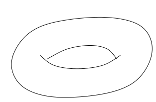
$$\mathbb{S}^d := \{x \in \mathbb{R}^{d+1} : ||x|| = 1\}.$$

This is a d*-dimensional manifold which does not admit a global parametrization. However we can cover the sphere by two coordinate systems: if we let* $N = (0, \dots, 0, 1)$ *and* $S = (0, \dots, 0, -1)$ *denote the north and south poles, then stereographic projection relative to* N *and* S *give two coordinate systems* $\pi_N : \mathbb{S}^d - \{N\} \to \mathbb{R}^d$ *and* $\pi_S : \mathbb{S}^d - \{S\} \to \mathbb{R}^d$*.*

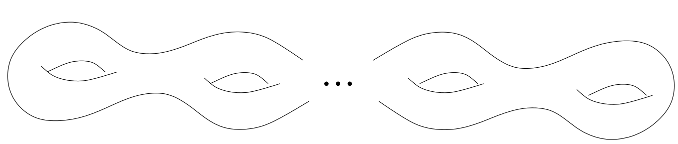


4. The only connected manifolds of dimension 1 are the line $\mathbb{R}$ *and the circle* $\mathbb{S}^1$*. What this statement means is that any connected manifold of dimension 1 is diffeomorphic to* $\mathbb{R}$ *or to* $\mathbb{S}^1$*.*

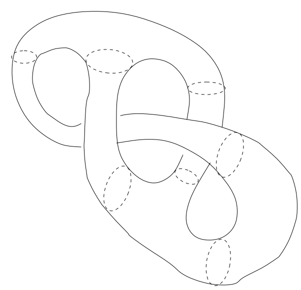
5. The manifolds of dimension 2 include the compact surfaces of genus g*. For* $g = 0$ *this is the sphere* $\mathbb{S}^2$*. For* $g = 1$ *this is the torus:*



For $g > 1$*, the* ***compact surface of genus*** g *is a* g*-holed torus:*



You should note, however, that a compact surface of genus g can be embedded in $\mathbb{R}^3$ in many forms. Here is one example (can you figure out what is the genus of this surface?):



You should note that in the definitions we have adopted so far in this section we have chosen the smooth category, where differentiable maps have all partial derivatives of all orders. We could have chosen other classes, such as continuous maps, C^k-maps, or analytic maps([1]). This would lead us to the categories of **topological manifolds**, C^k **manifolds** or **analytic manifolds**. Note that in each such category we have an appropriate notion of equivalence: for example, two topological manifolds X and Y are equivalent if and only if there exists a homeomorphism between them, i.e., a continuous bijection $f : X \to Y$ such that the inverse is also continuous.

Examples 0.4.

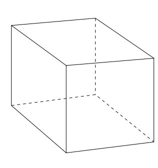
1. *Let $I = [-1, 1]$. The unit cube d-dimensional cube is the set:*

$$I^d = \{(x^1, \dots, x^d) \in \mathbb{R}^{d+1} : x^i \in I, \textit{ for all } i = 1, \dots, n\}.$$

The boundary of the cube

$$\partial I^d = \{(x^1, \dots, x^d) \in I^d : x^i = -1 \textbf{\textit{ or }} 1, \textit{ for some } i = 1, \dots, n\}.$$

is a topological manifold of dimension $d - 1$, which is not a smooth manifold.



[1]We shall also use the term C^k**-map**, $k = 1, 2, ..., +\infty$, for a map whose partial derivatives of all orders up to k exist and are continuous, and we make the conventions that a C^0**-map** is simply a continuous map and a C^ω**-map** means an analytic map. A C^k-map which is invertible and whose inverse is also a C^k-map is called a C^k**-equivalence** or a C^k**-isomorphism**.

2. *If $f : \mathbb{R}^d \to \mathbb{R}^l$ is any map of class C^k, its graph:*

$$\text{Graph}(f) := \{(x, f(x)) : x \in \mathbb{R}^d\} \subset \mathbb{R}^{d+l}$$

is a C^k-manifold of dimension d. Similarly, if f is any analytic map then Graph(f) *is an analytic manifold.*

Most of the times we will be working with smooth manifolds. However, there are many situations where it is desirable to consider other categories of manifolds, so you should keep them in mind.

You may wonder if the dimension d that appears in the definition of a manifold is a well defined integer, in other words if a manifold $M \subset \mathbb{R}^n$ could be of dimension d and d', for distinct integers $d \neq d'$. The reason that this cannot happen is due to the following important result:

Theorem 0.5 (Invariance of Domain)**.** *Let $U \subset \mathbb{R}^n$ be an open set and let $\phi : U \to \mathbb{R}^n$ be a 1:1, continuous map. Then $\phi(U)$ is open.*

The reason for calling this result "invariance of domain" is that a domain is a connected open set of $\mathbb{R}^n$, so the result says that the property of being a domain remains invariant under a continuous, 1:1 map. The proof of this result requires some methods from algebraic topology and so we will not give it here. We leave it as an exercise to show that the invariance of domain implies that the dimension of a manifold is a well defined integer.

Homework.

1. Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ and $Z \subset \mathbb{R}^p$. If $f : X \to Y$ and $g : Y \to Z$ are smooth maps, show that $g \circ f : X \to Z$ is also a smooth map.

2. Let $f : \mathbb{R}^d \to \mathbb{R}^m$ be a map of class C^k, $k = 0, \ldots, \omega$. Show that $\phi : \mathbb{R}^d \to$ Graph(f), $x \mapsto (x, f(x))$, is a C^k-equivalence.

3. Show that the sphere $\mathbb{S}^d$ and the boundary of the cube ∂I^{d+1} are equivalent topological manifolds.

4. Consider the set SL$(2, \mathbb{R})$ formed by all 2×2 matrices with real entries and determinant 1:

$$\text{SL}(2, \mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad - bc = 1 \right\} \subset \mathbb{R}^4.$$

Show that SL$(2, \mathbb{R})$ is a 3-dimensional smooth manifold.

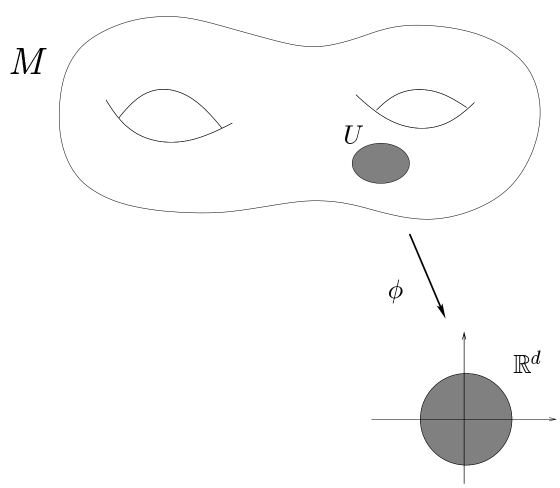
5. Use invariance of domain to show that the notion of dimension of a topological manifold is well defined.

1. Abstract Manifolds

In many situations manifolds do not arise naturally as subsets of Euclidean space. We will see several examples of this later. For that reason, the definition of manifold that we have seen in the previous section is often not the most useful one. We need a different definition of a manifold, where M is not assumed a priori to be a subset of some $\mathbb{R}^n$. For this more abstract definition of manifold we need the set M to have a notion of proximity, in other words, we need M to be furnished with a topology. At this point, it maybe useful to remind yourself of the basics of point set topology.

In this more general context, the definition of a topological manifold is very simple:

Definition 1.1. *A topological space M is called a* ***topological manifold of dimension*** *d if every $p \in M$ has a neighborhood $U \subset M$ homeomorphic to some open subset $V \subset \mathbb{R}^d$.*



Some times one also calls a topological manifold a **locally Euclidean space**. In this more general context we still use the same notation as before: we call $\phi : U \to \mathbb{R}^d$ a **system of coordinates** or a **chart**, and the functions $\phi^i = x^i \circ \phi$ are called **coordinate functions**. We shall denote a system of coordinates by (U, ϕ). Often we write x^i instead of ϕ^i for the coordinate functions, in which case we may denote the system of coordinates by $(U, x^1, \ldots, x^d)$. We say that a system of coordinates (U, ϕ) is *centered at a point* $p \in M$ if $\phi(p) = 0$.

Example 1.2.
On $\mathbb{R}^3 - \{0\}$ consider the equivalence relation $\sim$, where $v \sim w$ if and only if $v = \lambda w$ for some real number $\lambda \neq 0$. The set of equivalence classes

$$\mathbb{RP}^2 := (\mathbb{R}^3 - \{0\})/\sim$$

can be identified with the set of straight lines in $\mathbb{R}^3$ that pass through the origin and it is called the ***projective plane***. *Denoting by $[x:y:z]$ the equivalence class of $(x,y,z) \in \mathbb{R}^3 - \{0\}$, we have the quotient map:*

$$\pi : \mathbb{R}^3 - \{0\} \to \mathbb{RP}^2, \quad (x,y,z) \mapsto [x:y:z].$$

On $\mathbb{RP}^2$ we consider the quotient topology, so $U \subset \mathbb{RP}^2$ is open if and only if $\pi^{-1}(U) \subset \mathbb{R}^2 - \{0\}$ is open. The maps given by:

$$\begin{aligned}
\phi_1 : U_1 \to \mathbb{R}^2, &\quad [x:y:z] \mapsto \left(\frac{y}{x}, \frac{z}{x}\right), \quad U_1 := \{[x:y:z] \in \mathbb{RP}^2 : x \neq 0\}, \\
\phi_2 : U_2 \to \mathbb{R}^2, &\quad [x:y:z] \mapsto \left(\frac{x}{y}, \frac{z}{y}\right), \quad U_2 := \{[x:y:z] \in \mathbb{RP}^2 : y \neq 0\}, \\
\phi_3 : U_3 \to \mathbb{R}^2, &\quad [x:y:z] \mapsto \left(\frac{x}{z}, \frac{y}{z}\right), \quad U_3 := \{[x:y:z] \in \mathbb{RP}^2 : z \neq 0\}.
\end{aligned}$$

are homeomorphisms onto $\mathbb{R}^2$. Since $\{U_1, U_2, U_3\}$ is an open cover of $\mathbb{RP}^2$, we conclude that the projective plane is a topological manifold of dimension 2.

There is a **tacit assumption** about the underlying topology of a manifold, that we will also adopt here, and which is the following:

Manifolds are assumed to be Hausdorff and second countable

This assumption has significant implications, as we shall see shortly, which are very useful in the study of manifolds (e.g., existence of partitions of unity or of Riemannian metrics). On the other hand, it means that in any construction of a manifold we have to show that the underlying topology satisfies these assumptions. This is often easy since, for example, any metric space satisfies these assumptions.

It should be noted, however, that non-Hausdorff manifolds do appear sometimes, for example when one forms quotients of (Hausdorff) manifolds (see Section 9). Manifolds which are not second countable can also appear (e.g., in sheaf theory), although we will not meet them in the course of these sections. We limit ourselves here to give two such examples.

Examples 1.3.

1. *On $\mathbb{R}^2 - \{0\}$ consider the horizontal lines $y = c$. This defines a partition of $\mathbb{R}^2 - \{0\}$ and so defines an equivalence relation $\sim$. The quotient space $M = \mathbb{R}^2 - \{0\}/\sim$ (with the quotient topology) is a topological one-dimensional manifold: we can cover M by two open sets:*

$$U_+ = \{[(1,y)] : y \in \mathbb{R}\}, \quad U_- = \{[(-1,y)] : y \in \mathbb{R}\},$$

for which we have homeomorphisms:

$$\phi_\pm : U_\pm \to \mathbb{R}, \quad [(\pm 1, y)] \mapsto y.$$

However, M is not a Hausdorff topological space since the points $[(1,0)]$ and $[(-1,0)]$ cannot be separated. One calls M the **line with 2 origins**.

2. *Consider on $M = \mathbb{R}^2$ the topology generated by sets of the form $U \times \{y\}$ where $U \subset \mathbb{R}$ is open and $y \in \mathbb{R}$. This topology does not have a countable basis.*

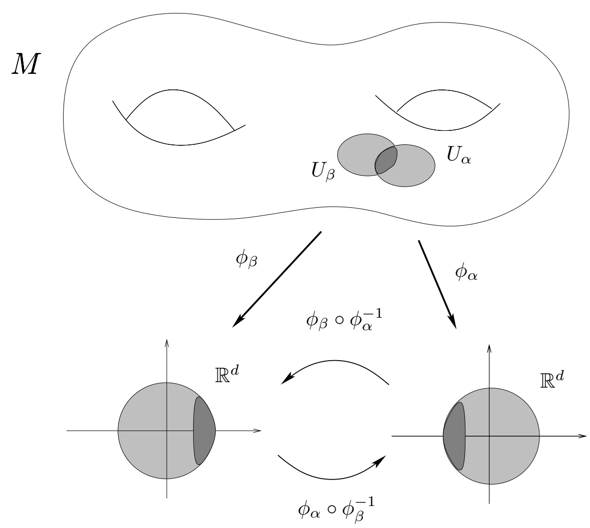
However, M is a topological one-dimensional manifold with charts $(U\times\{y\},\phi_y)$ given by $\phi_y(x,y)=x$. In this example, M is basically the disjoint, uncountable, union of copies of the real line, and it is not connected. It is possible to give examples of connected, Hausdorff, manifolds M which are not second countable, such as the **long line**.

Of course we are interested in smooth manifolds. The definition is slightly more involved:

Definition 1.4. *A* ***smooth structure*** *on a topological d-manifold M is a collection of coordinate systems $\mathcal{C}=\{(U_\alpha,\phi_\alpha):\alpha\in A\}$ which satisfies the following properties:*

(i) The collection $\mathcal{C}$ covers M: $\bigcup_{\alpha\in A}U_\alpha=M$;
(ii) For all $\alpha,\beta\in A$, the ***transition function*** *$\phi_\alpha\circ\phi_\beta^{-1}$ is a smooth map;*
(iii) The collection $\mathcal{C}$ is maximal: if (U,ϕ) is a coordinate system such that for all $\alpha\in A$ the maps $\phi\circ\phi_\alpha^{-1}$ and $\phi_\alpha\circ\phi^{-1}$ are smooth, then $(U,\phi)\in\mathcal{C}$.

The pair $(M,\mathcal{C})$ is called a ***smooth manifold*** *of dimension d.*



Given a topological manifold, a collection of coordinate systems which satisfies (i) and (ii) in the previous definition is called an **atlas**. Given an atlas $\mathcal{C}_0=\{(U_\alpha,\phi_\alpha):\alpha\in A\}$ there exists a unique maximal atlas $\mathcal{C}$ which contains $\mathcal{C}_0$: it is enough to define $\mathcal{C}$ to be the collection of all **smooth coordinate systems** relative to $\mathcal{C}$, i.e., all coordinate systems (U,ϕ) such that $\phi\circ\phi_\alpha^{-1}$ and $\phi_\alpha\circ\phi^{-1}$ are both smooth, for all $(U_\alpha,\phi_\alpha)\in\mathcal{C}_0$. For this reason, one often defines a smooth structure by specifying some atlas, and it is then implicit that the smooth structure is the one associated with the corresponding maximal atlas.

It should be clear from this definition that one can define in a similar fashion **manifolds of class** C^k for any $k = 1, \dots, +\infty, \omega$, by requiring the transition functions to be of class C^k. In these sections, we shall concentrate on the case $k = +\infty$.

EXAMPLES 1.5.

1. *The* ***standard differential structure on Euclidean space*** $\mathbb{R}^d$ *is the maximal atlas that contains the coordinate system* $(\mathbb{R}^d, i)$*, where* $i : \mathbb{R}^d \to \mathbb{R}^d$ *is the identity map. It is a non-trivial fact that the Euclidean space* $\mathbb{R}^4$ *has an infinite number of smooth structures, with the same underlying topology, but which are not equivalent to this one (in a sense to be made precise later). These are called* ***exotic smooth structures****. It is also known that* $\mathbb{R}^d$*, for* $d \neq 4$*, has no exotic smooth structures.*
2. *If* $M \subset \mathbb{R}^n$ *is a d-dimensional manifold in the sense of Definition 0.2, then* M *carries a natural smooth structure: the coordinate systems in Definition 0.2 form a maximal atlas (exercise) for the topology on* M *induced from the usual topology on* $\mathbb{R}^n$*. We shall see later in Section 7, that the Whitney Embedding Theorem shows that, conversely, any smooth manifold* M *arises in this way. Henceforth, we shall refer to a manifold* $M \subset \mathbb{R}^n$ *in the sense of Definition 0.2 as an* ***embedded manifold*** *in* $\mathbb{R}^n$*.*
3. *If* M *is a d-dimensional smooth manifold with smooth structure* $\mathcal{C}$ *and* $U \subset M$ *is an* ***open subset****, then* U *with the relative topology is also a smooth d-dimensional manifold with smooth structure given by:*
$$\mathcal{C}_U = \{(U_\alpha \cap U, \phi_\alpha|_{U_\alpha \cap U}) : (U, \phi_\alpha) \in \mathcal{C}\}.$$
4. *If* M *and* N *are smooth manifolds then the* ***Cartesian product*** $M \times N$*, with the product topology, is a smooth manifold: in* $M \times N$ *we consider the maximal atlas that contains all coordinate systems of the form* $(U_\alpha \times V_\beta, \phi_\alpha \times \psi_\beta)$*, where* (U_α, ϕ_α) *and* (V_β, ψ_β) *are smooth coordinate systems of* M *and* N*, respectively. It should be clear that* $\dim M \times N = \dim M + \dim N$*. More generally, if* $M_1, \dots, M_k$ *are smooth manifolds then* $M_1 \times \cdots \times M_k$ *is a smooth manifold of dimension* $\dim M_1 + \cdots + \dim M_k$*. For example, the* d***-torus*** $\mathbb{T}^d = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ *and the* ***cylinders*** $\mathbb{R}^n \times \mathbb{S}^m$ *are smooth manifolds of dimensions* d *and* $n+m$*, respectively.*
5. *Generalizing the projective plane, one defines the* ***real projective space*** *as the set*
$$\mathbb{RP}^d := \left\{ L \subset \mathbb{R}^{d+1} : L \textit{ is a straight line through the origin} \right\},$$
which we can think of as the quotient space of $\mathbb{R}^{d+1} - \{0\} / \sim$ *by the equivalence relation:*
$$(x^0, \dots, x^d) \sim (y^0, \dots, y^d) \textit{ if and only if } (x^0, \dots, x^d) = \lambda(y^0, \dots, y^d),$$
for some $\lambda \in \mathbb{R} - 0$*. On* $\mathbb{RP}^d$ *we take the quotient topology, so it becomes a topological manifold of dimension d: if we denote by* $[x^0 : \cdots : x^d]$ *the equivalence class of* $(x^0, \dots, x^d) \in \mathbb{R}^{d+1} - \{0\}$*, then for each* $\alpha = 0, \dots, n$ *we*

have the coordinate system (U_α, ϕ_α) where:

$$U_\alpha = \left\{[x^0 : \cdots : x^d] : x^\alpha \neq 0\right\},$$
$$\phi_\alpha : U_\alpha \to \mathbb{R}^d, \quad [x^0 : \cdots : x^d] \mapsto (\frac{x^0}{x^\alpha}, \ldots, \frac{\widehat{x^\alpha}}{x^\alpha}, \ldots, \frac{x^d}{x^\alpha})$$

(the symbol $\widehat{a}$ means that we omit the term a). We leave it as an exercise to check that the transition functions between these coordinate functions are smooth, so they form an atlas on $\mathbb{RP}^d$. Note that $\mathbb{RP}^d$ does not arise naturally as a subset of some Euclidean space.

We have established what are our objects. Now we turn to the morphisms.

Definition 1.6. *Let M and N be smooth manifolds.*

(i) A function $f : M \to \mathbb{R}$ is called a ***smooth function*** *if $f \circ \phi^{-1}$ is smooth for all smooth coordinate systems (U, ϕ) of M.*

(ii) A map $\Psi : M \to N$ is called a ***smooth map*** *if $\tau \circ \Psi \circ \phi^{-1}$ is smooth for all smooth coordinate systems (U, ϕ) of M and (V, τ) of N.*

A smooth map $\Psi : M \to N$ which is invertible and whose inverse is smooth is called a ***diffeomorphism****. In this case we say that M and N are* ***diffeomorphic manifolds****.*

Note that to check that a map $\Psi : M \to N$ is smooth, it is enough to verify that for each $p \in M$, there exist a smooth chart (U, ϕ) of M with $p \in U$ and a smooth chart (V, τ) of N with $\Psi(p) \in V$, such that $\tau \circ \Psi \circ \phi^{-1}$ is a smooth map. Also, a smooth function $f : M \to \mathbb{R}$ is just a smooth map where $\mathbb{R}$ has its standard smooth structure.

Clearly, the composition of two smooth maps, whenever defined, is a smooth map. The identity map is also a smooth map. So we have the **category of smooth manifolds**, whose objects are the smooth manifolds and whose morphisms are the smooth maps.

Just as we did for maps between subsets of Euclidean space, when $X \subset M$ and $Y \subset N$ are arbitrary subsets of some smooth manifolds, we will say that $\Psi : X \to Y$ is a **smooth map** if for each $p \in X$ there is an open neighborhood $U \subset M$ and a smooth map $F : U \to N$ such that $F|_{U \cap X} = \Psi|_{U \cap X}$.

The set of smooth maps from X to Y will be denoted $C^\infty(X; Y)$. When $Y = \mathbb{R}$, we use $C^\infty(X)$ instead of $C^\infty(X; \mathbb{R})$.

Examples 1.7.

1. *If $M \subset \mathbb{R}^n$ is an embedded manifold, any smooth function $F : U \to \mathbb{R}$ defined on an open $\mathbb{R}^n \supset U \supset M$ induces, by restriction, a smooth function $f : M \to \mathbb{R}$. Conversely, every smooth function $f : M \to \mathbb{R}$ is the restriction of some smooth function $F : U \to \mathbb{R}$ defined on some open set $\mathbb{R}^n \supset U \supset M$. To see this we will need the partitions of unity to be introduced in Section 3.*

   You should also check that if $M \subset \mathbb{R}^n$ and $N \subset \mathbb{R}^m$ are embedded manifolds then $\Psi : M \to N$ is a smooth map if and only if for every $p \in M$ there exists an open neighborhood $U \subset \mathbb{R}^n$ of p and a smooth map $F : U \to \mathbb{R}^m$ such that $\Psi|_{U\cap M} = F|_{U\cap M}$. This shows that the notion of smooth map in Definition 1.6 extends the notion we have introduced in the previous section.

2. *The map $\pi : \mathbb{S}^d \to \mathbb{RP}^d$ defined by:*

$$\pi(x^0, \dots, x^d) = [x^0 : \cdots : x^d],$$

is a smooth map. Moreover, any smooth function $F : \mathbb{S}^d \to \mathbb{R}$ which is invariant under inversion (i.e., $F(-x) = F(x)$), induces a smooth function $f : \mathbb{RP}^d \to \mathbb{R}$: the function f is the unique one that makes the following diagram commutative:

$$\begin{tikzcd} \mathbb{S}^d \arrow[r, "F"] \arrow[d, "\pi"'] & \mathbb{R} \\ \mathbb{RP}^d \arrow[ur, dashed, "f"'] & \end{tikzcd}$$

Conversely, every smooth function in $C^\infty(\mathbb{RP}^d)$ arises in this way.

If we are given two smooth structures $\mathcal{C}_1$ and $\mathcal{C}_2$ on the same manifold M we say that they are **equivalent smooth structures** if there is a diffeomorphism $\Psi : (M, \mathcal{C}_1) \to (M, \mathcal{C}_2)$.

Example 1.8.

On the line $\mathbb{R}$ the identity map $\mathbb{R} \to \mathbb{R}$, $x \mapsto x$, gives a chart which defines a smooth structure $\mathcal{C}_1$. We can also consider the chart $\mathbb{R} \to \mathbb{R}$, $x \mapsto x^3$, and this defines a distinct smooth structure $\mathcal{C}_2$ on $\mathbb{R}$ (why?). However, these two smooth structures are equivalent since the map $x \mapsto x^3$ gives a diffeomorphism from $(M, \mathcal{C}_2)$ to $(M, \mathcal{C}_1)$.

It is known that every topological manifold of dimension less or equal than 3 has a unique smooth structure. For dimension greater than 3 the situation is much more complicated, and not much is known. However, as we have mentioned before, the smooth structures on $\mathbb{R}^d$, compatible with the usual topology, are all equivalent if $d \neq 4$, and there are uncountably many inequivalent **exotic smooth structures** on $\mathbb{R}^4$. On the other hand, for the sphere $\mathbb{S}^d$ there are no exotic smooth structures for $d \leq 6$ but Milnor found that $\mathbb{S}^7$ has 27 inequivalent smooth structures. Its known, e.g., that $\mathbb{S}^{31}$ has more than 16 million inequivalent smooth structures!

Homework.

1. Let M be a topological manifold. Show that M is locally compact, i.e., every point of M has a compact neighborhood.

2. The Urysohn's Metrization Theorem states that a Hausdorff, regular, topological space with a countable basis is metrizable. Use this to show that every topological manifold M is metrizable.
Hint: A topological space X is called *regular* if given a closed set $A \subset X$ and a point $x \notin A$, there exist disjoint open sets U and V with $x \in U$ and $A \subset V$.

3. Let M be a *connected* topological manifold. Show that M is path connected. If, additionally, M is a smooth manifold, show that for any $p, q \in M$ there exists a smooth path $c : [0,1] \to M$ with $c(0) = p$ and $c(1) = q$.
Hint: Given any smooth path $c : [0,1] \to \mathbb{R}^n$ there is a smooth function $\tau : \mathbb{R} \to \mathbb{R}$, with $\tau(t) = 0$ if $t \leq 0$, $\tau(t) = 1$ if $t \geq 1$, and $\tau'(t) > 0$ if $t \in]0,1[$, so that $c_\tau := c \circ \tau : [0,1] \to \mathbb{R}^n$ is a new smooth path with the same image as c and $c'_\tau(0) = c'_\tau(1) = 0$.

4. Let $\phi : \mathbb{R}^m \to \mathbb{R}^n$ be a diffeomorphism. Use the chain rule to deduce that one must have $m = n$. Use this result to conclude that if M and N are diffeomorphic smooth manifolds then $\dim M = \dim N$, without appealing to invariance of domain.

5. Compute the transition functions for the atlas of real projective space $\mathbb{RP}^d$ and show that they are smooth. Show also that:
(a) $\mathbb{RP}^1$ is diffeomorphic to $\mathbb{S}^1$;
(b) $\mathbb{RP}^d - \mathbb{RP}^{d-1}$ is diffeomorphic to the open ball $B^n = \{x \in \mathbb{R}^d : ||x|| < 1\}$, where we identify $\mathbb{RP}^{d-1}$ with the subset $\{[x^0 : \cdots : x^d] : x^d = 0\} \subset \mathbb{RP}^d$.

6. The **complex projective d-dimensional space** is the set

$$\mathbb{CP}^d = \left\{L \subset \mathbb{C}^{d+1} : L \text{ is a complex line through the origin}\right\}.$$

Construct a smooth structure of $2d$-dimensional manifold on $\mathbb{CP}^d$ similar to the construction of a smooth structure on real projective space $\mathbb{RP}^d$.
Note: One identifies $\mathbb{C} \simeq \mathbb{R}^2$ by setting $(x + iy) \mapsto (x, y)$.

7. Show that if $M \subset \mathbb{R}^n$ is a d-dimensional manifold in the sense of Definition 0.2, then M carries a natural smooth structure.
Note: One sometimes says that M is an **embedded** manifold in $\mathbb{R}^n$ or a d-**surface** in $\mathbb{R}^n$. When $d = 1$, one says that M is a **curve**, when $d = 2$ one says that M is a **surface**, and when $k = n-1$ one says that M is an **hypersurface**.

8. Let $M \subset \mathbb{R}^n$ be a subset with the following property: for each $p \in M$, there exists an open set $U \subset \mathbb{R}^n$ containing p and diffeomorphism $\Phi : U \to V$ onto an open set $V \subset \mathbb{R}^n$, such that:

$$\Phi(U \cap M) = \left\{q \in V : q^{d+1} = \cdots = q^n = 0\right\}.$$

Show that M is a smooth manifold of dimension d (in fact, M is an embedded manifold or a d-surface in $\mathbb{R}^n$; see the previous exercise).

9. Let M be a set and assume that one has a collection $\mathcal{C} = \{(U_\alpha, \phi_\alpha) : \alpha \in A\}$, where $U_\alpha \subset M$ and $\phi_\alpha : U_\alpha \to \mathbb{R}^d$, satisfy the following properties:
   (a) For each $\alpha \in A$, $\phi_\alpha(U_\alpha) \subset \mathbb{R}^n$ is open and $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a bijection
   (b) For each $\alpha, \beta \in A$, the sets $\phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$ are open.
   (c) For each $\alpha, \beta \in A$, with $U_\alpha \cap U_\beta \neq \emptyset$ the map $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is smooth.
   (d) There is a countable set of U_α that cover M.
   (e) For any $p, q \in M$, with $p \neq q$, either there exists a U_α such that $p, q \in U_\alpha$, or there exists U_α and U_β, with $p \in U_\alpha$, $q \in U_\beta$ and $U_\alpha \cap U_\beta = \emptyset$.

   Show that there exists a unique smooth structure on M such that the collection $\mathcal{C}$ is an atlas.

10. Let $M = \mathbb{C} \cup \{\infty\}$. Let $U := M - \{\infty\} = \mathbb{C}$ and $\phi_U : U \to \mathbb{C}$ be the identity map and let $V := M - \{0\}$ and $\phi_V : V \to \mathbb{C}$ be the map $\phi_V(z) = 1/z$, with the convention that $\phi(\infty) = 0$. Use the previous exercise to show that M has a unique smooth structure with atlas $\mathcal{C} := \{(U, \phi_U), (V, \phi_V)\}$. Show that M is diffeomorphic to $\mathbb{S}^2$.
   Hint: Be careful with item (e)!

11. Let M and N be smooth manifolds and let $\Psi : M \to N$ be a map. Show that the following statements are equivalent:
   (i) $\Psi : M \to N$ is smooth.
   (ii) For every $p \in$M there are smooth coordinate systems (U, ϕ) of M and (V, τ) of N, with $p \in U$ and $\Psi(p) \in V$, such that $\tau \circ \Psi \circ \phi^{-1}$ is smooth.
   (iii) There exist atlases $\{(U_\alpha, \phi_\alpha) : \alpha \in A\}$ and $\{(U_\beta, \psi_\beta) : \beta \in B\}$ of M and N, such that for each $\alpha \in A$ and $\beta \in B$, $\psi_\beta \circ \Psi \circ \phi_\alpha^{-1}$ is smooth.

12. Let M and N be smooth manifolds and let $\Phi : M \to N$ be a map. Show that:
   (i) If Φ is smooth, then for every open set $U \subset M$ the restriction $\Phi|_U : U \to N$ is a smooth map.
   (ii) if every $p \in M$ has an open neighborhood U such that the restriction $\Phi|_U : U \to N$ is a smooth map, then $\Phi : M \to N$ is smooth.

2. Manifolds with Boundary

There are many spaces, such as the closed unit disk, a solid donought or the Möbius strip, which just fail to be a manifold because they have a "boundary". One can remedy this situation by trying to enlarge the notion of manifold so that it includes this possibility. The clue to be able to include boundary points is to understand what is the local model around points in the "boundary" and this turns out to be the **closed half-space** $\mathbb{H}^d$:

$$\mathbb{H}^d := \{(x^1, \dots, x^d) \in \mathbb{R}^d : x^d \geq 0\}.$$

We will denote the **open half-space** by:

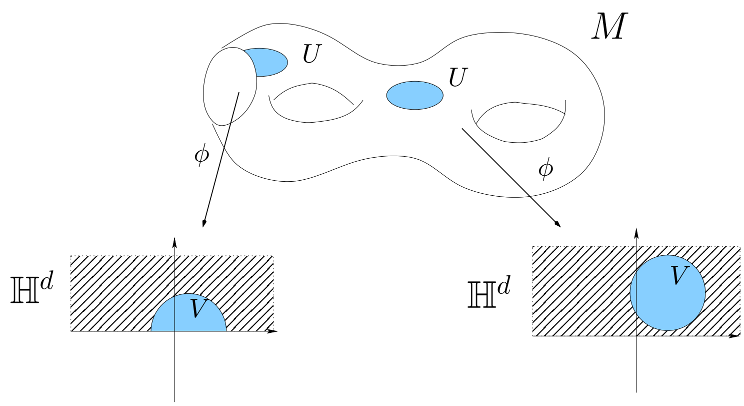
$$\operatorname{Int} \mathbb{H}^d =: \{(x^1, \dots, x^d) \in \mathbb{R}^d : x^d > 0\}.$$

and the **boundary of the closed half-space** by:

$$\partial\,\mathbb{H}^d =: \{(x^1,\dots,x^d)\in\mathbb{R}^d : x^d = 0\}.$$

When $n=0$, we have $\mathbb{H}^0=\mathbb{R}^0=\{0\}$, so $\mathrm{Int}\,H^0=\mathbb{R}^0$ and $\partial\,\mathbb{H}^0=\emptyset$.

Definition 2.1. *A **topological manifold with boundary of dimension** d is a topological space M such that every $p\in M$ has a neighborhood U which is homeomorphic to some open set $V\subset\mathbb{H}^d$.*



Just as we do for manifolds without boundary, we shall assume that all manifolds with boundary are Hausdorff and have a countable basis of open sets.

We shall use the same notations as before, so we call a homeomorphism $\phi : U\to V$ as in the definition a system of coordinates or a coordinate chart. Note that there are two types of open sets in $\mathbb{H}^d$ according to whether they intersect $\partial\,\mathbb{H}^d$ or not. These give rise to two types of coordinate systems $\phi : U\to V$, according to whether V intersects $\partial\,\mathbb{H}^d$ or not. In the first case, when $V\cap\partial\,\mathbb{H}^d=\emptyset$, we just have a coordinate system of the same sort as for manifolds without boundary, and we call it an **interior chart**. In the second case, when $V\cap\partial\,\mathbb{H}^d\neq\emptyset$, we call it a **boundary chart**.

Using Invariance of Domain (Theorem 0.5), one shows that:

Lemma 2.2. *Let M be a topological manifold with boundary of dimension d. If for some chart (U,ϕ) we have $\phi(p)\in\partial\,\mathbb{H}^d$, then this is also true for every other chart.*

Proof. Exercise. □

This justifies the following definition:

Definition 2.3. *Let M be a topological manifold with boundary of dimension d. A point $p\in M$ is called a **boundary point** if there exists some chart (U,ϕ) with $p\in U$, such that $\phi(p)\in\partial\,\mathbb{H}^d$. Otherwise, p is called an **interior point**.*

The set of boundary points of M will be denoted by ∂M and is called the **boundary of M** and the set of interior points of M will be denoted by $\operatorname{Int} M$ and is called the **interior of M**. If on both sets we consider the topology induced from M, we have:

Proposition 2.4. *Let M be a topological manifold with boundary of dimension $d > 0$. Then $\operatorname{Int} M$ and ∂M are topological manifolds without boundary of dimension d and $d-1$, respectively. If N is another manifold with boundary and $\Psi : M \to N$ is a homeomorphism then Ψ restricts to homeomorphisms $\Psi|_{\partial M} : \partial M \to \partial N$ and $\Psi|_{\operatorname{Int} M} : \operatorname{Int} M \to \operatorname{Int} N$.*

Proof. Let $p \in \operatorname{Int} M$ and let $\phi : U \to V$ be a chart with $p \in U$ and $V \subset \mathbb{H}$. Then if we set $V_0 := V - \partial \mathbb{H}$ and $U_0 := \phi^{-1}(V_0)$, we have that U_0 is an open neighborhood of M, V_0 is open in $\mathbb{R}^d$, and $\phi|_{U_0} : U_0 \to V_0$ is a homeomorphism. This shows that $\operatorname{Int} M$ is a topological manifold without boundary of dimension d.

On the other hand, let $p \in \partial M$ and let $\phi : U \to V$ be a chart with $p \in U$ and $\phi(p) \in \partial \mathbb{H}$. Then if we set $V_0 := V \cap \partial \mathbb{H}$ and $U_0 := \phi^{-1}(V_0)$, we have that $U_0 = U \cap \partial M$ is an open neighborhood of ∂M, V_0 is open in $\partial \mathbb{H} \simeq \mathbb{R}^{d-1}$, and $\phi|_{U_0} : U_0 \to V_0$ is a homeomorphism. This shows that ∂M is a topological manifold without boundary of dimension $d-1$. $\square$

It is important not to confuse the notions of interior and boundary point for manifolds with boundary with the usual notions of interior and boundary point of a subset of a topological space. If M happens to be a manifold with boundary embedded in some $\mathbb{R}^n$ then the two notions may or may not coincide, as shown by the following examples.

Examples 2.5.

1. *$M = \mathbb{H}^d$ is itself a topological manifold with boundary of dimension d, where $\operatorname{Int} M = \operatorname{Int} \mathbb{H}^d$ and $\partial M = \partial \mathbb{H}^d$, so our notations are consistent. If we think of $\mathbb{H}^d \subset \mathbb{R}^d$, then these notions coincide with the usual notions of boundary and interior of $\mathbb{H}^d$ as a topological subspace of $\mathbb{R}^d$.*
2. *The closed unit disk:*

$$D^k = \overline{B^d} := \{x \in \mathbb{R}^d : ||x|| \leq 1\},$$

is a topological manifold with boundary of dimension d with interior the open unit ball B^d and boundary the unit sphere $\mathbb{S}^{d-1}$. If we think of $D^d \subset \mathbb{R}^d$, then these notions coincide with the usual notions of boundary and interior of D^d as a topological subspace of $\mathbb{R}^d$.

3. *The cube I^d is a topological manifold with boundary of dimension d. I^d and D^d are homeomorphic topological manifolds with boundary.*
4. *The Möbius strip $M \subset \mathbb{R}^3$ is a topological manifold with boundary $\partial M = \mathbb{S}^1$. Note that, as a topological subspace of $\mathbb{R}^3$, all points of M are boundary points!*

Now that we have the notion of chart for a topological manifold with boundary, we can define a **smooth structure** on a topological d-manifold with boundary M by exactly the same procedure as we did for manifolds without boundary: it is a collection of charts $\mathcal{C} = \{(U_\alpha, \phi_\alpha) : \alpha \in A\}$ which satisfies the following properties:

(i) The collection $\mathcal{C}$ is an open cover of M: $\bigcup_{\alpha \in A} U_\alpha = M$;
(ii) For all $\alpha, \beta \in A$, the **transition function** $\phi_\alpha \circ \phi_\beta^{-1}$ is a smooth map;
(iii) The collection $\mathcal{C}$ is maximal: if (U, ϕ) any coordinate system such that $\phi \circ \phi_\alpha^{-1}$ and $\phi_\alpha \circ \phi^{-1}$ are smooth maps for all $\alpha \in A$, then $(U, \phi) \in \mathcal{C}$.

The pair $(M, \mathcal{C})$ is called a **smooth d-manifold with boundary**.

Again, given an atlas $\mathcal{C}_0 = \{(U_\alpha, \phi_\alpha) : \alpha \in A\}$ (i.e., a collection satisfying (i) and (ii)), there exists a unique maximal atlas $\mathcal{C}$ which contains $\mathcal{C}_0$: it is enough to define $\mathcal{C}$ to be the collection of all **smooth charts** relative to $\mathcal{C}$, i.e., all coordinate systems (U, ϕ) such that $\phi \circ \phi_\alpha^{-1}$ and $\phi_\alpha \circ \phi^{-1}$ are both smooth, for all $(U_\alpha, \phi_\alpha) \in \mathcal{C}_0$.

The notion of smooth map $\Psi : M \to N$ between two manifolds with boundary is also defined in exactly the same way as in the case of manifolds without boundary.

Proposition 2.6. *Let M be a smooth manifold with boundary of dimension $d > 0$. Then $\operatorname{Int} M$ and ∂M are smooth manifolds without boundary of dimension d and $d-1$, respectively. If N is another smooth manifold with boundary and $\Psi : M \to N$ is a diffeomorphism then Ψ restricts to diffeomorphisms $\Psi|_{\partial M} : \partial M \to \partial N$ and $\Psi|_{\operatorname{Int} M} : \operatorname{Int} M \to \operatorname{Int} N$.*

Proof. Exercise. □

You should check that the half space $\mathbb{H}^d$, the closed disk D^d or the Möbius strip, are all smooth manifolds with boundary, while the cube I^d is not.

Although often one can work with manifolds with boundary much the same way as one can work with manifolds without boundary, some care must be taken. For example, the Cartesian product of two half-spaces is not a manifold with boundary (it is rather a *manifold with corners*, a notion we will not discuss). So the cartesian product of manifolds with boundary may not be a manifold with boundary. However, we do have the following result:

Proposition 2.7. *If M is a smooth manifold without boundary and N is a smooth manifold with boundary, then $M \times N$ is a smooth manifold with $\partial(M \times N) = M \times \partial N$ and $\operatorname{Int}(M \times N) = M \times \operatorname{Int} N$.*

Proof. Exercise. □

Example 2.8.
If M is a manifold without boundary and $I = [0, 1]$ then $M \times I$ is a manifold with boundary for which:

$$\operatorname{Int}(M \times I) = M \times]0, 1[, \quad \partial(M \times I) = M \times \{0\} \cup M \times \{1\}.$$

It is very cumbersome to write always "manifold without boundary", so we agree to refer to these simply as "manifolds", and add the qualitative "with boundary", whenever that is the case. You should be aware that in the literature it is common to use **non-bounded manifold** for a manifold in our sense, and to call a **closed manifold** a compact non-bounded manifold and **open manifold** a non-bounded manifold with no compact component.

From now on we will be dealing almost exclusively with smooth manifolds. Hence, for a smooth manifold, we will use the term "chart" (or "coordinate system") to mean "smooth chart" (or "smooth coordinate system").

HOMEWORK.

1. Use Invariance of Domain to show that if for a chart (U, ϕ) of a topological manifold with boundary one has $\phi(p) \in \partial\,\mathbb{H}^d$, then this also holds for every other chart.

2. Let $M \subset \mathbb{R}^d$ have the induced topology. Show that if M is a closed subset and a d-dimensional manifold with boundary then the topological boundary of M coincides with ∂M. Give a counterexample to this statement when M is not a closed subset.

3. Give the details of the proofs of Propositions 2.6 and 2.7

4. Let $M = D^2 \times \mathbb{S}^1$ be the solid torus (a 3-manifold with boundary). What is the boundary of the solid torus? How does this generalize to dimension > 3?

3. Partitions of Unity

When M is a smooth manifold and $f \in C^\infty(M)$, we define the **support of** f to be the closed set:

$$\operatorname{supp} f \equiv \overline{\{p \in M : f(p) \neq 0\}}.$$

Also, given a collection $\mathcal{C} = \{U_\alpha : \alpha \in A\}$ of subsets of M we say that

- $\mathcal{C}$ is **locally finite** if, for all $p \in M$, there exists a neighborhood $p \in O \subset M$ such that $O \cap U_\alpha \neq \emptyset$ for only a finite number of $\alpha \in A$.
- $\mathcal{C}$ is a **cover** of M if $\bigcup_{\alpha\in A} U_\alpha = M$.
- $\mathcal{C}_0 = \{U_\beta : \beta \in B\}$ is a **subcover** if $\mathcal{C}_0 \subset \mathcal{C}$ and $\mathcal{C}_0$ still covers M.
- $\mathcal{C}' = \{V_i : i \in I\}$ is a **refinement of a cover** $\mathcal{C}$ if it is itself a cover and for each $i \in I$ there exists $\alpha_i = \alpha(i) \in A$ such that $V_i \subset U_{\alpha_i}$.

Definition 3.1. *A **partition of unity** in a smooth manifold M is a collection $\{\phi_i : i \in I\} \subset C^\infty(M)$ such that:*

(i) the collection of supports $\{\operatorname{supp} \phi_i : i \in I\}$ is locally finite;
(ii) $\phi_i(p) \geq 0$ and $\sum_{i\in I} \phi_i(p) = 1$ for every $p \in M$.

A partition of unity $\{\phi_i : i \in I\}$ is called subordinated *to a cover $\{U_\alpha : \alpha \in A\}$ of M if for each $i \in I$ there exists $\alpha_i \in A$ such that $\operatorname{supp} \phi_i \subset U_{\alpha_i}$.*

Notice that the sum in (ii) is actually finite: by (i), for each $p \in M$ there is only a finite number of functions ϕ_i with $\phi_i(p) \neq 0$.

The existence of partitions of unity is not obvious, but we will see in this section that there are many partitions of unity on a manifold.

Theorem 3.2 (Existence of Partitions of Unity)**.** *Let M be a smooth manifold and let $\{U_\alpha : \alpha \in A\}$ be an open cover of M. Then there exists a countable partition of unity $\{\phi_i : i = 1, 2, \dots\}$, subordinated to the cover $\{U_\alpha : \alpha \in A\}$ and with $\operatorname{supp} \phi_i$ compact for all i.*

If we do not care about compact supports, for any open cover we can get partitions of unity with the same set of indices:

Corollary 3.3. *Let M be a smooth manifold and let $\{U_\alpha : \alpha \in A\}$ be an open cover of M. Then there exists a partition of unity $\{\phi_\alpha : \alpha \in A\}$ such that $\operatorname{supp} \phi_\alpha \subset U_\alpha$ for each $\alpha \in A$.*

Proof. By Theorem 3.2 there exists a countable partition of unity

$$\{\psi_i : i = 1, 2, \dots\}$$

subordinated to the cover $\{U_\alpha : \alpha \in A\}$. For each i we can choose a $\alpha = \alpha(i)$ such that $\operatorname{supp} \psi_i \subset U_{\alpha(i)}$. Then the functions

$$\phi_\alpha = \begin{cases} \sum_{\alpha(i)=\alpha} \psi_i, & \text{if } \{i : \alpha(i) = \alpha\} \neq \emptyset, \\ \\ 0 & \text{otherwise,} \end{cases}$$

form a partition of unity with $\operatorname{supp} \phi_\alpha \subset U_\alpha$, for all $\alpha \in A$. □

EXAMPLE 3.4.
For the sphere $\mathbb{S}^d$, consider the cover with the two opens sets $U_N := \mathbb{S}^d - N$ and $U_S := \mathbb{S}^d - S$. Then the corollary says that there exists a partition of unity subordinated to this cover with the same indices, i.e., a pair of non-negative smooth functions $\phi_N, \phi_S \in C^\infty(\mathbb{S}^d)$ with $\operatorname{supp} \phi_N \subset U_N$ and $\operatorname{supp} \phi_S \subset U_S$, such that $\phi_N(p) + \phi_S(p) = 1$, for all $p \in \mathbb{S}^d$.

Corollary 3.5. *Let $A \subset O \subset M$, where O is an open subset and A is a closed subset of a smooth manifold M. There exists a smooth function $\phi \in C^\infty(M)$ such that:*

(i) $0 \leq \phi(p) \leq 1$ *for each* $p \in M$*;*
(ii) $\phi(p) = 1$ *if* $p \in A$*;*
(iii) $\operatorname{supp} \phi \subset O$.

Proof. The open sets $\{O, M - A\}$ give an open cover of M. Therefore, by the previous corollary, there is a partition of unity $\{\phi, \psi\}$ with $\sup \phi \subset O$ and $\sup \psi \subset M - A$. The function ϕ satisfies (i)-(iii). □

Roughly speaking, partitions of unity are used to "glue" local properties (i.e., properties that hold on domains of local coordinates), giving rise to global properties of a manifold, as shown in the proof of the following result.

Corollary 3.6 (Extension Lemma for smooth maps). *Let M be a smooth manifold, $A \subset M$ a closed subset and $\Psi : A \to \mathbb{R}^n$ a smooth map. For any open set $A \subset U \subset M$ there exists a smooth map $\widetilde{\Psi} : M \to \mathbb{R}^n$ such that $\widetilde{\Psi}|_A = \Psi$ and $\operatorname{supp} \widetilde{\Psi} \subset U$.*

Proof. For each $p \in A$ we can find an open neighborhood $U_p \subset M$, such that we can extend $\Psi|_{U_p \cap A}$ to a smooth function $\widetilde{\Psi}_p : U_p \to \mathbb{R}^n$. By replacing U_p by $U_p \cap U$ we can assume that $U_p \subset U$. The sets $\{U_p, M - A; p \in A\}$ form an open cover of M so we can find a partition of unit $\{\phi_p : p \in A\} \cup \{\phi_0\}$, subordinated to this cover with $\operatorname{supp} \phi_p \subset U_p$. Now define $\widetilde{\Psi} : M \to \mathbb{R}^n$ by setting

$$\widetilde{\Psi} := \sum_{p \in A} \phi_p \widetilde{\Psi}_p.$$

Clearly $\widetilde{\Psi}$ has the required properties. $\square$

We now turn to the proof of Theorem 3.2. There are two main ingredients in the proof. The first one is that topological manifolds are **paracompact**, i.e., every open cover has an open locally finite refinement. This is in fact a consequence of our assumption that manifolds are Hausdorff and second countable, and we will use the following more precise versions:

(a) Every open cover of a topological manifold M has a countable subcover.
(b) Every open cover of a topological manifold M has a countable, locally finite refinement consisting of open sets with compact closures.

The proofs are left to the exercises. The second ingredient is the existence of "very flexible" smooth functions, some times called **bump functions**:

(c) for any $\varepsilon > \delta > 0$, there exists a function $\phi \in C^\infty(\mathbb{R}^d)$ such that $\phi(x) = 1$, if $x \in \overline{B_\delta(0)}$, and $\phi(x) = 0$, if $x \in B_\varepsilon(0)^c$.

This can be proved by observing that:

- The function $f : \mathbb{R} \to \mathbb{R}$ defined by:
$$f(x) = \begin{cases} \exp(-\frac{1}{x^2}), & x \neq 0, \\ 0, & x = 0. \end{cases}$$
is a smooth function.
- If $\delta > 0$, the function $g : \mathbb{R} \to \mathbb{R}$ defined by:
$$g(x) = f(x) f(\delta - x),$$
is smooth, $g(x) > 0$ if $x \in]0, \delta[$ and $g(x) = 0$ otherwise.
- The function $h : \mathbb{R} \to \mathbb{R}$ defined by:
$$h(x) := \frac{\int_0^x g(t)\,\mathrm{d}t}{\int_0^\delta g(t)\,\mathrm{d}t},$$
is smooth, non-decreasing, $h(x) = 0$ if $x \leq 0$ and $h(x) = 1$ if $x \geq \delta$.

Using these functions you should now be able to show that (c) holds.

Proof of Theorem 3.2. By (b) above, we can assume that the open cover $\{U_\alpha : \alpha \in A\}$ is countable, locally finite, and the sets $\overline{U}_\alpha$ are compact. If $p \in U_\alpha$, we can choose a smooth chart (V_p, τ), centered in p, with $V_p \subset U_\alpha$, and such that $\overline{B_\varepsilon(0)} \subset \tau(V_p)$ for some $\varepsilon > 0$. Now if ϕ is the function defined in (c) above, we set:

$$\psi_p := \begin{cases} \phi \circ \tau, & \text{in } V_p, \\ \\ 0, & \text{in } M - V_p. \end{cases}$$

Then $\psi_p \in C^\infty(M)$ is a non-negative function, taking the value 1 in an open set $W_p \subset V_p$ which contains p. Since $\{W_p : p \in M\}$ is an open cover of M, by (a) above, there exists a countable subcover $\{W_{p_1}, W_{p_2}, \dots\}$ of M. Then the open cover $\{V_{p_1}, V_{p_2}, \dots\}$ is locally finite and subordinated to the cover $\{U_\alpha : \alpha \in A\}$. Moreover, the closures $\overline{V}_{p_i}$ are compact.

The sum $\sum_i \psi_{p_i}$ may not be equal to 1. To fix this we observe that

$$\psi = \sum_{i=1}^{+\infty} \psi_{p_i},$$

is well defined, of class C^∞ and $\psi(p) > 0$ for every $p \in M$. If we define:

$$\phi_i = \frac{\psi_{p_i}}{\psi},$$

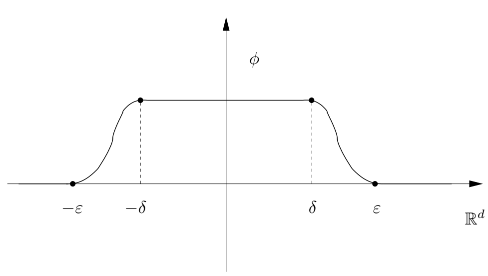
then the functions $\{\phi_1, \phi_2, \dots\}$ give a partition of unity, subordinated to the cover $\{U_\alpha : \alpha \in A\}$, with $\operatorname{supp} \phi_i$ compact for each $i = 1, 2, \dots$.

This completes the proof of Theorem 3.2. $\square$

Homework.

1. Show that $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = \exp(-1/x^2)$ is a smooth function.

2. Given any $\varepsilon > \delta > 0$, show that there exists a function $\phi \in C^\infty(\mathbb{R}^d)$ such that $0 \le \phi(x) \le 1$, $\phi(x) = 1$ if $|x| \le \delta$ and $\phi(x) = 0$ if $|x| > \varepsilon$.



3. Show that for a second countable topological space X, every open cover of X has a countable subcover.

Hint: If $\{U_\alpha : \alpha \in A\}$ is an open cover of X and $\mathcal{B} = \{V_j \in J\}$ is a countable basis of the topology of X, show that the collection $\mathcal{B}'$ formed by $V_j \in \mathcal{B}$ such that $V_j \subset U_\alpha$ for some α, is also a basis. Now, for each $V_j \in \mathcal{B}'$ choose some U_{α_j} containing V_j, and show that $\{U_{\alpha_j}\}$ is a countable subcover.

4. Show that a topological manifold is paracompact, in fact, show that every open cover of a topological manifold M has a countable, locally finite refinement consisting of open sets with compact closures.

   Hint: Show first that M can be covered by open sets $O_1, O_2, \ldots$, with compact closures and $\overline{O}_i \subset O_{i+1}$. Then given an arbitrary open cover $\{U_\alpha : \alpha \in A\}$ of M, choose for each $i \geq 3$ a finite subcover of the cover $\{U_\alpha \cap (O_{i+1} - \overline{O}_{i-2} : \alpha \in A\}$ of the compact set $\overline{O}_i - O_{i-1}$, and a finite subcover of the cover $\{U_\alpha \cap O_3 : \alpha \in A\}$ of the compact set $\overline{O}_2$. The collection of such open sets will do it.

5. Show that if $M \subset \mathbb{R}^n$ is an embedded manifold then a function $f : M \to \mathbb{R}$ is smooth if and only if there exists an open set $M \subset U \subset \mathbb{R}^n$ and a smooth function $F : U \to \mathbb{R}$ such that $F|_M = f$.

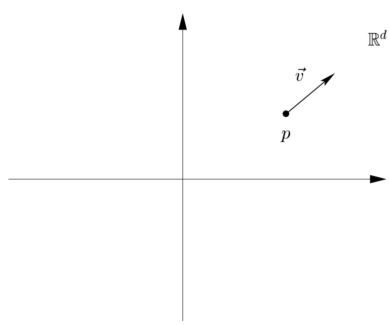
6. Show that the conclusion of the Extension Lemma for Smooth Maps may fail if $A \subset M$ is not assumed to be closed.

7. Show that Theorem 3.2 still holds for manifolds with boundary.

4. The Tangent Space

The tangent space to $\mathbb{R}^d$ at $p \in \mathbb{R}^d$ is by definition the set:

$$T_p\mathbb{R}^d := \left\{ (p, \vec{v}) : \vec{v} \in \mathbb{R}^d \right\}.$$



Note that this tangent space is a vector space over $\mathbb{R}$ where addition is defined by:

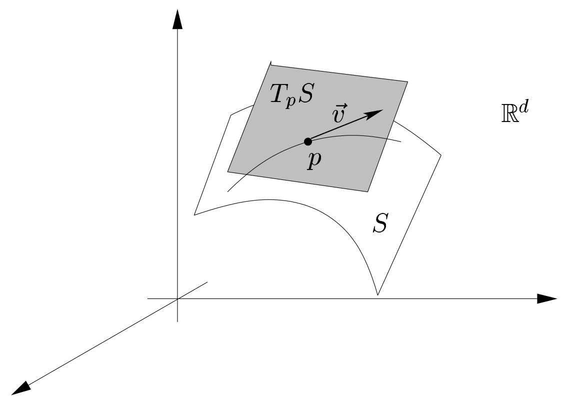
$$(p, \vec{v}_1) + (p, \vec{v}_2) \equiv (p, \vec{v}_1 + \vec{v}_2),$$

while multiplication is given by:

$$a(p, \vec{v}) \equiv (p, a\vec{v}).$$

Of course there is a natural isomorphism $T_p\mathbb{R}^d \simeq \mathbb{R}^d$, but in many situations it is better to think of $T_p\mathbb{R}^d$ as the set of vectors with origin at p.

This distinction is even more clear in the case of embedded manifolds, or d-surfaces, $S \subset \mathbb{R}^n$. In this case, we can define the tangent space to S at $p \in S$ to be the subspace $T_pS \subset T_p\mathbb{R}^n$ consisting of those tangent vectors $(p, \vec{v})$, for which there exists a smooth curve $c : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$, with $c(t) \in S$, $c(0) = p$ and $c'(0) = \vec{v}$.



A tangent vector $(p, \vec{v}) \in T_pS$ acts on smooth functions defined in a neighborhood of p: if $f : U \to \mathbb{R}$ is a smooth function defined on a open set U containing p then we can choose a smooth curve $c : (-\varepsilon, \varepsilon) \to U$, with $c(0) = p$ and $c'(0) = \vec{v}$, and set:

$$(p, \vec{v})(f) := \frac{d}{dt} f \circ c(0).$$

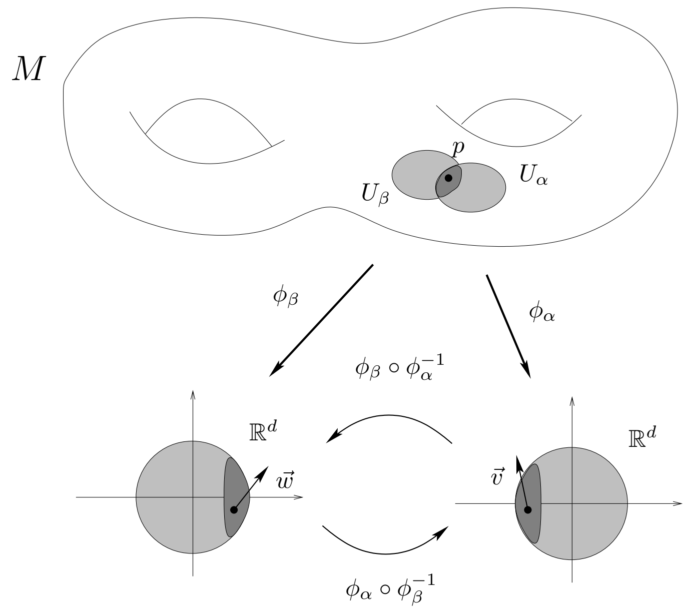
This operation does not depend on the choice of smooth curve c (exercise). In fact, this is just the usual notion of **directional derivative** of f at p in the direction $\vec{v}$.

We will now define the tangent space to an abstract manifold M at $p \in M$. There are several different approaches to define the tangent space at $p \in M$, which correspond to different points of view, all of them very useful. We shall give here three distinct descriptions and we leave it to the exercises to show that they are actually equivalent.

Description 1. Let M be a smooth d-dimensional manifold with an atlas $\mathcal{C} = \{(U_\alpha, \phi_\alpha) : \alpha \in A\}$. To each point $p \in M$ we would like to associate a copy of $\mathbb{R}^d$, so that each element $\vec{v} \in \mathbb{R}^d$ should represent a tangent vector. Of course if $p \in U_\alpha$, the system of coordinates ϕ_α gives an identification of an open neighborhood of p with $\mathbb{R}^d$. Distinct smooth charts will give different identifications, but they are all related by the transition functions.

This suggests one should consider triples $(p, \alpha, \vec{v}) \in M \times A \times \mathbb{R}^d$, with $p \in U_\alpha$, and that two such triples should be declared to be equivalent if

$$[p, \alpha, \vec{v}] = [q, \beta, \vec{w}] \qquad \text{iff} \qquad p = q \text{ and } (\phi_\alpha \circ \phi_\beta^{-1})'(\phi_\beta(p)) \cdot \vec{w} = \vec{v}.$$



Hence, we define a **tangent vector** to M at a point $p \in M$ to be an equivalence class $[p, \alpha, \vec{v}]$, and the **tangent space** at p to be the set of all such equivalence classes:

$$T_pM \equiv \left\{[p, \alpha, \vec{v}] : \alpha \in A, \vec{v} \in \mathbb{R}^d\right\}.$$

We leave it as an exercise to check that the operations:

$$[p, \alpha, \vec{v}_1] + [p, \alpha, \vec{v}_2] := [p, \alpha, \vec{v}_1 + \vec{v}_2], \quad a[p, \alpha, \vec{v}] := [p, \alpha, a\vec{v}],$$

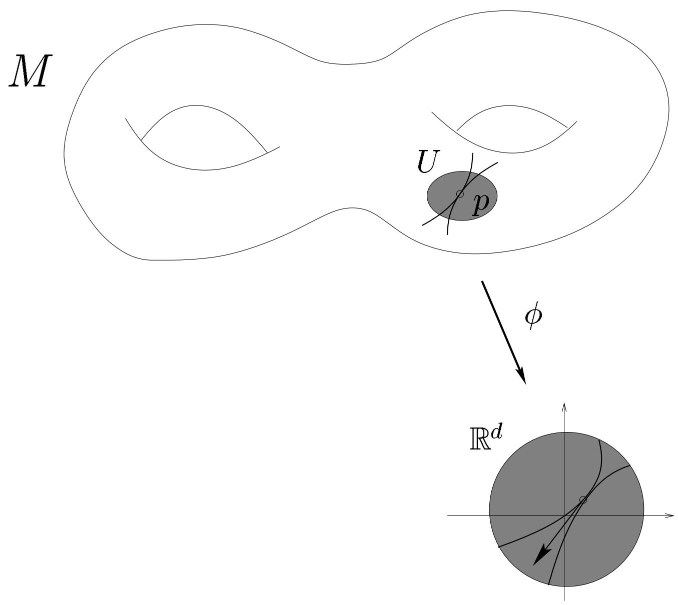
are well defined and give T_pM the structure of vector space over $\mathbb{R}$. Notice that we still have an isomorphism $T_pM \simeq \mathbb{R}^d$, but this isomorphism now depends on the choice of a chart.

Description 2. Again, fix $p \in M$. For this second description we will consider all smooth curves $c : (-\varepsilon, \varepsilon) \to M$, with $c(0) = p$. Two such smooth curves c_1 and c_2 will be declared equivalent if there exists some smooth chart (U, ϕ) with $p \in U$, such that

$$\frac{d}{dt}(\phi \circ c_1)(0) = \frac{d}{dt}(\phi \circ c_2)(0).$$

It should be clear that if this condition holds for some smooth chart around p, then it also holds for every other smooth chart around p belonging to the smooth structure.

We call a **tangent vector** at $p \in M$ an equivalence class of smooth curves $[c]$, and the set of all such classes is called the **tangent space** T_pM at the point p. Again, you should check that this tangent space has the structure of vector space over $\mathbb{R}$ and that T_pM is isomorphic to $\mathbb{R}^d$, through an isomorphism that depends on a choice of smooth chart.



Description 3. The two previous descriptions use smooth charts. Our third description has the advantage of not using charts, and it will be our final description of tangent vectors and tangent space.

Again we fix $p \in M$ and we look at the set of all smooth functions defined in some open neighborhood of p. Given two smooth functions $f : U \to \mathbb{R}$ and $g : V \to \mathbb{R}$, where U and V are open sets that contain p, we say that f and g define the same **germ at** p if there is an open set $W \subset U \cap V$ containing p and such that

$$f|_W = g|_W.$$

We denote by $\mathcal{G}_p$ the set of all germs at p. This set has the structure of an $\mathbb{R}$-algebra, where addition, product and multiplication by scalars are defined in the obvious way:

$$\begin{aligned} [f] + [g] &\equiv [f + g], \\ [f][g] &\equiv [fg], \\ a[f] &\equiv [af]. \end{aligned}$$

Notice also that it makes sense to talk of the value of a germ $[f] \in \mathcal{G}_p$ at the point p, which is $f(p)$. On the other hand, the value of $[f] \in \mathcal{G}_p$ at any other point $q \neq p$ is not defined.

Definition 4.1. *A* ***tangent vector*** *at a point $p \in M$ is a linear derivation of $\mathcal{G}_p$, i.e., a map $\mathbf{v} : \mathcal{G}_p \to \mathbb{R}$ satisfying:*

(i) $\mathbf{v}(a[f] + b[g]) = a\mathbf{v}([f]) + b\mathbf{v}([g])$, $a, b \in \mathbb{R}$;

(ii) $\mathbf{v}([f][g]) = \mathbf{v}([f])g(p) + f(p)\mathbf{v}([g])$;

The ***tangent space*** *at a point $p \in M$ is the set of all such tangent vectors and is denoted by T_pM.*

Since linear derivations can be added and multiplied by real numbers, it is clear that the tangent space T_pM has the structure of a real vector space.

Example 4.2.
Let $(U,\phi) = (U, x^1, \dots, x^d)$ be a coordinate system in M with $p \in U$. We define the tangent vectors $\frac{\partial}{\partial x^i}\big|_p \in T_pM$, $i = 1, \dots, d$, to be the derivations

$$\frac{\partial}{\partial x^i}\bigg|_p ([f]) = \frac{\partial(f \circ \phi^{-1})}{\partial x^i}\bigg|_{\phi(p)}.$$

Notice that the tangent vector $\frac{\partial}{\partial x^i}\big|_p$ corresponds to the direction one obtains by freezing all coordinates but the i-th coordinate.

In order to check that T_pM is a vector space with dimension equal to $\dim M$, consider the set of all germs that vanish at p:

$$\mathcal{M}_p = \{[f] \in \mathcal{G}_p : f(p) = 0\},$$

It is immediate to check that $\mathcal{M}_p \subset \mathcal{G}_p$ is a maximal ideal in $\mathcal{G}_p$. The k-th power of this ideal

$$\mathcal{M}_p^k = \underbrace{\mathcal{M}_p \cdots \mathcal{M}_p}_{k},$$

consists of germs that vanish to order k at p: if $[f] \in \mathcal{M}_p^k$ and (U,ϕ) is a coordinate system centered at p, then the smooth function $f \circ \phi^{-1}$ has vanishing partial derivatives at $\phi(p)$ up to order $k-1$. These powers form a tower of ideals

$$\mathcal{G}_p \supset \mathcal{M}_p \supset \mathcal{M}_p^2 \supset \cdots \supset \mathcal{M}_p^k \supset \dots$$

Theorem 4.3. *The tangent space T_pM is naturally isomorphic to $(\mathcal{M}_p/\mathcal{M}_p^2)^*$ and has dimension* $\dim M$.

Proof. First we check that if $[c] \in \mathcal{G}_p$ is the germ of the constant function $f(x) = c$ then $\mathbf{v}([c]) = 0$, for any tangent vector $\mathbf{v} \in T_pM$. In fact, we have that

$$\mathbf{v}([c]) = c\mathbf{v}([1]),$$

and that

$$\mathbf{v}([1]) = \mathbf{v}([1][1]) = 1\mathbf{v}([1]) + 1\mathbf{v}([1]) = 2\mathbf{v}([1]),$$

hence $\mathbf{v}([1]) = 0$.

Now if $[f] \in \mathcal{G}_p$ and $c = f(p)$, we remark that

$$\mathbf{v}([f]) = \mathbf{v}([f] - [c]),$$

so the derivation $\mathbf{v}$ is completely determined by its effect on $\mathcal{M}_p$. On the other hand, any derivation vanishes on $\mathcal{M}_p^2$, because if $f(p) = g(p) = 0$, then

$$\mathbf{v}([f][g]) = \mathbf{v}([f])g(p) + f(p)\mathbf{v}([g]) = 0.$$

We conclude that every tangent vector $\mathbf{v} \in T_pM$ determines a unique linear transformation $\mathcal{M}_p \to \mathbb{R}$, which vanishes on $\mathcal{M}_p^2$. Conversely, if

$L \in (\mathcal{M}_p/\mathcal{M}_p^2)^*$ is a linear transformation, we can define a linear transformation $\mathbf{v} : \mathcal{G}_p \to \mathbb{R}$ by setting

$$\mathbf{v}([f]) \equiv L([f] - [f(p)]).$$

This is actually a derivation (exercise), so we conclude that $T_pM \simeq (\mathcal{M}_p/\mathcal{M}_p^2)^*$.

In order to verify the dimension of T_pM, we choose some system of coordinates $(U, x^1, \dots, x^d)$ centered at p, and we show that the tangent vector

$$\left.\frac{\partial}{\partial x^i}\right|_p \in T_pM, \qquad i = 1, \dots, d,$$

form a basis for T_pM. If $f : U \to \mathbb{R}$ is any smooth function, then $f \circ \phi^{-1} : \mathbb{R}^d \to \mathbb{R}$ is smooth in a neighborhood of the origin. This function can be expanded as:

$$f \circ \phi^{-1}(x) = f \circ \phi^{-1}(0) + \sum_{i=1}^{d} \frac{\partial (f \circ \phi^{-1})}{\partial x^i}(0) x^i + \sum_{i,j} g_{ij}(x) x^i x^j,$$

where the g_{ij} are some smooth functions in a neighborhood of the origin. It follows that we have the expansion:

$$f(q) = f(p) + \sum_{i=1}^{d} \left.\frac{\partial (f \circ \phi^{-1})}{\partial x^i}\right|_{\phi(p)} x^i(q) + \sum_{i,j} h_{ij}(q) x^i(q) x^j(q),$$

where $h_{ij} \in C^\infty(U)$, valid for any $q \in U$. We conclude that for any tangent vector $\mathbf{v} \in T_pM$:

$$\mathbf{v}([f]) = \sum_{i=1}^{d} \left.\frac{\partial (f \circ \phi^{-1})}{\partial x^i}\right|_{\phi(p)} \mathbf{v}([x^i]).$$

In other words, we have:

$$\mathbf{v} = \sum_{i=1}^{d} a^i \left.\frac{\partial}{\partial x^i}\right|_p,$$

where $a^i = \mathbf{v}([x^i])$. This shows that the vectors $(\partial/\partial x^i)|_p \in T_pM$, $i = 1, \dots, \dim M$ form a generating set. We leave it as an exercise to show that they are linearly independent. $\square$

From now on, given $\mathbf{v} \in T_pM$ and a smooth function f defined in some neighborhood of $p \in M$ we set:

$$\mathbf{v}(f) \equiv \mathbf{v}([f]).$$

Note that $\mathbf{v}(f) = \mathbf{v}(g)$ if f and g coincide in a neighborhood of p and that:

$$\mathbf{v}(af + bg) = a\mathbf{v}(f) + b\mathbf{v}(g), \quad (a, b \in \mathbb{R}),$$
$$\mathbf{v}(fg) = f(p)\mathbf{v}(g) + \mathbf{v}(f)g(p),$$

where $af + bg$ and fg are only defined in the intersection of the domains of f and g.

The proof of Theorem 4.3 shows that if $(U,\phi)=(U,x^1,\dots,x^d)$ is a coordinate system around p, then any tangent vector $\mathbf{v}\in T_pM$ can be written as:

$$\mathbf{v}=\sum_{i=1}^d a^i \left.\frac{\partial}{\partial x^i}\right|_p .$$

The numbers $a^i=\mathbf{v}(x^i)$ are called the **components of tangent vector $\mathbf{v}$** in the coordinate system $(U,x^1,\dots,x^d)$. If we introduce the notation

$$\left.\frac{\partial f}{\partial x^i}\right|_p \equiv \left.\frac{\partial f\circ\phi^{-1}}{\partial x^i}\right|_{\phi(p)},$$

then:

$$\mathbf{v}(f)=\sum_{i=1}^d a^i \left.\frac{\partial f}{\partial x^i}\right|_p .$$

On the other hand, given another coordinate system $(V,y^1,\dots,y^d)$ we find that

$$\left.\frac{\partial}{\partial y^j}\right|_p=\sum_{i=1}^d \left.\frac{\partial x^i}{\partial y^j}\right|_p \left.\frac{\partial}{\partial x^i}\right|_p .$$

Hence, in this new coordinate system we have

$$\mathbf{v}=\sum_{j=1}^d b^j \left.\frac{\partial}{\partial y^j}\right|_p, \quad \text{with } b^j=\mathbf{v}(y^j),$$

where the new components b^j are related to the old components a^i by the transformation formula:

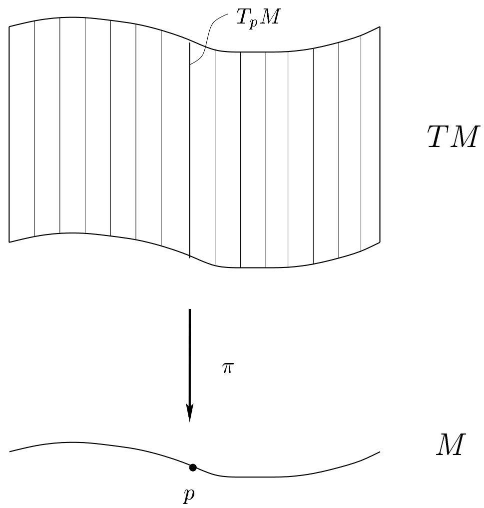
$$a^i=\sum_{j=1}^d \left.\frac{\partial x^i}{\partial y^j}\right|_p b^j. \tag{4.1}$$

Let us turn now to the question of how the tangent spaces vary from point to point. We define the **tangent bundle** to M as:

$$TM\equiv\bigcup_{p\in M} T_pM.$$

Notice that we have a natural projection $\pi: TM\to M$ which associates to a tangent vector $\mathbf{v}\in T_pM$ the corresponding base point $\pi(\mathbf{v})=p$. The term "bundle" comes from the fact that we can picture TM as a set of fibers (the spaces T_pM), juxtaposed with each other, forming a manifold:

Proposition 4.4. *TM has a natural smooth structure of manifolds of dimension $2\dim M$ such that the projection in the base is a smooth map.*



Proof. Let $\{(U_\alpha, \phi_\alpha) : \alpha \in A\}$ be an atlas for M. For each smooth chart $(U_\alpha, \phi_\alpha) = (U_\alpha, x^1, \dots, x^n)$, we define $\tilde{\phi}_\alpha : \pi^{-1}(U_\alpha) \to \mathbb{R}^{2d}$ by setting:

$$\tilde{\phi}_\alpha(\mathbf{v}) = (x^1(\pi(\mathbf{v})), \dots, x^d(\pi(\mathbf{v})), \mathbf{v}(x^1), \dots, \mathbf{v}(x^d)).$$

One checks easily that the collection:

$$\left\{\tilde{\phi}_\alpha^{-1}(O) : O \subset \mathbb{R}^{2d} \text{ open}, \alpha \in A\right\}$$

is a basis for a topology of TM, which is Hausdorff and second countable. Now, we have that:

(a) TM is a topological manifold with local charts $(\pi^{-1}(U_\alpha), \tilde{\phi}_\alpha)$.
(b) For any pair of charts $(\pi^{-1}(U_\alpha), \tilde{\phi}_\alpha)$ and $(\pi^{-1}(U_\beta), \tilde{\phi}_\beta)$, the transition functions $\tilde{\phi}_\beta \circ \tilde{\phi}_\alpha^{-1}$ are smooth.

We conclude that the collection $\left\{(\pi^{-1}(U_\alpha), \tilde{\phi}_\alpha) : \alpha \in A\right\}$ is an atlas, and so defines on TM the structure of a smooth manifold of dimension $\dim TM = 2\dim M$. Finally, the map $\pi : TM \to M$ is smooth because for each α we have that $\phi_\alpha \circ \pi \circ \tilde{\phi}_\alpha^{-1} : \mathbb{R}^{2d} \to \mathbb{R}^d$ is just the projection in the first d components. $\square$

We say that a d-dimensional manifold M has *trivial tangent bundle* if there is a diffeomorphism $\Psi : TM \to M \times \mathbb{R}^d$ commuting with the projections:

$$\begin{tikzcd} TM \arrow[rr, "\Psi"] \arrow[dr, "\pi"'] & & M \times \mathbb{R}^n \arrow[dl, "\mathrm{pr}_1"] \\ & M & \end{tikzcd}$$

whose restriction to each fiber is linear isomorphism $\Phi|_{T_pM} : T_pM \to \mathbb{R}^d$. For example, $\mathbb{R}^d$ and $\mathbb{T}^d$ have both trivial tangent bundles. However, we will see later that $\mathbb{S}^d$ has trivial tangent bundle if and only if $d = 1, 3$.

Homework.

1. Show that the 3 descriptions of tangent vectors given in this section are indeed equivalent.

2. In $\mathbb{R}^3$ consider the usual Cartesian coordinates (x, y, z). One defines **spherical coordinates** in $\mathbb{R}^3$ to be the smooth chart (U, ϕ), where $U = \mathbb{R}^3 - \{(x, 0, z) : x \geq 0\}$ and $\phi = (r, \theta, \varphi)$ is defined as usual by
   - $r(x, y, z) := \sqrt{x^2 + y^2 + z^2}$ is the distance to the origin;
   - $\theta(x, y, z)$ is the longitude, i.e., the angle in $]0, 2\pi[$ between the vector $(x, y, 0)$ and the x-axis;
   - $\varphi(x, y, z)$ is the co-latitude, i.e., the angle in $]0, \pi[$ between the vector (x, y, z) and the z-axis.

   Compute:
   (a) The components of the tangent vectors to $\mathbb{R}^3$ $\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \varphi}$ in Cartesian coordinates;
   (b) The components of the tangent vectors to $\mathbb{R}^3$ $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ in spherical coordinates.

3. Let $M \subset \mathbb{R}^n$ be an embedded d-manifold. Show that if $\psi : V \to M \cap U$ is a parameterization of a neighborhood of $p \in M$, then the tangent space T_pM can be identified with the subspace $\psi'(q)(\mathbb{R}^d) \subset \mathbb{R}^n$, where $p = \psi(q)$.

4. Let $(U, x^1, \ldots, x^d)$ be a local coordinate system in a manifold M. Show that the tangent vectors
$$\left.\frac{\partial}{\partial x^i}\right|_p \in T_pM, \qquad i = 1, \ldots, d,$$
are linearly independent.

5. Show that there is a canonical identification $T(M_1 \times M_2) \simeq TM_1 \times TM_2$ and use this to show that the torus $\mathbb{T}^d$ has a trivial tangent bundle.

5. The Differential

A smooth map between two smooth manifolds determines a linear transformation between the corresponding tangent spaces:

Definition 5.1. *Let $\Psi : M \to N$ be a smooth map. The* ***differential*** *of Ψ at $p \in M$ is the linear transformation $\mathrm{d}_p\Psi : T_pM \to T_{\Psi(p)}N$ defined by*
$$\mathrm{d}_p\Psi(\mathbf{v})(f) \equiv \mathbf{v}(f \circ \Psi),$$
where f is any smooth function defined in a neighborhood of $\Psi(p)$.

If $(U, \phi) = (U, x^1, \ldots, x^d)$ is a coordinate system around p and $(V, \psi) = (V, y^1, \ldots, y^e)$ is a coordinate system around $\Psi(p)$, we obtain
$$\mathrm{d}_p\Psi \cdot \left.\frac{\partial}{\partial x^i}\right|_p = \sum_{j=1}^{e} \left.\frac{\partial(\psi \circ \Psi \circ \phi^{-1})^j}{\partial x^i}\right|_{\phi(p)} \left.\frac{\partial}{\partial y^j}\right|_{\Psi(p)}.$$

The matrix formed by the partial derivatives $\frac{\partial(\psi\circ\Psi\circ\phi^{-1})^j}{\partial x^i}$ is often abbreviated to $\frac{\partial(y^j\circ\Psi)}{\partial x^i}$ and is called the **Jacobian matrix** of the smooth map Ψ relative to the specified system of coordinates.

The following result is an immediate consequence of the definitions and the usual chain rule for smooth maps between euclidean space:

Proposition 5.2 (Chain Rule)**.** *Let $\Psi : M \to N$ and $\Phi : N \to P$ be smooth maps. Then the composition $\Phi \circ \Psi$ is smooth and we have that:*

$$\mathrm{d}_p(\Phi\circ\Psi) = \mathrm{d}_{\Psi(p)}\Phi \circ \mathrm{d}_p\Psi.$$

Similarly, it is easy to prove the following proposition that generalizes a well known result:

Proposition 5.3. *If a smooth map $\Psi : M \to N$ has zero differential on a connected open set $U \subset M$, then Ψ is constant in U.*

A very important special case occurs when taking the differential of real valued smooth functions $f : M \to \mathbb{R}$, thought as smooth maps between M and the manifold $\mathbb{R}$, with its canonical smooth structure. In this case, the differential at p is a linear transformation $\mathrm{d}_pf : T_pM \to T_{f(p)}\mathbb{R}$. Since we have a canonical identification $T_x\mathbb{R} \simeq \mathbb{R}$, the differential d_pf is an element in the dual vector space to T_pM. Explicitly, it is given by:

$$\mathrm{d}_pf(\mathbf{v}) := \mathbf{v}(f).$$

Definition 5.4. *The **cotangent space** to M at a point p is the vector space T_p^*M dual to the tangent space T_pM:*

$$T_p^*M \equiv \{\omega : T_pM \to \mathbb{R},\ \textit{with}\ \omega\ \textit{linear}\}.$$

Of course we can define $\mathrm{d}_pf \in T_p^*M$ even if f is a smooth function *defined only* in a neighborhood of p. In particular, if choose a coordinate system $(U, x^1, \ldots, x^d)$ around p, we obtain elements

$$\left\{\mathrm{d}_px^1, \ldots, \mathrm{d}_px^d\right\} \subset T_p^*M.$$

It is then easy to check that

$$\mathrm{d}_px^i \cdot \left.\frac{\partial}{\partial x^j}\right|_p = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Hence:

Lemma 5.5. *For a coordinate system (U, x^i) of M around p, $\{\mathrm{d}_px^1, \ldots, \mathrm{d}_px^d\}$ is the basis of T_p^*M dual to the basis $\{\left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^d}\right|_p\}$ of T_pM.*

Therefore, once we have fixed a coordinate system $(U, x^1, \ldots, x^d)$ around p, every element $\omega \in T_p^*M$ can be written in the basis $\left\{\mathrm{d}_px^1, \ldots, \mathrm{d}_px^d\right\}$:

$$\omega = \sum_{i=1}^d a_i \mathrm{d}_px^i, \quad \text{with } a_i = \omega(\partial/\partial x^i\big|_p).$$

If $(V, y^1, \dots, y^d)$ is another coordinate system, we find

$$\omega = \sum_{j=1}^{d} b_j \mathrm{d}_p y^j, \quad \text{with } b_j = \omega(\partial/\partial y^j\big|_p),$$

and one checks easily that:

$$a_i = \sum_{j=1}^{d} \left.\frac{\partial y^j}{\partial x^i}\right|_p b_j. \tag{5.1}$$

This transformation formula for the components of elements of T_p^*M should be compared with the corresponding transformation formula (4.1) for the components of elements of T_pM.

Similarly to what we did for the tangent bundle, we can define the **cotangent bundle** to M as:

$$T^*M \equiv \bigcup_{p \in M} T_p^*M,$$

with a natural projection $\pi : T^*M \to M$ which associate to a tangent covector $\omega \in T_p^*M$ the corresponding base point $\pi(\omega) = p$. Again, T^*M has a natural smooth structure of manifold of dimension $2 \dim M$, such that the projection is a smooth map. The proof is entirely similar to the case of TM, so it is left as an exercise.

Let $\Psi : M \to N$ be a smooth map. We we will denote by $\mathrm{d}\Psi : TM \to TN$ the induced map on the tangent bundle which is defined by:

$$\mathrm{d}\Psi(\mathbf{v}) \equiv \mathrm{d}_{\pi(\mathbf{v})}\Psi(\mathbf{v}).$$

We call this map the **differential** of Ψ. We leave it as an exercise to check that $\mathrm{d}\Psi : TM \to TN$ is a smooth map between the smooth manifolds TM and TN.

If $f : M \to \mathbb{R}$ is a smooth function, then $\mathrm{d}f : TM \to T\mathbb{R}$. However, $T\mathbb{R} = \mathbb{R} \times \mathbb{R}$ so by projecting in the second factor, we consider $\mathrm{d}f$ as a map:

$$\mathrm{d}f : TM \to \mathbb{R}, \quad \mathrm{d}f(\mathbf{v}) \equiv \mathrm{d}_{\pi(\mathbf{v})}f(\mathbf{v}) = \mathbf{v}(f).$$

If $(U, x^1, \dots, x^d)$ is a system of coordinates around p, then from the definition we see that $\mathrm{d}_p f \in T_p^*M$ satisfies:

$$\mathrm{d}_p f \cdot \left.\frac{\partial}{\partial x^i}\right|_p = \left.\frac{\partial f}{\partial x^i}\right|_p.$$

It follows that the expression for $\mathrm{d}f$ in local coordinates $(x^1, \dots, x^d)$ is:

$$\mathrm{d}f|_U = \sum_{i=1}^{d} \frac{\partial f}{\partial x^i} \mathrm{d}x^i.$$

Notice that in this formula all terms have been precisely defined, in contrast with some formulas one often finds, where heuristic manipulations with $\mathrm{d}f$ are done without much justifications!

Remark 5.6. The definitions of tangent space, tangent bundle and differential, extend to manifolds with boundary. One defines the **tangent space to a manifold with boundary** of dimension d at some point $p \in M$ exactly as in Definition 4.1. The tangent space at any point $p \in M$, even at points of the boundary, has dimension d. The tangent bundle TM is now a manifold with boundary of dimension $2\dim M$. Similarly, one defines the differential of a smooth map $\Psi : M \to N$ between manifolds with boundary and this gives a smooth map between their tangent bundles $\mathrm{d}\Phi : TM \to TN$.

For a manifold with boundary M of dimension $d > 0$, the boundary ∂M is a smooth manifold of dimension $d-1$. Hence, if $p \in \partial M$ we have two tangent spaces: T_pM, which has dimension d, and $T_p(\partial M)$, which has dimension $d-1$. We leave it as an exercise to check that the inclusion $i : \partial M \hookrightarrow M$ is a smooth map and its differential $\mathrm{d}_p i : T_p(\partial M) \to T_pM$ is injective, at any point $p \in \partial M$. It follows that we can identify $T_p(\partial M)$ with its image in T_pM, so inside the tangent space to M at points of the boundary we have a well-defined subspace. It is common to denote this subspace also by $T_p(\partial M)$, a practice that we will also adopt.

Homework.

1. Show that the map $\Psi : \mathbb{RP}^2 \to \mathbb{R}^4$ given by:
$$\Psi([x:y:z]) = \frac{1}{x^2+y^2+z^2}(xy, xz, y^2-z^2, 2yz),$$
is smooth, injective and has differential $\mathrm{d}_p\Psi$ injective for all $p \in \mathbb{RP}^2$.

2. Let $\Psi : \mathbb{CP}^d \to \mathbb{R}^{d+1}$ be the smooth map given by:
$$\Psi([z^0 : \dots : z^d]) = \left(\frac{|z^0|^2}{|z^0|^2+\dots+|z^d|^2}, \dots, \frac{|z^d|^2}{|z^0|^2+\dots+|z^d|^2}\right).$$
Find the points $p \in \mathbb{CP}^d$ where the differential $\mathrm{d}_p\Phi$ vanishes.

3. Let $\pi : \mathbb{S}^d \to \mathbb{RP}^d$ be the map $(x^0, \dots, x^d) \mapsto [x^0 : \dots : x^d]$. Show that the differential $\mathrm{d}_p\pi$ is a linear isomorphism for all $p \in \mathbb{S}^2$.

4. Show that T^*M has a smooth structure of manifold of dimension $2\dim M$, for which the projection $\pi : T^*M \to M$ is a smooth map.

5. Check that if M and N are smooth manifolds and $\Psi : M \to N$ is a smooth map, then $\mathrm{d}\Psi : TM \to TN$ is also smooth.

6. Immersions, Submersions and Submanifolds

As we can expect from what we know from calculus in Euclidean space the properties of the differential of a smooth map between two smooth manifolds reflect the local behavior of the smooth map. In this section we will make this precise.

Definition 6.1. *Let $\Psi : M \to N$ be a smooth map:*

*(a) Ψ is called an **immersion** if $\mathrm{d}_p\Psi : T_pM \to T_{\Psi(p)}N$ is injective, for all $p \in M$;*

*(b) Ψ is called a **submersion** if $\mathrm{d}_p\Psi : T_pM \to T_{\Psi(p)}N$ is surjective, for all $p \in M$;*

*(a) Ψ is called an **étale**[2] if $\mathrm{d}_p\Psi : T_pM \to T_{\Psi(p)}N$ is an isomorphism, for all $p \in M$.*

Immersions, submersions and étales have local canonical forms. They are all consequences of the following general result:

Theorem 6.2 (Constant Rank Theorem)**.** *Let $\Psi : M \to N$ be a smooth map and $p \in M$. If $\mathrm{d}_q\Psi : T_qM \to T_{\Psi(q)}N$ has constant rank r, for all q in a neighborhood of p, then there are local coordinates $(U, \phi) = (U, x^1, \dots, x^m)$ for M centered at p and local coordinates $(V, \psi) = (V, y^1, \dots, y^n)$ for N centered at $\Psi(p)$, such that:*

$$\psi \circ \Psi \circ \phi^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^r, 0, \dots, 0).$$

Proof. Let $(\tilde{U}, \tilde{\phi})$ and $(\tilde{V}, \tilde{\psi})$ be local coordinates centered at p and $\Psi(p)$, respectively, with $\Psi(\tilde{U}) \subset \tilde{V}$. Then

$$\tilde{\psi} \circ \Psi \circ \tilde{\phi}^{-1} : \tilde{\phi}(\tilde{U}) \to \tilde{\psi}(\tilde{V})$$

is a smooth map from a neighborhood of zero in $\mathbb{R}^m$ to a neighborhood of zero in $\mathbb{R}^n$, whose differential has constant rank. Therefore, it is enough to consider the case where $\Psi : \mathbb{R}^m \to \mathbb{R}^n$ is a smooth map

$$(x^1, \dots, x^m) \mapsto (\Psi^1(x), \dots, \Psi^n(x)),$$

whose differential has constant rank in a neighborhood of the origin.

Let r be the rank of $\mathrm{d}\Psi$. Eventually after some reordering of the coordinates, we can assume that

$$\det \left[\frac{\partial \Psi^j}{\partial x^i}\right]_{i,j=1}^{r} (0) \neq 0.$$

It follows immediately from the Inverse Function Theorem, that the smooth map $\phi : \mathbb{R}^m \to \mathbb{R}^m$ defined by

$$(x^1, \dots, x^m) \to (\Psi^1(x), \dots, \Psi^r(x), x^{r+1}, \dots, x^m),$$

is a diffeomorphism from a neighborhood of the origin. We conclude that:

$$\Psi \circ \phi^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^r, \Psi^{r+1} \circ \phi^{-1}(x), \dots, \Psi^n \circ \phi^{-1}(x)).$$

Let q be any point in the domain of $\Psi \circ \phi^{-1}$. We can compute the Jacobian matrix of $\Psi \circ \phi^{-1}$ as:

$$\left[\begin{array}{c|c} I_r & 0 \\ \hline * & \frac{\partial(\Psi^j \circ \phi^{-1})}{\partial x^i}(q) \end{array}\right],$$

[2]We use this term provisionally. We shall see later in Corollary 6.5 that an étale map is the same thing as a local diffeomorphism.

where I_r is the $r \times r$ identity matrix and where in the lower right corner $i, j > r$. Since this matrix has exactly rank r, we conclude that:

$$\frac{\partial(\Psi^j \circ \phi^{-1})}{\partial x^i}(q) = 0, \text{ if } i, j > r.$$

In other words, the components of $\Psi^j \circ \phi^{-1}$, for $j > r$, do not depend on the coordinates $x^{r+1}, \dots, x^m$:

$$\Psi^j \circ \phi^{-1}(x) = \Psi^j \circ \phi^{-1}(x^1, \dots, x^r), \text{ if } j > r.$$

Let us consider now the map $\psi : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\psi(y^1, \dots, y^n) = (y^1, \dots, y^r, y^{r+1} - \Psi^{r+1} \circ \phi^{-1}(y), \dots, y^n - \Psi^n \circ \phi^{-1}(y)).$$

We see that ψ is a diffeomorphism since its Jacobian matrix at the origin is given by:

$$\left[\begin{array}{c|c} I_r & 0 \\ \hline * & I_{e-r} \end{array}\right],$$

which is non-singular. But now we compute:

$$\psi \circ \Psi \circ \phi^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^r, 0, \dots, 0).$$

□

An immediate corollary of this result is that an immersion of a m-manifold into a n-manifold, where necessarily $m \le n$, locally looks like the *inclusion* $\mathbb{R}^m \hookrightarrow \mathbb{R}^n$:

Corollary 6.3. *Let $\Psi : M \to N$ be an immersion. Then for each $p \in M$, there are local coordinates $(U, \phi) = (U, x^1, \dots, x^m)$ for M centered at p and local coordinates $(V, \psi) = (V, y^1, \dots, y^n)$ for N centered at $\Psi(p)$, such that:*

$$\psi \circ \Psi \circ \phi^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^m, 0, \dots, 0).$$

Similarly, we conclude that a submersion of a m-manifold into a n-manifold, where necessarily $m \ge n$, locally looks like the *projection* $\mathbb{R}^m \twoheadrightarrow \mathbb{R}^n$:

Corollary 6.4. *Let $\Psi : M \to N$ be a submersion. Then for each $p \in M$, there are local coordinates $(U, \phi) = (U, x^1, \dots, x^m)$ for M centered at p and local coordinates $(V, \psi) = (V, y^1, \dots, y^n)$ for N centered at $\Psi(p)$, such that:*

$$\psi \circ \Psi \circ \phi^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^n).$$

Since an étale is a smooth map which is simultaneously an immersion and a submersion, we conclude that an étale is just a local diffeomorphism:

Corollary 6.5. *Let $\Psi : M \to N$ be an étale. Then for each $p \in M$, there are local coordinates $(U, \phi) = (U, x^1, \dots, x^d)$ for M centered at p and local coordinates $(V, \psi) = (V, y^1, \dots, y^d)$ for N centered at $\Psi(p)$, such that:*

$$\psi \circ \Psi \circ \phi^{-1}(x^1, \dots, x^d) = (x^1, \dots, x^d).$$

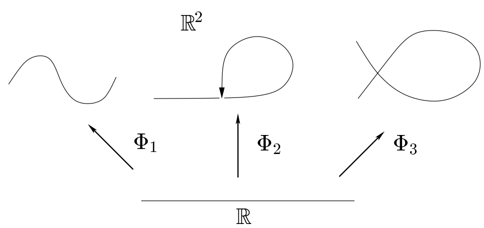
Let us now turn to the study of subobjects in the category of smooth manifolds:

Definition 6.6. *A **submanifold** of a manifold M is a pair (N, Φ) where N is a manifold and $\Phi : N \to M$ is an injective immersion. When $\Phi : N \to \Phi(N)$ is a homeomorphism, where on $\Phi(N)$ one takes the relative topology, one calls the pair (N, Φ) an **embedded submanifold** and Φ an **embedding**.*

One sometimes uses the term **immersed submanifold** to emphasize that $\Phi : N \to M$ is only an immersion and reserves the term *submanifold* for embedded submanifolds. However, in these notes we will use the term submanifold to denote immersed submanifolds that may fail to be embedded.

Examples 6.7.

1. *The next picture illustrates various immersions of $N = \mathbb{R}$ in $M = \mathbb{R}^2$. Notice that $(\mathbb{R}, \Phi_1)$ is an embedded submanifold of $\mathbb{R}^2$, while $(\mathbb{R}, \Phi_2)$ is only an immersed submanifold of $\mathbb{R}^2$. On the other hand, Φ_3 is an immersion but it is not injective, so $(\mathbb{R}, \Phi_3)$ is not a submanifold of $\mathbb{R}^2$.*



2. *According to a problem in the previous section, the map $\Psi : \mathbb{RP}^2 \to \mathbb{R}^4$ given by:*

$$\Psi([x : y : z]) = \frac{1}{x^2 + y^2 + z^2}(xy, xz, y^2 - z^2, 2yz),$$

is smooth, injective and has differential $\mathrm{d}_p\Psi$ injective for all $p \in \mathbb{RP}^2$. Since $\mathbb{RP}^2$ is compact, this map is an embedding (see the problems at the end of this section). It follows that $\mathbb{RP}^2$ can be realized as an embedded submanifold of $\mathbb{R}^4$.

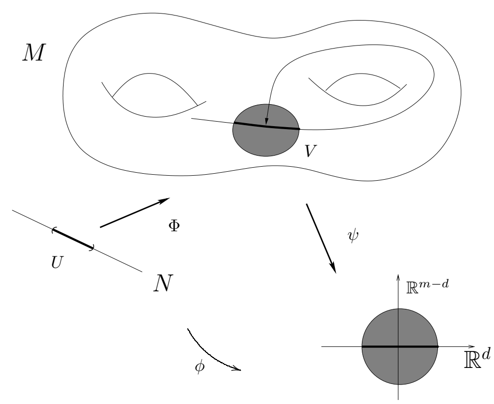
If (N, Φ) is a submanifold of M, then for each $p \in N$, the linear map $\mathrm{d}_p\Phi : T_pN \to T_{\Phi(p)}M$ is injective. Hence, we can always identify the tangent space T_pN with its image $\mathrm{d}_p\Phi(T_pN)$, which is a subspace of $T_{\Phi(p)}M$. From now on, we will use this identification, so that T_pN will always be interpreted as a subspace of $T_{\Phi(p)}M$.

The local canonical form (Corollary 6.3) yields the following:

Proposition 6.8 (Local normal form for immersed submanifolds)**.** *Let (N, Φ) be a submanifold of dimension d of a manifold M. Then for all $p \in N$, there exists a neighborhood U of p and a coordinate system $(V, x^1, \ldots, x^m)$ for M*

centered at $\Phi(p)$ *such that:*

$$\Phi(U) = \left\{ q \in V : x^{d+1}(q) = \cdots = x^m(q) = 0 \right\}.$$



Proof. By Corollary 6.3, for any $p \in N$ we can choose coordinates (U, ϕ) for N centered at p and coordinates $(V, \psi) = (V, x^1, \dots, x^m)$ for M centered at $\Phi(p)$, such that $\psi \circ \Phi \circ \phi^{-1} : \mathbb{R}^d \to \mathbb{R}^m$ is the inclusion. But then $\psi \circ \Phi(U)$ is exactly the set of points in $\psi(V) \subset \mathbb{R}^m$ with the last $m - d$ coordinates equal to 0. $\square$

You should notice (using the same notation as in the proposition) that, in general, $\Phi(N) \cap V \neq \Phi(U)$, so there could exist points in $\Phi(N) \cap V$ which *do not belong* to the slice $\{q \in V : x^{d+1}(q) = \cdots = x^m(q) = 0\}$.

However, whenever (N, Φ) is an embedded submanifold we find:

Corollary 6.9 (Local normal form for embedded submanifolds)**.** *Let* (N, Φ) *be an embedded submanifold of dimension* d *of a manifold* M*. For each* $p \in N$*, there exists a chart* $(V, x^1, \dots, x^m)$ *of* M *centered at* $\Phi(p)$*, such that:*

$$\Phi(N) \cap V = \left\{ q \in V : x^{d+1}(q) = \cdots = x^m(q) = 0 \right\}.$$

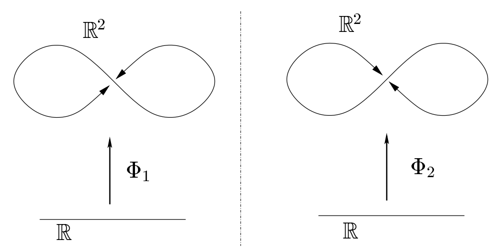
Proof. Fix $p \in N$ and choose a neighborhood U of p and a chart $(V', x^1, \dots, x^m)$ centered at $\Phi(p)$, as in the proposition. Since (N, Φ) is assumed to be embedded, $\Phi(U)$ is an open subset of $\Phi(N)$ for the relative topology: there exists an open set $V'' \subset M$ such that $\Phi(U) = V'' \cap \Phi(N)$. If we set $V = V' \cap V''$ the restrictions of the x^i to V, yield a coordinate system $(V, x^1, \dots, x^m)$ such that:

$$\Phi(N) \cap V = \left\{ q \in V : x^{d+1}(q) = \cdots = x^m(q) = 0 \right\}.$$

$\square$

We would like to think of submanifolds of a manifold M simply as subsets of M. However, this in general is not possible, as illustrated by the following example.

Example 6.10.
There are two injective immersions $\Phi_i : \mathbb{R} \to \mathbb{R}^2$, $i = 1, 2$, whose images in $\mathbb{R}^2$ coincide with the infinite symbol:



The example of the infinity symbol shows that one must be careful when we think of a submanifold of M as a subset. In order to see what can go wrong, we introduce the following equivalence relation:

Definition 6.11. *We say that (N_1, Φ_1) and (N_2, Φ_2) are* ***equivalent submanifolds*** *of M if there exists a diffeomorphism $\Psi : N_1 \to N_2$ such that the following diagram commutes:*

$$\begin{tikzcd} N_1 \arrow[r, "\Phi_1"] \arrow[dr, dashed, "\Psi"'] & M \\ & N_2 \arrow[u, "\Phi_2"'] \end{tikzcd}$$

If (N, Φ) is a submanifold of M we can consider the image $\Phi(N) \subset M$ with the unique smooth structure for which $\hat{\Phi} : N \to \Phi(N)$ is a diffeomorphism. Obviously, if we take this smooth structure on $\Phi(N)$, the inclusion $i : \Phi(N) \hookrightarrow M$ is an injective immersion and the following diagram commutes:

$$\begin{tikzcd} N \arrow[r, "\Phi"] \arrow[dr, "\hat{\Phi}"'] & M \\ & \Phi(N) \arrow[u, "i"'] \end{tikzcd}$$

Therefore, *every* submanifold (N, Φ) has a unique representative (A, i), where $A \subset M$ is a subset and $i : A \hookrightarrow M$ is the inclusion. We then say that $A \subset M$ is a submanifold.

Example 6.12.
If $A \subset M$ is an arbitrary subset, in general, there will be no smooth structure on A for which the inclusion $i : A \hookrightarrow M$ is an immersion. For example, the subset $A = \{(x, |x|) : x \in \mathbb{R}\} \subset \mathbb{R}^2$ does not admit such a smooth structure (exercise).

On the other hand, if A admits a smooth structure such that the inclusion $i : A \hookrightarrow M$ is an immersion, this smooth structure may not be unique: this is exactly what we saw Example 6.10.

Still, we have the following result:

Theorem 6.13. *Let $A \subset M$ be some subset of a smooth manifold and $i : A \hookrightarrow M$ the inclusion. Then:*

(i) For each choice of a topology in A there exists at most one smooth structure compatible with this topology and such that (A, i) is a submanifold of M.

(ii) If for the relative topology in A there exists a compatible smooth structure such that (A, i) is a submanifold of M, then this is the only topology in A for which there exists a compatible smooth structure such that (A, i) is a submanifold of M.

Example 6.14.
The sphere $\mathbb{S}^7 \subset \mathbb{R}^8$ is an embedded submanifold. We have mentioned before that the sphere $\mathbb{S}^7$ has smooth structures compatible with the usual topology but which are not equivalent to the standard smooth structure on the sphere. It follows that for these exotic smooth structures, $\mathbb{S}^7$ is not a submanifold of $\mathbb{R}^8$.

In order to prove Theorem 6.13, we observe that if (N, Φ) is a submanifold of M and $\Psi : P \to M$ is a smooth map such that $\Psi(P) \subset \Phi(N)$, the fact that Φ is 1:1 implies that Ψ factors through a map $\hat{\Psi} : P \to N$, i.e., we have a commutative diagram:

$$\begin{tikzcd} P \arrow[r, "\Psi"] \arrow[dr, dashed, "\hat{\Psi}"'] & M \\ & N \arrow[u, "\Phi"'] \end{tikzcd}$$

However, the problem is that, in general, the map $\hat{\Psi}$ *is not* smooth, as shown by the example of the infinite symbol.

Example 6.15.
Let $\Phi_i : \mathbb{R} \to \mathbb{R}^2$, $i = 1, 2$, be the two injective immersions whose images in $\mathbb{R}^2$ coincide with the infinite symbol, as in Example 6.10. Since $\Phi_1(\mathbb{R}) = \Phi_2(\mathbb{R})$, we have unique maps $\hat{\Phi}_1 : \mathbb{R} \to \mathbb{R}$ and $\hat{\Phi}_2 : \mathbb{R} \to \mathbb{R}$ such that $\Phi_2 \circ \hat{\Phi}_1 = \Phi_1$ and $\Phi_1 \circ \hat{\Phi}_2 = \Phi_2$. It is easy to check that $\hat{\Phi}_1$ and $\hat{\Phi}_2$ are not continuous, hence they are not smooth.

The next result shows that what may fail is precisely the continuity of the map $\hat{\Psi}$:

Proposition 6.16. *Let (N, Φ) be a submanifold of M, $\Psi : P \to M$ a smooth map such that $\Psi(P) \subset \Phi(N)$ and $\hat{\Psi} : P \to N$ the induced map.*

(i) If $\hat{\Psi}$ is continuous, then it is smooth.
(ii) If Φ is an embedding, then $\hat{\Psi}$ is continuous (hence smooth).

Proof. Assume first that $\hat{\Psi}$ is continuous. For each $p \in N$, choose $U \subset N$ and $(V, \phi) = (V, x^1, \ldots, x^m)$ as in Proposition 6.8, and consider the smooth map

$$\psi = \pi \circ \phi \circ \Phi : U \to \mathbb{R}^d,$$

where $\pi : \mathbb{R}^m \to \mathbb{R}^d$ is the projection $(x^1, \ldots, x^m) \mapsto (x^1, \ldots, x^d)$. The pair (U, ψ) is a smooth coordinate system for N centered at p. On the other hand, we see that

$$\psi \circ \hat{\Psi} = \pi \circ \phi \circ \Phi \circ \hat{\Psi} = \pi \circ \phi \circ \Psi,$$

is smooth in the open set $\hat{\Psi}^{-1}(U)$. Since the collection of all such open sets $\hat{\Psi}^{-1}(U)$ covers P, we conclude that $\hat{\Psi}$ is smooth, so (i) holds.

Now if Φ is an embedding, then every open set $U \subset N$ is of the form $\Phi^{-1}(V)$, where $V \subset M$ is open. Hence, $\hat{\Psi}^{-1}(U) = \hat{\Psi}^{-1}(\Phi^{-1}(V)) = \Psi^{-1}(V)$ is also open. We conclude that $\hat{\Psi}$ is continuous, so (ii) also holds. $\square$

Proof of Theorem 6.13. (i) follows immediately from Proposition 6.16 (i).

On the other hand, to prove (ii), let (N, Φ) be a submanifold with $\Phi(N) = A$ and consider the following diagram:

$$\begin{tikzcd} N \arrow[r, "\Phi"] \arrow[dr, "\hat{\Phi}"'] & M \\ & A \arrow[u, "i"'] \end{tikzcd}$$

Since A is assume to have the relative topology, by Proposition 6.16 (ii), $\hat{\Phi}$ is smooth. Hence, $\hat{\Phi}$ is an invertible immersion so it is a diffeomorphism (exercise). We conclude that (N, Φ) is equivalent to (A, i), so (ii) holds. $\square$

The previous discussion justifies considering the following class of submanifolds, which lies inbetween the classes of immersed submanifolds and embedded submanifolds:

Definition 6.17. *A **initial submanifold** of M is a submanifold (N, Φ) such that every smooth map $\Psi : P \to M$ with $\Psi(P) \subset \Phi(N)$ factors through a smooth map $\hat{\Psi} : P \to N$:*

$$\begin{tikzcd} P \arrow[r, "\Psi"] \arrow[dr, dashed, "\hat{\Psi}"'] & M \\ & N \arrow[u, "\Phi"'] \end{tikzcd}$$

Sometimes initial submanifolds are also called **regular immersed submanifolds** or **weakly embedded submanifolds**. The two different immersions of the infinity symbol that we saw above are not initial submanifols. On the other hand, Proposition 6.16 (ii) shows that embedded submanifolds are initial submanifolds. But be aware that there are many examples of initial submanifolds which are not embedded.

EXAMPLE 6.18.
In the 2-torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ we have a family of submanifolds $(\mathbb{R}, \Phi_a)$, depending on the parameter $a \in \mathbb{R}$, defined by:

$$\Phi_a(t) = (e^{it}, e^{iat}).$$

If $a = m/n$ is rational, this is a closed curve, which turns m times in one torus direction and n times in the other torus direction, so this is an embedding.

If $a \notin \mathbb{Q}$ then the curve is dense in the 2-torus, so this is only an immersed submanifold. However, if $\hat{\Psi} : P \to \mathbb{R}$ is a map such that the composition $\Phi_a \circ \hat{\Psi}$ is smooth, then we see immediately that $\hat{\Psi} : P \to \mathbb{R}$ is continuous. By Proposition 6.16, we conclude that $\hat{\Psi}$ is smooth. Hence, (N, Φ_a) is a initial submanifold.

HOMEWORK.

1. Show that $\{(x, |x|) : x \in \mathbb{R}\}$ *is not* the image of an immersion $\Phi : \mathbb{R} \to \mathbb{R}^2$.

2. Show that $\mathbb{S}^3$ has trivial tangent bundle, i.e., there exists a diffeomorphism $\Psi : T\mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{R}^3$, which makes the following diagram commutative:

$$\begin{tikzcd} T\mathbb{S}^3 \arrow[rr, "\Psi"] \arrow[dr, "\pi"'] & & \mathbb{S}^3 \times \mathbb{R}^3 \arrow[dl, "\mathrm{pr}_{\mathbb{S}^3}"] \\ & \mathbb{S}^3 & \end{tikzcd}$$

and where the restriction $\Psi : T_p\mathbb{S}^3 \to \mathbb{R}^3$ is linear for every $p \in \mathbb{S}^3$.
Hint: The 3-sphere is the set of quaternions of norm 1.

3. Let $\{y^1, \ldots, y^e\}$ be some set of smooth functions on a manifold M. Show that:
   (a) If $\{\mathrm{d}_p y^1, \ldots, \mathrm{d}_p y^e\} \subset T_p^*M$ is a linearly independent set, then the functions $\{y^1, \ldots, y^e\}$ is a part of a coordinate system around p.
   (b) If $\{\mathrm{d}_p y^1, \ldots, \mathrm{d}_p y^e\} \subset T_p^*M$ is a generating set, then a subset of $\{y^1, \ldots, y^e\}$ is a coordinate system around p.
   (c) If $\{\mathrm{d}_p y^1, \ldots, \mathrm{d}_p y^e\} \subset T_p^*M$ is a basis, then the functions $\{y^1, \ldots, y^e\}$ form a coordinate system around p.

4. Show that a submersion is an open map. What can you say about an immersion?

5. Let $\Phi : \mathbb{RP}^2 \to \mathbb{R}^3$ be the map defined by

$$\Phi([x, y, z]) = \frac{1}{x^2 + y^2 + z^2}(yz, xz, xy).$$

Show that Φ is smooth and show that it only fails to be an immersion at 6 points. Make a sketch of the image of Φ.

6. Let M be a manifold, $A \subset M$, and $i : A \hookrightarrow M$ the inclusion. Show that (A, i) is a an embedded submanifold of M of dimension d, if and only if for each $p \in A$ there exists a coordinate system $(U, x^1, \dots, x^m)$ centered at p such that

$$A \cap U = \left\{p \in U : x^{d+1}(p) = \cdots = x^m(p) = 0\right\}.$$

7. Show that a subset $M \subset \mathbb{R}^n$ is a d-surface (i.e., satisfies Definition 0.2) if and only it is an embedded submanifold (so this justifies us calling M an embedded manifold in $\mathbb{R}^n$).

8. One says that a subset S of a manifold M has **zero measure** if for every coordinate system (U, ϕ) of M, the set $\phi(S \cap U) \subset \mathbb{R}^d$ has zero measure. Show that:
   (a) A smooth map $\Phi : M \to N$ maps zero measure sets to zero measure sets;
   (b) If $\Phi : N \to M$ is an immersion and $\dim N < \dim M$, then $\Phi(N)$ has zero measure.

9. Show that for a submanifold (N, Φ) of a smooth manifold M the following are equivalent:
   (a) $\Phi(N) \subset M$ is a closed subset and (N, Φ) is embedded.
   (b) $\Phi : N \to M$ is a closed map (i.e, $\Phi(A)$ is closed whenever $A \subset N$ is a closed subset).
   (c) $\Phi : N \to M$ is a proper map (i.e., $\Phi^{-1}(K) \subset N$ is compact, whenever $K \subset M$ is compact).

   Use this to conclude that a submanifold (N, Φ) with N compact, is always an embedded submanifold.

10. Show that an invertible immersion $\Phi : N \to M$ is a diffeomorphism. Give a counterexample to this statement if N does not have a countable basis.

11. Let $\pi : \widetilde{M} \to M$ be a covering space of a smooth manifold M, where $\widetilde{M}$ is a second countable topological space. Show that $\widetilde{M}$ has unique smooth structure for which the covering map π is a local diffeomorphism.

7. Embeddings and Whitney's Theorem

Definition 7.1. *Let $\Psi : M \to N$ be a smooth map*

*(i) One calls $p \in M$ a **regular point** of Ψ if $\mathrm{d}_p\Psi : T_pM \to T_{\Psi(p)}N$ is surjective. Otherwise one calls p a **singular point** of Ψ;*

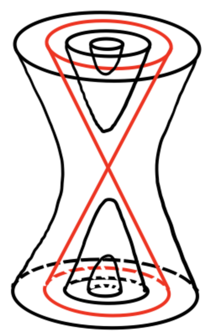
*(ii) One calls $q \in N$ a **regular value** of Ψ if every $p \in \Psi^{-1}(q)$ is a regular point. Otherwise one calls q a **singular value** of Ψ.*

The following example gives some evidence for the use of the terms "regular" and "singular".

Example 7.2.
Let $\Psi : \mathbb{R}^3 \to \mathbb{R}$ be the map defined by

$$\Psi(x, y, z) = x^2 + y^2 - z^2.$$

This map has Jacobian matrix $[2x\ 2y\ {-2z}]$. Therefore, every $(x, y, z) \neq (0, 0, 0)$ is a regular point of Ψ and $(0, 0, 0)$ is a singular point of Ψ. On the other hand, 0 is a singular value of Ψ, while every other value is a regular value of Ψ.



If we consider a regular value c, the level set $\Psi^{-1}(c)$ is a submanifold of $\mathbb{R}^2$ (either a 1 sheet or a 2 sheets hyperboloid). On the other hand, for the singular value 0, we see that $\Psi^{-1}(0)$ is a cone, which is not a manifold at the origin.

In fact, the level sets of regular values are always submanifolds:

Theorem 7.3. *Let $\Psi : M \to N$ be a smooth map and let $q \in N$ be a regular value of Ψ. Then $\Psi^{-1}(q) \subset M$ is an embedded submanifold of dimension $\dim M - \dim N$ and for all $p \in \Psi^{-1}(q)$ we have:*

$$T_p(\Psi^{-1}(q)) = \operatorname{Ker} \mathrm{d}_p\Psi.$$

Proof. If $q \in N$ is a regular value of Ψ there exists an open set $\Psi^{-1}(q) \subset O \subset M$ such that $\Psi|_O$ is a submersion. Therefore, for any $p \in \Psi^{-1}(q)$ we can choose coordinates $(U, x^1, \dots, x^m)$ around p and coordinates $(V, y^1, \dots, y^n)$ around q, such that Ψ is represented in these local coordinates by the projection

$$\mathbb{R}^m \to \mathbb{R}^n : \ (x^1, \dots, x^m) \mapsto (x^1, \dots, x^n).$$

Therefore, we see that

$$\Psi^{-1}(q) \cap U = \left\{p \in U : x^1(p) = \cdots = x^n(p) = 0\right\}.$$

It follows that $\Psi^{-1}(q)$ is an embedded submanifold of dimension $m - n = \dim M - \dim N$ (see Exercise 6 in the previous section). The statement about the tangent space to $\Psi^{-1}(q)$ is left as an exercise.

□

EXAMPLE 7.4.
Let $M = \mathbb{R}^{d+1}$ and let $\Psi : \mathbb{R}^{d+1} \to \mathbb{R}$ be the smooth map:

$$\Psi(x) = ||x||^2.$$

The Jacobian matrix of Ψ at x is given by:

$$\Psi'(x) = [2x^1, \dots, 2x^{d+1}].$$

Since $\Psi'(x)$ has rank one if $||x|| > 0$, it follows that any $c = R^2 > 0$ is a regular value of Ψ. The theorem above then asserts that the spheres $\mathbb{S}^d_R = \Psi^{-1}(R^2)$ are embedded submanifolds of $\mathbb{R}^{d+1}$ of codimension 1. Note that for the differential structure on $\mathbb{S}^d$ that we have defined before, $\mathbb{S}^d$ is also an embedded submanifold of $\mathbb{R}^{d+1}$. Hence, that differential structure coincides with this one.

Not every embedded submanifold $S \subset M$ is of the form $\Psi^{-1}(q)$, for a regular value of some smooth map $\Psi : M \to N$. There are global obstructions that we will study later. Also, what happens at singular values can be very wild: using a partition of unity argument it is possible to show that for any closed subset $A \subset M$ of a smooth manifold, there exists a smooth function $f : M \to \mathbb{R}$ such that $f^{-1}(0) = A$.

If $N \subset M$ is a submanifold we call the **codimension** of N in M the integer $\dim M - \dim N$. Since a set with a single point is a manifold of dimension 0, the previous result can be restated as saying that if q is a regular value of Ψ, then $\Psi^{-1}(q)$ is an embedded submanifold with $\operatorname{codim} \Psi^{-1}(q) = \operatorname{codim} \{q\}$. In this form, the previous result can be generalized in the following very useful way:

Theorem 7.5. *Let $\Psi : M \to N$ be a smooth map and let $Q \subset N$ be an embedded submanifold. Assume that for all $p \in \Psi^{-1}(Q)$ one has:*

$$\operatorname{Im} \mathrm{d}_p\Psi + T_{\Psi(p)}Q = T_{\Psi(p)}N. \tag{7.1}$$

Then $\Psi^{-1}(Q) \subset M$ is an embedded submanifold with

$$\operatorname{codim} \Psi^{-1}(Q) = \operatorname{codim} Q$$

and for all $p \in \Psi^{-1}(Q)$ we have:

$$T_p(\Psi^{-1}(Q)) = (\mathrm{d}_p\Psi)^{-1}(T_{\Psi(p)}Q).$$

Proof. Choose $p_0 \in \Psi^{-1}(Q)$ and set $q_0 = \Psi(p_0)$. Since $Q \subset N$ is assumed to be an embedded submanifold, we can choose a coordinate system $(V, \phi) = (V, y^1, \dots, y^n)$ for N around q_0, such that

$$Q \cap V = \left\{ q \in V : y^{l+1}(q) = \cdots = y^n(q) = 0 \right\},$$

where $l = \dim Q$. Define a smooth map $\Phi : \Psi^{-1}(V) \to \mathbb{R}^{n-l}$ by

$$\Phi = (y^{l+1} \circ \Psi, \dots, y^n \circ \Psi).$$

Then we see that $U = \Psi^{-1}(V)$ is an open subset of M which contains p_0 and such that $\Psi^{-1}(Q) \cap U = \Phi^{-1}(0)$. If we can show that 0 is a regular value of

Φ, then by Theorem 7.3 it follows that for all $p_0 \in \Psi^{-1}(Q)$, there exists an open set $U \subset M$ such that $\Psi^{-1}(Q) \cap U$ is an embedded submanifold of M of codimension $n - l = \operatorname{codim} Q$. This implies that $\Psi^{-1}(Q)$ is an embedded submanifold of M, as claimed.

To check that 0 is a regular value of Φ note that $\Phi = \pi \circ \phi \circ \Psi$, where $\pi : \mathbb{R}^n \to \mathbb{R}^{n-l}$ is the projection in the last $n - l$ components. Since π is a submersion, ϕ is a diffeomorphism and $\ker \mathrm{d}_q(\pi \circ \phi) = T_q Q$, for all $q \in Q \cap V$, it follows from (7.1), that $\mathrm{d}_p \Phi = \mathrm{d}_{\Psi(p)}(\pi \circ \phi) \cdot \mathrm{d}_p \Psi$ is surjective, for all $p \in \Psi^{-1}(Q) \cap U = \Phi^{-1}(0)$, i.e., 0 is a regular value of Φ.

The statement about the tangent space to $\Psi^{-1}(Q)$ is left as an exercise. □

The condition (7.1) appearing in the statement of the theorem is so important that one has a special name for it.

Definition 7.6. *Let $\Psi : M \to N$ be a smooth map. We say that Ψ is* ***transversal*** *to a submanifold $Q \subset N$, and we write $\Psi \pitchfork Q$, if:*

$$\operatorname{Im} \mathrm{d}_p \Psi + T_{\Psi(p)} Q = T_{\Psi(p)} N, \quad \forall p \in \Psi^{-1}(Q).$$

Notice that submersions $\Psi : M \to N$ are specially nice: they are transverse to every submanifold $Q \subset N$! So for a submersion the theorem shows that the inverse image of any submanifold is a submanifold.

A special case that justifies the use of the term "transversal" is when $M \subset N$ is a submanifold and $\Psi : M \hookrightarrow N$ is the inclusion. In this case, $\Psi^{-1}(Q) = M \cap Q$ and the transversality condition reduces to:

$$T_q M + T_q Q = T_q N, \quad \forall q \in M \cap Q.$$

Note that this condition is symmetric in M and Q. So in this case we simply say that M and Q **intersect transversely** and we write $M \pitchfork Q$.

Corollary 7.7. *If $M, Q \subset N$ are embedded submanifolds such that $M \pitchfork Q$. Then $M \cap Q$ is an embedded submanifold of N with:*

$$\dim M \cap Q = \dim M + \dim Q - \dim N,$$

and for all $q \in M \cap Q$ we have:

$$T_q(M \cap Q) = T_q M \cap T_q Q.$$

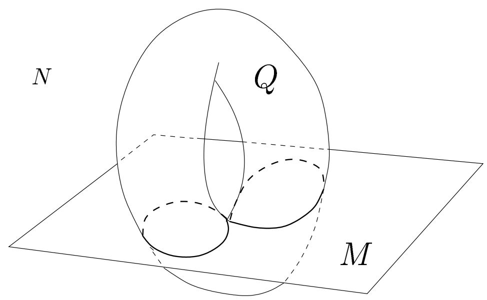
Although Theorem 7.5 and its corollary were stated for embedded submanifolds, you are asked in an exercise in this Section to check that these results still hold for immersed submanifolds.

Transversality plays an important role because of the following properties:

- *Transversality is a stable property:* If $\Phi : M \to N$ is transverse to Q then any map $\Psi : M \to N$ close enough to Φ is also transverse to Q.
- *Transversality is a generic property:* Any smooth map $\Phi : M \to N$ can be approximated by $\widetilde{\Phi} : M \to N$ transverse to Q.

We shall not attempt to make precise these two statements, since we would need to introduce and study appropriate topologies on the space of smooth maps $C^\infty(M,N)$. In fact, transversality is an important topic studied in *Differential Topology.*

On the other hand, when two submanifolds do not intersect transversally, in general, the intersection is not a manifold as illustrated by the following figure.



Examples 7.8.

1. *Let $M = \mathbb{S}^1 \times \mathbb{R}$ be a cylinder. We can embed M in $\mathbb{R}^3$ as follows: we define a smooth map $\Phi : M \to \mathbb{R}^3$ by:*

$$\Phi(\theta,t) = (R\cos\theta, R\,\mathrm{sen}\,\theta, t),$$

where we identify $\mathbb{S}^1 = [0,2\pi]/2\pi\mathbb{Z}$. This map is injective and its Jacobian matrix $\Phi'(\theta,t)$ has rank 2, hence Φ is an injective immersion.

The image of Φ is the subset of $\mathbb{R}^3$:

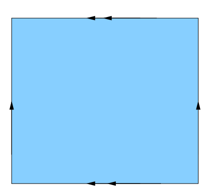
$$\left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = R^2\right\} = \Psi^{-1}(c),$$

where $c = R^2$ and $\Psi : \mathbb{R}^3 \to \mathbb{R}$ is the smooth map

$$\Psi(x,y,z) = x^2+y^2.$$

Since $\Psi'(x,y,z) = [2x,2y,0] \neq 0$ if $x^2+y^2 = c \neq 0$, we conclude that any $c \neq 0$ is a regular value of Ψ, so we have an embedding of of $\mathbb{S}^1 \times \mathbb{R}$ in $\mathbb{R}^3$.

2. *The 2-**torus** $M = \mathbb{S}^1 \times \mathbb{S}^1$ can also be embedded in $\mathbb{R}^3$ as follows. First, we can think of the the two torus as $\mathbb{S}^1 \times \mathbb{S}^1 = [0,2\pi]/2\pi\mathbb{Z} \times [0,2\pi]/2\pi\mathbb{Z}$. Note that this amounts to think of the torus as a square of side 2π where we identify the sides of the square, as in the following figure:*



Now define $\Phi : M \to \mathbb{R}^3$ by:

$$\Phi(\theta, \phi) = ((R + r\cos\phi)\cos\theta, (R + r\cos\phi)\,\mathrm{sen}\,\theta, r\,\mathrm{sen}\,\phi).$$

It is easy to check that if $R > r > 0$, then Φ is an injective immersion whose image is the subset of $\mathbb{R}^3$:

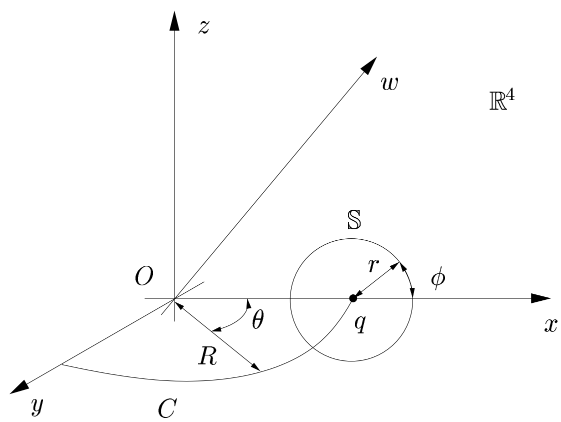
$$\left\{(x, y, z) \in \mathbb{R}^3 : (x^2 + y^2 + z^2 - R^2 - r^2)^2 + 4R^2z^2 = 4R^2r^2\right\} = \Psi^{-1}(c),$$

where $c = 4R^2r^2$ and $\Psi : \mathbb{R}^3 \to \mathbb{R}$ is the smooth map

$$\Psi(x, y, z) = (x^2 + y^2 + z^2 - R^2 - r^2)^2 + 4R^2z^2.$$

We leave it as an exercise to check that every $c \neq 0$ is a regular value of Ψ, so this gives an embedding of $\mathbb{S}^1 \times \mathbb{S}^1$ in $\mathbb{R}^3$.

3. *The **Klein bottle** is the subset $K \subset \mathbb{R}^4$ defined as follows: Let Ox, Oy, Oz, and Ow, be the coordinate axes in $\mathbb{R}^4$ and denote by C a circle of radius R in the plane xOy. Let θ be the angle coordinate on this circle (say, measured from the Ox-axis).*

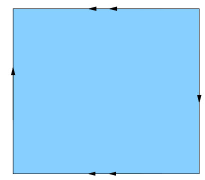


If $\mathbb{S}^1$ is a circle of radius r in the plane xOz, with centre at $q \in C$, then K is the figure obtained by rotating this circle around the Oz axis so that when its center $q \in C$ is rotated an angle θ, the plane where $\mathbb{S}^1$ lies has rotated an angle $\theta/2$ around the Oq-axis in the 3-space $OqOzOw$. Let ϕ be the angle coordinate in the circle $\mathbb{S}^1$ (say, measured from the Oq-axis).

Note that the points of K with $\theta \neq 0$ and $\phi \neq 0$ can be parameterized by: $\Phi_1 :]0, 2\pi[\times]0, 2\pi[\to \mathbb{R}^4$:

$$\Phi_1(\theta, \phi) = ((R + r\cos\phi)\cos\theta, (R + r\cos\phi)\,\mathrm{sen}\,\theta, r\,\mathrm{sen}\,\phi\cos\theta/2, r\,\mathrm{sen}\,\phi\,\mathrm{sen}\,\theta/2).$$

We can change the origin of θ and ϕ, obtaining new parameterizations, which all together cover K. We leave it as an exercise to show that 3 parameterizations Φ_1, Φ_2 and Φ_3 are enough to cover K. Since for these parameterizations the transitions $\Phi_i \circ \Phi_j^{-1}$ are C^∞, we see that K is a 2-surface in $\mathbb{R}^4$. Also, we remark that these parameterizations amount to think of K as a square of side 2π where we identify the sides of the square, as in the following figure:



Just like for the 2-torus, one checks that K is given by:

$$K = \Psi^{-1}(c, 0),$$

where $c = 4R^2r^2$ and $\Psi : \mathbb{R}^4 \to \mathbb{R}^2$ is the smooth map

$$\Psi(x, y, z) = ((x^2 + y^2 + z^2 + w^2 - R^2 - r^2)^2 + 4R^2(z^2 + w^2), y(z^2 - w^2) - 2xzw).$$

For $c \neq 0$, one checks that $(c, 0)$ is a regular value of Ψ, so we conclude that K is an embedded submanifold of $\mathbb{R}^4$.

Actually, any manifold can always be embedded in a Euclidean space of large enough dimension.

Theorem 7.9 (Whitney). *Let M be a compact manifold. There exists an embedding $\Phi : M \to \mathbb{R}^m$, for some integer m.*

Proof. Since M is compact, we can find a finite collection of coordinate systems $\{(U_i, \phi_i) : i = 1, \dots, N\}$ such that:
(a) $\overline{B_1(0)} \subset \phi_i(U_i)$;
(b) $\bigcup_{i=1}^{N} \phi_i^{-1}(B_1(0)) = M$.
Let $\lambda_i : M \to \mathbb{R}$, $i = 1, \dots, N$, be smooth functions such that

$$\lambda_i(p) = \begin{cases} 1 \text{ if } p \in \phi_i^{-1}(B_1(0)), \\ \\ 0 \text{ if } p \notin U_i. \end{cases}$$

Also, let $\psi_i : M \to \mathbb{R}^d$, $i = 1, \dots, N$, be smooth maps defined by:

$$\psi_i(p) = \begin{cases} \lambda_i \phi_i(p) \text{ if } p \in U_i, \\ \\ 0 \text{ if } p \notin U_i. \end{cases}$$

We claim that the smooth map $\Phi : M \to \mathbb{R}^{Nd+N}$ defined by:

$$\Phi(p) = (\psi_1(p), \lambda_1(p), \dots, \psi_N(p), \lambda_N(p))$$

is the desired embedding. In fact, we have that

(i) Φ is an immersion: if $p \in M$ then $p \in \phi_i^{-1}(B_1(0))$, for some i. Hence, we have that $\psi_i = \phi_i$ in a neighborhood p. We conclude that $\mathrm{d}_p\psi_i = \mathrm{d}_p\phi_i$ is injective. This shows that $\mathrm{d}_p\Phi$ is injective.

(ii) Φ is injective: Let $p, q \in M$, $p \neq q$, and choose i such that $p \in \lambda_i^{-1}(1)$. If $q \notin \lambda_i^{-1}(1)$, then $\lambda_i(p) \neq \lambda_i(q)$ so that $\Phi(p) \neq \Phi(q)$. On the other hand, if $q \in \lambda_i^{-1}(1)$, then $\psi_i(p) = \phi_i(p) \neq \phi_i(q) = \psi_i(q)$, since ϕ_i is injective. In any case, $\Phi(p) \neq \Phi(q)$, so Φ is injective.

Since M is compact, we conclude that Φ is an embedding.

$\square$

The previous result also holds for non-compact manifolds (see the exercises in this section) and is valid also for manifolds with boundary. It is a weaker version of the following result:

- (Whitney) Any smooth manifold (compact or not) of dimension d can be *embedded* in $\mathbb{R}^{2d}$.

As the example of the Klein bottle shows, there are smooth manifolds of dimension d which cannot be embedded in $\mathbb{R}^{2d-1}$. On the other hand, for $d > 1$, Whitney also showed that any manifold of dimension d can be *immersed* in $\mathbb{R}^{2d-1}$.

These results are not the best possible: Ralph Cohen in 1985 showed that a compact manifold of dimension d can be *immersed* in $\mathbb{R}^{2d-a(d)}$ where $a(d)$ is the number of 1's in the binary expression of d, and this is the best possible!! (e.g., every compact 5-manifold can immersed in $\mathbb{R}^8$, but there are compact 5-manifolds which cannot be immersed in $\mathbb{R}^7$). On the other hand, the best optimal *embedding* dimension is only known for a few dimensions.

Homework.

1. Consider the following sets of $n \times n$ matrices:
   - $O(n) = \left\{A : AA^T = I\right\}$ (orthogonal matrices);
   - $S(n) = \left\{A : A = A^T\right\}$ (symmetric matrices).

   Show that $O(n)$ and $S(n)$ are embedded submanifolds of the space $\mathbb{R}^{n^2}$ of all $n \times n$ matrices and check that they intersect transversely at I. Use this to conclude that there is a neighborhood of I where the only $n \times n$-matrix which is both orthogonal and symmetric is I itself.

2. Furnish the details of the example of the Klein bottle K and show that K is a 2-surface in $\mathbb{R}^4$.

3. Let $\Psi : M \to N$ be a smooth map and let $q \in N$ be a regular value of Ψ. Show that
$$T_p\Psi^{-1}(q) = \{\mathbf{v} \in T_pM : \mathrm{d}_p\Psi \cdot \mathbf{v} = 0\}.$$

4. Let $\Psi : M \to N$ be a smooth map which is transversal to a submanifold $Q \subset N$ (not necessarily embedded). Show that $\Psi^{-1}(Q)$ is a submanifold of M (not necessarily embedded) and that
$$T_p\Psi^{-1}(Q) = \left\{\mathbf{v} \in T_pM : \mathrm{d}_p\Psi \cdot \mathbf{v} \in T_{\Psi(p)}Q\right\}.$$

5. Let M and N be smooth manifolds and let $S \subset M \times N$ be a submanifold. Denote by $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ the projections on each factor. Show that the following are equivalent:
 (a) S is the graph of a smooth map $\Phi : M \to N$;
 (b) $\pi_M|_S$ is a diffeomorphism from S onto M;
 (c) For each $p \in M$, the submanifolds S and $\{p\} \times N = \pi_M^{-1}(p)$ intersect transversely and the intersection consists of a single point.

Moreover, if any of these hold then S is an embedded submanifold.

6. Extend Theorem 7.5 to the case where $\Psi : M \to N$ is a smooth map between manifolds with boundary such that $\Psi(\partial M) = \partial N$. Show that the conclusion of the theorem may fail if this last condition is omitted.

The next sequence of exercises give a sketch of the proof of the weak Whitney's Embedding Theorem for non-compact manifolds. It uses the following result which we will not discuss in these lectures:

Sard's Theorem: *The set of singular values of any smooth map $\Psi : M \to N$ has zero measure.*

7. Using Sard's Theorem, show that if $\Phi : M \to N$ is a smooth map between smooth manifolds and $\dim M < \dim N$ then $\Phi(M)$ has zero measure.

8. Let $M \subset \mathbb{R}^n$ be a smooth submanifold of dimension d. Given $v \in \mathbb{R}^n - \mathbb{R}^{n-1}$ denote by $\pi_v : \mathbb{R}^n \to \mathbb{R}^{n-1}$ the linear projection with kernel $\mathbb{R}v$. Show that if $n > 2d + 1$ there is a dense set of vectors $v \in \mathbb{R}^n - \mathbb{R}^{n-1}$ for which $\pi_v|_M$ is an injective immersion of M in $\mathbb{R}^{n-1}$. Conclude that any compact manifold with boundary of dimension d can be embedded in $\mathbb{R}^{2d+1}$.

HINT: Check that the proof given in the text of Whitney's embedding theorem is valid for compact manifolds with boundary. Then apply Sard's theorem in a clever way.

9. Using a smooth exhaustion function, show that any smooth manifold M of dimension d can be embedded in $\mathbb{R}^{2d+1}$.

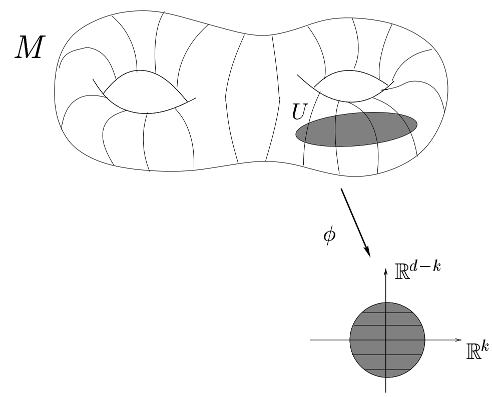
HINT: If $f : M \to \mathbb{R}$ is a smooth exhaustion function, then by Sard's Theorem, in each interval $[i, i+1[$, the function f has a regular value a_i. It follows that the sets $E_0 = f^{-1}(]-\infty, a_2]$, $E_i = f^{-1}([a_{i-1}, a_{i+1}]$ $(i = 1, 2, \dots)$, are all compact submanifolds of M of dimension d to which the previous result can be applied. Now use a partition of unity to build an embedding of M in $\mathbb{R}^{2d+1}$.

8. Foliations

A foliation is a nice decomposition of a manifold into submanifolds:

Definition 8.1. *Let M be a manifold of dimension d. A* ***foliation*** *of dimension k of M is a decomposition $\{L_\alpha : \alpha \in A\}$ of M into disjoint path-connected subsets satisfying the following property: for any $p \in M$ there* *exists a smooth chart* $\phi = (x^1, \dots, x^k, y^1, \dots, y^{d-k}) : U \to \mathbb{R}^d = \mathbb{R}^k \times \mathbb{R}^{d-k}$, *such the connected components of* $L_\alpha \cap U$ *are the sets of*

$$\{p \in U : y^1(p) = const., \dots, y^{d-k}(p) = const.\}.$$



We will denote a foliation by $\mathcal{F} = \{L_\alpha : \alpha \in A\}$. The connected sets L_α are called **leaves** of $\mathcal{F}$ and a chart (U, ϕ) as in the definition is called a **foliated coordinate chart**. The connected components of $U \cap L_\alpha$ are called **plaques**.

A **path of plaques** is a collection of plaques $P_1, \dots, P_l$ such that $P_i \cap P_{i+1} \neq \emptyset$, for all $i = 1, \dots, l-1$. The integer l is called the **length of the path of plaques**. Two points $p, q \in M$ belong to the same leaf if and only if there exists a path of plaques $P_1, \dots, P_l$, with $p \in P_1$ and $q \in P_l$.

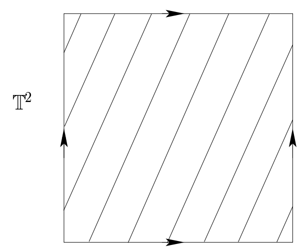
Each leaf of a k-dimensional foliation of M is a submanifold of M of dimension k. In general, these are only immersed submanifolds: a leaf can intersect a foliated coordinate chart an infinite number of times and accumulate overt itself. Before we check that leaves are submanifolds, let us look at some examples.

EXAMPLES 8.2.

1. *Let* $\Phi : M \to N$ *be a submersion. By the local normal form for submersions, the connected components of the fibers* $\Phi^{-1}(q)$*, where* $q \in N$*, form a foliation of* M *of codimension equal to the dimension of* N*. In this case, all leaves are actually embedded submanifolds.*
2. *In* $\mathbb{R}^2$*, take the foliation by straight lines with a fixed slope* $a \in \mathbb{R}$*. This is just a special case of the previous example, where* $\Phi : \mathbb{R}^2 \to \mathbb{R}$*, is given by:*

$$\Phi(x, y) = y - ax.$$

Now let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ *be the torus. Then we have an induced foliation on* $\mathbb{T}^2$*, and there are two possibilities. If* $a \in \mathbb{Q}$*, the leaves are closed curves, hence they are embedded submanifolds. However, if* $a \notin \mathbb{Q}$*, then the leaves are dense in the torus, so they are only immersed submanifolds.*



3. *Let $\Phi : \mathbb{R}^3 \to \mathbb{R}$ be the smooth map defined by*

$$\Phi(x, y, z) = f(x^2 + y^2)e^{-z},$$

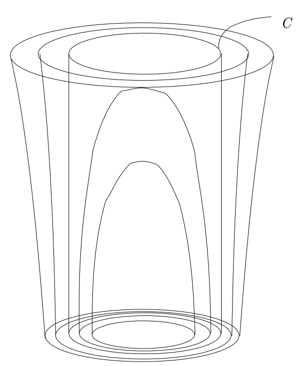
where $f \in C^\infty(\mathbb{R})$ is a smooth function with $f(0) = 1$, $f(1) = 0$ and $f'(t) < 0$. It is easy to check that Φ is a submersion and so determines a foliation $\mathcal{F}$ of $\mathbb{R}^3$ whose leaves are the pre-images $\{\Phi^{-1}(c)\}_{c\in\mathbb{R}}$. When $c = 0$ we obtain as leaf the cylinder $C = \{(x, y, z) : x^2 + y^2 = 1\}$. This cylinder splits the leaves into two classes:

- *The leaves with $c > 0$ lying in the interior of the cylinder C, which are all diffeomorphic to $\mathbb{R}^2$;*
- *The leaves with $c < 0$ lying in the exterior of the cylinder C which are all diffeomorphic to C;*

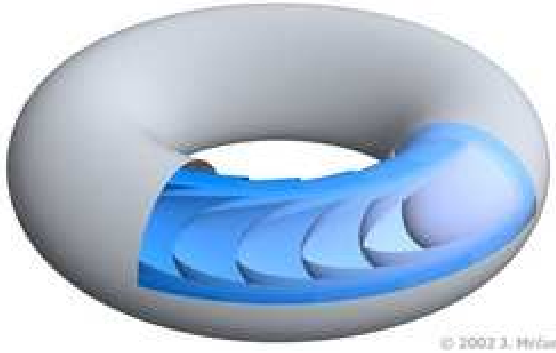
An explicit parameterization of the leaves with $c \neq 0$ is given by:

$$(x, y) \mapsto (x, y, \log(c/f(x^2 + y^2))).$$

For the first type of leaves, $c > 0$ and $x^2 + y^2 < 1$, while for the second type of leaves $c < 0$ and $x^2 + y^2 > 1$.



4. *The foliation in the previous example is invariant under translations in the Oz-axis direction. If we identify $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$, we obtain a foliation in the quotient $\mathbb{R}^2 \times \mathbb{S}^1 = \mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$. If we restrict this foliation to $\operatorname{Int} D^2 \times \mathbb{S}^1$, where $D^2 = \{(x, y) : x^2 + y^2 \leq 1\}$, we obtain a foliation of the solid 2-torus. This example suggests that foliations of manifolds with boundary are also interesting. We will not pursue this topic, but you should be aware of the existence of foliations on manifolds with boundary.*



5. The 3-sphere $\mathbb{S}^3$ *can be obtained by "gluing" two solid 2-torus along its boundary:*

$$\mathbb{S}^3 = T_1 \cup_\Phi T_2,$$

where $\Phi : \partial T_1 \to \partial T_2$ *is a diffeomorphism that takes the meridians of* ∂T_1 *in the circles of latitude of* ∂T_2*, and vice-versa. Explicitly, if* $\mathbb{S}^3 = \{(x, y, z, w) : x^2 + y^2 + z^2 + w^2 = 1\}$*, then we can take:*

$$T_1 = \{(x, y, z, w) \in \mathbb{S}^3 : x^2 + y^2 \leq 1/2\},$$
$$T_2 = \{(x, y, z, w) \in \mathbb{S}^3 : x^2 + y^2 \geq 1/2\}.$$

Each of these solid 2-torus admits a 2-dimensional foliation as in the previous example. One then obtains a famous 2-dimensional foliation of the sphere $\mathbb{S}^3$*, called the* ***Reeb foliation*** *of* $\mathbb{S}^3$*.*

Proposition 8.3. *Let* $\mathcal{F}$ *be a* k*-dimensional foliation of a smooth manifold* M*. Every leaf* $L \in \mathcal{F}$ *is a initial submanifold of dimension* k*.*

Proof. Let L be a leaf of $\mathcal{F}$. On each plaque of L we consider the relative topology, and we furnish L with the topology generated by the open sets in the plaques of L. For each plaque P, associated with a foliated chart $(U, \phi) = (U, x^1, \dots, x^k, y^1, \dots, y^{d-k})$, we consider the map $\psi : P \to \mathbb{R}^k$ obtain by choosing the first k-components:

$$\psi(p) = (x^1(p), \dots, x^k(p)).$$

The pairs (P, ψ) give charts for L, which turn L into a Hausdorff topological manifold. The transition functions for these charts are clearly smooth, so we can consider the maximal atlas that contains all the charts (U, ψ). To check that L is a manifold, we only need to check that the topology admits a countable basis. For that we apply the following lemma:

Lemma 8.4. *Let* L *be a leaf of* $\mathcal{F}$ *and* $\{U_n : n \in \mathbb{Z}\}$ *a covering of* M *by domains of foliated charts. The number of plaques of* L *in this covering, i.e., the number of connected components of* $L \cap U_n$*,* $n \in \mathbb{Z}$*, is countable.*

Fix a plaque P_0 of L in the covering $\{U_n : n \in \mathbb{Z}\}$. If a plaque P' belongs to L then there exists a **path of plaques** $P_1, \ldots, P_l$ in the covering, with $P_i \cap P_{i+1} \neq \emptyset$ which connects P' to P_0. Therefore it is enough to check that the collection of such paths is countable.

For each path of plaques $P_1, \ldots, P_l$ let us call l the length of the path. Using induction on n, we show that the collection of paths of length less or equal to n is countable:

- The collection of paths of length 1 has only one element hence is countable.
- Assume that the collection of paths of length $n-1$ is countable. Let $P_1, \ldots, P_{n-1}$ be a path of length $n-1$, corresponding to domains of foliated charts $U_1, \ldots, U_{n-1}$. In order to obtain a path of plaques of length n, we choose a domain of a foliated chart $U_n \neq U_{n-1}$ and we consider the plaques P', which are connected components of $L \cap U_n$, such that the intersection with P_{n-1} is non-empty. Now observe that:

$$(L \cap U_n) \cap P_{n-1} = U_n \cap P_{n-1},$$

  intersections form an open cover of the plaque P_{n-1}. This cover has a countable subcover, so the collection of all such P' is countable. It follows that the collection of paths of length less or equal than n is countable.

We leave it as an exercise to check that the leaves are actually initial submanifolds. □

Corollary 8.5. *Each leaf of a foliation intersects the domain of a foliated chart at most a countable number of times.*

There are few constructions which allows one to obtain new foliations out of other foliations. The details of these constructions are left for the exercises.

Product of foliations. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be foliations of M_1 and M_2, respectively. Then the **product foliation** $\mathcal{F}_1 \times \mathcal{F}_2$ is a foliation of $M_1 \times M_2$ defined as follows: if $\mathcal{F}_1 = \{L^{(1)}_\alpha\}_{\alpha\in A}$ and $\mathcal{F}_2 = \{L^{(2)}_\beta\}_{\beta\in B}$, then

$$\mathcal{F}_1 \times \mathcal{F}_2 = \{L^{(1)}_\alpha \times L^{(2)}_\beta\}_{(\alpha,\beta)\in A\times B}.$$

It should be clear that $\dim(\mathcal{F}_1 \times \mathcal{F}_2) = \dim \mathcal{F}_1 + \dim \mathcal{F}_2$ and, hence, that $\operatorname{codim}(\mathcal{F}_1 \times \mathcal{F}_2) = \operatorname{codim} \mathcal{F}_1 + \operatorname{codim} \mathcal{F}_2$

Pull-back of a foliation. Let $\Phi : M \to N$ be a smooth map between smooth manifolds. If $\mathcal{F}$ is a foliation of N we will say that Φ **is transversal to** $\mathcal{F}$ and write $\Phi \pitchfork \mathcal{F}$ if Φ is transversal to every leaf L of $\mathcal{F}$:

$$\mathrm{d}_p\Phi(T_pM) + T_{\Phi(p)}L = T_{\Phi(p)}N, \quad \forall p \in M.$$

Whenever $\Phi \pitchfork \mathcal{F}$ one defines the **pull-back foliation** $\Phi^*(\mathcal{F})$ to be the foliation of M whose leaves are the connected components of $\Phi^{-1}(L)$, where $L \in \mathcal{F}$. It should be clear that $\operatorname{codim} \Phi^*(\mathcal{F}) = \operatorname{codim} \mathcal{F}$.

Suspension of a difeomorphism. The manifold $\mathbb{R} \times M$ has a foliation $\mathcal{F}$ of codimension 1: the leaves are the sets $\{t\} \times M$, where $t \in \mathbb{R}$ (or if your prefer, the fibers of the projection $\pi : \mathbb{R} \times M \to \mathbb{R}$). A difeomorphism $\Phi : M \to M$ induces an action of $\mathbb{Z}$ on $\mathbb{R} \times M$ by setting

$$n \cdot (t, p) = (t + n, \Phi^n(p)).$$

This action takes leaves of $\mathcal{F}$ into leaves. The quotient $N = (\mathbb{R} \times M)/\mathbb{Z}$ is a manifold called the **suspension** or **mapping cylinder** of the diffeomorphism Φ. It carries a codimension 1 foliation $\tilde{\mathcal{F}}$ whose leaves are the equivalence classes $[L]$ in N, where $L \in \mathcal{F}$.

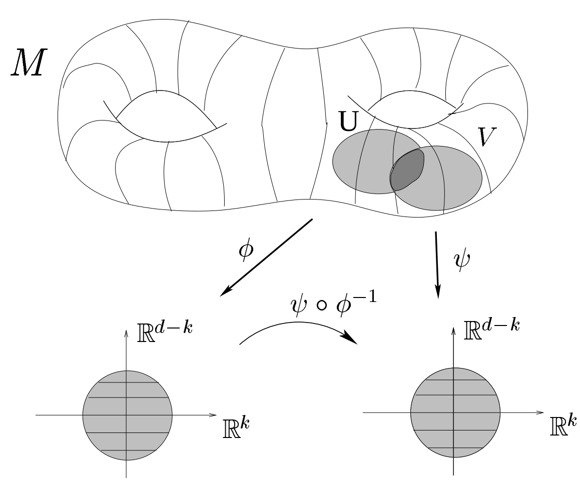
It is convenient to have alternative characterizations of foliations.

Foliations via smooth $\mathcal{G}^k_d$-structures. Let $\mathcal{F} = \{L_\alpha : \alpha \in A\}$ be a k-dimensional foliation of M. If (U, ϕ) and (V, ψ) are foliated charts then the change of coordinates $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is of the form:

$$\mathbb{R}^k \times \mathbb{R}^{d-k} \ni (x, y) \mapsto (h_1(x, y), h_2(y)) \in \mathbb{R}^k \times \mathbb{R}^{d-k}.$$

In other words, we have that the transition functions satisfy:

$$\frac{\partial(\psi \circ \phi^{-1})^j}{\partial x^i} = 0, \quad (i = 1, \dots, k,\ j = k + 1, \dots, d). \tag{8.1}$$



Conversely, denote by $\mathcal{G}^k_d$ the diffeomorphisms $\mathbb{R}^d \to \mathbb{R}^d$ defined on some open set, that satisfy this condition. We can refine the notion of smooth structure by requiring that in Definition 1.4 the transition functions belong to $\mathcal{G}^k_d$, and we then speak of a **smooth $\mathcal{G}^k_d$-structure**. An ordinary smooth structure on M is just a $\mathcal{G}_d^d$-structure: the leaves are the connected components of M.

We have the following alternative description of a foliation:

Proposition 8.6. *Let M be a smooth d-dimensional manifold. Given a foliation $\mathcal{F} = \{L_\alpha : \alpha \in A\}$ of M of dimension k the collection of all foliated charts $\mathcal{C} = \{(U, \phi)\}$ defines a smooth $\mathcal{G}_d^k$-structure. Conversely, for every smooth $\mathcal{G}_d^k$-structure $\mathcal{C}$ on a topological space M, there is smooth structure that makes M into a d-dimensional manifold and there exists a foliation $\mathcal{F}$ of M of dimension k, for which the foliated charts are the elements of $\mathcal{C}$.*

Proof. We have shown above that every k-dimensional foliation of a d-dimensional manifold determines a smooth $\mathcal{G}_d^k$-structure. We will show that, conversely, given a smooth $\mathcal{G}_d^k$-structure $\mathcal{C} = \{(U, \phi)\}$ we can associate to it a smooth structure on M of dimension d and a k-dimensional foliation $\mathcal{F}$ of M.

It should be clear that a smooth $\mathcal{G}_d^k$-structure $\mathcal{C} = \{(U, \phi)\}$ determines a smooth structure on M of dimension d, since it is in particular an atlas. In order to build $\mathcal{F}$, we first observe that we can choose an atlas defining $\mathcal{C}$ with the property that the slices $\phi^{-1}(\mathbb{R}^k \times \{c\})$, for $c \in \mathbb{R}^{d-k}$, are connected. We call these slices *plaques* and note that M is covered by all such plaques. Hence, we can define an equivalence relation in M by:

- $p \sim q$ if there exists a path of plaques $P_1, \ldots, P_l$ with $p \in P_1$ and $q \in P_l$.

Let $\mathcal{F}$ be the set of equivalence classes of $\sim$. We will show that $\mathcal{F}$ is a foliation of M.

Let $p_0 \in M$ and consider a plaque P_0 which contains p_0. Then,

$$P_0 = \phi^{-1}(\mathbb{R}^k \times \{c_0\}),$$

for some smooth chart $(U, \phi) \in \mathcal{C}$ with $\phi(p_0) = (a_0, c_0) \in \mathbb{R}^k \times \mathbb{R}^{d-k}$. We claim that (U, ϕ) is a foliated chart: let $L \in \mathcal{F}$ be an equivalence class that intersects U. If $p \in U \cap L$, then $\phi(p) = (a, c) \in \mathbb{R}^k \times \mathbb{R}^{d-k}$, so we see that that the plaque

$$P = \phi^{-1}(\mathbb{R}^k \times \{c\}),$$

is contained in L. Since P is connected, it is clear that P is contained in the connected component of $L \cap U$ that contains p. We claim that this connected component is actually P, from which it will follow that (U, ϕ) is a foliated chart.

Let $q \in L \cap U$ be some point in the connected component of $L \cap U$ containing p. We claim that $q \in P$. By the definition of $\sim$, there exists a path of plaques $P_1, \ldots, P_l$, with $p \in P_1$ and $q \in P_l$, and such that $P_i \subset U$. Each plaque P_i is associated to a smooth chart $(U_i, \phi_i) \in \mathcal{C}$ such that

$$P_i = \phi_i^{-1}(\mathbb{R}^k \times \{c_i\}).$$

We can assume also that $U_1 = U$, $\phi_1 = \phi$, $P_1 = P$ and $c_1 = c$. Since $\phi_2 \circ \phi^{-1} \in \mathcal{G}_d^k$, we have that:

$$\phi_2^{-1}(\mathbb{R}^k \times \{c_2\}) \subset \phi_2^{-1} \circ \phi_2 \circ \phi^{-1} \circ (\mathbb{R}^k \times \{\bar{c}_2\}) = \phi^{-1}(\mathbb{R}^k \times \{\bar{c}_2\}),$$

for some $\bar{c}_2 \in \mathbb{R}^{d-k}$. Since $P_2 \cap P_1 \neq \emptyset$ and the plaques $\phi^{-1} \circ (\mathbb{R}^k \times \{c\})$ are disjoint, we conclude that $\bar{c}_2 = c_1$ and $P_2 \subset P_1 = P$. By induction, $P_i \subset P$ so $q \in P$, as claimed. □

Foliations via Haefliger cocycles. We saw before that the connected components of the fibers of a submersion is an example of a foliation. Actually, every foliation is locally of this form: if $\mathcal{F} = \{L_\alpha\}_{\alpha \in A}$ is a foliation of M of dimension k, for any foliated chart:

$$\phi = (x^1, \dots, x^k, y^1, \dots, y^{d-k}) : U \to \mathbb{R}^d,$$

the projection in the last $(d-k)$-components gives a submersion:

$$\psi = (y^1, \dots, y^{d-k}) : U \to \mathbb{R}^{d-k},$$

whose fibers are the connected components of $L_\alpha \cap U$. Given another foliated chart:

$$\bar{\phi} = (\bar{x}^1, \dots, \bar{x}^k, \bar{y}^1, \dots, \bar{y}^{d-k}) : \bar{U} \to \mathbb{R}^d,$$

with $U \cap \bar{U} \neq \emptyset$, for the corresponding submersion

$$\bar{\psi} = (\bar{y}^1, \dots, \bar{y}^{d-k}) : \bar{U} \to \mathbb{R}^{d-k},$$

we have a change of coordinates of the form

$$\bar{\phi} \circ \phi^{-1}(x, y) = (h_1(x, y), h_2(y)),$$

where h_2 has Jacobian matrix

$$\left[\frac{\partial h_2^j}{\partial y^i}\right]_{i,j=1}^{d-k}$$

with rank $d-k$. We conclude that the submersions ψ and $\bar{\psi}$ differ by a local diffeomorphism: for every $p \in U \cap \bar{U}$ there exists an open neighborhood $p \in U_p \subset U \cap \bar{U}$ and a local diffeomorphism $\Psi : \mathbb{R}^{d-k} \to \mathbb{R}^{d-k}$, such that:

$$\bar{\psi}|_{U_p} = \Psi \circ \psi|_{U_p}.$$

This suggests another way of defining foliations:

Proposition 8.7. *Let M be a d-dimensional manifold. Every k-dimensional foliation $\mathcal{F}$ of M determines a maximal collection $\{\psi_i\}_{i \in I}$ of submersions $\psi_i : U_i \to \mathbb{R}^{d-k}$, where $\{U_i\}_{i \in I}$ is an open cover of M, and which satisfies the following property: for every $i, j \in I$ and $p \in U_i \cap U_j$, there exists a local diffeomorphism ψ_{ji}^p of $\mathbb{R}^{d-k}$, such that:*

$$\psi_j = \psi_{ji}^p \circ \psi_i,$$

in an open neighborhood U_p of p. Conversely, every such collection determines a foliation of M.

We have already seen how to a foliation we can associate a collection of submersions. We leave it as an exercise to prove the converse.

Given a collection of submersions $\{\psi_i\}_{i\in I}$, as in the proposition, we consider for each pair $i, j \in I$, the map

$$\psi_{ij} : U_i \cap U_j \to \mathrm{Diff}_{\mathrm{loc}}(\mathbb{R}^{d-k}),\ p \longmapsto \psi_{ij}^p.$$

This map satisfies:

$$(\psi_{ji})^{-1} = \psi_{ji} \text{ in } U_i \cap U_j, \tag{8.2}$$

and the *cocycle condition*:

$$\psi_{ij} \circ \psi_{jk} \circ \psi_{ki} = 1 \text{ in } U_i \cap U_j \cap U_k. \tag{8.3}$$

We will see later, in Part IV of these notes, when we study the theory of fiber bundles, that these cocycles, called **Haefliger cocycles**, play a very important role.

Foliations appear naturally in many problems in differential geometry, and we shall see many other examples of foliations during the course of these sections.

Homework.

1. Show that the leaves of a foliation are initial submanifolds.

2. Let $\mathcal{F}$ be the Reeb foliation of S^3 and let $\Phi : S^3 \to N$ be a continuous map whose restriction to each leaf of $\mathcal{F}$ is constant. Show that Φ is constant.

3. Proof Proposition 8.7.

4. Let $\mathcal{F}_1 = \{L^{(1)}_\alpha\}_{\alpha\in A}$ and $\mathcal{F}_2 = \{L^{(2)}_\beta\}_{\beta\in B}$ be foliations. Using your favorite definition of a foliation, show that the product $\mathcal{F}_1 \times \mathcal{F}_2$ is a foliation:
$$\mathcal{F}_1 \times \mathcal{F}_2 := \{L^{(1)}_\alpha \times L^{(2)}_\beta\}_{(\alpha,\beta)\in A\times B}.$$

5. Let $\Phi : M \to N$ be a smooth map and $\mathcal{F} = \{L_\alpha\}_{\alpha\in A}$ a foliation of N such that $\Phi \pitchfork \mathcal{F}$. Using your favorite definition of a foliation, show that the pull-back $\Phi^*(\mathcal{F})$ is a foliation:
$$\Phi^*(\mathcal{F}) := \{\text{connected components of } \Phi^{-1}(L_\alpha)\}_{\alpha\in A}.$$

6. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two foliations of a smooth manifold M such that $\mathcal{F}_1 \pitchfork \mathcal{F}_2$, i.e., such that
$$T_pM = T_pL^{(1)} + T_pL^{(2)}, \qquad \forall p \in M,$$
where $L^{(1)}$ and $L^{(2)}$ are the leaves of $\mathcal{F}_1$ and $\mathcal{F}_2$ through p. Show that there exists a foliation $\mathcal{F}_1 \cap \mathcal{F}_2$ of M whose leaves are the connected components of $L^{(1)}_\alpha \cap L^{(2)}_\beta$, and which satisfies $\operatorname{codim}\mathcal{F} = \operatorname{codim}\mathcal{F}_1 + \operatorname{codim}\mathcal{F}_2$.

7. Given a foliation $\mathcal{F}$ of M, one denotes by $M/\mathcal{F}$ the space of leaves of $\mathcal{F}$ with the quotient topology. Try to describe for each of the examples given in the text their space of leaves.

9. Quotients

We have seen before several constructions that produce new manifolds out of old manifolds, such as the product of manifolds or the pullback of sub manifolds under transversal maps. We will now study another important, but more delicate, construction: forming quotients of manifolds.

Let X be a topological space. If $\sim$ is an equivalence relation on X, we will denote by $X/\sim$ the set of equivalence classes of $\sim$ and by $\pi : X \to X/\sim$ the *quotient map* which associates to each $x \in X$ its equivalence class $\pi(x) = [x]$. In $X/\sim$ we consider the *quotient topology*: a subset $V \subset X/\sim$ is open if and only if $\pi^{-1}(V)$ is open. This is the largest topology in $X/\sim$ for which the quotient map $\pi : M \to M/\sim$ is continuous. We have the following basic result about the quotient topology which we leave as an exercise:

Lemma 9.1. *Let X be a Hausdorff topological space and let $\sim$ be an equivalence relation on X such that $\pi : X \to X/\sim$ is an open map. Then $X/\sim$ is Hausdorff if and only if the graph of $\sim$:*

$$R = \{(x, y) \in X \times X : x \sim y\},$$

is a closed subset of $X \times X$.

Let M be a smooth manifold and let $\sim$ be an equivalence relation on M. We would like to known when there exists a smooth structure on $M/\sim$, compatible with the quotient topology, such that $\pi : M \to M/\sim$ becomes a submersion. Before we can state a result that gives a complete answer to this question, we need one definition.

Recall that a continuous map $\Phi : X \to Y$, between two Hausdorff topological spaces is called a **proper map** if $\Phi^{-1}(K) \subset X$ is compact whenever $K \subset Y$ is compact. A proper map is always a closed map.

Definition 9.2. *A **proper submanifold** of M is a submanifold (N, Φ) such that $\Phi : N \to M$ is a proper map.*

By an exercise in Section 6, any proper submanifold is an embedded submanifold. Also, if $\Phi : N \to M$ is proper, then its image $\Phi(N)$ is a closed subset of M. Conversely, every embedded closed submanifold of M is a proper submanifold.

Theorem 9.3 (Godement's Criterion)**.** *Let M be a smooth manifold and let $\sim$ be an equivalence relation on M. The following statements are equivalent:*

- *(i) There exists a smooth structure on $M/\sim$, compatible with the quotient topology, such that $\pi : M \to M/\sim$ is a submersion.*
- *(ii) The graph R of $\sim$ is a proper submanifold of $M \times M$ and the restriction of the projection $p_1 : M \times M \to M$ to R is a submersion.*

$$\begin{tikzcd} R \arrow[r, hook] & M \times M \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ M & & M \end{tikzcd}$$

Proof. We must show both implications:

(i) $\Rightarrow$ (ii). The graph of the quotient map, as for every smooth map, is a closed embedded submanifold:

$$\mathcal{G}(\pi) = \{(p, \pi(p)) : p \in M\} \subset M \times M/\sim,$$

Since $I \times \pi : M \times M \to M \times M/\sim$ is a submersion and

$$R = (Id \times \pi)^{-1}(\mathcal{G}(\pi)),$$

we conclude that $R \subset M \times M$ is an embedded closed submanifold, i.e., is a proper submanifold.

On the other hand, the map $(I \times \pi)|_R : R \to \mathcal{G}(\pi)$ is a submersion while $\mathcal{G}(\pi) \to M$, $(p, \pi(p)) \mapsto p$ is a diffeomorphism, hence their composition $p_1|R$ is a submersion.

(ii) $\Rightarrow$ (i). We split the proof into several lemmas. The first of these lemmas states that we can "straighten out" $\sim$:

Lemma 9.4. *For every $p \in M$, there exists a local chart $(U, (x^1, \ldots, x^d))$ centered at p, such that*

$$\forall q, q' \in U,\ q \sim q' \ \text{ if and only if } \ x^{k+1}(q) = x^{k+1}(q'), \ldots, x^d(q) = x^d(q'),$$

where k is an integer independent of p.

To prove this lemma, let $\Delta \subset M \times M$ be the diagonal. Note that $\Delta \subset R \subset M \times M$, and since Δ and R are both embedded submanifolds of $M \times M$, we have that Δ is an embedded submanifold of R. Therefore, for each $p \in M$, there exists a neighborhood O of (p,p) in $M \times M$ and a submersion $\Phi : O \to \mathbb{R}^{d-k}$, where $d - k = \operatorname{codim} R$, such that:

$$(q, q') \in O \cap R \ \text{ if and only if } \Phi(q, q') = 0.$$

We have that $k \geq 0$, since $\Delta \subset R$ and $\operatorname{codim} \Delta = d$.

Next we observe that the differential of the map $q \mapsto \Phi(q,p)$ has maximal rank at $q = p$: in fact, after identifying $T_{(p,p)}(M \times M) = T_pM \times T_pM$, we see that $\mathrm{d}_{(p,p)}\Phi$ is zero precisely in the subspace formed by pairs $(\mathbf{v}, \mathbf{v}) \in T_pM \times T_pM$, and this subspace is complementary to the subspace formed by elements of the form $(\mathbf{v}, 0) \in T_pM \times T_pM$. We conclude that there exists a neighborhood V' of p such that $V' \times V' \subset O$, and the map $q \mapsto \Phi(q, p)$ is a submersion in V'. By the local canonical form for submersions, there exist a chart $(V, \phi) = (V, (u^1, \ldots, u^k, v^1, \ldots, v^{d-k}))$ centered at p, with $V \subset V'$, such that

$$\Phi \circ (\phi^{-1} \times \phi^{-1})(u^1, \ldots, u^k, v^1, \ldots, v^{d-k}, 0, \ldots, 0) = (v^1, \ldots, v^{d-k}).$$

In the domain of this chart, the points $q \in V$ such that $q \sim p$ are precisely the points satisfying $v^1(q) = 0, \ldots, v^{d-k}(q) = 0$.

Now set $\widehat{\Phi} = \Phi \circ (\phi^{-1} \times \phi^{-1})$. The smooth map

$$\mathbb{R}^d \times \mathbb{R}^{d-k} \to \mathbb{R}^{d-k},\ (u, v, w) \mapsto \widehat{\Phi}((u, v), (0, w)),$$

satisfies

$$\widehat{\Phi}((u,v),(0,0)) = v.$$

so the matrix of partial derivatives $\partial\widehat{\Phi}^i/\partial v^j$, $(i,j = 1,\dots,d-k)$ is non-degenerate. We can apply the Implicit Function Theorem to conclude that there exists a local defined smooth function $\mathbb{R}^k \times \mathbb{R}^{d-k} \to \mathbb{R}^{d-k}$, $(u,w) \mapsto v(u,w)$, such that:

$$\widehat{\Phi}((u,v),(0,w)) = 0 \ \text{ if and only if } v = v(u,w).$$

Since $v(0,w) = w$ is a solution, uniqueness implies that:

$$\phi^{-1}(0,w) \sim \phi^{-1}(0,w') \ \text{ if and only if } w = w'.$$

This shows that the map $(u,w) \mapsto (u, v(u,w))$ is a local diffeomorphism. Hence, there exists an open set U where

$$(x^1,\dots,x^d) = (u^1,\dots,u^k,w^1,\dots,w^{d-k})$$

are local coordinates and in these coordinates:

$$\forall q,q' \in U,\ q \sim q' \ \text{ if and only if } \ x^{k+1}(q) = x^{k+1}(q'),\dots,x^d(q) = x^d(q'),$$

so the lemma follows.

Since the functions $x^{k+1},\dots,x^d$ given by this lemma induce well-defined functions $\bar{x}^{k+1},\dots,\bar{x}^d$ on the quotient $M/\sim$, we consider the pairs of the form $(\pi(U), \bar{x}^{k+1},\dots,\bar{x}^d)$:

Lemma 9.5. *The collection $\mathcal{C} = \{(\pi(U), \bar{x}^{k+1},\dots,\bar{x}^d)\}$ gives $M/\sim$, with the quotient topology, the structure of a topological manifold of dimension $d-k$.*

First note that $\pi : M \to M/\sim$ is an open map: in fact, for any $V \subset M$, we have that

$$\pi^{-1}(\pi(V)) = p_1|_R((p_2|_R)^{-1}(V)).$$

By assumption, $p_1|_R$ is a submersion hence is an open map. Therefore, if $V \subset M$ is open then $\pi^{-1}(\pi(V))$ is also open, so $\pi(V) \subset M/\sim$ is open.

This shows that $\pi(U)$ is open. Since the map

$$(x^{k+1},\dots,x^d) : U \to \mathbb{R}^{d-k}$$

is both continuous and open, it follows that the induced map

$$(\bar{x}^{k+1},\dots,\bar{x}^d) : \pi(U) \to \mathbb{R}^{d-k}$$

is continuous, open and injective, i.e., is a homeomorphism onto its image.

Now we show that:

Lemma 9.6. *The family $\mathcal{C} = \{(\pi(U), \bar{x}^{k+1},\dots,\bar{x}^d)\}$ is an atlas generating a smooth structure for $M/\sim$ such that $\pi : M \to M/\sim$ is a submersion.*

Take two pairs in $\mathcal{C}$:

$$(\pi(U), \bar{\phi}) = (\pi(U), \bar{x}^{k+1}, \ldots, \bar{x}^d),$$
$$(\pi(V), \bar{\psi}) = (\pi(V), \bar{y}^{k+1}, \ldots, \bar{y}^d),$$

which correspond to two charts in M:

$$(U, \phi) = (U, x^1, \ldots, x^d),$$
$$(V, \psi) = (V, y^1, \ldots, y^d).$$

The corresponding transition function:

$$\bar{\psi} \circ \bar{\phi}^{-1} : \mathbb{R}^{d-k} \to \mathbb{R}^{d-k},$$

composed with the projection $p : \mathbb{R}^d \to \mathbb{R}^{d-k}$ in the last $d-k$ components is given by:

$$\bar{\psi} \circ \bar{\phi}^{-1} \circ p = p \circ \psi \circ \phi^{-1}.$$

Since the right-hand side is a smooth map $\mathbb{R}^d \to \mathbb{R}^{d-k}$ it follows that $\bar{\psi} \circ \bar{\phi}^{-1}$ is smooth.

In order to check that $\pi : M \to M/\sim$ is a submersion, it is enough to observe that in the charts $(U, x^1, \ldots, x^d)$ for M and $(\pi(U), \bar{x}^{k+1}, \ldots, \bar{x}^d)$ for $M/\sim$, this map corresponds to the projection $p : \mathbb{R}^d \to \mathbb{R}^{d-k}$.

To finish the proof of Theorem 9.3, we check that

Lemma 9.7. *The quotient topology $M/\sim$ is Hausdorff and second countable.*

It is obvious that if M has a countable basis, then the quotient topology also has a countable basis. Since the graph R of $\sim$ is closed in $M \times M$, M is Hausdorff and π is an open map, it follows from Lemma 9.1 that $M/\sim$ is Hausdorff.

□

Remark 9.8. The proof shows that if we assume that R is embedded, not closed, and $p_1|_R : R \to M$ is a submersion, then the quotient $M/\sim$ is a smooth manifold, second countable, but not Hausdorff (see Exercise 4 for an example).

We will now study two important examples of quotients.

Leaf spaces of foliations. Let $\mathcal{F}$ be a foliation of a smooth manifold M. Since $\mathcal{F}$ is a partition of M, it determines an equivalence relation on M, namely:

$$p \sim q \quad \text{if and only if } p \text{ and } q \text{ belong to same leaf.}$$

The set of equivalence classes:

$$M/\mathcal{F} := M/\sim$$

is the collection of all leaves of $\mathcal{F}$ and hence is called the *leaf space* of the foliation.

In general, the leaf space of a foliation does not carry a smooth structure compatible with the quotient topology, but we can use Godement's Criterion to find an answer to this question:

Corollary 9.9. *Let $\mathcal{F}$ be a foliation of a smooth manifold M. The following statements are equivalent:*

(i) There exists a smooth structure on $M/\mathcal{F}$, compatible with the quotient topology, such that $\pi : M \to M/\mathcal{F}$ is a submersion.

(ii) The leaf space $M/\mathcal{F}$ is Hausdorff and there is a cover of M by foliated charts with the property that each leaf of $\mathcal{F}$ intersects each chart at most once.

A foliation satisfying either of the equivalent conditions in this corollary is called a a **simple foliation**. We leave the proof as an exercise.

You may notice that the proof that Godement's Criterion yields a smooth quotient manifold actually amounts to show that the equivalence classes of R form a simple foliation of M.

Orbit spaces of discrete group actions. A very important class of equivalence relations on manifolds is given by actions of groups of diffeomorphisms. If G is a group, we recall that an **action** of G on a set M is a group homomorphism $\widehat{\Psi}$ from G to the group of bijections of M. One can also view an action as a map $\Psi : G \times M \to M$, which we write as $(g,p) \mapsto g \cdot p$, if one sets:

$$g \cdot p \equiv \widehat{\Psi}(g)(p).$$

Since $\widehat{\Psi}$ is a group homomorphism, it follows that:

(a) $e \cdot p = p$, for all $p \in M$;

(b) $g \cdot (h \cdot p) = (gh) \cdot p$, for all $g, h \in G$ and $p \in M$.

Conversely, any map $\Psi : G \times M \to M$ satisfying (a) and (b), determines a homomorphism $\widehat{\Psi}$. From now on, we will denote an action by $\Psi : G \times M \to M$, and for each $g \in G$ we denote by Ψ_g the bijection:

$$\Psi_g : M \to M, \quad p \mapsto g \cdot p$$

Assume now that M is a manifold. We say that that a group G *acts on M by diffeomorphims* if, for each $g \in G$, $\Psi_g : M \to M$ is a diffeomorphism. This means that we have a group homomorfismo $\widehat{\Psi} : G \to \mathrm{Diff}(M)$, where $\mathrm{Diff}(M)$ is the group of all diffeomorphisms of M. We can also express this condition by saying that the map $\Psi : G \times M \to M$ is smooth, where G is viewed as a smooth 0-dimensional manifold with the discrete topology. So we will also say in this case that the *discrete group G acts smoothly on M*.

Given any action of G on M the quotient $G\backslash M$ is, by definition, the set of equivalence classes determined by the **orbit equivalence relation**:

$$p \sim q \iff \exists g \in G : \; q = g \cdot p.$$

Let us see conditions on an action by diffeomorphisms for the quotient $G\backslash M$ to be a manifold.

We recall that a **free action** is an action $G \times M \to M$ such that each $g \in G - \{e\}$ acts without fixed points, i.e.,

$$g \cdot p = p \text{ for some } p \in M \quad \Longrightarrow \quad g = e.$$

Denoting by G_p the **isotropy subgroup** of $p \in M$, i.e.,

$$G_p = \{g \in G : g \cdot p = p\},$$

an action is free if and only if $G_p = \{e\}$, for all $p \in M$.

Definition 9.10. *A smooth action $\Psi : G \times M \to M$ of a discrete group G on a smooth manifold M is said to be* ***proper*** *if the map:*

$$G \times M \to M \times M, \quad (g, p) \mapsto (g \cdot p, p),$$

is a proper map.

Examples 9.11.

1. *Actions of finite groups are always proper (exercise). For example, the $\mathbb{Z}_2$-action on $M = \mathbb{S}^d$, defined by:*

$$\pm 1 \cdot (x^1, \ldots, x^{d+1}) := \pm(x^1, \ldots, x^{d+1}).$$

is a free and proper action.

2. *Let $\mathbb{Z}^d$ act on $\mathbb{R}^d$ by translations:*

$$(n_1, \ldots, n_d) \cdot (x^1, \ldots, x^d) := (x^1 + n_1, \ldots, x^d + n_d).$$

It is easy to see that his action is also free and proper. So there are proper actions of discrete groups G, where G is not finite.

3. *A proper smooth action $G \times M \to M$ of a discrete group must have finite isotropy groups (exercise). So, for example, the orthogonal group $O(d)$ consisting of orthogonal matrices of size d acts smoothly on the sphere $\mathbb{S}^d$ by matrix multiplication:*

$$A \cdot x := Ax.$$

The isotropy group of x_0 consists of those orthogonal matrices fixing x_0 and contains, e.g., the rotations with axis the line through x_0 and the origin. Since there are infinite isotropy groups the action is not proper (note that $O(d)$ is considered here as a discrete group).

Godement's Criterion allows us to find conditions for an orbit space to be smooth:

Corollary 9.12. *Let $\Psi : G \times M \to M$ be a free and proper smooth action of a discrete group G on M. There exists a unique smooth structure on $G \backslash M$ such that $\pi : M \to G \backslash M$ is a local diffeomorphism.*

Proof. We check that condition (ii) of Theorem 9.3 holds.

We claim that $R \subset M \times M$ is a proper submanifold. Since the action if free and proper, one has (see Exercise 7) for each $p_0 \in M$ an open set $p_0 \in U$, such that:

$$g \cdot U \cap U = \emptyset, \quad \forall g \in G - \{e\}.$$

For such an open set, if $g_0 \in G$, we have:

$$(U \times g_0 \cdot U) \cap R = \{(q, g_0 \cdot q) : q \in U\}.$$

so the map

$$U \to (U \times g_0 \cdot U) \cap R, \quad q \mapsto (q, g_0 \cdot q),$$

is a parameterization of $O \cap R$, with $O \subset M \times M$ open. It follows that R can be covered by open sets $O \cap R$ embedded in $M \times M$, so that R is an embedded submanifold. Also, the action being proper, the inclusion

$$R = \{(p, g \cdot p) : p \in M, g \in G\} \hookrightarrow M \times M$$

is a proper map.

Now we observe that the projection $p_1 : M \times M \to M$ restricted to R is an inverse to the parameterizations of R constructed above, hence $p_1|_R$ is a local diffeomorphism. □

Under the conditions of this corollary, it is easy to check that the projection $\pi : M \to G\backslash M$ is in fact a covering map. Therefore, if M is simply connected, then M is the universal covering space of $G\backslash M$ and we conclude that $\pi_1(G\backslash M) \simeq G$.

Examples 9.13.

1. *The action $\mathbb{Z}_2 \times \mathbb{S}^d \to \mathbb{S}^d$ defined in Example 9.11.1 is free and proper so the orbit space $\mathbb{Z}_2\backslash\mathbb{S}^d$ is a manifold. We claim that this manifold is diffeomorphic to $\mathbb{RP}^d$: the map $\mathbb{S}^d \to \mathbb{RP}^d$ given by $(x^1, \dots, x^d) \mapsto [x^1 : \cdots : x^d]$ induces a diffeomorphism $\mathbb{Z}_2\backslash\mathbb{S}^d \to \mathbb{RP}^d$ such that the following diagram commutes:*

$$\begin{tikzcd} \mathbb{S}^d \arrow[d] \arrow[dr] & \\ \mathbb{Z}_2\backslash\mathbb{S}^d \arrow[r] & \mathbb{RP}^d \end{tikzcd}$$

For $d > 1$, $\mathbb{S}^d$ is simply connected, so we conclude also that the quotient map is a covering map and that $\pi_1(\mathbb{RP}^d) = \mathbb{Z}_2$.

2. *The action $\mathbb{Z}^d \times \mathbb{R}^d \to \mathbb{R}^d$ defined in Example 9.11.2 is also free and proper, so the orbit space $\mathbb{Z}^d\backslash\mathbb{R}^d$ is a smooth manifold. This manifold is diffeomorphic to d-torus $\mathbb{T}^d$: the map $\mathbb{R}^d \to \mathbb{T}^d$ given by $(x^1, \dots, x^d) \to (e^{2\pi i x^1}, \dots, e^{2\pi i x^1})$ induces a diffeomorphism $\mathbb{Z}^d\backslash\mathbb{R}^d \to \mathbb{T}^d$ such that the following diagram commutes:*

$$\begin{tikzcd} \mathbb{R}^d \arrow[d] \arrow[dr] & \\ \mathbb{Z}^d\backslash\mathbb{R}^d \arrow[r] & \mathbb{T}^d \end{tikzcd}$$

Since $\mathbb{R}^d$ is simply connected, we conclude also that the quotient map is a covering map and that $\pi_1(\mathbb{T}^d) = \mathbb{Z}^n$.

3. *Let $(\mathbb{R}, +)$ act on $\mathbb{R}^2$ by translations in the x-direction:*

$$\lambda \cdot (x^1, x^2) = (x^1 + \lambda, x^2).$$

This is a free but non-proper action of a discrete group. However, the orbits of this action form a simple foliation of $\mathbb{R}^2$ so that $\mathbb{R}\backslash\mathbb{R}^2$ inherits a smooth structure. The quotient $\mathbb{R}\backslash\mathbb{R}^2$ is diffeomorphic to $\mathbb{R}$.

The issue in the last example is that one should consider on the group $(\mathbb{R}, +)$ the usual topology, instead of the discrete topology. Later we will study *Lie groups*, which are groups carrying a compatible smooth structure (of positive dimension). Their smooth actions give rise to many other examples of quotients.

HOMEWORK.

1. Let X be a Hausdorff topological space and $\sim$ an equivalence relation in X such that $\pi : X \to X/\sim$ is an open map, for the quotient topology. Show that $X/\sim$ with the quotient topology is Hausdorff if and only if the graph of $\sim$ is closed in $X \times X$.

2. Let $\pi : M \to Q$ be a surjective submersion, $\Phi : M \to N$ and $\Psi : Q \to N$ any maps into a smooth manifold N such that the following diagram commutes:

$$\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "\Phi"] & \\ Q \arrow[r, "\Psi"'] & N \end{tikzcd}$$

Show that Φ is smooth if and only if Ψ is smooth. Use this to conclude that if M is a manifold, $\sim$ is an equivalence relation satisfying any of the conditions of Theorem 9.3, and $\Phi : M \to N$ is a smooth map such that $\Phi(x) = \Phi(y)$ whenever $x \sim y$, then there is an induced smooth map $\overline{\Phi} : M/\sim \to N$ such that the following diagram commutes:

$$\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "\Phi"] & \\ M/\sim \arrow[r, dashed, "\overline{\Phi}"'] & N \end{tikzcd}$$

3. Use Godement's Criterion to prove Corollary 9.9 characterizing simple foliations.

4. Let $\mathcal{F}$ be the foliation of $M = \mathbb{R}^2 - \{0\}$ whose leaves are the connected components of the horizontal lines $y = \text{const}$. Show that the leaf space $M/\mathcal{F}$ has a non-Hausdorff smooth structure (this non-Hausdorff manifold is sometimes called the **line with two origins**).

5. Let $(\mathbb{R}, +)$ act on $\mathbb{R}^2$ by translations in the x-direction:

$$\lambda \cdot (x^1, x^2) = (x^1 + \lambda, x^2).$$

Show that:

(a) The action is not proper when one considers the discrete topology on $\mathbb{R}$.

(b) The orbits of the action give a simple foliation of $\mathbb{R}^2$ and $\mathbb{R}\backslash\mathbb{R}^2$ is diffeomorphic to $\mathbb{R}$ with its usual smooth structure.

6. Show that any smooth action $G \times M \to M$ of a finite group G on a manifold M is proper.

7. A smooth action $\Psi : G \times M \to M$ of a discrete group G is said to be **properly discontinuous** if the following two conditions are satisfied:

(a) For every $p \in M$, there exists a neighborhood U of p, such that:

$$g \cdot U \cap U = \emptyset, \quad \forall g \in G - G_p.$$

(b) If $p, q \in M$ do not belong to the same orbit, then there are open neighborhoods U of p and V of q, such that

$$g \cdot U \cap V = \emptyset, \quad \forall g \in G.$$

Show that a free action of a discrete group is proper if and only if it is properly discontinuous.

8. Show that for a proper and free action of a discrete group $G \times M \to M$, the projection $\pi : M \to G\backslash M$ is a covering map.

Part 2. Lie Theory

In the first part of these sections we have introduced and study some elementary concepts about manifolds. We will now initiate the study of the local differential geometry of smooth manifolds. The main concept and ideas that we will introduce in this second part are the following:

- Section 10: the notion of *vector field* and the related concepts of *integral curve* and *flow* of a vector field.
- Section 11: the *Lie bracket* of vector fields and the *Lie derivative*, which allows to differentiate along vector fields.
- Section 12: an important generalization of vector fields, called *distributions*. The *Frobenius Theorem* says that foliations are the global objects associated with *involutive distributions*.
- Section 13: *Lie groups* and their infinitesimal versions called *Lie algebras*.
- Section 14: how to *integrate* Lie algebras to Lie groups.
- Section 15: the *exponential map* from the Lie algebra to the Lie group, generalizing the exponential of matrices.
- Section 16: *groups of transformations* which are concrete realizations of Lie groups.

10. Vector Fields and Flows

Definition 10.1. *A **vector field** on a manifold M is a section of the tangent bundle $\pi : TM \to M$, i.e., a map $X : M \to TM$ such that $\pi \circ X = I$. We say that the vector field X is smooth or C^∞, if the map $X : M \to TM$ is smooth. We will denote by $\mathfrak{X}(M)$ the set of smooth vector fields on a manifold M.*

If X is a vector field on M, we denote by X_p, rather than $X(p)$, the value of X at $p \in M$. For each $p \in M$, X_p is a derivation, hence, given any $f \in C^\infty(M)$ we can define a new function $X(f) : M \to \mathbb{R}$ by setting

$$X(f)(p) \equiv X_p(f).$$

If you recall the definition of the differential of a function, you see immediately that this definition is equivalent to:

$$X(f) = \mathrm{d}f(X).$$

Also, from the definition of a tangent vector as a derivation, we see that $f \mapsto X(f)$ satisfies:

(i) $X(af + bg) = aX(f) + bX(g)$;
(ii) $X(fg) = X(f)g + fX(g)$;

for any $a, b \in \mathbb{R}$ and smooth functions f, g.

Let $(U, x^1, \dots, x^d)$ be a coordinate system on M. Then we have the vector fields $\frac{\partial}{\partial x^i} \in \mathfrak{X}(U)$ defined by:

$$\frac{\partial}{\partial x^i}(p) \equiv \left.\frac{\partial}{\partial x^i}\right|_p, \qquad (i = 1, \dots, d).$$

At each $p \in U$ these vector fields yield a basis of T_pM, so if $X \in \mathfrak{X}(M)$ is any vector field on M, its restriction to the open set U, denoted by $X|_U$, can be written in the form:

$$X|_U = \sum_{i=1}^{d} X^i \frac{\partial}{\partial x^i},$$

where $X^i : U \to \mathbb{R}$ are certain functions which we call the **components of the vector field** X with respect to the chart $(x^1, \dots, x^d)$.

Lemma 10.2. *Let X be a vector field on M. The following statements are equivalent:*

(i) The vector field X is C^∞;
(ii) For any chart $(U, x^1, \dots, x^d)$, the components X^i of X with respect to this chart are C^∞;
(iii) For any $f \in C^\infty(M)$, the function $X(f)$ is C^∞.

Proof. We show that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

To show that (i) $\Rightarrow$ (ii), note that if X is C^∞ and U is an open set, the restriction $X|_U$ is also C^∞. Hence, if $(U, x^1, \dots, x^d)$ is any chart, we have that $\mathrm{d}x^i(X|_U) := \mathrm{d}x^i \circ X|_U$ is C^∞. But:

$$\mathrm{d}x^i(X|_U) = \mathrm{d}x^i(\sum_{j=1}^d X^j \frac{\partial}{\partial x^j}) = X^i.$$

To show that (ii) $\Rightarrow$ (iii), note that $f \in C^\infty(M)$ if and only if $f|_U \in C^\infty(U)$, for every domain U of a chart. But:

$$X(f)|_U = \sum_{i=1}^d X^i \frac{\partial f}{\partial x^i} \in C^\infty(U).$$

To show that (iii) $\Rightarrow$ (i), it is enough to show that $X|_U$ is C^∞, for every domain U of a chart. Recall that if $(U, x^1, \dots, x^d)$ is a chart then

$$(\pi^{-1}(U), (x^1 \circ \pi, \dots, x^d \circ \pi, \mathrm{d}x^1, \dots, \mathrm{d}x^d))$$

is a coordinate systems in TM. Since:

$$\begin{aligned} x^i \circ \pi \circ X|_U &= x^i \in C^\infty(U), \\ \mathrm{d}x^i \circ X|_U &= X(x^i) \in C^\infty(U), \end{aligned}$$

we conclude that $X|_U$ is C^∞. $\square$

We conclude from this lemma, that a vector field $X \in \mathfrak{X}(M)$ defines a linear derivation $D_X : C^\infty(M) \to C^\infty(M)$, $f \mapsto X(f)$. Conversely, we have:

Lemma 10.3. *Every linear derivation $D : C^\infty(M) \to C^\infty(M)$ determines a vector field $X \in \mathfrak{X}(M)$ through the formula:*

$$X_p(f) = D(f)(p).$$

Proof. We only need to show that $X_p(f)$ only depends on the germ $[f] \in \mathcal{G}_p$, i.e., if $f, g \in \mathcal{C}^\infty(M)$ are two function which agree in some neighborhood U of p, then $D(f)(p) = D(g)(p)$. This follows from the fact that derivations are local: if D is a derivation and $f \in C^\infty(M)$ is zero on some open set $U \subset M$, then $D(f)$ is also zero in U. To see this, let $p \in U$ and choose $g \in C^\infty(M)$ such that $g(p) > 0$ and $\operatorname{supp} g \subset U$. Since $gf \equiv 0$, we have that:

$$0 = D(gf) = D(g)f + gD(f).$$

If we evaluate both sides at p, we obtain $D(f)(p) = 0$. Hence, $D(f)|_U = 0$ as claimed. $\square$

From now on we will not distinguish between a vector field and the associated derivation of $C^\infty(M)$, so we will use the same letter to denote them.

Recall that a **path** in a manifold M is a continuous map $\gamma : I \to M$, where $I \subset \mathbb{R}$ is an interval. A **smooth path** is a path for which γ is C^∞. Note that if $\partial I \neq \emptyset$, i.e., is not an open interval, then γ is smooth if and only if it has a smooth extension to a smooth path defined in an open interval $J \supset I$. If $\gamma : I \to M$ is a smooth path, its **derivative** is:

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) \equiv \mathrm{d}\gamma \cdot \left.\frac{\partial}{\partial t}\right|_t \in T_{\gamma(t)}M, \quad (t \in I)$$

We often abbreviate writing $\dot\gamma(t)$ instead of $\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t)$. The derivative $t \mapsto \dot\gamma(t)$ is a smooth path in the manifold TM.

Definition 10.4. *Let $X \in \mathfrak{X}(M)$ be a vector field. A smooth path $\gamma : I \to M$ is called an* ***integral curve*** *of X if*

$$\dot\gamma(t) = X_{\gamma(t)}, \quad \forall t \in I. \tag{10.1}$$

In a chart $(U, (x^1, \dots, x^d))$, a path $\gamma(t)$ is determined by its components $\gamma^i(t) = x^i(\gamma(t))$. Its derivative is then given by

$$\dot\gamma(t) = \mathrm{d}\gamma \cdot \frac{\partial}{\partial t} = \sum_{i=1}^{d} \frac{\mathrm{d}\gamma^i}{\mathrm{d}t}\frac{\partial}{\partial x^i}.$$

It follows that the integral curves of a vector field X, which has components X^i in the local chart $(x^1, \dots, x^d)$, are the solutions of the system of o.d.e.'s:

$$\frac{\mathrm{d}\gamma^i}{\mathrm{d}t} = X^i(\gamma^1(t), \dots, \gamma^d(t)), \quad (i = 1, \dots, d). \tag{10.2}$$

This system is the local form of the equation (10.1). Note that it is common to write $x^i(t)$ for the components $\gamma^i(t) = x^i(\gamma(t))$ so that this system of equations becomes:

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = X^i(x^1(t), \dots, x^d(t)), \quad (i = 1, \dots, d).$$

EXAMPLE 10.5.
In $\mathbb{R}^2$ consider the vector field $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$. The equations for the integral curves (10.2) *are:*

$$\begin{cases} \dot x(t) = -y(t), \\ \dot y(t) = x(t). \end{cases}$$

Hence, the curves $\gamma(t) = (R\cos t, R\sin t)$ are integral curves of this vector field.

This vector field is tangent to the submanifold $\mathbb{S}^1 = \{(x, y) : x^2 + y^2 = 1\}$, so defines a vector field on the circle: $Y = X|_{\mathbb{S}^1}$. If we consider the angle coordinate θ on the circle, the smooth functions $C^\infty(\mathbb{S}^1)$ can be identified with the 2π-periodic smooth functions $f(\theta) = f(\theta + 2\pi)$. It is easy to see that the vector field Y as a derivation is given by:

$$Y(f)(\theta) = f'(\theta).$$

Hence we will write this vector field as:

$$Y = \frac{\partial}{\partial\theta},$$

although the function θ is not a globally defined smooth coordinate on $\mathbb{S}^1$.

Now consider the cylinder $M = \mathbb{S}^1 \times \mathbb{R}$, with coordinates (θ, x). We have a well defined vector field:

$$Z = \frac{\partial}{\partial \theta} + x\frac{\partial}{\partial x}.$$

You should try to plot this vector field on a cylinder and verify that the integral curve of Z through a point (θ_0, x_0) is given by

$$\gamma(t) = (\theta_0 + t, x_0 e^t).$$

If $x_0 = 0$, this is just a circle around the cylinder. If $x_0 \neq 0$ this is a spiral that approaches the circle when $t \to -\infty$ and goes to infinity when $t \to +\infty$.

Standard results about existence, uniqueness and maximal interval of definition of solutions a system of o.d.e.'s lead to the following proposition:

Proposition 10.6. *Let $X \in \mathfrak{X}(M)$ be a vector field. For each $p \in M$, there exist real numbers $a_p, b_p \in \mathbb{R} \cup \{\pm\infty\}$ and a smooth path $\gamma_p :]a_p, b_p[\to M$, such that:*

(i) $0 \in]a_p, b_p[$ *and* $\gamma_p(0) = p$*;*
(ii) γ_p *is an integral curve of* X*;*
(iii) If $\eta :]c, d[\to M$ *is any integral curve of* X *with* $\eta(0) = p$*, then* $]c, d[\subset]a_p, b_p[$ *and* $\gamma_p|_{]c,d[} = \eta$*.*

We call the integral curve γ_p given by this proposition the **maximal integral curve** of X through p. For each $t \in \mathbb{R}$, we define the domain $D_t(X)$ consisting of those points for which the integral curve through p exists at least until time t:

$$D_t(X) = \{p \in M : t \in]a_p, b_p[\}.$$

If it is clear the vector field we are referring to, we will write D_t instead of $D_t(X)$. The **flow of the vector field** $X \in \mathfrak{X}(M)$ is the map $\phi_X^t : D_t \to M$ given by

$$\phi_X^t(p) \equiv \gamma_p(t).$$

Proposition 10.7. *Let $X \in \mathfrak{X}(M)$ be a vector field with flow ϕ_X^t. Then:*

(i) For each $p \in M$, there exists a neighborhood U of p and $\varepsilon > 0$, such that the map $(-\varepsilon, \varepsilon) \times U \to M$ given by:

$$(t, q) \mapsto \phi_X^t(q),$$

is well defined and smooth;

(ii) For each $t \in \mathbb{R}$, D_t is open and $\bigcup_{t>0} D_t = M$;

(iii) For each $t \in \mathbb{R}$, $\phi_X^t : D_t \to D_{-t}$ is a diffeomorphism and:

$$(\phi_X^t)^{-1} = \phi_X^{-t};$$

(iv) For each $s, t \in \mathbb{R}$, the domain of $\phi_X^t \circ \phi_X^s$ is contained in D_{t+s} and:

$$\phi_X^{t+s} = \phi_X^t \circ \phi_X^s.$$

Proof. Exercise. □

One calls a vector field X **complete** if $D_t(X) = M$, for every $t \in \mathbb{R}$, i.e., if the maximal integral curve through any $p \in M$ is defined for all $t \in]-\infty, +\infty[$. In this case the flow of X is a map:

$$\mathbb{R} \times M \to M, \quad (t, p) \mapsto \phi_X^t(p).$$

The properties above then say that this map gives an action of the group $(\mathbb{R}, +)$ in M. In other words, the map

$$\mathbb{R} \to \mathrm{Diff}(M), \quad t \mapsto \phi_X^t,$$

is a group homomorphism from $(\mathbb{R}, +)$ to the group $(\mathrm{Diff}(M), \circ)$ of diffeomorphisms of M. One often says that ϕ_X^t is a *1-parameter group of transformations* of M. In the non-complete case, one also says that ϕ_X^t is a 1-parameter group of *local* transformations of M.

Examples 10.8.

1. *The vector field $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ in $\mathbb{R}^2$ is complete (see Example 10.5) and is flow is given by:*

$$\phi_X^t(x, y) = (x\cos t - y\sin t, x\sin t + y\cos t).$$

2. *The vector field $Y = -x^2\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ in $\mathbb{R}^2$ is not complete: the integral curve through a point (x_0, y_0) is the solution to the system of odes:*

$$\begin{cases} \dot{x}(t) = -x^2, & x(0) = x_0, \\ \dot{y}(t) = -y, & y(0) = y_0. \end{cases}$$

After solving this system, ones obtains the flow of Y:

$$\phi_X^t(x, y) = \left(\frac{x}{xt+1}, ye^{-t}\right).$$

It follows that the flow through points $(0, y)$ exist for all t. But for points (x, y), with $x \neq 0$, the flow exists only for $t \in]-1/x, +\infty[$ if $x > 0$ and for $t \in]-\infty, -1/x[$ if $x > 0$. The domain of the flow is then given by:

$$D_t(Y) = \begin{cases} \{(x, y) \in \mathbb{R}^2 : x > -1/t\} & \text{if } t > 0, \\ \mathbb{R}, & \text{if } t = 0, \\ \{(x, y) \in \mathbb{R}^2 : x < -1/t\} & \text{if } t < 0 \end{cases}$$

Let $\Phi : M \to N$ be a smooth map. In general, given a vector field X in M, it is not possible to use Φ to map X to obtain a vector field Y in N. However, given two vector fields, one in M and one in N, we can say when they are related by this map:

Definition 10.9. *Let $\Phi : M \to N$ be a smooth map. A vector field $X \in \mathfrak{X}(M)$ is said to be **Φ-related** to a vector field $Y \in \mathfrak{X}(N)$ if*

$$Y_{\Phi(p)} = \mathrm{d}\Phi(X_p), \quad \forall p \in M.$$

If X and Y are Φ-related vector fields then, as derivations of $C^\infty(M)$:

$$Y(f)\circ\Phi = X(f\circ\Phi), \quad \forall f\in C^\infty(N).$$

When Y is determined from X via Φ we write $Y=\Phi_*(X)$, and call $\Phi_*(X)$ the **push forward** of X by Φ. This is the case, for example, when Φ is a diffeomorphism, in which case:

$$\Phi_*(X)(f) = X(f\circ\Phi)\circ\Phi^{-1}, \quad \forall f\in C^\infty(N).$$

The integral curves of vector fields which are Φ-related are also Φ-related. The proof is a simple exercise applying the chain rule:

Proposition 10.10. *Let $\Phi: M\to N$ be a smooth map and let $X\in\mathfrak{X}(M)$ and $Y\in\mathfrak{X}(N)$ be Φ-related vector fields. If $\gamma: I\to M$ is an integral curve of X, then $\Phi\circ\gamma: I\to N$ is an integral curve of Y. In particular, we have that $\Phi(D_t(X))\subset D_t(Y)$ and that the flows of X and Y are related by:*

$$\begin{array}{ccc} D_t(X) & \xrightarrow{\Phi} & D_t(Y) \\ {\scriptstyle \phi^t_X}\downarrow & & \downarrow{\scriptstyle \phi^t_Y} \\ D_{-t}(X) & \xrightarrow{\Phi} & D_{-t}(Y) \end{array}$$

If $X\in\mathfrak{X}(M)$ is a vector field and $f\in C^\infty(M)$, we already know that $X(f)\in C^\infty(M)$. The expression for $X(f)$ is local coordinates show that X is a first order differential operator. If we iterate, we obtain the powers X^k, which are kth-order differential operators:

$$X^{k+1}(f)\equiv X(X^k(f)).$$

Proposition 10.11 (Taylor Formula)**.** *Let $X\in\mathfrak{X}(M)$ be a vector field and $f\in C^\infty(M)$. For each positive integer k, one has the expansion*

$$f\circ\phi^t_X = f + tX(f) + \frac{t^2}{2!}X^2(f)+\cdots+\frac{t^k}{k!}X^k(f)+0(t^{k+1}),$$

where for each $p\in M$, $t\mapsto 0(t^{k+1})(p)$ denotes a real smooth function defined in a neighborhood of $t=0$ whose derivatives of order $\leq k$ all vanish at $t=0$.

Proof. By the usual Taylor formula for real functions applied to $t\mapsto f(\phi^t_X(p))$, it is enough to show that:

$$\left.\frac{\mathrm{d}^k}{\mathrm{d}t^k}f(\phi^t_X(p))\right|_{t=0} = X^k(f)(p).$$

To prove this, we show by induction that:

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}f(\phi^t_X(p)) = X^k(f)(\phi^t_X(p)).$$

When $k=1$, this follows because:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}f(\phi^t_X(p)) &= \mathrm{d}_pf\cdot X_{\phi^t_X(p)} \\ &= X_{\phi^t_X(p)}(f) = X(f)(\phi^t_X(p)).\end{aligned}$$

On the other hand, if we assume that the formula is valid for $k-1$, we obtain:

$$\begin{aligned}\frac{\mathrm{d}^k}{\mathrm{d}t^k} f(\phi_X^t(p)) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}} f(\phi_X^t(p))\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t} X^{k-1}(f)(\phi_X^t(p)) \\ &= X(X^{k-1}(f))(\phi_X^t(p)) = X^k(f)(\phi_X^t(p)).\end{aligned}$$

□

Another common notation for the flow of a vector field, which is justified by the previous result, is the exponential notation:

$$\exp(tX) \equiv \phi_X^t.$$

In this notation, the properties of the flow are written as:

$$\exp(tX)^{-1} = \exp(-tX), \quad \exp((t+s)X) = \exp(tX) \circ \exp(sX),$$

while the Taylor expansion takes the following suggestive form:

$$f(\exp(tX)) = f + tX(f) + \frac{t^2}{2!}X^2(f) + \cdots + \frac{t^k}{k!}X^k(f) + 0(t^{k+1}).$$

We will not use this notation in these sections.

If $X \in \mathfrak{X}(M)$ is a vector field, a point $p \in M$ is called a **singular point** or an **equilibrium point** of X if $X_p = 0$. It should be obvious that the integral curve through a singular point of X is the constant path: $\phi_X^t(p) = p$, for all $t \in \mathbb{R}$. On the other hand, for non-singular points we have a unique local canonical form X:

Theorem 10.12 (Flow Box Theorem)**.** *Let $X \in \mathfrak{X}(M)$ be a vector field and $p \in M$ a non-singular point: $X_p \neq 0$. There are local coordinates $(U, (x^1, \dots, x^d))$ centered at p, such that:*

$$X|_U = \frac{\partial}{\partial x^1}.$$

Proof. First we choose a chart $(V, (y^1, \dots, y^d)) = (V, \psi)$, centered at p, such that:

$$X|_p = \left.\frac{\partial}{\partial y^1}\right|_p.$$

The map $\sigma : \mathbb{R}^d \to M$ given by

$$\sigma(t_1, \dots, t_d) = \phi_X^{t_1}(\psi^{-1}(0, t_2, \dots, t_d)),$$

is well defined and C^∞ in a neighborhood of the origin. Its differential at the origin is given by:

$$\begin{aligned}
\mathrm{d}_0\sigma \cdot \left.\frac{\partial}{\partial t_1}\right|_0 &= \left.\frac{\mathrm{d}}{\mathrm{d}t_1}\phi_X^{t_1}(\psi^{-1}(0,0,\ldots,0))\right|_{t_1=0} = X_p = \left.\frac{\partial}{\partial y^1}\right|_p, \\
\mathrm{d}_0\sigma \cdot \left.\frac{\partial}{\partial t_i}\right|_0 &= \left.\frac{\partial}{\partial t_i}\psi^{-1}(0,t_2,\ldots,t_d))\right|_0 = \left.\frac{\partial}{\partial y^i}\right|_p.
\end{aligned}$$

We conclude that σ is a local diffeomorphism in a neighborhood of the origin. Hence, there exists an open set U containing p such that $\phi = \sigma^{-1} : U \to \mathbb{R}^d$ is a chart. If we write $(U,\phi) = (U,(x^1,\ldots,x^d))$, we have:

$$\begin{aligned}
\left.\frac{\partial}{\partial x^1}\right|_{\sigma(t_1,\ldots,t_d)} &= \mathrm{d}\sigma \cdot \left.\frac{\partial}{\partial t_1}\right|_{(t_1,\ldots,t_d)} \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\phi_X^t(\psi^{-1}(0,t_2,\ldots,t_d))\right|_{t=t_1} \\
&= X(\phi_X^{t_1}(\psi^{-1}(0,t_2,\ldots,t_d))) = X_{\sigma(t_1,\ldots,t_d)}.
\end{aligned}$$

□

Homework.

1. Let M be a connected manifold. Show that for any pair of points $p,q \in M$, with $p \neq q$, there exists a smooth path $\gamma : [0,1] \to M$ such that
   (a) $\gamma(0) = p$ and $\gamma(1) = q$;
   (b) $\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) \neq 0$, for every $t \in [0,1]$;
   (c) γ is simple (i.e., γ is injective).
   Use this to prove that any connected manifold of dimension 1 is diffeomorphic to either $\mathbb{R}$ or $\mathbb{S}^1$.

2. Let $X \in \mathfrak{X}(M)$ be a vector field and $f \in C^\infty(M)$ a nowhere vanishing function. What is the relationship between the integral curves of X and the integral curves of fX?

3. Verify the properties of the flow of a vector field given by Proposition 10.7.

4. Determine the flow of the vector field $X = y\partial/\partial x - x\partial/\partial y$ in $\mathbb{R}^3$ with coordinates (x,y,z).

5. Give an example of a manifold M and two vector fields X_1 and X_2 which are complete but for which their sum $X_1 + X_2$ is not complete. On the other hand, show that if M is compact then every vector field $X \in \mathfrak{X}(M)$ is complete.
   Hint: Show that if $K \subset M$ is a compact set then there exists $a > 0$ such that for every $x \in K$ the maximal integral curve through x exists for $t \in [-a,a]$.

6. Let $A \subset M$. Call a map $X : A \to TM$ a *vector field along* A if $X_p \in T_pM$ for all $p \in A$. Show that if $A \subset O \subset M$, with A closed and O open, then every smooth vector field X along A can be extended to a smooth vector field in M such that $X_p = 0$ for $p \notin O$.

7. Let $X \in \mathfrak{X}(M)$ be a vector field without singular points. Show that the integral curves of X form a foliation $\mathcal{F}$ of M of dimension 1. Conversely, show that locally the leaves of a foliation of dimension 1 are the orbits of a vector field. What about globally?

8. A Riemannian structure on a manifold M is a smooth choice of an inner product $\langle\ ,\ \rangle_p$ in each tangent space T_pM. Here by smooth we mean that for any vector fields $X, Y \in \mathfrak{X}(M)$, the function $p \mapsto \langle X(p), Y(p)\rangle_p$ is C^∞. Show that every smooth manifold admits a Riemannian structure M.

11. Lie Bracket and Lie Derivative

Definition 11.1. *Let $X, Y \in \mathfrak{X}(M)$ be smooth vector fields. The **Lie bracket** of X and Y is the vector field $[X, Y] \in \mathfrak{X}(M)$ given by:*

$$[X, Y](f) = X(Y(f)) - Y(X(f)), \quad \forall f \in C^\infty(M).$$

Note that the formula for the Lie bracket $[X, Y]$ shows that it is a differential operator of order ≤ 2. A simple computation shows that $[X, Y]$ is a linear derivation of $C^\infty(M)$:

$$[X, Y](fg) = [X, Y](f)g + f[X, Y](g), \quad \forall f, g \in C^\infty(M).$$

In order words, the terms of 2nd order cancel each other and we have in fact that $[X, Y] \in \mathfrak{X}(M)$.

In a local chart we can compute the Lie bracket in a straightforward way if we think of vector fields as differential operators. This is illustrated in the next example.

Example 11.2.
Let $M = \mathbb{R}^3$ with coordinates (x, y, z), and consider the vector fields:

$$\begin{aligned} X &= z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \\ Y &= x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \\ Z &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \end{aligned}$$

Then we compute:

$$\begin{aligned} [X, Y] &= \left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right)\left(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\right) - \left(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\right)\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right) \\ &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} = Z. \end{aligned}$$

We leave it as an exercise the computation of the other Lie brackets:

$$[Y, Z] = X, \quad [Z, X] = Y.$$

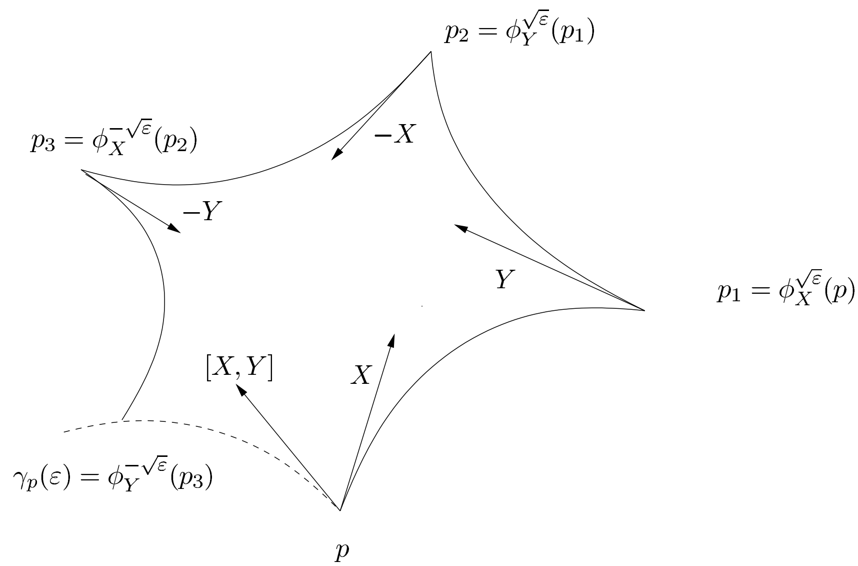
Our next result shows that the Lie bracket $[X,Y]$ measures the failure in the commutativity of the flows of X and Y.

Proposition 11.3. *Let $X, Y \in \mathfrak{X}(M)$ be vector fields. For each $p \in M$, the commutator*

$$\gamma_p(\varepsilon) \equiv \phi_Y^{-\sqrt{\varepsilon}} \circ \phi_X^{-\sqrt{\varepsilon}} \circ \phi_Y^{\sqrt{\varepsilon}} \circ \phi_X^{\sqrt{\varepsilon}}(p)$$

is well defined for a small enough $\varepsilon \geq 0$, and we have:

$$[X,Y]_p = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\gamma_p(\varepsilon)\right|_{\varepsilon=0^+}.$$



Proof. Fix a local chart $(U, x^1, \ldots, x^d)$, centered at p, and write:

$$X = \sum_{i=1}^{d} X^i \frac{\partial}{\partial x^i}, \quad Y = \sum_{i=1}^{d} Y^i \frac{\partial}{\partial x^i}.$$

The Lie bracket of X and Y is given by:

$$[X,Y](x^i) = X(Y^i) - Y(X^i).$$

Consider the points p_1, p_2 and p_3 defined by (see figure above):

$$p_1 = \phi_X^{\sqrt{\varepsilon}}(p), \quad p_2 = \phi_Y^{\sqrt{\varepsilon}}(p_1), \quad p_3 = \phi_X^{-\sqrt{\varepsilon}}(p_2),$$

Then $\gamma_p(\varepsilon) = \phi_Y^{-\sqrt{\varepsilon}}(p_3)$, and Taylor's formula (see Proposition 10.11), applied to each coordinate x^i, yields:

$$x^i(p_1) = x^i(p) + \sqrt{\varepsilon}X^i(p) + \frac{1}{2}\varepsilon X^2(x^i)(p) + O(\varepsilon^{\frac{3}{2}})$$

Similarly, we have:

$$\begin{aligned} x^i(p_2) &= x^i(p_1) + \sqrt{\varepsilon} Y^i(p_1) + \frac{1}{2}\varepsilon Y^2(x^i)(p_1) + O(\varepsilon^{\frac{3}{2}}) = \\ &= x^i(p) + \sqrt{\varepsilon} X^i(p) + \frac{1}{2}\varepsilon X^2(x^i)(p) + \\ &\quad + \sqrt{\varepsilon} Y^i(p_1) + \frac{1}{2}\varepsilon Y^2(x^i)(p_1) + O(\varepsilon^{\frac{3}{2}}) \end{aligned}$$

The last two terms can also be estimated using again Taylor's formula:

$$\begin{aligned} Y^i(p_1) &= Y^i(\phi_X^{\sqrt{\varepsilon}}(p)) = Y^i(p) + \sqrt{\varepsilon} X(Y^i)(p) + O(\varepsilon) \\ Y^2(x^i)(p_1) &= Y^2(x^i)(\phi_X^{\sqrt{\varepsilon}}(p)) = Y^2(x^i)(p) + \sqrt{\varepsilon} X(Y^2(x^i))(p) + O(\varepsilon) \end{aligned}$$

hence, we have:

$$\begin{aligned} x^i(p_2) =& x^i(p) + \sqrt{\varepsilon}(Y^i(p) + X^i(p)) + \\ &+ \varepsilon\left(\frac{1}{2}Y^2(x^i)(p) + X(Y^i)(p) + \frac{1}{2}X^2(x^i)(p)\right) + O(\varepsilon^{\frac{3}{2}}) \end{aligned}$$

Proceeding in a similar fashion, we can estimate $x^i(p_3)$ and $x^i(\gamma_p(\varepsilon))$, obtaining:

$$\begin{aligned} x^i(p_3) &= x^i(p_2) - \sqrt{\varepsilon} X^i(p_2) + \frac{1}{2}\varepsilon X^2(x^i)(p_2) + O(\varepsilon^{\frac{3}{2}}) \\ &= x^i(p) + \sqrt{\varepsilon} Y^i(p) + \varepsilon\left(X(Y^i)(p) - Y(X^i)(p) + \frac{1}{2}Y^2(x^i)(p)\right) + O(\varepsilon^{\frac{3}{2}}) \\ x^i(\gamma_p(\varepsilon)) &= x^i(p_3) - \sqrt{\varepsilon} Y^i(p_3) + \frac{1}{2}\varepsilon Y^2(x^i)(p_3) + O(\varepsilon^{\frac{3}{2}}) \\ &= x^i(p) + \varepsilon\left(X(Y^i)(p) - Y(X^i)(p)\right) + O(\varepsilon^{\frac{3}{2}}) \end{aligned}$$

Therefore:

$$\lim_{\varepsilon \to 0^+} \frac{x^i(\gamma_p(\varepsilon)) - x^i(p)}{\varepsilon} = X(Y^i)(p) - Y(X^i)(p) = [X, Y]_p(x^i).$$

□

The following proposition gives the most basic properties of the Lie bracket of vector fields. The proof is elementary and is left as an exercise.

Proposition 11.4. *The Lie bracket satisfies the following properties:*

(i) Skew-symmetry: $[X, Y] = -[Y, X]$*;*
(ii) Bi-linearity: $[aX + bY, Z] = a[X, Z] + b[Y, Z]$, $\forall a, b \in \mathbb{R}$*;*
(iii) Jacobi identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$*;*
(iv) Leibniz identity: $[X, fY] = X(f)Y + f[X, Y]$, $\forall f \in C^\infty(M)$.

Moreover, if $\Phi : M \to N$ *is a smooth map,* X *and* $Y \in \mathfrak{X}(M)$ *are* Φ*-related with, respectively,* Z *and* $W \in \mathfrak{X}(N)$*, then* $[X, Y]$ *is* Φ*-related with* $[Z, W]$*.*

The geometric interpretation of the Lie bracket given by Proposition 11.3 shows that the Lie bracket and the flow of vector fields are intimately related. There is another form of this relationship which we now explain. For that, we need the following definition:

Definition 11.5. *Let $X \in \mathfrak{X}(M)$ be a vector field.*

*(i) The **Lie derivative of a function** $f \in C^\infty(M)$ along X is the smooth function $\mathcal{L}_X f$ given by:*

$$(\mathcal{L}_X f)(p) = \lim_{t\to 0} \frac{1}{t}\left(f(\phi_X^t(p)) - f(p)\right).$$

*(ii) The **Lie derivative of a vector field** $Y \in \mathfrak{X}(M)$ along X is the smooth vector field $\mathcal{L}_X Y$ given by:*

$$(\mathcal{L}_X Y)_p = \lim_{t\to 0} \frac{1}{t}\left(\mathrm{d}\phi_X^{-t} \cdot Y_{\phi_X^t(p)} - Y_p\right).$$

One can "unify" these two definitions observing that a diffeomorphism $\Phi : M \to M$ acts on functions $C^\infty(M)$ by:

$$(\Phi^* f)(p) = f(\Phi(p)),$$

and it acts on vector fields $Y \in \mathfrak{X}(M)$:

$$(\Phi^* Y)_p = \mathrm{d}\Phi^{-1} \cdot Y_{\Phi(p)}.$$

Note that $\Phi^* Y = (\Phi^{-1})_* Y$, so the two operations are related by:

$$\Phi^* Y(f) = Y((\Phi^{-1})^* f).$$

It follows that the Lie derivative of an object P (a function or a vector field) is given by:

$$\mathcal{L}_X P = \left.\frac{\mathrm{d}}{\mathrm{d}t}(\phi_X^t)^* P\right|_{t=0} = \lim_{t\to 0} \frac{1}{t}\left((\phi_X^t)^* P - P\right). \tag{11.1}$$

We will see later that one can take Lie derivatives of other objects using precisely this definition.

Theorem 11.6. *Let $X \in \mathfrak{X}(M)$ be a vector field.*

(i) For any functions $f \in C^\infty(M)$: $\mathcal{L}_X f = X(f)$.

(ii) For any vector field $Y \in \mathfrak{X}(M)$: $\mathcal{L}_X Y = [X, Y]$.

Proof. To prove (i), we simply observe that:

$$\mathcal{L}_X f = \left.\frac{\mathrm{d}}{\mathrm{d}t} f \circ \phi_X^t\right|_{t=0} = \mathrm{d}f \cdot X = X(f).$$

To prove (ii), we note first that:

$$\begin{aligned}(\mathcal{L}_X Y)(f)(p) &= \lim_{t\to 0} \frac{1}{t}\left(\mathrm{d}\phi_X^{-t} \cdot Y_{\phi_X^t(p)} - Y_p\right)(f)\\ &= \lim_{t\to 0} \frac{1}{t}\left(Y_{\phi_X^t(p)}(f \circ \phi_X^{-t}) - Y_p(f)\right).\end{aligned}$$

On the other hand, Taylor's formula gives::

$$f \circ \phi_X^{-t} = f - tX(f) + O(t^2),$$

hence, using also (i), we find:

$$\begin{aligned}
(\mathcal{L}_X Y)(f)(p) &= \lim_{t \to 0} \frac{1}{t} \left(Y_{\phi_X^t(p)}(f) - tY_{\phi_X^t(p)}(X(f)) - Y_p(f) \right) \\
&= \lim_{t \to 0} \frac{1}{t} \left(Y_{\phi_X^t(p)}(f) - Y_p(f) \right) - Y_p(X(f)) \\
&= X_p(Y(f)) - Y_p(X(f)) = [X, Y](f)(p).
\end{aligned}$$

□

Homework.

1. Complete the computation of the Lie brackets in Example 11.2 and show that all 3 vector fields X,Y and Z are tangent to the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. Show that there are unique vector fields $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ on $\mathbb{RP}^2$ such that $\pi_* X = \tilde{X}$, $\pi_* Y = \tilde{Y}$ and $\pi_* Z = \tilde{Z}$ where $\pi : \mathbb{S}^2 \to \mathbb{RP}^2$ is the projection. What are the Lie brackets between $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$?

2. Find 3 everywhere linearly independent vector fields X,Y and Z on the sphere $\mathbb{S}^3$ such that $[X, Y] = Z$, $[Y, Z] = X$ and $[Z, X] = Y$.

   Hint: Recall that $\mathbb{S}^3$ can be identified with the unit quaternions.

3. Check the properties of the Lie bracket given in Proposition 11.4.

4. In $\mathbb{R}^2$ consider the vector fields $X = \frac{\partial}{\partial x}$ and $Y = x\frac{\partial}{\partial y}$. Compute the Lie bracket $[X, Y]$ in two distinct ways: (i) using the definition and (ii) using the flows of X and Y, as in Proposition 11.3.

5. Let $X, Y \in \mathfrak{X}(M)$ be complete vector fields with flows ϕ_X^t and ϕ_Y^s. Show that $[X, Y] = 0$ if and only if $\phi_X^t \circ \phi_Y^s = \phi_Y^s \circ \phi_X^t$ for all s and t.

   Note: If the vector fields are not assumed complete then this still holds if the last condition is replaced by $\phi_X^t \circ \phi_Y^s(p) = \phi_Y^s \circ \phi_X^t(p)$ for all $p \in M$ and all s and t sufficiently small (which may depend on p).

6. Let $X_1, \dots, X_k \in \mathfrak{X}(M)$ be vector fields such that:
   (a) $\{X_1|_p, \dots, X_k|_p\}$ are linearly independent, for all $p \in M$;
   (b) $[X_i, X_j] = 0$, for all $i, j = 1, \dots, k$.
   Show that for each $p \in M$ there exists a neighborhood U of p and a unique k-dimensional foliation $\mathcal{F}$ of U such that:
   $$T_q L = \langle X_1|_q, \dots, X_k|_q \rangle, \quad \forall q \in U,$$
   where $L \in \mathcal{F}$ is the leaf containing q.

   Hint: Use the previous exercise to show that the leaf L is obtained by flowing from q along the flows of the vector fields $X_1, \dots, X_k$.

12. Distributions and the Frobenius Theorem

A vector field $X \in \mathfrak{X}(M)$ which is nowhere vanishing determines a partition of M into 1 dimensional submanifolds:

$$\mathcal{F} = \{\gamma(I) : \gamma : I \to M \text{ a maximal integral curve of } X\}.$$

By the Flow Box Theorem, this is a 1-dimensional foliation of M. Notice that if $Y \in \mathfrak{X}(M)$ is another vector field such that $Y = fX$, for some nowhere vanishing smooth function $f \in C^\infty(M)$, then Y determines the same foliation of M. So this foliation only depends on the family of 1-dimensional subspaces

$$M \ni p \mapsto \langle X_p \rangle \subset T_pM.$$

We will now generalize all this to higher dimensions.

Definition 12.1. *Let M be a smooth manifold of dimension d and let $1 \leq k \leq d$ be an integer. A k-dimensional **distribution** D in M is a map*

$$M \ni p \mapsto D_p \subset T_pM,$$

which associates to each $p \in M$ a subspace $D_p \subset T_pM$ of dimension k. We say that a distribution D is of class C^∞ if for each $p \in M$ there exists a neighborhood U of p and smooth vector fields $X_1, \dots, X_k \in \mathfrak{X}(U)$, such that:

$$D_q = \langle X_1|_q, \dots, X_k|_q \rangle, \quad \forall q \in U.$$

If D is a distribution in M we consider the set of **vector fields tangent to** D:

$$\mathfrak{X}(D) := \{X \in \mathfrak{X}(M) : X_p \in D_p, \forall p \in M\}.$$

Note that $\mathfrak{X}(D)$ is a module over the ring $C^\infty(M)$: if $f \in C^\infty(M)$ and $X \in \mathfrak{X}(D)$ then $fX \in \mathfrak{X}(D)$.

Examples 12.2.

1. *Every nowhere vanishing smooth vector field X defines a 1-dimensional smooth distribution by:*

$$D_p := \langle X_p \rangle = \{\lambda X_p : \lambda \in \mathbb{R}\}.$$

We have that $Y \in \mathfrak{X}(D)$ if and only $Y = fX$ for some uniquely defined smooth function $f \in C^\infty(M)$.

2. *A set of smooth vector fields $X_1, \dots, X_k$ which at each $p \in M$ are linearly independent define a k-dimensional smooth distribution by:*

$$D_p := \langle X_1|_p, \dots, X_k|_p \rangle.$$

We have that a vector field $X \in \mathfrak{X}(D)$ if and only if

$$X = f_1X_1 + \cdots + f_kX_k,$$

for uniquely defined functions $f_i \in C^\infty(M)$.

For example, in $M = \mathbb{R}^3$, we have the 2-dimensional smooth distribution $D = \langle X_1, X_2\rangle$ generated by the vector fields:

$$X_1 = \frac{\partial}{\partial x} + z^2\frac{\partial}{\partial y},$$
$$X_2 = \frac{\partial}{\partial y} + z^2\frac{\partial}{\partial z}.$$

and every vector field $X \in \mathfrak{X}(D)$ is a linear combination $aX_1 + bX_2$, where the smooth functions $a = a(x, y)$ and $b = b(x, y)$ are uniquely determined.

3. *More generally, a set of smooth vector fields $X_1, \dots, X_s$ which at each $p \in M$ span a k-dimensional subspace define a k-dimensional smooth distribution by:*

$$D_p := \langle X_1|_p, \dots, X_s|_p\rangle.$$

We have that $X \in \mathfrak{X}(D)$ if and only if

$$X = f_1X_1 + \cdots + f_sX_s,$$

for some smooth functions $f_i \in C^\infty(M)$. The difference from the previous example is that the functions f_i are not uniquely defined. Moreover, we may not be able to find k vector fields tangent to D which globally generate D.

For example, in $M = \mathbb{R}^3 - \{0\}$ consider the vector fields X, Y and Z defined in Example 11.2. The matrix whose columns are the components of the vector fields X, Y and Z relative to the usual coordinates (x, y, z) of $\mathbb{R}^3$ is:

$$\begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}$$

and has rank 2 everywhere. Hence, we have the 2-dimensional distribution $D = \langle X, Y, Z\rangle$. We leave it as an exercise to check that this distribution is not globally generated by only 2 vector fields.

We can think of a distribution as a generalization of the notion of a vector field. In this sense, the concept of an integral curve of a vector field is replaced by the following:

Definition 12.3. *Let D be a distribution in M. A connected submanifold (N, Φ) of M is called an* ***integral manifold*** *of D if:*

$$\mathrm{d}_p\Phi(T_pN) = D_{\Phi(p)}, \forall p \in N.$$

Note that if D is a k-dimensional distribution, its integral manifolds, if they exist, are k-dimensional manifolds.

Examples 12.4.

1. *Consider the 2-distribution of $\mathbb{R}^3$ given in Example 12.2.2. The plane $N = \{z = 0\}$ is an integral manifold of this distribution, since it is a connected submanifold and*

$$D_{(x,y,0)} = \left\langle \left.\frac{\partial}{\partial x}\right|_{(x,y,0)}, \left.\frac{\partial}{\partial y}\right|_{(x,y,0)} \right\rangle = T_{(x,y,0)}N.$$

2. Consider the 2-distribution D of $\mathbb{R}^3 - \{0\}$ defined by the vector fields X, Y and Z in Example 12.2.3. The spheres

$$S_c = \{(x, y, z) \in \mathbb{R}^3 - 0 : x^2 + y^2 + z^2 = c\},$$

are integral manifolds of D: each sphere is a connected submanifold, has dimension 2 and:

$$X_p, Y_p.Z_p \in T_pS_c, \quad \forall p \in S_c.$$

Since D has dimension 2, we have $T_pS_c = D_p$, for all $p \in S_c$.

As suggested by the last example, given a smooth k-dimensional foliation $\mathcal{F}$ of a manifold M, we associate to it a k-dimensional distribution d defined by:

$$D_p := T_pL,$$

where $L \in \mathcal{F}$ denotes the leaf containing the point $p \in M$. Henceforth, we will denote this distribution by $T\mathcal{F}$ and we will write $T_p\mathcal{F}$ instead of $(T\mathcal{F})_p$. The existence of foliated charts shows that $T\mathcal{F}$ is a smooth distribution. A vector field is tangent to $T\mathcal{F}$ if and only if it is tangent to each leaf of the foliation.

Definition 12.5. *A smooth distribution D in M is called **integrable** if there exists a foliation $\mathcal{F}$ in M such that $D = T\mathcal{F}$.*

A distribution D in M may fail to be integrable. In fact, there may not even exist integral manifolds through each point of M. The following proposition gives a necessary condition for this to happen:

Proposition 12.6. *Let D be a smooth distribution in M. If there exists an integral manifold of D through $p \in M$, then for any $X, Y \in \mathfrak{X}(D)$ we must have that $[X, Y]_p \in D_p$.*

Proof. Let $X, Y \in \mathfrak{X}(D)$ and fix $p \in M$. Assume there exists an integral manifold (N, Φ) of D through p and choose $q \in N$, such that $\Phi(q) = p$. For any $q' \in N$, the map $\mathrm{d}_{q'}\Phi : T_{q'}N \to T_{\Phi(q')}M$ is injective and its image is $D_{\Phi(q')}$. By the local normal form for submanifolds, there exist smooth vector fields $\tilde{X}, \tilde{Y} \in \mathfrak{X}(N)$ which are Φ-related with X and Y, respectively. It follows that $[\tilde{X}, \tilde{Y}]$ is also Φ-related with $[X, Y]$ and we must have

$$[X, Y]_p = \mathrm{d}_{q_0}\Phi([\tilde{X}, \tilde{Y}]_q) \in \mathrm{d}_q\Phi(T_qN) = D_p.$$

□

Example 12.7.

For the smooth distribution $D = \langle X_1, X_2\rangle$ of $\mathbb{R}^3$ given in Example 12.2.2, we saw that the plane $z = 0$ is an integral manifold. On the other hand, we find that:

$$[X_1, X_2] = -2z^3\frac{\partial}{\partial y}.$$

If $z \neq 0$ this vector field is not tangent to the distrubution. Hence, the only points through which there exist integral manifolds are the points in the plane $z = 0$.

For an integrable distribution $D = T\mathcal{F}$ we have an integral manifold through every point. Hence, for any pair of vector fields $X, Y \in \mathfrak{X}(T\mathcal{F})$ we must have $[X, Y] \in \mathfrak{X}(T\mathcal{F})$. This suggests:

Definition 12.8. *A smooth distribution D in M is called* ***involutive*** *if for any $X, Y \in \mathfrak{X}(D)$ one has $[X, Y] \in \mathfrak{X}(D)$.*

The following important result says that the lack of involutivity is the only obstruction to integrability of a distribution:

Theorem 12.9 (Frobenius)**.** *A smooth distribution D is integrable if and only if it is involutive. In this case, the integral foliation tangent to D is unique.*

Proof. Proposition 12.6 show that one of the implications hold. To check the other implication we assume that D is an involutive distribution.

We claim that, for each $p \in M$, there exist vector fields $X_1, \dots, X_k \in \mathfrak{X}(U)$, defined in an open neighborhood U of p, such that:

(a) $D|_U = \langle X_1, \dots, X_k \rangle$;
(b) $[X_i, X_j] = 0$, for every $i, j = 1, \dots, k$.

Then, by Exercise 6 in Section 11, we obtain an open cover $\{U_i\}_{i \in I}$ of M, such that for each $i \in I$ there exists a unique foliation $\mathcal{F}_i$ in U_i which satisfies $T\mathcal{F}_i = D|_{U_i}$. By uniqueness, whenever $U_i \cap U_j \neq 0$, we obtain $\mathcal{F}_i|_{U_i \cap U_j} = \mathcal{F}_j|_{U_i \cap U_j}$. Hence, there exists a unique foliation $\mathcal{F}$ of M such that $\mathcal{F}|_{U_i} = \mathcal{F}_i$.

To prove the claim, fix $p \in M$. Since D is smooth, there exist vector fields $Y_1, \dots, Y_k$ defined in some neighborhood V of p, such that $D|_V = \langle Y_1, \dots, Y_k \rangle$. We can also assume that V is the domain of some coordinate system $(x^1, \dots, x^d)$ of M, so that

$$Y_i = \sum_{l=1}^{d} a_{il} \frac{\partial}{\partial x^l}, \quad (i = 1, \dots, k),$$

where $a_{il} \in C^\infty(V)$. The matrix $A(q) = [a_{il}(q)]_{i,l=1}^{k,d}$ has rank k at p and we can assume, eventually after some relabeling of the the coordinates, that the $k \times k$ minor formed by the first k rows and k columns of A has non-zero determinant in a smaller open neighborhood U of p. Let B be the $k \times k$

inverse matrix of this minor, and define vector fields $X_1, \ldots, X_k \in \mathfrak{X}(U)$ by:

$$\begin{aligned} X_i &= \sum_{j,l=1}^{k,d} b_{ij} a_{jl} \frac{\partial}{\partial x^l} \\ &= \frac{\partial}{\partial x^i} + \sum_{l=k+1}^{d} c_{il} \frac{\partial}{\partial x^l}, \quad (i = 1, \ldots, k), \end{aligned}$$

where $c_{il} \in C^\infty(U)$. On the one hand, we have that

$$D|_U = \langle Y_1, \ldots, Y_k \rangle = \langle X_1, \ldots, X_k \rangle,$$

so (a) is satisfied. On the other hand, a simple computation shows that:

$$[X_i, X_j] = \sum_{l=k+1}^{d} d_l^{ij} \frac{\partial}{\partial x^l}, \quad (i, j = 1, \ldots, k),$$

for certain functions $d_l^{ij} \in C^\infty(U)$. Since D is involutive, this commutator must be a $C^\infty(M)$-linear combination of $X_1, \ldots, X_k$. Therefore, the functions d_l^{ij} must be identically zero, so (b) also holds. $\square$

The Frobenius Theorem establishes a one-to-one correspondence:

$$\left\{ \begin{matrix} \text{involutive distributions } D \\ \text{on } M \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{foliations } \mathcal{F} \\ \text{on } M \end{matrix} \right\}$$

This is an example of an *integrability theorem*: a distribution D is an infinitesimal object on M while a foliation $\mathcal{F}$ is a global object on M and the integrability condition is the involutivity of D.

Homework.

1. Give an example of a smooth distribution D of dimension 1 on the cylinder $\mathbb{S}^1 \times \mathbb{R}$ which is not globally generated by a vector field.

2. Show that the 2-dimensional distribution D in Example 12.2.3 is not globally generated by only 2 vector fields.

3. Show that the 2-dimensional distribution in $\mathbb{R}^3$ defined by the vector fields

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = e^{-x} \frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

has no integral manifolds.

4. Consider the distribution D in $\mathbb{R}^3$ generated by the vector fields:

$$\frac{\partial}{\partial x} + \cos x \cos y \frac{\partial}{\partial z}, \qquad \frac{\partial}{\partial y} - \sin x \sin y \frac{\partial}{\partial z}.$$

Check that D is involutive and determine the foliation $\mathcal{F}$ that integrates it.

5. Consider the 3-sphere:

$$\mathbb{S}^3 = \{(x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + w^2 = 1\}.$$

Check that the vector field in $\mathbb{R}^3$ given by:

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - w\frac{\partial}{\partial z} + z\frac{\partial}{\partial w},$$

restricts to a nowhere vanishing vector field on $\mathbb{S}^3$, so determines a 1-dimensional distribution D. Find the foliation $\mathcal{F}$ integrating this distribution.

13. Lie Groups and Lie Algebras

The next definition axiomatizes some of the properties of the Lie bracket of vector fields (see Proposition 11.4).

Definition 13.1. *A **Lie algebra** is a vector space $\mathfrak{g}$ with a binary operation $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the **Lie bracket**, which satisfies:*

(i) Skew-symmetry: $[X, Y] = -[Y, X]$;
(ii) Bilinearity: $[aX + bY, Z] = a[X, Z] + b[Y, Z]$, $\forall a, b \in \mathbb{R}$;
(iii) Jacobi identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

We can also define Lie algebras over the complex numbers ($\mathfrak{g}$ is a complex vector space) or over other fields. Note also, that $\mathfrak{g}$ can have infinite dimension, but we will be mainly interested in finite dimensional Lie algebras.

Examples 13.2.

1. *$\mathbb{R}^d$ with the zero Lie bracket $[\ ,\ ] \equiv 0$ is a Lie algebra, called the **abelian Lie algebra** of dimension d.*
2. *In $\mathbb{R}^3$, we can define a Lie algebra structure where the Lie bracket is the vector product:*
$$[\vec{v}, \vec{w}] = \vec{v} \times \vec{w}.$$
3. *If V is any vector space, the vector space of all linear transformations $T : V \to V$ is a Lie algebra with Lie bracket the commutator:*
$$[T, S] = T \circ S - S \circ T.$$
*This Lie algebra is called the **general linear Lie algebra** and denoted $\mathfrak{gl}(V)$. When $V = \mathbb{R}^n$, we denote it by $\mathfrak{gl}(n)$. After fixing a basis, we can identitify $\mathfrak{gl}(n)$ with the space of all $n \times n$ matrices, and the Lie bracket becomes the commutator of matrices.*
4. *If $\mathfrak{g}_1, \dots, \mathfrak{g}_k$ are Lie algebras, their cartesian product $\mathfrak{g}_1 \times \cdots \times \mathfrak{g}_k$ is a Lie algebra with Lie bracket:*
$$[(X_1, \dots, X_k), (Y_1, \dots, Y_k)] = ([X_1, Y_1]_{\mathfrak{g}_1}, \dots, [X_k, Y_k]_{\mathfrak{g}_k}).$$

We shall see shortly that Lie algebras are the "infinitesimal versions" of groups with a smooth structure:

Definition 13.3. *A **Lie group** is a group G with a smooth structure such that its structure maps are smooth:*

$$\mu : G \times G \to G,\ (g, h) \mapsto gh \quad \textit{(multiplication)},$$
$$\iota : G \to G,\ g \mapsto g^{-1} \quad \textit{(inverse)}.$$

One can also define topological groups, analytic groups, etc.

EXAMPLES 13.4.

1. *Any countable group with the discrete topology is a Lie group of dimension 0 (we need it to be countable so that the discrete topology is second countable).*
2. *$\mathbb{R}^d$ with the usual addition of vectors is an abelian Lie group. The groups of all non-zero real numbers $\mathbb{R}^*$ and all non-zero complex numbers $\mathbb{C}^*$, with the usual multiplication operations, are also abelian Lie groups. Note that $\mathbb{C}^*$ is also a complex Lie group (thinking of $\mathbb{C}^*$ as a complex manifold), but we will restrict ourselves always to real Lie groups.*
3. *The circle $\mathbb{S}^1 = \{z \in \mathbb{C} : ||z|| = 1\} \subset \mathbb{C}^*$ with the usual complex multiplication is also an abelian Lie group. The unit quaternions $\mathbb{S}^3$, with quaternionic multiplication, is a non-abelian Lie group. It can be shown that the only spheres $\mathbb{S}^d$ that admit Lie group structures are $d = 0, 1, 3$.*
4. *If V is a finite dimensional vector space, the set of all invertible linear transformations $T : V \to V$ is a Lie group with multiplication composition of transformations. It is called the **general linear group** and denoted by $GL(V)$. If $V = \mathbb{R}^n$ we can identify $GL(V)$ with the group of all invertible $n \times n$ matrices with matrix multiplication and we denote it by $GL(n)$.*
5. *If $G_1, \dots, G_k$ are Lie groups their cartesian product $G \times \cdots \times G_k$ is also a Lie group. For example, the torus $\mathbb{T}^d = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ is a (abelian) Lie group.*
6. *If G is a Lie group, its connected component of the identity, is a Lie group denoted by G^0. For example, the connected component of the identity of the Lie group $(\mathbb{R}^*, \times)$ is the group of positive real numbers $(\mathbb{R}_+, \times)$.*

In a Lie group G, a **left invariant vector field** is a vector field X such that:

$$(L_g)_* X = X, \quad \forall g \in G,$$

where $L_g : G \to G$, $h \mapsto gh$ denotes the **left translation** by g. One defines analogously a **right invariant vector field** using the **right translation** $R_g : G \to G$, $h \mapsto hg$. As a matter of choice, we use left invariant vector fields, and we denote the set of all smooth left invariant vector fields by:

$$\mathfrak{X}_{\text{L-inv}}(G) = \{X \in \mathfrak{X}(G) : (L_g)_* X = X,\ \forall g \in G\}.$$

Proposition 13.5. *Let G be a Lie group.*

(i) Every left invariant vector field if smooth;
(ii) If $X, Y \in \mathfrak{X}_{L\text{-}inv}(G)$ then $[X, Y] \in \mathfrak{X}_{L\text{-}inv}(G)$;
(iii) $\mathfrak{X}_{L\text{-}inv}(G) \subset \mathfrak{X}(G)$ is a finite dimensional subspace of dimension $\dim G$.

Proof. We leave the proof of (i) as an exercise. To check (ii), it is enough to observe that if $X, Y \in \mathfrak{X}_{\text{L-inv}}(G)$ then:

$$(L_g)_*[X,Y] = [(L_g)_*X, (L_g)_*Y] = [X,Y], \quad \forall g \in G.$$

Hence, $[X,Y] \in \mathfrak{X}_{\text{L-inv}}(G)$.

Now to see that (iii) holds, it is clear from the definition of a left invariant vector field that $\mathfrak{X}_{\text{L-inv}}(G) \subset \mathfrak{X}(G)$ is a linear subspace. On the other hand, the restriction map

$$\mathfrak{X}_{\text{L-inv}}(G) \to T_eG, \ X \mapsto X_e,$$

is a linear isomorphism: if $\mathbf{v} \in T_eG$ we can define a left invariant vector field X in G with $X_e = \mathbf{v}$ by setting

$$X_g = \mathrm{d}L_g \cdot \mathbf{v}.$$

Hence, the restriction $\mathfrak{X}_{\text{L-inv}}(G) \to T_eG$ is invertible. We conclude that:

$$\dim \mathfrak{X}_{\text{L-inv}}(G) = \dim T_eG = \dim G.$$

□

This proposition show that for a Lie group G the set $\mathfrak{X}_{\text{L-inv}}(G)$ forms a Lie algebra. We call it the **Lie algebra of the Lie group** G and denote it by $\mathfrak{g}$. The proof also shows that $\mathfrak{g}$ can be identified with T_eG.

Examples 13.6. .

1. *The Lie algebra of a discrete Lie group G is the zero dimensional vector space $\mathfrak{g} = \mathbb{R}^0 = \{0\}$.*
2. *Let $G = (\mathbb{R}^d, +)$. A vector field in $\mathbb{R}^d$ is left invariant if and only if it is constant: $X = \sum_{i=1}^d a_i \frac{\partial}{\partial x^i}$, with $a_i \in \mathbb{R}$. The Lie bracket of any two such constant vector fields is zero, hence the Lie algebra of G is the abelian Lie algebra of dimension d.*
3. *The Lie algebra of the cartesian product $G \times H$ of two Lie groups, is the cartesian product $\mathfrak{g} \times \mathfrak{h}$ of their Lie algebras. For example, the Lie algebra of $\mathbb{S}^1$ has dimension 1, hence it is abelian. It follows that the Lie algebra of the torus $\mathbb{T}^d$ is also the abelian Lie algebra of dimension d.*
4. *The tangent space at the identity to the general linear group $G = GL(n)$ can be identified with $\mathfrak{gl}(n)$. The restriction map $\mathfrak{g} \to \mathfrak{gl}(n)$, maps the commutator of left invariant vector fields to the commutator of matrices (exercise). Hence, we can identify the Lie algebra of $GL(n)$ with $\mathfrak{gl}(n)$.*

Remark 13.7. The space $\mathfrak{X}(M)$ formed by all vector fields in a manifold M is a Lie algebra. One may wonder if the Lie algebra $\mathfrak{X}(M)$ is associated with some Lie group. Since this Lie algebra is infinite dimensional (if $\dim M > 0$), this Lie group must be infinite dimension: it is the group $\mathrm{Diff}(M)$ of all diffeomorphisms of M under composition. The study of such infinite dimensional Lie groups is an important topic which is beyond the scope of this course.

We have seen that to each Lie group there is associated a Lie algebra. Similarly, to each homomorphism of Lie groups there is associated a homomorphism of their Lie algebras.

Definition 13.8.

*(i) A **homomorphism of Lie algebras** is a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ between two Lie algebras which preserves the Lie brackets:*

$$\phi([X,Y]_{\mathfrak{g}}) = [\phi(X), \phi(Y)]_{\mathfrak{h}}, \quad \forall X, Y \in \mathfrak{g}.$$

*(ii) A **homomorphism of Lie groups** is a smooth map $\Phi : G \to H$ between two Lie groups which is also a group homomorphism:*

$$\Phi(gh^{-1}) = \Phi(g)\Phi(h)^{-1}, \quad \forall g, h \in G.$$

If $\Phi : G \to H$ is a homomorphism of Lie groups we have an induced map $\Phi_* : \mathfrak{g} \to \mathfrak{h}$: if $X \in \mathfrak{g}$, then $\Phi_*(X) \in \mathfrak{h}$ is the unique left invariant vector field such that $\Phi_*(X)|_e = \mathrm{d}_e\Phi \cdot X_e$.

Proposition 13.9. *If $\Phi : G \to H$ is a Lie group homomorphism, then:*

(i) For all $X \in \mathfrak{g}$, $\Phi_* X$ is Φ-related with X;
(ii) $\Phi_* : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.

Proof. Part (ii) follows from (i), since the Lie bracket of Φ-related vector fields is preserved. In order to show that (i) holds, we observe that since Φ is a group homomorphism, $\Phi \circ L_g = L_{\Phi(g)} \circ \Phi$. Hence:

$$\begin{aligned} \Phi_*(X)_{\Phi(g)} &= \mathrm{d}_e L_{\Phi(g)} \cdot \mathrm{d}_e\Phi \cdot X_e \\ &= \mathrm{d}_e(L_{\Phi(g)} \circ \Phi) \cdot X_e \\ &= \mathrm{d}_e(\Phi \circ L_g) \cdot X_e \\ &= \mathrm{d}_g\Phi \cdot \mathrm{d}_e L_g \cdot X_e = \mathrm{d}_g\Phi \cdot X_g. \end{aligned}$$

□

EXAMPLES 13.10.

1. *Let $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$. For each $a \in \mathbb{R}$ we have the Lie group homomorphism $\Phi_a : \mathbb{R} \to \mathbb{T}^2$ given by:*

$$\Phi_a(t) = (e^{it}, e^{iat}).$$

If a is rational, the image Φ_a is a closed curve, while if a is irrational the image is dense curve in the torus. The induced Lie algebra homomorphism $(\Phi_a)_* : \mathbb{R} \to \mathbb{R}^2$ is given by:

$$(\Phi_a)_*(X) = (X, aX).$$

2. *The determinant defines a Lie group homomorphim* $\det : GL(n) \to \mathbb{R}^*$. *The induced Lie algebra homomorphism is the trace* $\mathrm{tr} = (\det)_* : \mathfrak{gl}(n) \to \mathbb{R}$.

3. Each invertible matrix $A \in GL(n)$ determines a Lie group automorphism $\Phi_A : GL(n) \to GL(n)$ given by conjugation:

$$\Phi_A(B) = ABA^{-1}.$$

Since this map is linear, the associated Lie algebra automorphism $(\Phi_A)_* : \mathfrak{gl}(n) \to \mathfrak{gl}(n)$ is also given by:

$$(\Phi_A)_*(X) = AXA^{-1}.$$

4. More generally, for any Lie group G we can consider conjugation by a fix $g \in G$: $i_g : G \to G$, $h \mapsto ghg^{-1}$. This is a Lie group automorphism and the induced Lie algebra automorphism is denoted by $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$:

$$\mathrm{Ad}_g(X) = (i_g)_* X.$$

Let us continue our study of the Lie group/algebra correspondence. We show now that to each subgroup of a Lie group G corresponds a Lie sub algebra of the Lie algebra $\mathfrak{g}$ of G.

Definition 13.11. *A subspace $\mathfrak{h} \subset \mathfrak{g}$ is called a* ***Lie subalgebra*** *if, for all $X, Y \in \mathfrak{h}$, we have $[X, Y] \in \mathfrak{h}$.*

Examples 13.12.

1. Any subspace of the abelian Lie algebra $\mathbb{R}^d$ is a Lie subalgebra.

2. In the Lie algebra $\mathfrak{gl}(n)$ we have the Lie subalgebra formed by all matrices of zero trace:

$$\mathfrak{sl}(n) = \{X \in \mathfrak{gl}(n) : \operatorname{tr} X = 0\},$$

and also the Lie subalgebra formed by all skew-symmetric matrices:

$$\mathfrak{o}(n) = \{X \in \mathfrak{gl}(n) : X + X^T = 0\}.$$

3. The complex $n \times n$ matrices, denoted by $\mathfrak{gl}(n, \mathbb{C})$, can be seen as a real Lie algebra. It has the Lie subalgebra of all skew-Hermitean matrices:

$$\mathfrak{u}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) : X + \bar{X}^T = 0\},$$

and the Lie subalgebra of all skew-Hermitean matrices of trace zero:

$$\mathfrak{su}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) : X + \bar{X}^T = 0, \operatorname{tr} X = 0\}.$$

4. If $\phi : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras, then its kernel is a Lie subalgebra of $\mathfrak{g}$ and its image is a Lie subalgebra of $\mathfrak{h}$.

A notion of a Lie subgroup is defined similarly:

Definition 13.13. *A* ***Lie subgroup*** *of G is a submanifold (H, Φ) of G such that:*

(i) H is Lie group;
(ii) $\Phi : H \to G$ is a Lie group homomorphism.

As we discussed in Section 6, we can always replace the submanifold (H, Φ) by the subset $\Phi(H) \subset G$, and the immersion Φ by the inclusion i. Since $\Phi(H)$ is a subgroup of G, in the definition of a Lie subgroup we can assume that $H \subset G$ is a a subgroup and that Φ is the inclusion. On the other hand, since the induced map $\Phi_* : \mathfrak{h} \to \mathfrak{g}$ is injective, we can assume that the Lie algebra of a Lie subgroup $H \subset G$ is a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

Examples 13.14.

1. *In Example 13.10.1, for each $a \in \mathbb{R}$ we have a Lie subgroup $\Phi_a(\mathbb{R})$ of $\mathbb{T}^2$. If a is rational, this Lie subgroup is embedded, while if a is irrational this Lie subgroup is only immersed.*

2. *The general linear group $GL(n)$ has the following (embedded) subgroups:*
   (i) *The* ***special linear group*** *of all matrices of determinant 1:*
   $$SL(n) = \{A \in GL(n) : \det A = 1\}.$$
   To this subgroup corresponds the Lie subalgebra $\mathfrak{sl}(n)$.
   (ii) *The* ***orthogonal group*** *of all orthogonal matrices:*
   $$O(n) = \{A \in GL(n) : AA^T = I\}.$$
   To this subgroup corresponds the Lie subalgebra $\mathfrak{o}(n)$.
   (iii) *The* ***special orthogonal group*** *of all orthogonal matrices of positive determinant:*
   $$SO(n) = \{A \in O(n) : \det A = 1\}.$$
   To this subgroup corresponds the Lie subalgebra $\mathfrak{so}(n) = \mathfrak{o}(n)$.

3. *The (real) Lie group $GL(n, \mathbb{C})$ has the following (embedded) subgroups:*
   (i) *The* ***unitary group*** *of all unitary matrices:*
   $$U(n) = \{A \in GL(n, \mathbb{C}) : A\bar{A}^T = I\}.$$
   To this subgroup corresponds the Lie subalgebra $\mathfrak{u}(n)$.
   (ii) *The* ***special unitary group*** *of all unitary matrices of determinant 1:*
   $$SU(n) = \{A \in U(n) : \det A = 1\}.$$
   To this subgroup corresponds the Lie subalgebra $\mathfrak{su}(n)$.

4. *Let $\Phi : G \to H$ is a Lie group homomorphism and let $(\Phi)_* : \mathfrak{g} \to \mathfrak{h}$ the induced Lie algebra homomorphism. Then $\operatorname{Ker} \Phi \subset G$ and $\operatorname{Im} \Phi \subset H$ are Lie subgroups whose Lie algebras coincide with $\operatorname{Ker}(\Phi)_* \subset \mathfrak{g}$ and $\operatorname{Im}(\Phi)_* \subset \mathfrak{h}$, respectively.*

Homework.

1. Show that in the definition of a Lie group, it is enough to assume that:
   (a) The inverse map $G \to G$, $g \mapsto g^{-1}$ is smooth, or that
   (b) The map $G \times G \to G$, $(g, h) \mapsto gh^{-1}$, is smooth.

2. Show that every left invariant vector field in a Lie group G is smooth and complete.

3. Show that the tangent space at the identity of $GL(n)$ can be identified with $\mathfrak{gl}(n)$. Show also that, under this identification, the linear isomorphism $\mathfrak{g} \to \mathfrak{gl}(n)$ takes the Lie bracket of left invariant vector fields to the commutator of matrices.

4. Show that the tangent bundle TG of a Lie group G is trivial, i.e., there exist vector fields $X_1, \dots, X_d \in \mathfrak{X}(G)$ which at each $g \in G$ give a basis for T_gG. Conclude that an even dimension sphere $\mathbb{S}^{2n}$ does not admit the structure of a Lie group.

5. Show that the Lie algebra homomorphism induced by the determinant $\det : GL(n) \to \mathbb{R}^*$ is the trace: $\operatorname{tr} = (\det)_* : \mathfrak{gl}(n) \to \mathbb{R}$.

6. Consider $\mathbb{S}^3 \subset \mathbb{H}$ as the set of quaternions of norm 1. Show that $\mathbb{S}^3$, with the product of quaternions, is a Lie group and determine its Lie algebra.

7. Show that $\mathbb{S}^3$ and $SU(2)$ are isomorphic Lie groups.

HINT: For any pair of complex numbers $z, w \in \mathbb{C}$ with $|z|^2 + |w|^2 = 1$, the matrix:

$$\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$$

is an element in $SU(2)$.

8. Identify the vectors $v \in \mathbb{R}^3$ with the purely imaginary quaternions. For each quaternion $q \in \mathbb{S}^3$ of norm 1 define a linear map $T_q : \mathbb{R}^3 \to \mathbb{R}^3$ by $v \mapsto qvq^{-1}$. Show that T_q is a special orthogonal transformation and that the map $\mathbb{S}^3 \to SO(3)$, $q \mapsto T_q$, is a Lie group homomorphism. Is this map surjective? Injective?

9. Let G be a Lie group. Show that the connected component of the identity is a Lie group G^0 whose Lie algebra is isomorphic to the Lie algebra of G.

10. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Show that G is abelian if and only if $\mathfrak{g}$ is abelian. What can you say if G is not connected?

11. Show that a compact connected abelian Lie group G is isomorphic to a torus $\mathbb{T}^d$.

12. Let (H, Φ) be a Lie subgroup of G. Show that Φ is an embedding if and only if $\Phi(H)$ is closed in G.

13. Let $A \subset G$ be a subgroup of a Lie group G. Show that if (A, i) has a smooth structure making it into a submanifold of G, then this smooth structure is unique and that for that smooth structure A is a Lie group and (A, i) a Lie subgroup.

HINT: Show that (A, i) is an initial submanifold.

14. Integrations of Lie Algebras

We saw in the previous section that:

- To each Lie group corresponds a Lie algebra;
- To each Lie group homomorphism corresponds a Lie algebra homomorphism;
- To each Lie subgroup corresponds a Lie subalgebra.

It is natural to wonder about the inverse to these correspondences. We have seen that two distinct Lie groups can have isomorphic Lie algebras (e.g., $\mathbb{R}^n$ and $\mathbb{T}^n$, $O(n)$ and $SO(n)$, or $SU(2)$ and $SO(3)$). There are indeed topological issues that one must take care of when studying the inverse correspondences.

We start with the following result that shows that a connected Lie group is determined by a neighborhood of the identity:

Proposition 14.1. *Let G be a connected Lie group and U a neighborhood of the identity $e \in G$. Then,*

$$G = \bigcup_{n=1}^{\infty} U^n,$$

where $U^n = \{g_1 \cdots g_n : g_i \in U, i = 1, \dots, n\}$.

Proof. If $U^{-1} = \{g^{-1} : g \in U\}$ and $V = U \cap U^{-1}$, then V is a neighborhood of the origin such that $V = V^{-1}$. Let:

$$H = \bigcup_{n=1}^{\infty} V^n \subset \bigcup_{n=1}^{\infty} U^n.$$

To complete the proof it is enough to show that $H = G$. For that we note:

(i) H is a subgroup: if $g, h \in H$, then $g = g_1 \dots g_n$ and $h = h_1 \dots h_m$, with $g_i, h_j \in V$. Hence,

$$gh^{-1} = g_1 \dots g_n h_m^{-1} \dots h_1^{-1} \in V^{n+m} \subset H.$$

(ii) H is open: if $g \in H$ then $gV \subset gH = H$ is an open set containing g.

(iii) H is closed: for each $g \in G$, gH is an open set and we have

$$H^c = \bigcup_{g \notin H} gH.$$

Since G is connected and $H \neq \emptyset$ is open and closed, we conclude that $H = G$. □

We can now prove:

Theorem 14.2. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Given a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, there exists a unique connected Lie subgroup $H \subset G$ with Lie algebra $\mathfrak{h}$.*

Proof. A Lie subalgebra $\mathfrak{h}$ defines a distribution in G by setting:

$$D : g \mapsto D_g \equiv \{X_g : X \in \mathfrak{h}\}.$$

This distribution is smooth and involutive: if $X_1, \dots, X_k$ is a basis for $\mathfrak{h}$, then these vector fields are smooth and generate D everywhere, hence D is smooth C^∞. On the other hand, if $Y, Z \in \mathfrak{X}(D)$, then

$$Y = \sum_{i=1}^{k} a_i X_i, \quad Z = \sum_{j=1}^{k} b_j X_j.$$

so using that $\mathfrak{h}$ is a Lie subalgebra it follows that:

$$[Y, Z] = \sum_{i,j=1}^{k} a_i b_j [X_i, X_j] + a_i X_i(b_j) X_j - b_j X_j(a_i) X_i \in \mathfrak{X}(D),$$

proving that D is involutive.

Let (H, i) be the leaf of this distribution that contains the identity $e \in G$, where $i : H \hookrightarrow G$ denotes the inclusion. We claim that (H, i) is the desired Lie subgroup.

If $g \in H$, then $(H, L_{g^{-1}} \circ i)$ is also an integral manifold of D which contains e, since:

$$\mathrm{d}_h(L_{g^{-1}} \circ i)(T_h H) = \mathrm{d}_h L_{g^{-1}}(D_h) = D_{g^{-1}h}.$$

Hence, $L_{g^{-1}} \circ i(H) \subset i(H)$, so we conclude that for all $g, h \in H$, we have $g^{-1}h \in H$, proving that H is a subgroup of G.

To verify that (H, i) is a Lie subgroup, it remains to prove that the map $\hat{\nu} : H \times H \to H$, $(g, h) \mapsto g^{-1}h$, is smooth. For this we observe that the map $\nu : H \times H \to G$, $(g, h) \mapsto i(g)^{-1} i(h)$ is smooth, being the composition of smooth maps, so that the following diagram is commutative:

$$\begin{tikzcd} H \times H \arrow[r, "\nu"] \arrow[dr, "\hat{\nu}"'] & G \\ & H \arrow[u, "i"'] \end{tikzcd}$$

Since the leaves of any foliation are initial submanifolds, we conclude that $\hat{\nu} : H \times H \to H$ is smooth.

Uniqueness follows from Proposition 14.1 (exercise). □

The question of deciding if every finite dimensional Lie algebra $\mathfrak{g}$ is associated with some Lie group G is a much harder question which is beyond these notes. There are several ways to proceed to prove that this is indeed true. One way, is to first show that any finite dimensional Lie algebra is isomorphic to a matrix Lie algebra. This requires developing the structure theory of Lie algebras and can be stated as follows:

Theorem 14.3 (Ado)**.** *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. There exists an integer n and an injective Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{gl}(n)$.*

Remark 14.4. A **representation of a Lie algebra** $\mathfrak{g}$ in a vector space V is a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. A representation (V, ρ) is called **faithful** if ρ is injective. In this language, Ado's Theorem states that every finite dimensional Lie algebra has a faithful representation.

Since $\mathfrak{gl}(n)$ is the Lie algebra of $GL(n)$, as a corollary of Ado's Theorem and Theorem 14.2 we obtain:

Theorem 14.5. *For any finite dimensional Lie algebra* $\mathfrak{g}$ *there exists a Lie group* G *with Lie algebra isomorphic to* $\mathfrak{g}$.

The previous theorem gives a matrix group integrating any finite dimensional Lie algebra. Note however, in spite of what Ado's Theorem may suggest, that there are Lie groups which *are not* isomorphic to any matrix group. This happens because, as we know, there can be several Lie groups integrating the same Lie algebra.

In order to clarify the issue of multiple Lie groups integrating the same Lie algebra, recall that if $\pi : N \to M$ is a covering of a manifold M, then there is a unique differentiable structure on N for which the covering map is a local diffeomorphism. In particular, if M is connected then the **universal covering space** of M, which is characterized as the 1-connected (i.e., connected and simply connected) covering of M, is a manifold. For Lie groups this leads to:

Proposition 14.6. *Given a connected Lie group* G *its universal covering space* $\widetilde{G}$ *has a unique Lie group structure for which the covering map* $\pi : \widetilde{G} \to G$ *is a Lie group homomorphism. Moreover, the Lie algebras of* G *and* $\widetilde{G}$ *are isomorphic and* $\ker \pi \subset \widetilde{G}$ *is a discrete, normal, subgroup of the center of* $\widetilde{G}$. *In particular,* $\pi_1(G) \simeq \ker \pi$ *is abelian.*

Proof. Recall that we can identify the universal covering space as:

$$\widetilde{G} = \{[\gamma] \mid \gamma : [0,1] \to G, \gamma(0) = e\},$$
$$\pi : \widetilde{G} \to G, \quad [\gamma] \mapsto \gamma(1),$$

where $[\gamma]$ denotes the homotopy class of the path γ relative to end points. We define a group structure in $\widetilde{G}$ as follows:

(a) The product $[\gamma][\eta]$ in $\widetilde{G}$ is the homotopy class of the path $t \mapsto \gamma(t)\eta(t)$.
(b) The Identity $\tilde{e} \in \widetilde{G}$ is homotopy class of the constant path based at the identity $\gamma(t) = e$.
(c) The inverse map $i : \widetilde{G} \to \widetilde{G}$ associated to an element $[\gamma]$ the homotopy class of the path $t \mapsto \gamma(t)^{-1}$.

It is clear that we these choices the covering map $\pi : \widetilde{G} \to G$ is a group homomorphism.

We consider on $\widetilde{G}$ the unique smooth structure for which the covering map is a local diffeomorphism. To check that $\widetilde{G}$ is a Lie group, we observe that the map $\tilde{\nu} : \widetilde{G} \times \widetilde{G} \to \widetilde{G}$, $(g,h) \to g^{-1}h$, is smooth since it fits into a commutative diagram:

$$\begin{tikzcd} \widetilde{G} \times \widetilde{G} \arrow[r, "\tilde{\nu}"] \arrow[d, "\pi\times\pi"'] & \widetilde{G} \arrow[d, "\pi"] \\ G \times G \arrow[r, "\nu"'] & G \end{tikzcd}$$

where the vertical arrows are local diffeomorphisms and ν is differentiable. Since $\pi : \widetilde{G} \to G$ is a local diffeomorphism it induces an isomorphism between the Lie algebras of $\widetilde{G}$ and G.

Uniqueness follows, because the condition that $\pi : \widetilde{G} \to G$ induces an isomorphism between the Lie algebras of $\widetilde{G}$ and G implies that π is a local diffeomorphism, so both the smooth structure and the group structure are uniquely determined.

We leave as an exercise the remaining statements in the theorem. $\square$

EXAMPLE 14.7.

The special unitary group $SU(2)$ is formed by the matrices:

$$SU(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}.$$

Therefore $SU(2)$ is isomorphic as a manifold to $\mathbb{S}^3$, hence it is 1-connected. In fact, by an exercise in the previous section, $SU(2)$ is isomorphic, as a Lie group, to the group $\mathbb{S}^3$ consisting of the quaternions of length 1.

The Lie algebra of $SU(2)$ consists of the skew-hermitean matrices of trace zero:

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} i\alpha & \beta \\ -\bar{\beta} & -i\alpha \end{pmatrix} : \alpha \in \mathbb{R}, \beta \in \mathbb{C} \right\}.$$

Setting $x = \frac{\alpha}{\sqrt{2}}$, $y = \frac{\operatorname{Re}\beta}{\sqrt{2}}$, $z = \frac{\operatorname{Im}\beta}{\sqrt{2}}$, we obtain identifications

$$\begin{pmatrix} i\alpha & \beta \\ -\bar{\beta} & -i\alpha \end{pmatrix} \quad \longleftrightarrow \quad \begin{pmatrix} 0 & -x & y \\ x & 0 & -z \\ -y & z & 0 \end{pmatrix} \quad \longleftrightarrow \quad (x, y, z),$$

gives Lie algebra isomorphisms $\mathfrak{su}(2) \simeq \mathfrak{so}(3) \simeq \mathbb{R}^3$, where on $\mathbb{R}^3$ the Lie bracket is given by the vector product.

Let us consider on $\mathfrak{su}(2)$ the inner product arising from this identification with $\mathbb{R}^3$ with the standard euclidean inner product. Then for each $g \in SU(2)$ we have the linear transformation $\mathrm{Ad}_g : \mathfrak{su}(2) \to \mathfrak{su}(2)$ (see Example 13.10.3), and we leave it as an exercise to check that:

(a) The linear transformation Ad_g preserves the inner product and the usual orientation, hence determines an element in $SO(3)$.

(b) $\mathrm{Ad} : SU(2) \to SO(3)$ is a surjective group homomorphism with kernel the group $\mathbb{Z}_2 = \{\pm I\}$.

It follows that $\mathrm{Ad} : SU(2) \to SO(3)$ is a covering map. Since $SU(2) \simeq \mathbb{S}^3$ is 1-connected, we conclude that $SU(2)$ is the universal covering space of $SO(3)$. The covering map identifies the antipodal points in the sphere, so we can identify $SO(3)$ with the real projective space $\mathbb{RP}^3$ and $\pi_1(SO(3)) = \mathbb{Z}_2$.

Let us consider now the question of integrating homomorphisms of Lie algebras to homomorphisms of Lie groups. Notice that again there are topological obstructions. For example, the identity $\phi : \mathbb{R} \to \mathbb{R}$ is a Lie algebra isomorphism between the Lie algebras of the Lie groups $\mathbb{S}^1$ and $(\mathbb{R}, +)$. However, the only Lie group homomorphisms $\Phi : \mathbb{S}^1 \to \mathbb{R}$ is the trivial one because the image $\Phi(\mathbb{S}^1)$ is a compact subgroup of $(\mathbb{R}, +)$, and $\{0\}$ is the only such subgroup. Therefore, there is no Lie group homomorphism $\Phi : \mathbb{S}^1 \to \mathbb{R}$ with $\Phi_* = \phi$.

The problem in this example is that $\mathbb{S}^1$ is not simply connected. In fact, we have:

Theorem 14.8. *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. If G is 1-connected then for every Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ there exists a unique Lie group homomorphism $\Phi : G \to H$ such that $\Phi_* = \phi$.*

Proof. Let $\mathfrak{k} = \{(X, \phi(X)) : X \in \mathfrak{g}\} \subset \mathfrak{g} \times \mathfrak{h}$ be the graph of ϕ. Since ϕ is a Lie algebra homomorphism, $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g} \times \mathfrak{h}$. Hence, there exists a unique connected Lie subgroup $K \subset G \times H$ with Lie algebra $\mathfrak{k}$. Let us consider the restriction to K of the projections on each factor:

$$
\begin{tikzcd}
 & K \subset G \times H \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
G & & H
\end{tikzcd}
$$

The restriction of the first projection $\pi_1|_K : K \to G$ gives a Lie group homomorphism such that:

$$(\pi_1)_*(X, \phi(X)) = X.$$

Hence, the map $(\pi_1|_K)_* : \mathfrak{k} \to \mathfrak{g}$ is a Lie algebra isomorphism and it follows that $\pi_1|_K : K \to G$ is a covering map (see the Exercises). Since G is 1-connected, we conclude that $\pi_1|_K$ is a Lie isomorphism. Then, the composition

$$\Phi = \pi_2 \circ (\pi_1|_K)^{-1} : G \to H$$

is a Lie group homomorphism and we have that:

$$\begin{aligned}(\Phi)_*(X) &= (\pi_2)_* \circ (\pi_1|_K)_*^{-1}(X) \\ &= (\pi_2)_*(X, \phi(X)) = \phi(X).\end{aligned}$$

We leave the proof of uniqueness as an exercise. □

We summarize the previous results in the following statements, sometimes known as Lie's Theorems:

Theorem 14.9 (Lie I)**.** *If G is a connected Lie group with Lie algebra $\mathfrak{g}$, there is a unique (up to isomorphism) 1-connected Lie group $\widetilde{G}$ with Lie algebra $\mathfrak{g}$ and a surjective Lie group morphism $\Phi : \widetilde{G} \to G$.*

Theorem 14.10 (Lie II). *Let G and H be two Lie groups, with Lie algebras denoted $\mathfrak{g}$ and $\mathfrak{h}$, respectively. If G is 1-connected, then a Lie algebra morphism $\phi : \mathfrak{g} \to \mathfrak{h}$ integrates to a unique Lie group morphism $\Phi : G \to H$.*

Theorem 14.11 (Lie III). *Any finite dimensional Lie algebra $\mathfrak{g}$ is integrable.*

Note also that, given a finite dimensional Lie algebra $\mathfrak{g}$, we can obtain any connected Lie group G integrating it (up to isomorphism) as follows:

(i) Construct the 1-connected Lie group $\widetilde{G}$ integrating $\mathfrak{g}$;
(ii) Find a discrete normal subgroup N of the center of $\widetilde{G}$;
(iii) $G = \widetilde{G}/N$ is a connected Lie group integrating $\mathfrak{g}$.

If one drops the condition of G being connected this problem is not solvable since it would include as a special case the classification of all finite groups, a problem which is well-known not to have any reasonable solution.

Homework.

1. Let $\Phi : G \to H$ be a Lie group homomorphism between connected Lie groups G and H such that $(\Phi)_* : \mathfrak{g} \to \mathfrak{h}$ is an isomorphism. Show that Φ is a covering map.

2. Complete the proof of Theorem 14.2 by showing that the integrating Lie subgroup is unique.

3. Let G be a Lie group and let $\pi : H \to G$ be a covering map. Show that H is a Lie group.

4. Let $SL(2, \mathbb{C})$ be the group of complex 2×2 matrices with determinant 1. Show that $SL(2, \mathbb{C})$ is 1-connected.
(Hint: Show that a matrix in $SL(2, \mathbb{C})$ can be written uniquely as a product AB, where $A \in SU(2)$ and B is upper triangular with determinant 1.)

5. Show that any homomorphism of Lie algebras $\phi : \mathfrak{sl}(2) \to \mathfrak{gl}(n)$ integrates to a unique homomorphism of Lie groups $\Phi : SL(2) \to GL(n)$.
(Hint: Consider the complexification $\phi^c : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{gl}(n, \mathbb{C})$ of ϕ and use the previous exercise.)

6. Let G be a connected Lie group and let $D \subset G$ be a discrete normal subgroup. Show that D is contained in the center of G, so in particular it must be abelian. Conclude that the any connected Lie group has abelian fundamental group.

7. Find all (up to isomorphism) connected Lie groups integrating the abelian Lie algebra $\mathfrak{g} = \mathbb{R}^d$.

8. Find all (up to isomorphism) connected Lie groups integrating the Lie algebra $\mathfrak{so}(3)$.

15. The Exponential Map

We will now construct the *exponential map* for Lie groups/algebras, which generalizes the exponential of matrices.

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Given a left invariant vector field $X \in \mathfrak{g}$, the map

$$\phi_X : \mathbb{R} \to \mathfrak{g}, \quad t \mapsto tX,$$

is a Lie algebra homomorphism. Since $\mathbb{R}$ is 1-connected it follows that there exists a unique Lie group homomorphism

$$\Phi_X : \mathbb{R} \to G, \text{ with } (\Phi_X)* = \phi_X.$$

We note that

$$\begin{aligned}
\Phi_X(0) &= e \\
\Phi_X(t+s) &= \Phi_X(t)\Phi_X(s) = L_{\Phi_X(t)}\Phi_X(s), \\
\frac{\mathrm{d}}{\mathrm{d}t}\Phi_X(t) &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\Phi_X(t+s)\right|_{s=0} \\
&= \mathrm{d}_e L_{\Phi_X(t)} \cdot \left.\frac{\mathrm{d}}{\mathrm{d}s}\Phi_X(s)\right|_{s=0} \\
&= \mathrm{d}_e L_{\Phi_X(t)} \cdot X_e = X_{\Phi_X(t)}.
\end{aligned}$$

This means that $t \mapsto \Phi_X(t)$ is actually the integral curve of X through $e \in G$. Recalling that ϕ_X^t denotes the flow of the vector field X, we have:

Definition 15.1. *The* ***exponential map*** $\exp : \mathfrak{g} \to G$ *is the map*

$$\exp(X) = \phi_X^1(e).$$

The following proposition lists the main properties of the exponential map. Its proof is left for the exercises.

Proposition 15.2. *The exponential map* $\exp : \mathfrak{g} \to G$ *satisfies:*

(i) $\exp((t+s)X) = \exp(sX)\exp(tX)$*;*
(ii) $\exp(-tX) = [\exp(tX)]^{-1}$*;*
(iii) $\exp$ *is a smooth map and* $\mathrm{d}_0 \exp = I$*;*
(iv) For any Lie group homomorphism $\Phi : G \to H$ *the following diagram is commutative:*

$$\begin{tikzcd}
G \arrow[r, "\Phi"] & H \\
\mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\Phi_*"'] & \mathfrak{h} \arrow[u, "\exp"']
\end{tikzcd}$$

Property (iii) implies that that the exponential is a diffeomorphism from a neighborhood of $0 \in \mathfrak{g}$ to a neighborhood of $e \in G$. In geral, the exponential $\exp : \mathfrak{g} \to G$ is neither surjective, nor injective. Also, it may fail to be a local diffeomorphism at other points of G. There are however examples of Lie groups/algebras in which some of these properties do hold (see also the exercises).

EXAMPLE 15.3.
Recall that the Lie algebra of $G = GL(n)$ *can be identified with* $\mathfrak{gl}(n)$. *If* $A \in \mathfrak{gl}(n)$, *the left invariant vector field associated with the Lie algebra element* $A = (a_{ij})$ *is:*

$$X_A = \sum_{ijk} x_{ik} a_{kj} \frac{\partial}{\partial x_{ij}}.$$

Hence, the integral curves if this vector field are the solutions of the system of ode's:

$$\dot{x}_{ij} = \sum_k x_{ik} a_{kj},$$

These are given by:

$$x_{ij}(t) = \sum_k x_{ik}(0)(e^{tA})_{kj},$$

where the matrix exponential is defined as usual by:

$$e^A = \sum_{k=0}^{+\infty} \frac{A^n}{n!}.$$

We conclude that the exponential map $\exp : \mathfrak{gl}(n) \to GL(n)$ *coincides with the usual matrix exponential.*

By item (iv) in Proposition 15.2, it follows from the previous example that if $\mathfrak{h} \subset \mathfrak{gl}(n)$ is a Lie subalgebra and $H \subset GL(n)$ is the associated connected Lie subgroup, then the exponential map $\exp : \mathfrak{h} \to H$ also coincides with the matrix exponential.

Note, however, although Ado's Theorem states that every Lie algebra is isomorphic to a Lie algebra of matrices, there are Lie groups which are not isomorphic to any group of matrices. Hence one needs the abstract definition of the exponential map. As an application of the integration of morphisms we given an example of such Lie group.

EXAMPLE 15.4.
Consider the special linear group

$$SL(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\}.$$

To exhibit its topological structure, it is convenient to perform the change of variables $(a, b, c, d) \mapsto (p, q, r, s)$ *defined by*

$$a = p + q, \quad d = p - q, b = r + s, c = r - s.$$

Then

$$ad - bc = 1 \quad \iff p^2 + s^2 = q^2 + r^2 + 1.$$

Hence we see that we can also describe $SL(2)$ *as:*

$$SL(2) = \{(p, q, r, s) \in \mathbb{R}^4 : p^2 + s^2 = q^2 + r^2 + 1\}$$

so we conclude that $SL(2)$ *is diffeomorphic to* $\mathbb{R}^2 \times \mathbb{S}^1$. *In particular,*

$$\pi_1(SL(2)) = \pi_1(\mathbb{S}^1) = \mathbb{Z}.$$

Let $\widetilde{SL(2)}$ be the universal covering group of $SL(2)$. We claim that $\widetilde{SL(2)}$ is not isomorphic to any group of matrices. By an exercise in the previous section, we have:

- *Given a Lie algebra morphism $\phi : \mathfrak{sl}(2) \to \mathfrak{gl}(n)$, there exists a unique Lie group morphism $\Phi : SL(2) \to GL(n)$ such that $\Phi_* = \phi$.*

Now assume that, for some n, there exists an injective Lie group homomorphism:

$$\widetilde{\Phi} : \widetilde{SL(2)} \to GL(n).$$

This leads to a contradiction. Indeed, $\widetilde{\Phi}$ induces a morphism of Lie algebras $\phi := (\widetilde{\Phi})_* : \mathfrak{sl}(2) \to \mathfrak{gl}(n)$, so there exists a unique Lie group homomorphism $\Phi : SL(2) \to GL(n)$ such that $\Phi_* = \phi$ and we obtain a commutative diagram:

$$\begin{tikzcd}
\widetilde{SL(2)} \arrow[r, "\widetilde{\Phi}"] \arrow[d, "\pi"'] & GL(n) \\
SL(2) \arrow[ur, "\Phi"'] &
\end{tikzcd}$$

In this diagram the morphism π is not injective, while the morphism $\widetilde{\Phi}$ is injective, which is a contradiction.

The exponential map is very useful in the study of Lie groups and Lie algebras since it provides a direct link between the Lie algebra (the infinitesimal object) and the Lie group (the global object). For example, we have the following result whose proof is left as an exercise:

Proposition 15.5. *Let H be a subgroup of a Lie group G and let $\mathfrak{h} \subset \mathfrak{g}$ be a subspace of the Lie algebra of G. If $U \subset \mathfrak{g}$ is a neighborhood of 0 which is diffeomorphic via the exponential map to a neighborhood $V \subset G$ of e, and*

$$\exp(\mathfrak{h} \cap U) = H \cap V,$$

then, for the relative topology, H is a Lie subgroup of G with Lie algebra $\mathfrak{h}$.

Using this proposition one can then proof the following important result:

Theorem 15.6. *Let G be a Lie group and $H \subset G$ a closed subgroup. Then H, with the relative topology, is a Lie subgroup.*

Sketch of the proof. The idea of the proof is the consider the set:

$$\mathfrak{h} := \{X \in \mathfrak{g} : \exp(tX) \in H, \forall t \in \mathbb{R}\}$$

and apply the previous proposition.

Clearly, the set $\mathfrak{h}$ is closed under multiplication by scalars. On the other hand, if $X, Y \in \mathfrak{g}$ one shows that:

$$\lim_{n \to +\infty} \left(\exp(\frac{t}{n}X) \exp(\frac{t}{n}Y) \right)^n = \exp(t(X+Y)),$$

and then it follows that $\mathfrak{h}$ is also closed under addition, since H is a closed subset. Hence, $\mathfrak{h}$ is a linear subspace.

Finally, arguing by contradiction using again that H is closed in G, one shows that there exists neighborhoods $U \subset \mathfrak{g}$ of 0 and $V \subset G$ of e, such that $\exp : U \to V$ is a diffeomorphism and:

$$\exp(\mathfrak{h} \cap U) = H \cap V.$$

$\square$

Example 15.7.
The previous results allows one to check quickly if subgroups of $GL(n)$ are Lie subgroups and to determine their Lie algebras. For example, consider the subgroup $SL(n) \subset GL(n)$. It is a closed subgroup, so by Theorem 15.6 it is a Lie subgroup. To find its Lie algebra, one observes first that the set $\mathfrak{sl}(n)$ of matrices of trace zero is a subspace of $\mathfrak{gl}(n)$ and second that we have the well-known formula:

$$\det(e^X) = e^{\operatorname{tr} X}.$$

Hence, we see that $\exp(X) \in SL(n)$ *if and only if* $\operatorname{tr} X = 0$. *By Proposition 15.5, we conclude that the Lie algebra of* $\mathrm{SL}(n)$ *is* $\mathfrak{sl}(n)$.

Homework.

1. Verify the properties of the exponential map given in Proposition 15.2.

2. Show that the exponential map $\exp : \mathfrak{gl}(2) \to GL(2)$ is not surjective.

3. Let $N \subset GL(n)$ be the subgroup formed by all upper triangular matrices with diagonal elements all equal to 1. Show that N is a Lie subgroup, find its Lie algebra $\mathfrak{n}$ and prove that the exponential map $\exp : \mathfrak{n} \to N$ is a bijection.

4. Let G be a compact Lie group. Show that $\exp : \mathfrak{g} \to G$ is surjective. (Hint: Use the fact, to be proved later, that any compact Lie group has a bi-invariant metric, i.e., a metric invariant under both right and left translations.)

5. Let G and H be Lie groups. Show that:
   (a) Every continuous homomorphism $\Phi : \mathbb{R} \to G$ is smooth;
   (b) Every continuous homomorphism $\Phi : G \to H$ is smooth;
   (c) If G and H are isomorphic as topological groups, then G and H are isomorphic as Lie groups.

6. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $H \subset G$ be a Lie subgroup with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. Show that $X \in \mathfrak{g}$ belongs to $\mathfrak{h}$ if and only if $\exp(tX) \in H$ for all $t \in \mathbb{R}$.

7. Prove Proposition 15.5.
   (Hint: Show that H has a smooth structure compatible with the relative topology making (H, i) a submanifold of G, by considering the charts:

$$\{(H \cap hV, \exp^{-1} \circ L_h) : h \in H\}.$$

   Then check that multiplication in H is smooth and use the previous exercise to complete the proof.)

16. Groups of Transformations

Let G be a group. Recall (Section 9) that we denote an action of G on a set M by a map $\Psi : G \times M \to M$, which we write as $(g, p) \mapsto g \cdot p$, and satisfies:

(a) $e \cdot p = p$, for all $p \in M$;
(b) $g \cdot (h \cdot p) = (gh) \cdot p$, for all $g, h \in G$ and $p \in M$.

An action can also be viewed as a group homomorphism $\widehat{\Psi}$ from G to the group of bijections of M. For each $g \in G$ we denote by Ψ_g the bijection:

$$\Psi_g : M \to M, \quad p \mapsto g \cdot p$$

When G is a Lie group, M is a smooth manifold and the map $\Psi : G \times M \to M$ is smooth, we say that we have a **smooth action**. In this case each $\Psi_g : M \to M$ is a diffeomorphism of M, so one also says that G is a **group of transformations** of M. Note that for a smooth action, for each $p \in M$, the isotropy subgroup

$$G_p \equiv \{g \in G : g \cdot p = p\}.$$

is a closed subgroup, hence it is an (embedded) Lie subgroup of G (see Theorem 15.6).

The results in Section 9 concerning smooth structures on orbits spaces of discrete group actions extend to arbitrary smooth actions of Lie groups. First, we call a smooth action $\Psi : G \times M \to M$ a **proper action** if the map:

$$G \times M \to M \times M, \quad (g, p) \mapsto (p, g \cdot p),$$

is proper.

Examples 16.1.

1. *The action by translations of a Lie group G on itself, $G \times G \to G$, $(g, h) \mapsto gh$, is always proper.*
2. *Smooth actions of compact Lie groups on manifolds are always proper. Also, for any proper action $G \times M \to M$ the isotropy groups G_p are all compact. So, for example, the action of $O(n)$ on $\mathbb{R}^n$ by matrix multiplication is proper (since $O(n)$ is compact), while the action of $SL(n)$ on $\mathbb{R}^n$ by matrix multiplication is not proper (since the isotropy group of 0 is $SL(n)$ which is not compact).*
3. *Given a smooth proper action $G \times M \to M$ and a closed subgroup $H \subset G$, the restricted action $H \times M \to M$ is still a smooth proper action. For example, restricting the action by translations of $(\mathbb{R}^n, +)$ on itself, we obtain the smooth proper action of $(\mathbb{R}, +)$ on $\mathbb{R}^n$ given by:*

$$t \cdot (x^1, \dots, x^n) := (x^1 + t, x^2, \dots, x^n).$$

Next, recall that an action is **free** if the isotropy groups G_p are trivial, for all $p \in M$. We leave as an exercise to check that:

Lemma 16.2. *Given a smooth free action $G \times M \to M$ and $p \in M$ the map*

$$\Psi_p : G \to M, \quad g \mapsto g \cdot p$$

is an injective immersion. In particular, the orbits of a smooth, free action $G \times M \to M$ are submanifolds of M diffeomorphic to G.

Note that, in general, the orbits are not embedded submanifolds: for example, the irrational lines on the torus $\mathbb{T}^2$ are the orbits of a free, smooth, action of $(\mathbb{R}, +)$. We will see later that the orbits of *any* action are immersed submanifolds.

For proper actions the geometry of the orbits is much nicer. In particular, for proper and free actions we have:

Theorem 16.3. *Let $\Psi : G \times M \to M$ be a smooth action of a Lie group G on a manifold M. If the action is free and proper, then $G\backslash M$ has a unique smooth structure, compatible with the quotient topology, such that $\pi : M \to G\backslash M$ is a submersion. In particular,*

$$\dim G\backslash M = \dim M - \dim G.$$

In particular, the orbits of a smooth, proper and free action of G are embedded submanifolds diffeomorphic to G.

Proof. We apply Theorem 9.3 to the orbit equivalence relation defined by the action. This means that we need to verify that its graph:

$$R = \{(p, g \cdot p) : p \in M, g \in G\} \subset M \times M,$$

is a proper submanifold of $M \times M$ and that the restriction of the projection $p_1|_R : R \to M$ is a submersion.

Let us consider the map:

$$\Phi : G \times M \to M \times M, \quad (g, p) \mapsto (p, g \cdot p),$$

whose image is precisely R. Since the action is assumed to be free, we see that Φ is injective. The differential $\mathrm{d}_{(g,p)}\Phi : T_gG \times T_pM \to T_pM \times T_{g \cdot p}M$ is given by:

$$(\mathbf{v}, \mathbf{w}) \mapsto (\mathbf{w}, \mathrm{d}\Psi_p \cdot \mathbf{v} + \mathrm{d}\Psi_g \cdot \mathbf{w}).$$

Since this differential is injective we conclude that Φ is an injective immersion with image R. Since, by assumption, Φ is proper, it follows that R is a proper submanifold of $M \times M$.

To verify that $p_1|_R : R \to M$ is a submersion, it is enough to show that the composition $p_1 \circ \Phi : G \times M \to M$ is a submersion. But this composition is just the projection $(g, p) \mapsto p$, which is obviously a submersion. $\square$

EXAMPLE 16.4.
Consider the action of $\mathbb{S}^1 = \{w \in \mathbb{C} : |w| = 1\}$ on the 3-sphere $\mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$, defined by:

$$w \cdot (z_1, z_2) = (wz_1, wz_2).$$

This action is free and proper. Hence, the orbits of this action are embedded submanifolds of $\mathbb{S}^3$ diffeomorphic to $\mathbb{S}^1$. The orbit space $\mathbb{S}^1\backslash\mathbb{S}^3$ is a smooth manifold. We will see later that this manifold is diffeomorphic to $\mathbb{S}^2$.

Let G be a Lie group and consider the action of G on itself by left translations:

$$G \times G \to G, \quad (g, h) \mapsto gh.$$

This action is free and proper. If $H \subset G$ is a closed subgroup, then H is a Lie subgroup and the action of H on G, by left translation is also free and proper. The orbit space for this action consist of the right cosets:

$$H\backslash G = \{Hg : g \in G\}.$$

From Theorem 16.3, we conclude that:

Corollary 16.5. *Let G be a Lie group and let $H \subset G$ be a closed subgroup. Then $H\backslash G$ has a unique smooth structure, compatible with the quotient topology, such that $\pi : G \to H\backslash G$ is a submersion. In particular,*

$$\dim H\backslash G = \dim G - \dim H.$$

Remark 16.6. So far we have discussed **left actions**. We can also discuss **right actions** $M \times G \to M$, $(m, g) \to m \cdot g$, where axioms (a) and (b) are replaced by:

(a) $p \cdot e = p$, for all $p \in M$;
(b) $(p \cdot h) \cdot g = p \cdot (hg)$, for all $g, h \in G$ and $p \in M$.

Given a left action $(g, m) \mapsto g \cdot m$ one obtains a right action by setting $(m, g) \mapsto g^{-1} \cdot m$, and conversely. Hence, every result about left actions yields a result about right actions, and conversely. For example, if G is a Lie group and $H \subset G$ is a closed subgroup, the right action of H on G by right translations is free and proper. Hence, the set of left cosets

$$G/H = \{gH : g \in G\},$$

also has a natural smooth structure.

Given two G-actions, $G \times M \to M$ and $G \times N \to N$, a **G-equivariant map** is a map $\Phi : M \to N$ such that:

$$\Phi(g \cdot p) = g \cdot \Phi(p), \quad \forall g \in G, p \in M.$$

We say that we have **equivalent actions** is there exists a G-equivariant bijection between them.

Given any action $\Psi : G \times M \to M$, for each $p \in M$ the map

$$\Psi_p : G \to M, \quad g \mapsto g \cdot p,$$

induces a bijection $\bar{\Psi}_p$ between G/G_p and the orbit through p. Notice that G acts on the set of right cosets by left translations:

$$G \times G/G_p \to G/G_p, \quad (h, gG_p) \mapsto (hg)G_p.$$

The map $\bar{\Psi}_p$ is a G-equivariant bijection between the set of right cosets G/G_p and the orbit through p.

If we have a smooth action $\Psi : G \times M \to M$ we can use the results above with $H = G_p$ to conclude that G/G_p has a smooth structure and that the map:

$$\bar{\Psi}_p : G/G_p \to M, \quad gG_p \mapsto g \cdot p,$$

is an injective immersion. Since the image of this map is the orbit through p, we conclude that:

Theorem 16.7. *Let $\Psi : G \times M \to M$ be a smooth action of a Lie group G on a manifold M. The orbits of the action are initial submanifolds of M. Moreover, for every $p \in M$, the map*

$$\bar{\Psi}_p : G/G_p \to M, \quad gG_p \mapsto g \cdot p,$$

is a G-equivariant diffeomorphism between G/G_p and the orbit through p.

Proof. Since G_p is a closed subgroup, by Corollary 16.5, G/G_p has a smooth structure. The map:

$$\bar{\Psi}_p : G/G_p \to M, \quad gG_p \mapsto g \cdot p,$$

is an injective immersion whose image is the orbit through p. This makes the orbit an immersed submanifold and we leave it as an exercise to show that it is initial.

This smooth structure on the orbit does not depend on the choice of $p \in M$: two points $p, q \in M$ which belong to the same orbit have conjugate isotropy groups:

$$q = g \cdot p \quad \Longrightarrow \quad G_q = gG_pg^{-1}.$$

It follows that $\Phi : G/G_p \to G/G_q$, $hG_p \mapsto ghg^{-1}G_q$, is an equivariant diffeomorphism which makes the following diagram commute:

$$\begin{tikzcd} G/G_p \arrow[r, "\bar{\Psi}_p"] \arrow[d, "\Phi"'] & M \arrow[d, "\Psi_g"] \\ G/G_q \arrow[r, "\bar{\Psi}_q"] & M \end{tikzcd}$$

Since $\Psi_g : M \to M$, $m \mapsto g \cdot m$, is a diffeomorphism, it is clear that the two immersions give equivalent smooth structures on the orbit. □

A **transitive action** $\Psi : G \times M \to M$ is an action with only one orbit. This means that for any pair of points $p, q \in M$, there exists $g \in G$ such that $q = g \cdot p$. In this case, fixing any point $p \in M$, we obtain an equivariant bijection $G/G_p \to M$. When the action is smooth, this gives an equivariant diffeomorphism between M and the quotient G/G_p. In this case, one also calls M a **homogeneous space**.

The homogeneous G-spaces are just the manifolds of the form G/H where $H \subset G$ is a closed subgroup. In the homogenous space G/H we have the natural G-action, induced from the action of G on itself by left translations. Homogenous spaces are particularly nice examples of manifolds. The next examples will show that a manifold can be a homogeneous G-space for different choices of Lie groups.

EXAMPLES 16.8.

1. *Let $\mathbb{S}^3$ be the unit quarternions. Identifying $\mathbb{R}^3$ with the purely imaginary quaternions, we obtain an action of $\mathbb{S}^3$ on $\mathbb{R}^3$:*
$$q \cdot v = qvq^{-1}.$$
It is easy to see that the orbits of this action are the spheres of radius r and the origin. Let us restrict the action to $\mathbb{S}^2$, the sphere of radius 1. An easy computation shows that the isotropy group of $p = (1,0,0)$ is the subgroup $\mathbb{S}^1 = (\mathbb{S}^3)_p \subset \mathbb{S}^3$ formed by quaternions of the form $q_0 + iq_1 + 0j + 0k$. It follows that the sphere is diffeomorphic to the homogeneous space $\mathbb{S}^3/\mathbb{S}^1$. The surjective submersion $\pi : \mathbb{S}^3 \to \mathbb{S}^2$, $q \mapsto q \cdot (1,0,0)$, whose fibers are diffeomorphic to $\mathbb{S}^1$, is known as the ***Hopf fibration****.*

2. *Let $O(d+1) \times \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ be the standard action by matrix multiplication:*
$$(A, \vec{v}) \mapsto A\vec{v}.$$
The orbits of this action are the spheres $(x^0)^2 + \cdots + (x^d)^2 = r^2$ and the origin. Again, we consider the sphere $\mathbb{S}^d$ of radius 1 and we let $p_N = (0, \dots, 0, 1) \in \mathbb{S}^d$, the north pole. The isotropy group at p_N consists of matrices of the form:
$$\left(\begin{array}{c|c} B & 0 \\ \hline 0 & 1 \end{array}\right) \in O(d+1),$$
so we can identify it with $O(d)$. It follows that the map
$$O(d+1)/O(d) \to \mathbb{S}^d, \quad A\ O(d) \mapsto A \cdot p_N,$$
is a diffeomorphism. A similar reasoning shows that $\mathbb{S}^d$ is also diffeomorphic to the homogeneous space $SO(d+1)/SO(d)$.

3. *Let $\mathbb{RP}^d$ be the real projective space and denote by $\pi : \mathbb{R}^{d+1} - \{0\} \to \mathbb{RP}^d$ the map*
$$\pi(x^0, \dots, x^d) = [x^0 : \cdots : x^d].$$
The action $SO(d+1) \times \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ by matrix multiplication, induces a smooth transitive action $SO(d+1) \times \mathbb{RP}^d \to \mathbb{RP}^d$. The isotropy subgroup of the point $[0 : \cdots : 0 : 1]$ consist of matrices of the form:
$$\left(\begin{array}{c|c} B & 0 \\ \hline 0 & \det B \end{array}\right) \in SO(d+1),$$
so we can identify it with $O(d)$. We conclude that $\mathbb{RP}^d$ is diffeomorphic to the homogeneous space $SO(d+1)/O(d)$.

   A similar reasoning shows that the complex projective space $\mathbb{CP}^d$ is diffeomorphic to the homogeneous space $SU(d+1)/U(d)$.

4. Let $G_k(\mathbb{R}^d)$ denote that set of all linear subspaces of $\mathbb{R}^d$ of dimension k. The usual action of the orthogonal group $O(d)$ on $\mathbb{R}^d$ by matrix multiplication induces an action $O(d) \times G_k(\mathbb{R}^d) \to G_k(\mathbb{R}^d)$: an invertible linear transformation takes linear subspaces of dimension k to linear subspaces of dimension k. It is easy to check that given any two k-dimensional linear subspaces $S_1, S_2 \subset \mathbb{R}^d$ there exists $A \in O(d)$ mapping S_1 onto S_2. This means that the action $O(d) \times G_k(\mathbb{R}^d) \to G_k(\mathbb{R}^d)$ is transitive.

We fix the point $S_0 \in G_k(\mathbb{R}^d)$ to be the subspace $\mathbb{R}^k \times \{0\} \subset \mathbb{R}^d$, then

$$O(d)_{S_0} = \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right) \in O(d) : A \in O(k), B \in O(d-k) \right\}.$$

so we have a bijection

$$O(d)/O(k) \times O(d-k) \to G_k(V).$$

On $G_k(\mathbb{R}^d)$ we can consider the unique smooth structure for which this bijection becomes a diffeomorphism. This gives $G_k(\mathbb{R}^d)$ the structure of a manifold of dimension $k(d-k) = \dim O(d) - (\dim O(k) + \dim O(d-k))$. One can show that this smooth structure is independent of the choice of base point S_0. The manifold $G_k(\mathbb{R}^d)$ is called the ***Grassmannian manifold*** *of k-planes in $\mathbb{R}^d$.*

Since Lie groups have infinitesimal counterparts, it should come as no surprise that Lie group actions also have an infinitesimal counterpart. Let $\Psi : G \times M \to M$ be a smooth action, which we can view as "Lie group" homomorphism:

$$\widehat{\Psi} : G \to \mathrm{Diff}(M).$$

We think of $\mathrm{Diff}(M)$ as a Lie group with Lie algebra $\mathfrak{X}(M)$, then there must exist a homomorphism of Lie algebras

$$\psi = (\widehat{\Psi})_* : \mathfrak{g} \to \mathfrak{X}(M).$$

In fact, if $X \in \mathfrak{g}$ and $p \in M$, the curve

$$t \mapsto \exp(tX) \cdot p,$$

goes through p at $t = 0$, and it is defined and smooth for $t \in \mathbb{R}$. We define the vector field $\psi(X)$ in M, by:

$$\psi(X)_p \equiv \left. \frac{\mathrm{d}}{\mathrm{d}t} \exp(tX) \cdot p \right|_{t=0}.$$

The proof of the following lemma is left as an exercise:

Lemma 16.9. *For each $X \in \mathfrak{g}$, $\psi(X)$ is a smooth vector field and the map $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ is linear and satisfies:*

$$\psi([X,Y]_{\mathfrak{g}}) = -[\psi(X), \psi(Y)], \quad \forall X, Y \in \mathfrak{g}.$$

Remark 16.10. An *anti-homomorphism* of Lie algebras is a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ which satisfies:

$$\phi([X,Y]) = -[\phi(X), \phi(Y)], \quad \forall X, Y \in \mathfrak{g}.$$

The appearance of a minus sign in the lemma is easy to explain: with our conventions, where the Lie algebra of a Lie group is formed by the left invariant vector fields, the Lie algebra of the group of diffeomorphisms $\mathrm{Diff}(M)$ is formed by the vector fields $\mathfrak{X}(M)$ with a Lie bracket which is the *simmetric* of the usual Lie bracket of vector fields. One can see this, for example, by determining the 1-parameter subgroups of the group of diffeomorphims. We could have defined the Lie bracket of vector fields with the opposite sign, but this would lead to the presence of negative signs in other formulas.

The lemma above suggests the following definition:

Definition 16.11. *An **infinitesimal action** of a Lie algebra $\mathfrak{g}$ on a manifold M is an anti-homomorphism of Lie algebras $\psi : \mathfrak{g} \to \mathfrak{X}(M)$.*

Example 16.12.
The Lie algebra $\mathfrak{so}(3)$ has a basis consisting of the skew-symmetric matrices:

$$X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In this basis, we have the following Lie bracket relations:

$$[X, Y] = -Z, \quad [Y, Z] = -X, \quad [Z, X] = -Y.$$

For the usual action of $SO(3)$ on $\mathbb{R}^3$ by rotations, we can compute the infinitesimal action as follows. First, we compute the exponential

$$\exp(tX) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{bmatrix}.$$

Then:

$$\psi(X)_{(x,y,z)} = \frac{\mathrm{d}}{\mathrm{d}t} \exp(tX) \cdot (x, y, z)\Big|_{t=0} = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}.$$

Similarly, we compute:

$$\psi(Y) = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \quad \psi(Z) = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

*The vector fields $\{\psi(X), \psi(Y), \psi(Z)\}$ are called the **infinitesimal generators** of the action. Using that ψ is an anti-homomorphism of Lie algebras, one recovers the Lie brackets of Example 11.2.*

A smooth action $\Psi : G \times M \to M$ determines an infinitesimal action $\psi : \mathfrak{g} \to \mathfrak{X}(M)$. The converse does not necessarily hold, as shown in the next examples.

Examples 16.13.

1. *Consider the infinitesimal Lie algebra action of $\mathfrak{so}(3)$ on $\mathbb{R}^3$ given in Example 16.12. We can restrict this action to $M = \mathbb{R}^3 - \{p_0\}$ by taking for each $X \in \mathfrak{g}$, the restriction of $\psi(X)$ to M. This defines an infinitesimal action of*

$\mathfrak{so}(3)$ on M which, if $p_0 \neq 0$, is not induced from a Lie group action of $SO(3)$ in M.

2. *Any non-zero vector field X on a manifold M determines an infinitesimal action of the Lie algebra $\mathfrak{g} = \mathbb{R}$ on M by setting $\psi(\lambda) := \lambda X$. This infinitesimal action integrates to a Lie group action of $G = (\mathbb{R}, +)$ on M if and only if the vector field X is complete. The Lie group $\mathbb{S}^1$ also has Lie algebra $\mathbb{R}$, but even if the vector field is complete, there will be no action $\Psi : \mathbb{S}^1 \times M \to M$ with $\Psi_* = \psi$, since the orbits of X may not be periodic.*

Obviously, for any infinitesimal Lie algebra action $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ which is induced from a Lie group action $G \times M \to M$ the infinitesimal generators $\psi(X) \in \mathfrak{X}(M)$ are all complete vector fields. If we assume that G is 1-connected, the converse also holds but the proof is beyond the scope of these notes:

Theorem 16.14. *Let $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ be an infinitesimal Lie algebra action such that $\psi(X)$ is complete, for all $X \in \mathfrak{g}$. Then there exists a smooth action $\Psi : G \to \mathrm{Diff}(M)$ with $\Psi_* = \psi$, where G is the 1-connected Lie group with Lie algebra $\mathfrak{g}$.*

For example, if M is a compact manifold then every infinitesimal Lie algebra action $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ integrates to a smooth Lie group action $\Psi : G \times M \to M$, where G is the 1-connected Lie group with Lie algebra $\mathfrak{g}$.

Example 16.15.

*A **representation of a Lie group** G in a vector space V is a Lie group homomorphism $\widehat{\Psi} : G \to GL(V)$. Since $GL(V) \subset \mathrm{Diff}(V)$, this is the same as a smooth linear action $\Psi : G \times V \to V$.*

*A vector field X on a vector space V is called a **linear vector field** if for any linear function $l \in V^*$ the function $X(l)$ is a also linear. A linear vector field X determines a linear map $X : V^* \to V^*$, so its transpose is an element of $\mathfrak{gl}(V)$. The converse also holds, so linear maps $T : V \to V$ are in 1:1 correspondence with linear vector fields $X_T \in \mathfrak{X}(V)$ and one has:*

$$X_T(l) = l \circ T,$$

for any linear function $l : V \to \mathbb{R}$. You should check that:

$$[X_{T_1}, X_{T_2}] = -X_{[T_1,T_2]},$$

so the inclusion $\mathfrak{gl}(V) \hookrightarrow \mathfrak{X}(V)$ reverses the sign of the Lie brackets.

*A **representation of a Lie algebra** $\mathfrak{g}$ is a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. Composing ρ with the natural inclusion $\mathfrak{gl}(V) \hookrightarrow \mathfrak{X}(V)$, we obtain an anti-Lie algebra homomorphism $\psi : \mathfrak{g} \to \mathfrak{X}(V)$, whose image consists of linear vector fields. Conversely, every Lie algebra homomorphism $\psi : \mathfrak{g} \to \mathfrak{X}(V)$ whose image consists of linear vector fields arises from a Lie algebra representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$.*

We conclude that a representation $\widehat{\Psi} : G \to GL(V)$ is the same thing as a linear action of G. It yields by differentiation a Lie algebra representation

$\widehat{\Psi}_* : \mathfrak{g} \to \mathfrak{gl}(V)$ which is the same thing as an infinitesimal Lie algebra action $\psi : \mathfrak{g} \to \mathfrak{X}(V)$ by linear vector fields.

Conversely, since a linear vector field on a vector space is complete, any Lie algebra representation $\mathfrak{g} \to \mathfrak{gl}(V)$ integrates to a Lie group representation $G \to GL(V)$ of the 1-connected Lie group G with Lie algebra G.

Homework.

1. Prove Lemma 16.2.

2. Let $\Psi : G \times M \to M$ be a proper and free smooth action and denote by $B = G\backslash M$ its orbit space. Show that the projection $\pi : M \to B$ is locally trivial, i.e., for any $b \in B$ there exists a neighborhood $b \in U \subset B$ and diffeomorphism
$$\sigma : \pi^{-1}(U) \to G \times U, \quad q \mapsto (\chi(q), \pi(q)),$$
such that:
$$\sigma(g \cdot q) = (g\chi(q), \pi(q)), \quad \forall q \in \pi^{-1}(U), g \in G.$$

3. Show that the orbits of a smooth action are initial submanifolds.

4. Let G be a connected Lie group and $H \subset G$ a closed connected subgroup. Show that:
   (a) H is a normal subgroup of G if and only if its Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ is an **ideal**, i.e.,
   $$\forall X \in \mathfrak{g}, Y \in \mathfrak{h}, \quad [X,Y] \in \mathfrak{h}.$$
   (b) If H is normal in G, then G/H is a Lie group and $\pi : G \to G/H$ is a Lie group homomorphism.

5. Let G be a Lie group and let $H \subset G$ be a closed subgroup. Show that if G/H and H are both connected then G is connected. Conclude from this that the groups $SO(d)$, $SU(d)$ and $U(d)$ are all connected. Show that $O(d)$ and $GL(d)$ have two connected components.

6. Let $\Psi : G \times M \to M$ be a smooth transitive action with M connected. Show that:
   (a) The connected component of the identity G^0 also acts transitively on M;
   (b) For all $p \in M$, G/G^0 is diffeomorphic to $G_p/(G_p \cap G^0)$;
   (c) If G_p is connected for some $p \in M$, then G is connected.

7. For any Lie group G, recall that its **adjoint representation** $\mathrm{Ad} : G \to GL(\mathfrak{g})$, $g \mapsto \mathrm{Ad}_g$, is defined by $\mathrm{Ad}_g := \mathrm{d}_e i_g$, where $i_g : G \to G$ is given by $i_g(h) = ghg^{-1}$. Show that the induced Lie algebra representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is given by:
$$\mathrm{ad}_X(Y) = [X,Y], \quad \forall X, Y \in \mathfrak{g}.$$

8. Find the orbits and the isotropy groups for the adjoint representations of the 3 dimensional Lie groups $SL(2)$, $SO(3)$ and $SU(2)$.

9. For a vector space V of dimension d denote by $S_k(V)$ the set of all k-frames of V:

$$S_k(V) = \{(\mathbf{v}_1, \dots, \mathbf{v}_k) \in V \times \cdots \times V : \mathbf{v}_1, \dots, \mathbf{v}_k \text{ are linearly independent}\}.$$

Show that $S_k(V)$ is a homogenous space of dimension dk. $S_k(V)$ is called the **Stiefel manifold** of k-frames of V.
(HINT: Fix a base of V and consider the action $GL(d)$ in V by matrix multiplication.)

10. Give a proof of Lemma 16.9.
(HINT: If G is a Lie group with Lie algebra $\mathfrak{g}$, for each $X \in \mathfrak{g}$ denoted by $\overline{X} \in \mathfrak{X}(G)$ the *right invariant* vector field in G which takes the value X_e at the identity. Show that:

$$[\overline{X}, \overline{Y}] = -\overline{[X, Y]}, \quad \forall X, Y \in \mathfrak{g},$$

and express the infinitesimal action $\phi : \mathfrak{g} \to \mathfrak{X}(M)$ in terms of right invariant vector fields.)

11. Let $\Psi : G \times M \to M$ be a smooth action with associated infinitesimal action $\psi : \mathfrak{g} \to \mathfrak{X}(M)$. If G_p is the isotropy group at p, show that its Lie algebra is the **isotropy subalgebra**:

$$\mathfrak{g}_p = \{X \in \mathfrak{g} : \psi(X)_p = 0\}.$$

12. Let $\Psi : G \times M \to M$ be a smooth action with associated infinitesimal action $\psi : \mathfrak{g} \to \mathfrak{X}(M)$. We call $p_0 \in M$ a **fixed point of the action** if:

$$g \cdot p_0 = p_0, \forall g \in G.$$

Show that if p_0 is a fixed point of the action then:
(a) Ψ induces a representation $\Xi : G \to GL(T_{p_0}M)$;
(b) ψ induces a representation $\xi : \mathfrak{g} \to \mathfrak{gl}(T_{p_0}M)$;
(c) The representation Ξ of G integrates the representation ξ of $\mathfrak{g}$: $(\Xi)_* = \xi$.

Part 3. Differential Forms

Differential forms are the objects that can be integrated over a manifold. For this reason, they play a crucial role when passing from *local* to *global* aspects of manifolds. In this third part of the sections, we will introduce differential forms and we will see how effective they are in the study of global properties of manifolds.

The main concept and ideas that we will introduce in this round of sections are the following:

- In Section 17: the notion of *differential form* and, more generally, of tensor fields. The elementary operations with differential forms: *exterior product*, *inner product* and *pull-back*.
- In Section 18: the *differential* and the *Lie derivative* of differential forms, which give rise to the *Cartan calculus* on differential forms.
- In Section 19: the *integration* of differential forms on manifolds and *Stokes Theorem*.
- In Section 20: the *de Rham complex* formed by the differential forms and its *cohomology*, an important invariant of a differentiable manifold.
- In Section 21: the relationship between de Rham cohomology and *singular cohomology*, which shows that de Rham cohomology is a topological invariant.
- In Section 22: the basic properties of de Rham cohomology: *homotopy invariance* and the *Mayer-Vietoris sequence*.
- In Section 23: applications of the Mayer-Vietoris sequence to deduce further properties of cohomology like finite dimensionality and *Poincaré duality*. How to define and compute the *Euler characteristic* of a manifold.
- In Section 24: the *degree* of a map and the *index of a zero* of a vector field.

17. Differential Forms and Tensor Fields

For a finite dimensional vector space V, we denote the dual vector space by V^*:

$$V^* = \{\alpha : \alpha : V \to \mathbb{R} \text{ is a linear map}\}.$$

Its **tensor algebra** is:

$$\bigotimes V^* = \bigoplus_{k=0}^{+\infty} \otimes^k V^*,$$

and is furnished with the **tensor product** $\otimes : \otimes^k V^* \times \otimes^l V^* \to \otimes^{k+l} V^*$. Its **exterior algebra** is:

$$\bigwedge V^* = \bigoplus_{k=0}^{d} \wedge^k V^*$$

and is furnished with the **exterior product** $\wedge : \wedge^k V^* \times \wedge^l V^* \to \wedge^{k+l} V^*$. If $\alpha_1, \dots, \alpha_k \in V^*$ and $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$, our convention is that:

$$\alpha_1 \wedge \cdots \wedge \alpha_k(\mathbf{v}_1, \dots, \mathbf{v}_k) = \det(\alpha_i(\mathbf{v}_j))_{i,j=1}^k.$$

It maybe worth to recall that one can identify $\otimes^k V^*$ (respectively, $\wedge^k V^*$) with the space of k-multilinear (respectively, k-multilinear and alternating) maps $V \times \cdots \times V \to \mathbb{R}$.

If $T : V \to W$ is a linear transformation between two finite dimensional vector spaces, its transpose is the linear transformation $T^* : W^* \to V^*$ defined by:

$$T^*\alpha(\mathbf{v}) = \alpha(T\mathbf{v}).$$

Similarly, there exists an induced application $T^* : \wedge^k W^* \to \wedge^k V^*$ defined by:

$$T^*\omega(\mathbf{v}_1, \dots, \mathbf{v}_k) = \omega(T\mathbf{v}_1, \dots, T\mathbf{v}_k).$$

This is the restriction of a similarly defined map $T^* : \otimes^k W^* \to \otimes^k V^*$.

Let now M be a smooth manifold. If $(x^1, \dots, x^d)$ are local coordinates around $p \in M$, we know that the tangent vectors

$$\left.\frac{\partial}{\partial x^i}\right|_p \quad (i = 1, \dots, d),$$

form a base for T_pM. Similarly, the forms

$$\mathrm{d}_p x^i \quad (i = 1, \dots, d),$$

form a base for T_p^*M. These basis are dual to each other. If we take tensor products and exterior products of elements of these basis, we obtain basis for $\otimes^k T_pM$, $\wedge^k T_pM$, $\otimes^k T_p^*M$, $\wedge^k T_p^*M$, etc. For example, the space $\wedge^k T_p^*M$ has the base

$$\mathrm{d}_p x^{i_1} \wedge \cdots \wedge \mathrm{d}_p x^{i_k} \quad (i_1 < \cdots < i_k).$$

As in the case of the tangent and cotangent spaces, we are interested in the spaces $\otimes^k T_pM$, $\wedge^k T_pM$, $\otimes^k T_p^*M$, $\wedge^k T_p^*M$, etc., when p varies. For example, we define

$$\wedge^k T^*M := \bigcup_{p\in M} \wedge^k T_p^*M.$$

and we have a projection $\pi : \wedge^k T^*M \to M$. We call $\wedge^k T^*M$ the **k-exterior bundle** of M. We leave as an exercise to check that, just like the case of the tangent bundle, one has a smooth structure on this bundle.

Proposition 17.1. *There exists a canonical smooth structure on $\wedge^k T^*M$ such that the canonical projection in M is a submersion.*

One has also smooth structures on the bundles $\wedge^k TM$, $\otimes^k T^*M$, $\otimes^k TM$, $\otimes^k TM \otimes^s T^*M$, etc.

For any map $\pi : E \to M$ a **section** is a map $s : M \to E$ such that $\pi \circ s(p) = p$, for all $p \in M$.

Definition 17.2. *Let M be a manifold.*

*(i) A **differential form of degree** k is a section of $\pi : \wedge^k T^*M \to M$.*
*(ii) A **multivector field of degree** k is a section of $\pi : \wedge^k TM \to M$.*
*(iii) A **tensor field of degree** (k,s) is a section of $\pi : \otimes^k TM \otimes^s T^*M \to M$.*

We will consider only smooth differential forms, smooth multivector fields and smooth tensor fields, meaning that the corresponding sections are smooth maps. Note that $\wedge^k TM$ and $\wedge^k T^*M$ are submanifolds of $\otimes^k TM \otimes^s T^*M$, so a multivector field of degree k and a differential form of degree k are examples of tensor fields of degree $(k,0$ and $(0,k)$, respectively. Of course, there are tensor fields of degree $(k,0$ and $(0,k)$ which are not alternating: for example, a Riemannian metric is a tensor field of degree $(0,2)$ which is symmetric, rather than alternating.

If $(U,\phi) = (U,x^1,\dots,x^d)$ is a chart then a tensor field θ of degree (k,s) takes the local expression:

$$\theta|_U = \sum_{i_1,\dots,i_k,j_1,\dots,j_s} \theta^{i_1,\dots,i_k}_{j_1,\dots,j_s} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_k}.$$

It should be clear that θ is smooth if and only if for any open cover by charts the components $\theta^{i_1,\dots,i_k}_{j_1,\dots,j_s}$ are smooth function in $C^\infty(U)$.

On the other hand, a smooth differential form ω of degree k can be written in a local chart in the forms:

$$\begin{aligned}\omega|_U &= \sum_{i_1<\cdots<i_k} \omega_{i_1\cdots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\\ &= \sum_{i_1\cdots i_k} \frac{1}{k!} \omega_{i_1\cdots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k},\end{aligned}$$

where the components $\omega_{i_1\cdots i_k} \in C^\infty(U)$ are alternating: for every permutation $\sigma \in S_k$ one has

$$\omega_{\sigma(i_1)\cdots\sigma(i_k)} = (-1)^{\operatorname{sgn}\sigma}\omega_{i_1\cdots i_k}.$$

Similarly, a smooth multivector field π of degree k can be written in a local chart in the forms:

$$\begin{aligned}\pi|_U &= \sum_{i_1<\cdots<i_k} \pi^{i_1\cdots i_k}\frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_k}} \\ &= \sum_{i_1\cdots i_k} \frac{1}{k!}\pi^{i_1\cdots i_k}\frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_k}},\end{aligned}$$

where the components $\pi^{i_1\cdots i_k} \in C^\infty(U)$ are alternating.

If $(U, \phi) = (U, x^1, \ldots, x^d)$ and $(V, \psi) = (V, y^1, \ldots, y^d)$ are two local charts, then we have two local coordinate expressions for a differential form $\omega \in \Omega^k(M)$:

$$\begin{aligned}\omega|_U &= \sum_{i_1<\cdots<i_k} \omega_{i_1\cdots i_k}\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \\ \omega|_V &= \sum_{j_1<\cdots<j_k} \overline{\omega}_{j_1\cdots j_k}\mathrm{d}y^{j_1} \wedge \cdots \wedge \mathrm{d}y^{j_k}.\end{aligned}$$

If $U \cap V \neq \emptyset$, the transformation formulas that we obtain before;

$$\frac{\partial}{\partial x^i} = \sum_{j=1}^{d} \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^i}, \qquad \mathrm{d}x^i = \sum_{j=1}^{d} \frac{\partial x^i}{\partial y^j}\mathrm{d}y^j.$$

then lead to transformations for the components of the forms on the overlap $U \cap V$ of the two charts:

$$\overline{\omega}_{j_1\cdots j_k}(y) = \sum_{i_1<\cdots<i_k} \omega_{i_1\cdots i_k}(\phi \circ \psi^{-1}(y))\frac{\partial(x^{i_1}\cdots x^{i_k})}{\partial(y^{j_1}\cdots y^{j_k})}(y).$$

The symbol in the right side of this expression is an abbreviation for the minor consisting of the rows $i_1, \ldots, i_k$ and the columns $j_1, \ldots, j_k$ of the Jacobian matrix of the change of coordinates $\phi\circ\psi^{-1} : \psi(U\cap V) \to \phi(U\cap V)$.

We leave it as an exercise to determine the formulas of transformation of variables for multivector fields and tensor fields.

Remark 17.3. One maybe intrigued with the relative positions of the indices, as subscripts and superscripts, in the different objects. The convention that we follow is such that an index is only summed if it appears in a formula repeated both as a subscript and as a superscript. With this convention, one can even omit the summation sign from the formula, with the agreement that one sums over an index whenever that index is repeated. This convention is called the **Einstein convention sum**.

From now on we will concentrate on the study of differential forms. Although other objects, such as multivector fields and tensor fields, are also interesting, differential forms play a more fundamental role because they are the objects one can integrate over a manifold.

We will denote the vector space of smooth differential forms of degree k on a manifold M by $\Omega^k(M)$. Given a differential form $\omega \in \Omega^k(M)$ its value at a point $\omega_p \in \wedge^k T_p^* M$ can be seen as an alternating, multilinear map

$$\omega_p : T_pM \times \cdots \times T_pM \to \mathbb{R}.$$

Hence, if $X_1, \ldots, X_k \in \mathfrak{X}(M)$ are smooth vector fields M we obtain a smooth function $\omega(X_1, \ldots, X_k) \in C^\infty(M)$:

$$p \mapsto \omega_p(X_1|_p, \ldots, X_k|_p).$$

Therefore every differential form $\omega \in \Omega^k(M)$ can be seen as a map

$$\omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M).$$

This map is $C^\infty(M)$-multilinear and alternating. Conversely, every $C^\infty(M)$-multilinear, alternating, map $\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M)$ defines a smooth differential form. This is usually the simplest way to specify a smooth differential form.

We consider now several basic operations with differential forms.

Exterior product of differential forms. The exterior (or wedge) product $\wedge$ in the exterior algebra $\wedge T_p^* M$ induces an exterior (or wedge) product of differential forms

$$\wedge : \Omega^k(M) \times \Omega^s(M) \to \Omega^{k+s}(M),\ (\omega \wedge \eta)_p \equiv \omega_p \wedge \eta_p.$$

If we consider the space of all differential forms:

$$\Omega(M) = \bigoplus_{k=0}^{d} \Omega^k(M).$$

where we convention that $\Omega^0(M) = C^\infty(M)$ and $f\omega = f \wedge \omega$, the exterior product turns $\Omega(M)$ into a **Grassmann algebra** over the ring $C^\infty(M)$, i.e., the following properties hold:

(a) $(f\omega + g\eta) \wedge \theta = f\omega \wedge \theta + g\eta \wedge \theta$.
(b) $\omega \wedge \eta = (-1)^{\deg\omega \deg\eta} \eta \wedge \omega$.
(c) $(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta)$.

Moreover, if $\alpha_1, \ldots, \alpha_k \in \Omega^1(M)$ and $X_1, \ldots, X_k \in \mathfrak{X}(M)$, according to our conventions we have:

(d) $\alpha_1 \wedge \cdots \wedge \alpha_k(X_1, \ldots, X_k) = \det\left[\alpha_i(X_j)\right]_{i,j=1}^k$.

These 4 properties is all that we need to know to compute exterior products in local coordinates, as we illustrate in the next example:

EXAMPLE 17.4.
In $\mathbb{R}^4$, with coordinates (x, y, z, w), consider the differential forms of degree 2:

$$\omega = (x + w^2)\mathrm{d}x \wedge \mathrm{d}y + e^z \mathrm{d}x \wedge \mathrm{d}w + \cos x \mathrm{d}y \wedge \mathrm{d}z,$$
$$\eta = x\mathrm{d}y \wedge \mathrm{d}z - e^z \mathrm{d}z \wedge \mathrm{d}w.$$

Then:

$$\begin{aligned}\omega \wedge \eta &= -(x + w^2)e^z \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}w + xe^z \mathrm{d}x \wedge \mathrm{d}w \wedge \mathrm{d}y \wedge \mathrm{d}z \\ &= -w^2 e^z \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}w.\end{aligned}$$

Also, if we would like to compute, e.g., η on the vector fields $X = y\frac{\partial}{\partial z} - \frac{\partial}{\partial y}$ and $Y = e^z \frac{\partial}{\partial w}$ we proceed as follows:

$$\begin{aligned}\eta(X, Y) &= x\mathrm{d}y \wedge \mathrm{d}z(X, Y) - e^z \mathrm{d}z \wedge \mathrm{d}w(X, Y) \\ &= x \begin{vmatrix} \mathrm{d}y(X) & \mathrm{d}y(Y) \\ \mathrm{d}z(X) & \mathrm{d}z(Y) \end{vmatrix} - e^z \begin{vmatrix} \mathrm{d}z(X) & \mathrm{d}z(Y) \\ \mathrm{d}w(X) & \mathrm{d}w(Y) \end{vmatrix} \\ &= x \begin{vmatrix} -1 & 0 \\ y & 0 \end{vmatrix} - e^z \begin{vmatrix} y & 0 \\ 0 & e^z \end{vmatrix} = -ye^{2z}\end{aligned}$$

Pull-back of differential forms. Let $\Phi : M \to N$ be a smooth map. For each $p \in M$, the transpose of the differential

$$(\mathrm{d}_p\Phi)^* : T^*_{\Phi(p)}N \to T^*_p M.$$

induces a linear map

$$(\mathrm{d}_p\Phi)^* : \wedge^k T^*_{\Phi(p)}N \to \wedge^k T^*_p M.$$

The **pull-back of differential forms** $\Phi^* : \Omega^k(N) \to \Omega^k(M)$ is defined as:

$$\begin{aligned}(\Phi^*\omega)(X_1, \dots, X_k)_p &= ((\mathrm{d}_p\Phi)^*\omega)(X_1|_p, \dots, X_k|_p) \\ &= \omega_{\Phi(p)}(\mathrm{d}_p\Phi \cdot X_1|_p, \dots, \mathrm{d}_p\Phi \cdot X_k|_p).\end{aligned}$$

This defines a $C^\infty(M)$-multilinear, alternating, map $\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M)$, hence $\Phi^*\omega$ is a smooth differential form of degree k in M.

It is easy to check that for any smooth map $\Phi : M \to N$, the pull-back $\Phi^* : \Omega(N) \to \Omega(M)$ is a homomorphism of Grassmann algebras, i.e., the following properties hold:

(a) $\Phi^*(a\omega + b\eta) = a\Phi^*\omega + b\Phi^*\eta$, $a, b \in \mathbb{R}$;
(b) $\Phi^*(\omega \wedge \eta) = \Phi^*\omega \wedge \Phi^*\eta$;
(c) $\Phi^*(f\omega) = (f \circ \Phi)\Phi^*\omega$, $f \in C^\infty(M)$;

Note that if $f : N \to \mathbb{R}$ is a smooth function then the differential $\mathrm{d}f$ can be viewed as a differential form of degree 1. The chain rule and the definition above also gives:

(d) $\Phi^*(\mathrm{d}f) = \mathrm{d}(f \circ \Phi)$.

These properties is all that it is needed to compute pull-backs in local coordinates, as we illustrate in the next example:

EXAMPLE 17.5.
Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^4$ be the smooth map:

$$\Phi(u,v) = (u+v, u-v, v^2, \frac{1}{1+u^2}).$$

In order to compute the pull-back under Φ of the form:

$$\eta = x\mathrm{d}y \wedge \mathrm{d}z - e^z \mathrm{d}z \wedge \mathrm{d}w \in \Omega^2(\mathbb{R}^4),$$

we proceed as follows:

$$\begin{aligned}
\Phi^*\eta &= (x \circ \Phi)\mathrm{d}(y \circ \Phi) \wedge \mathrm{d}(z \circ \Phi) - e^{(z\circ\Phi)}\mathrm{d}(z \circ \Phi) \wedge \mathrm{d}(w \circ \Phi) \\
&= (u+v)\mathrm{d}(u-v) \wedge \mathrm{d}(v^2) - e^{v^2}\mathrm{d}(v^2) \wedge \mathrm{d}(\frac{1}{1+u^2}) \\
&= (u+v)\mathrm{d}u \wedge 2v\mathrm{d}v - 2ve^{v^2}\mathrm{d}v \wedge \frac{-2u\mathrm{d}u}{(1+u^2)^2} \\
&= \left(2v(u+v) - \frac{4uve^{v^2}}{(1+u^2)^2}\right)\mathrm{d}u \wedge \mathrm{d}v.
\end{aligned}$$

In other words, to compute the pull-back $\Phi^*\eta$, one replaces in η, the coordinates (x,y,z,w) by its expressions in terms of the coordinates (u,v).

Remark 17.6. When (N,i) is a submanifold of M the pull-back of a differential form $\omega \in \Omega^k(M)$ by the inclusion map $i : N \hookrightarrow M$ is called the **restriction of the differential form** ω to N. Often one denotes the restriction $\omega|_N$ instead of $i^*\omega$.

For example, for the sphere

$$\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\},$$

we can write

$$\omega = (x\mathrm{d}y \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y)|_{\mathbb{S}^2},$$

meaning that ω is the pull-back by the inclusion $i : \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ of the differential form $x\mathrm{d}y \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y \in \Omega^2(\mathbb{R}^3)$. Sometimes, one even drops the restriction sign.

One should also notice that if $\Phi : M \to N$ and $\Psi : N \to Q$ are smooth maps, then $\Psi \circ \Phi : M \to Q$ is a smooth map and we have:

$$(\Psi \circ \Phi)^*\omega = \Phi^*(\Psi^*\omega).$$

In categorical language, we have a *contravariant functor* from the category of smooth manifolds to the category of Grassmann algebras, which to a smooth manifold M associates the algebra $\Omega(M)$ and to a smooth map $\Phi : M \to N$ associates a homomorphism $\Phi^* : \Omega(N) \to \Omega(M)$.

Interior Product. Given a vector field $X \in \mathfrak{X}(M)$ and a differential form $\omega \in \Omega^k(M)$, the **interior product** of ω by X, denoted $i_X\omega \in \Omega^{k-1}(M)$, is the the differential form of degree $(k-1)$ defined by:

$$i_X\omega(X_1, \dots, X_{k-1}) = \omega(X, X_1, \dots, X_{k-1}).$$

Since $i_X\omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M)$ is a $C^\infty(M)$-multilinear, alternating, map, it is indeed a smooth differential form of degree $k-1$.

It is easy to check that the following properties hold:

(a) $i_X(f\omega + g\theta) = f i_X\omega + g i_X\theta$;
(b) $i_X(\omega \wedge \theta) = (i_X\omega) \wedge \theta + (-1)^{\deg\omega}\omega \wedge (i_X\theta)$;
(c) $i_{(fX+gY)}\omega = f i_X\omega + g i_Y\omega$;
(d) $i_X(\mathrm{d}f) = X(f)$.

Again, these properties is all that it is needed to compute interior products in local coordinates.

Example 17.7.
Let $\omega = e^x \mathrm{d}x \wedge \mathrm{d}y + e^z \mathrm{d}y \wedge \mathrm{d}z \in \Omega^2(\mathbb{R}^3)$, and $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{R}^3)$. Then:

$$\begin{aligned}
i_{\frac{\partial}{\partial x}}(\mathrm{d}x \wedge \mathrm{d}y) &= (i_{\frac{\partial}{\partial x}}\mathrm{d}x) \wedge \mathrm{d}y - \mathrm{d}x \wedge (i_{\frac{\partial}{\partial y}}\mathrm{d}y) = \mathrm{d}y,\\
i_{\frac{\partial}{\partial y}}(\mathrm{d}x \wedge \mathrm{d}y) &= (i_{\frac{\partial}{\partial y}}\mathrm{d}x) \wedge \mathrm{d}y - \mathrm{d}x \wedge (i_{\frac{\partial}{\partial y}}\mathrm{d}y) = -\mathrm{d}x,\\
i_{\frac{\partial}{\partial x}}(\mathrm{d}y \wedge \mathrm{d}z) &= (i_{\frac{\partial}{\partial x}}\mathrm{d}y) \wedge \mathrm{d}z - \mathrm{d}y \wedge (i_{\frac{\partial}{\partial x}}\mathrm{d}z) = 0,\\
i_{\frac{\partial}{\partial y}}(\mathrm{d}y \wedge \mathrm{d}z) &= (i_{\frac{\partial}{\partial y}}\mathrm{d}y) \wedge \mathrm{d}z - \mathrm{d}y \wedge (i_{\frac{\partial}{\partial y}}\mathrm{d}z) = \mathrm{d}z.
\end{aligned}$$

Hence, we conclude that:

$$i_X\omega = -xe^x\mathrm{d}x - ye^x\mathrm{d}y + xe^z\mathrm{d}z.$$

Remark 17.8. One can extend the interior product in a more or less obvious way to other objects (multivector fields, tensor fields, etc.). For these objects it is frequent to use the designation **contraction**, instead of interior product. For example, one can define the contraction of a differential form ω of degree k by a multivector field π of degree $l < k$, to be a differential form $i_\pi\omega$ of degree $k-l$. In a local chart $(U, x^1, \dots, x^d)$, if

$$\omega|_U = \sum_{i_1\cdots i_k} \omega_{i_1\cdots i_k}\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}, \quad \pi|_U = \sum_{j_1\cdots j_l} \pi^{j_1\cdots j_l}\frac{\partial}{\partial x^{j_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{j_l}},$$

then:

$$(i_\pi\omega)|_U = \sum_{i_1\cdots i_k} \omega_{i_1\cdots i_k}\pi^{i_1\cdots i_l}\mathrm{d}x^{i_{l+1}} \wedge \cdots \wedge \mathrm{d}x^{i_k}.$$

As a first application of differential forms, we are going to formalize the notion of *orientation* of a manifold.

Recall that if V is a linear vector space of dimension d and $\mu \in \wedge^d(V^*)$ is a non-zero element, then for any base $\{\mathbf{v}_1, \dots, \mathbf{v}_d\}$ of V we have

$$\mu(\mathbf{v}_1, \dots, \mathbf{v}_d) \neq 0.$$

This implies that μ splits the ordered basis of V into two classes: a base $\{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$ has positive (respectively, negative) μ-orientation if this number is positive (respectively, negative). Hence, μ determines a orientation for V.

EXAMPLE 17.9.
Let $V = \mathbb{R}^d$ then we have a canonical element $\mu_0 \in \wedge^d(\mathbb{R}^d)^*$, namely the determinant:

$$\mu_0(\mathbf{v}_1, \ldots, \mathbf{v}_d) = \det[\mathbf{v}_i^j]_{i,j=1}^n.$$

The standard basis of $\mathbb{R}^d$ is positively oriented for this canonical choice. Note also that $|\mu_0(\mathbf{v}_1, \ldots, \mathbf{v}_d)|$ represents the usual volume of the parallelepiped span by the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_d$. For an arbitrary vector space V there is no such canonical choice of orientation and one needs to choose an element $\mu \in \wedge^d(V^*)$ to orient its bases.

Definition 17.10. *For a smooth manifold M of dimension d, we call a differential form $\mu \in \Omega^d(M)$ a **volume form** if $\mu_p \neq 0$, for all $p \in M$. A manifold M is said to be **orientable** if it admits a volume form.*

Notice that if $\mu \in \Omega^d(M)$ is a volume form then any other differential form of degree d in M is of the form $f\mu$ for a smooth function $f \in C^\infty(M)$. In particular, if $\mu_1, \mu_2 \in \Omega^d(M)$ are two volume forms then there exists a unique smooth *non-vanishing* function $f \in C^\infty(M)$ such that $\mu_2 = f\mu_1$.

Let M be an orientable manifold of dimension d. If $\mu_1, \mu_2 \in \Omega^d(M)$ are volumes forms we say that μ_1 and μ_2 define the same orientation if for all $p \in M$ and any ordered base $\{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$ of T_pM, one has:

$$\mu_1(\mathbf{v}_1, \ldots, \mathbf{v}_d)\mu_2(\mathbf{v}_1, \ldots, \mathbf{v}_d) > 0.$$

Note that if μ_1 and μ_2 define the same orientation, then a base is μ_1-positive if and only if it is μ_2-positive. We leave the proof of the following lemma as an exercise:

Lemma 17.11. *Let M be manifold of dimension d. Two volume forms $\mu_1, \mu_2 \in \Omega^d(M)$ define the same orientation if and only if $\mu_2 = f\mu_1$ for a smooth everywhere positive function $f \in C^\infty(M)$.*

The property "define the same orientation" is an equivalence relation on the set of volume forms in an orientable manifold M.

Definition 17.12. *An **orientation** for an orientable manifold M is a choice of an equivalence class $[\mu]$. A pair $(M, [\mu])$ is called an **oriented manifold**.*

Notice that an orientation $[\mu]$ for a manifold M (if it exists!) amounts to a choice of orientation for each tangent space T_pM varying smoothly with p. Note also that a connected orientable manifold has two orientations. More generally, an orientable manifold with k connected components has 2^k orientations.

EXAMPLES 17.13.

1. *The euclidean space* $\mathbb{R}^d$ *is orientable. The canonical orientation of* $\mathbb{R}^d$ *is the orientation defined by the volume form* $\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^d$*. For this canonical orientation, the canonical base of* $T_p\mathbb{R}^d \simeq \mathbb{R}^d$ *has positive orientation.*
2. *A Lie group* G *is always orientable. If* $\{\alpha_1, \ldots, \alpha_d\}$ *is a base of left invariant 1-forms then* $\mu = \alpha_1 \wedge \cdots \wedge \alpha_d$ *a left invariant volume form.*
3. *The sphere* $\mathbb{S}^d$ *is an orientable manifold. A volume form is given by:*
$$\omega = \sum_{i=1}^{d+1} (-1)^i x^i \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^{d+1} \Big|_{\mathbb{S}^d}.$$
We leave it as an exercise to check that this form never vanishes.
4. *The real projective space* $\mathbb{RP}^2$ *is not orientable. To see this let* $\mu \in \Omega^2(\mathbb{RP}^2)$ *be any differential 2-form. If* $\pi : \mathbb{S}^2 \to \mathbb{RP}^2$ *is the quotient map, then the pull-back* $\pi^*\mu$ *is a differential 2-form in* $\mathbb{S}^2$*. It follows from the previous example that*
$$\pi^*\mu = f\omega,$$
for some smooth function $f \in C^\infty(\mathbb{S}^2)$*.*
Let $\Phi : \mathbb{S}^2 \to \mathbb{S}^2$ *be the anti-podal map:* $p \mapsto -p$*. Since* $\pi \circ \Phi = \pi$*, we have:*
$$\Phi^*(\pi^*\mu) = (\pi \circ \Phi)^*\mu = \pi^*\mu.$$
On the other, it is easy to check that $\Phi^*\omega = -\omega$*. Hence:*
$$\begin{aligned} f\omega = \pi^*\mu &= \Phi^*(\pi^*\mu) \\ &= \Phi^*(f\omega) = (f \circ \Phi)\Phi^*(\omega) = -(f \circ \Phi)\omega. \end{aligned}$$
We conclude that $f(-p) = -f(p)$*, for all* $p \in \mathbb{S}^2$*. But then we must have* $f(p_0) = 0$*, at some* $p_0 \in \mathbb{S}^2$*. Hence,* $\pi^*\mu$ *vanishes at some point. Since* π *is a local diffeomorphism, we conclude that every differential form* $\mu \in \Omega^2(\mathbb{RP}^2)$ *vanishes at some point, so* $\mathbb{RP}^2$ *has no volume forms, and it is non-orientable.*

Let $(M, [\mu_M])$ and $(N, [\mu_N])$ be oriented manifolds. We say that a diffeomorphism $\Phi : M \to N$ **preserves orientations** or that it is **positive**, if $[\Phi^*\mu_N] = [\mu_M]$.

EXAMPLE 17.14.
Let $[\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^d]$ *be the standard orientation for* $\mathbb{R}^d$*. Given a diffeomorphism* $\phi : U \to V$*, where* U, V *are open sets in* $\mathbb{R}^d$*, we have:*
$$\phi^*(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^d) = \det[\phi'(x)]\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^d.$$
Hence ϕ *preserves the standard orientation if and only if* $\det[\phi'(x)] > 0$*, for all* $x \in \mathbb{R}^d$*.*

One can also express the possibility of orienting a manifold in terms of an atlas, as shown by the following proposition.

Proposition 17.15. *Let M be a manifold of dimension d. The following statements are equivalents:*

(i) M is orientable, i.e., M has a volume form.

(ii) There exists an atlas $\{(U_i, \phi_i)\}_{i\in I}$ for M such that for all $i, j \in I$ the transition functions preserve the standard orientation of $\mathbb{R}^d$.

In particular, if $[\mu_M]$ is an orientation for M, then there exists an atlas $\{(U_i, \phi_i)\}_{i\in I}$ for M such that each chart $\phi_i : U_i \to \mathbb{R}^d$ is positive, where in $\mathbb{R}^d$ we consider the canonical orientation.

The proof is left as an exercise.

Homework.

1. Construct the natural differentiable structure on $\wedge^k T^*M$, for which the canonical projection $\pi : \wedge^k T^*M \to M$ is a submersion.

2. Determine the formulas of transformation of variables for multivector fields and tensor fields.

3. Show that a Riemannian structure on a manifold M (see Exercise 8 in Section 10) defines a symmetric tensor field of degree (0,2).
Note: In a chart (U, x^i), a symmetric tensor field of degree (0,2) is written as
$$g|_U = \sum_{i,j} g_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j,$$
where the components $g_{ij} \in C^\infty(U)$ satisfy $g_{ij} = g_{ji}$.

4. Prove the basic properties of the pull-back and interior product of differential forms.

5. Let $\Phi : M \to N$ be a smooth map and let $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ be Φ-related smooth vector fields. Show that
$$\Phi^*(i_Y \omega) = i_X \Phi^* \omega, \quad \forall \omega \in \Omega(N).$$

6. Prove Proposition 17.15.

7. Show that for any orientable manifolds M and N the product $M \times N$ is orientable. Conclude that the torus $\mathbb{T}^d$ is orientable. Give an example of a volume form in $\mathbb{T}^d$.

8. Show that the real projective space $\mathbb{RP}^d$ is orientable if and only if d is odd.

9. Verify that the Klein bottle (see Example 7.8.4) is a non-orientable manifold.

10. Show that every oriented manifold $(M, [\mu])$ has an atlas whose transition functions preserve the standard orientation of $\mathbb{R}^d$.
Hint: If $(U, \phi) = (U, x^1, \dots, x^d)$ is a negative chart, then
$$(U, \bar{\phi}) := (U, -x^1, x^2, \dots, x^d)$$
is a positive chart.

11. Let (M, g) be a Riemannian manifold of dimension d. Show that:

(a) For each $p \in M$, the map $T_pM \to T_p^*M$, $v \mapsto g(v, \cdot)$, is an isomorphism, so the inner product on the tangent space T_pM induces an inner product on the cotangent space T_p^*M.

(b) For each $p \in M$, there exists a neighborhood U of p and orthonormal smooth vector fields $X_1, \dots, X_d \in \mathfrak{X}(U)$:

$$\langle X_i, X_j \rangle = \delta_{ij} \text{ (Kronecker symbol).}$$

The set $\{X_1, \dots, X_d\}$ is called a **(local) orthonormal frame**.

(c) For each $p \in M$, there exists a neighborhood U of p and orthonormal differential forms $\alpha_1, \dots, \alpha_d \in \Omega^1(U)$:

$$\langle \alpha_i, \alpha_j \rangle = \delta_{ij} \text{ (Kronecker symbol).}$$

The set $\{\alpha_1, \dots, \alpha_d\}$ is called a **(local) orthonormal coframe**.

(d) Assume further that $(M, [\mu])$ is oriented. Show that there exists a volume form $\mu_0 \in \Omega^d(M)$ such that:

$$\mu_0|_U = \alpha_1 \wedge \cdots \wedge \alpha_d,$$

for every local orthonormal coframe $\alpha_1, \dots, \alpha_d \in \Omega^1(U)$ which is positive (i.e., $\alpha_1 \wedge \cdots \wedge \alpha_d$ is positive). One call μ_0 the **canonical volume form of the oriented Riemannian manifold** $(M, g, [\mu])$.

12. Let $(M, g, [\mu])$ be an oriented Riemannian manifold of dimension d. Show that there exists a unique linear map $* : \Omega^k(M) \to \Omega^{d-k}(M)$ such that for every local orthonormal coframe $\alpha_1, \dots, \alpha_d$ which is positive (i.e., $\alpha_1 \wedge \cdots \wedge \alpha_d$ is positive) the following properties hold:

(a) $*1 = \alpha_1 \wedge \cdots \wedge \alpha_d$ and $*(\alpha_1 \wedge \cdots \wedge \alpha_d) = 1$;

(b) $*(\alpha_1 \wedge \cdots \wedge \alpha_k) = \alpha_{k+1} \wedge \cdots \wedge \alpha_d$.

Show also that:

$$* * \omega = (-1)^{k(d-k)}\omega, \text{ where } k = \deg \omega.$$

$*$ is called the **Hodge star operator**.

18. Differential and Cartan Calculus

We will introduce now two important differentiation operations on differential forms: the *differential* of forms, which is an intrinsic derivative, and the *Lie derivative* of differential forms, which is a derivative along vector fields. These differential operations together with the algebraic operations on differential forms that we studied in the previous section, are the basis of a calculus on differential forms on which is usually called *Cartan Calculus*.

Let $\omega \in \Omega^k(M)$. The **differential** of ω is the differential form of degree $k+1$, denoted $\mathrm{d}\omega$, defined by:

$$\begin{aligned}(18.1)\quad \mathrm{d}\omega(X_0,\ldots,X_k) = &\sum_{i=0}^{k}(-1)^i X_i(\omega(X_0,\ldots,\widehat{X}_i,\ldots,X_k))+\\ &+\sum_{0\le i<j\le k}(-1)^{i+j}\omega([X_i,X_j],X_0,\ldots,\widehat{X}_i,\ldots,\widehat{X}_j\ldots,X_k),\end{aligned}$$

for any smooth vector fields $X_0,\ldots,X_k \in \mathfrak{X}(M)$. This formula defines a $C^\infty(M)$-multilinear, alternating, map $\mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)\to C^\infty(M)$, so that $\mathrm{d}\omega$ is indeed a smooth differential (k+1)-form.

A smooth function $f \in \mathcal{C}^\infty(M)$ is a degree 0 form. In this case, formula (18.1) gives:

$$\mathrm{d}f(X) = X(f).$$

Therefore this definition matches our previous definition of the differential of a smooth function.

Our next result shows that the differential is the only operation on the forms which extends the differential of functions in a reasonable way:

Theorem 18.1. *The differential*

$$\mathrm{d}:\Omega^\bullet(M)\to\Omega^{\bullet+1}(M)$$

is the only operation on forms satisfying the following properties:

(i) d *is* $\mathbb{R}$*-linear:*

$$\mathrm{d}(a\omega+b\theta) = a\mathrm{d}\omega + b\mathrm{d}\theta.$$

(ii) d *is a derivation:*

$$\mathrm{d}(\omega\wedge\theta) = (\mathrm{d}\omega)\wedge\theta + (-1)^{\deg\omega}\omega\wedge(\mathrm{d}\theta).$$

(iii) d *extends the differential of smooth functions: if* $f\in C^\infty(M)$*, then*

$$\mathrm{d}f(X) = X(f), \forall X\in\mathfrak{X}(M).$$

(iv) $\mathrm{d}^2 = 0$.

Moreover, if $\Phi: M\to N$ *is a smooth map, then for every* $\omega\in\Omega^k(N)$*:*

$$\Phi^*\mathrm{d}\omega = \mathrm{d}\Phi^*\omega.$$

Proof. We leave it for the exercises to check that d, as defined by (18.1), satisfies properties (i) through (iv). To prove uniqueness, we need to check that given $\omega\in\Omega^k(M)$, then $\mathrm{d}\omega$ is determined by properties (i)–(iv).

Since d is a derivation, it is local: if $\omega|_U = 0$ on an open set U then $(\mathrm{d}\omega)|_U = 0$. In fact, let $p\in U$ and $f\in C^\infty(M)$ with $f(p)>0$ and $\operatorname{supp} f\subset U$. Since $f\omega\equiv 0$, we find that:

$$0 = \mathrm{d}(f\omega) = \mathrm{d}f\wedge\omega + f\mathrm{d}\omega.$$

If we evaluate both sides of this identity at p, we conclude that $f(p)(\mathrm{d}\omega)_p = 0$. Hence $\mathrm{d}\omega|_U = 0$, as claimed.

Therefore, to prove uniqueness, it is enough to consider the case where $\omega \in \Omega^k(U)$, where U is the domain of some local chart $(x^1, \dots, x^d)$. In this case we have:

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1 \cdots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}.$$

Using only properties (i)–(iv) we find:

$$\begin{aligned}
\mathrm{d}\omega &= \sum_{i_1 < \cdots < i_k} \mathrm{d}(\omega_{i_1 \cdots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}) && \text{(by (i))} \\
&= \sum_{i_1 < \cdots < i_k} \mathrm{d}(\omega_{i_1 \cdots i_k}) \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} && \text{(by (ii) and (iv))} \\
&= \sum_{i_1 < \cdots < i_k} \sum_i \frac{\partial \omega_{i_1 \cdots i_k}}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} && \text{(by (iii))}.
\end{aligned}$$

The last expression defines a differential form of degree $k+1$ in U. Hence, $\mathrm{d}\omega$ is completely determine by properties (i)–(iv), as claimed.

The proof that the differential commutes with pull-backs also follows if one proves it for every local chart. We leave the (easy) computation to the exercises. □

As this proof shows, one can compute the differential of a form in local coordinates using only properties (i)–(iv). This is often much more efficient than applying directly the formula (18.1).

Example 18.2.
Let $\omega = e^y \mathrm{d}x \wedge \mathrm{d}z + e^z \mathrm{d}y \wedge \mathrm{d}z \in \Omega^2(\mathbb{R}^3)$. Using properties (i)–(iv), we find:

$$\begin{aligned}
\mathrm{d}\omega &= \mathrm{d}(e^y \mathrm{d}x \wedge \mathrm{d}z + e^z \mathrm{d}y \wedge \mathrm{d}z) \\
&= (\mathrm{d}e^y) \wedge \mathrm{d}x \wedge \mathrm{d}z + \mathrm{d}(e^z) \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&= e^y \mathrm{d}y \wedge \mathrm{d}x \wedge \mathrm{d}z + e^z \mathrm{d}z \wedge \mathrm{d}y \wedge \mathrm{d}z = -e^y \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.
\end{aligned}$$

The operation $\mathrm{d} : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$ is also referred to as *exterior differentiation*, since it increases the degree of a form. There is another type of differentiation of a form which preserves the degree:

Definition 18.3. *The* ***Lie derivative*** *of a differential form $\omega \in \Omega^k(M)$ along a vector $X \in \mathfrak{X}(M)$ is the differential form $\mathcal{L}_X \omega \in \Omega^k(M)$ defined by:*

$$\mathcal{L}_X \omega = \frac{\mathrm{d}}{\mathrm{d}t} (\phi_X^t)^* \omega \Big|_{t=0} = \lim_{t \to 0} \frac{1}{t} \left((\phi_X^t)^* \omega - \omega \right).$$

Example 18.4.
Let $\omega = e^y \mathrm{d}x \wedge \mathrm{d}z + e^z \mathrm{d}y \wedge \mathrm{d}z \in \Omega^2(\mathbb{R}^3)$ and $X = x \frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^3)$. The flow of X is given by $\phi_X^t(x, y, z) = (x, y + tx, z)$. Hence, we find that:

$$\begin{aligned}
(\phi_X^t)^* \omega &= e^{y+tx} \mathrm{d}x \wedge \mathrm{d}z + e^z \mathrm{d}(y + tx) \wedge \mathrm{d}z \\
&= e^{y+tx} \mathrm{d}x \wedge \mathrm{d}z + e^z \mathrm{d}y \wedge \mathrm{d}z + te^z \mathrm{d}x \wedge \mathrm{d}z
\end{aligned}$$

Then:

$$\begin{aligned}\mathcal{L}_X\omega &= \frac{\mathrm{d}}{\mathrm{d}t}(\phi_X^t)^*\omega\Big|_{t=0}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big(e^{y+tx}\mathrm{d}x\wedge\mathrm{d}z + e^z\mathrm{d}y\wedge\mathrm{d}z + te^z\mathrm{d}x\wedge\mathrm{d}z\big)\\ &= xe^y\mathrm{d}x\wedge\mathrm{d}z + e^z\mathrm{d}x\wedge\mathrm{d}z.\end{aligned}$$

In most examples, it is impossible to find explicitly the flow of a vector field. Still the basic properties of the Lie derivative listed in the next proposition allow one to find the Lie derivative without knowledge of the flow. The proof is left as an exercise:

Proposition 18.5. *Let $X \in \mathfrak{X}(M)$ and $\omega, \eta \in \Omega^\bullet(M)$. Then:*

(i) $\mathcal{L}_X(a\omega + b\eta) = a\mathcal{L}_X\omega + b\mathcal{L}_X\eta$ *for all* $a, b \in \mathbb{R}$.
(ii) $\mathcal{L}_X(\omega\wedge\eta) = \mathcal{L}_X\omega\wedge\eta + \omega\wedge\mathcal{L}_X\eta$.
(iii) $\mathcal{L}_X(f) = X(f)$, *if* $f \in \Omega^0(M) = \mathcal{C}^\infty(M)$.
(iv) $\mathcal{L}_X\mathrm{d}\omega = \mathrm{d}\mathcal{L}_X\omega$.

Example 18.6.
Let us redo Example 18.4 using only properties (i)-(iv) in the previous proposition:

$$\begin{aligned}\mathcal{L}_X\omega &= \mathcal{L}_X(e^y\mathrm{d}x\wedge\mathrm{d}z + e^z\mathrm{d}y\wedge\mathrm{d}z)\\ &= \mathcal{L}_X(e^y)\mathrm{d}x\wedge\mathrm{d}z + e^y\mathcal{L}_X(\mathrm{d}x)\wedge\mathrm{d}z + e^y\mathrm{d}x\wedge\mathcal{L}_X(\mathrm{d}z) +\\ &\quad + \mathcal{L}_X(e^z)\mathrm{d}y\wedge\mathrm{d}z + e^z\mathcal{L}_X(\mathrm{d}y)\wedge\mathrm{d}z + e^z\mathrm{d}y\wedge\mathcal{L}_X(\mathrm{d}z)\\ &= X(e^y)\mathrm{d}x\wedge\mathrm{d}z + e^z\mathrm{d}X(y)\wedge\mathrm{d}z\\ &= xe^y\mathrm{d}x\wedge\mathrm{d}z + e^z\mathrm{d}x\wedge\mathrm{d}z.\end{aligned}$$

There is still another efficient way to compute the Lie derivative, by applying a formula which relates all three basic operations on forms: Lie derivative, exterior differential and interior product. This "magic" formula often plays an unexpected role.

Theorem 18.7 (Cartan's Magic Formula)**.** *Let $X \in \mathfrak{X}(M)$ and $\omega \in \Omega(M)$. Then:*

$$\mathcal{L}_X\omega = i_X\mathrm{d}\omega + \mathrm{d}i_X\omega. \tag{18.2}$$

Proof. By Proposition 18.5, $\mathcal{L}_X : \Omega(M) \to \Omega(M)$ is a derivation. The properties of d and i_X give that $i_X\mathrm{d} + \mathrm{d}i_X : \Omega(M) \to \Omega(M)$ is also a derivation. Hence, it is enough to check that both derivations take the same values on differential forms of the type $\omega = f$ and $\omega = \mathrm{d}g$, where $f, g \in C^\infty(M)$.

On the one hand, the properties in Proposition 18.5, give:

$$\mathcal{L}_X(f) = X(f), \quad \mathcal{L}_X(\mathrm{d}g) = \mathrm{d}\mathcal{L}_X g = \mathrm{d}(X(g)).$$

On the other hand, the properties of d and i_X yield:

$$(i_X\mathrm{d} + \mathrm{d}i_X)f = i_X\mathrm{d}f = X(f),$$
$$(i_X\mathrm{d} + \mathrm{d}i_X)\mathrm{d}g = \mathrm{d}(i_X\mathrm{d}g) = \mathrm{d}(X(g)).$$

□

Example 18.8.
Let us redo Example 18.4 using Cartan's Magic Formula:

$$\begin{aligned}\mathcal{L}_X\omega &= i_X\mathrm{d}\omega + \mathrm{d}i_X\omega \\ &= i_X(-e^y\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z) + \mathrm{d}(xe^z\mathrm{d}z) \\ &= xe^y\mathrm{d}x \wedge \mathrm{d}z + e^z\mathrm{d}x \wedge \mathrm{d}z.\end{aligned}$$

Homework.

1. Show that d defined by formula (18.1), satisfies properties (i)–(iv) in Theorem 18.1.

2. Let $\Phi : M \to N$ be a smooth map. Show that for any form $\omega \in \Omega^k(M)$:

$$\Phi^*\mathrm{d}\omega = \mathrm{d}\Phi^*\omega.$$

3. Let $I \subset \Omega(M)$ be an ideal generated by k linearly independent differential forms $\alpha_1, \dots, \alpha_k \in \Omega^1(M)$ (i.e., such that $\{\alpha_1|_p, \dots, \alpha_k|_p\}$ is a linearly independent set for every $p \in M$). Show that the following statements are equivalent:
   (a) I is a differential ideal, i.e., if $\alpha \in I$ then $\mathrm{d}\alpha \in I$;
   (b) $\mathrm{d}\alpha_i = \sum_j \omega_{ij} \wedge \alpha_j$, for some 1-forms $\omega_{ij} \in \Omega^1(M)$;
   (c) If $\omega = \alpha_1 \wedge \cdots \wedge \alpha_k$, then $\mathrm{d}\omega = \alpha \wedge \omega$, for some 1-form $\alpha \in \Omega^1(M)$.
   (d) The distribution $D = \bigcap_{i=1}^k \ker \alpha_i$ is involutive.

4. Prove the properties of the Lie derivative given in Proposition 18.5.

5. Let $X, Y \in \mathfrak{X}(M)$ be vector fields and $\omega \in \Omega(M)$ a differential form. Show that:

$$\mathcal{L}_{[X,Y]}\omega = \mathcal{L}_X(\mathcal{L}_Y\omega) - \mathcal{L}_Y(\mathcal{L}_X\omega).$$

6. Let $\Phi : M \to N$ be smooth. Show that if $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are Φ-related vector fields, then

$$\Phi^*(\mathcal{L}_Y\omega) = \mathcal{L}_X(\Phi^*\omega),$$

for every differential form $\omega \in \Omega(N)$.

7. Let $X \in \mathfrak{X}(M)$ and $\omega \in \Omega^k(M)$. Show that:

$$\mathcal{L}_X(\omega(X_1, \dots, X_k)) = \mathcal{L}_X\omega(X_1, \dots, X_k) + \sum_{i=1}^k \omega(X_1, \dots, \mathcal{L}_X X_i, \dots, X_k). \tag{18.3}$$

8. Let M be a manifold equipped with a volume form μ. Given a vector field X, the **divergence** of X is the unique function $\operatorname{div}_\mu(X) \in C^\infty(M)$ that satisfies:

$$\mathcal{L}_X \mu = \operatorname{div}_\mu(X)\mu.$$

Show that:

(a) a complete vector field $X \in \mathfrak{X}(M)$ is divergence free (i.e., $\operatorname{div}_\mu(X) = 0$) if and only the flow of X preserves the volume form μ, i.e., if and only if:

$$(\phi_X^t)^*\mu = \mu, \quad \forall t \in \mathbb{R}.$$

(b) if $\mu = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^d$ is the canonical volume form on $M = \mathbb{R}^d$ then for a vector field $X = \sum_{i=1}^d X^i \frac{\partial}{\partial x^i}$ one has:

$$\operatorname{div}_\mu(X) = \sum_{i=1}^d \frac{\partial X^i}{\partial x^i}.$$

(c) if $(M, g, [\mu])$ is an oriented Riemannian manifold with associated volume form μ and Hodge-star operator $*$ (see Exercises 17.11 and 17.12), then:

$$\operatorname{div}_\mu(X) = * \,\mathrm{d} * X.$$

9. Let (M, g) be a Riemannian manifold. Given a function $f \in C^\infty(M)$ one defines its **gradient** to be the unique vector field $\operatorname{grad} f \in \mathfrak{X}(M)$ satisfying:

$$g(\operatorname{grad} f, v) = \mathrm{d}f(v), \quad \forall v \in TM.$$

If $(M, g, [\mu])$ is also oriented, one defines the **laplacian** of $f : M \to \mathbb{R}$ to be the function $\Delta f : M \to \mathbb{R}$ given by:

$$\Delta f := -\operatorname{div}(\operatorname{grad} f).$$

Let $M = \mathbb{R}^3$ with its canonical Riemannian structure and canonical orientation. Find the gradient, the divergence and the laplacian in cylindrical and in spherical coordinates.

10. In a smooth manifold M denote by $\mathfrak{X}^k(M)$ the vector space of multivector fields of degree k. Show that there exists a unique $\mathbb{R}$-bilinear operation $[\ ,\ ] : \mathfrak{X}^{p+1}(M) \times \mathfrak{X}^{q+1}(M) \to \mathfrak{X}^{p+q+1}(M)$ which coincides with the usual Lie bracket of vector fields when $p = q = 0$ and satisfies:

(a) $[P, Q] = -(-1)^{pq}[Q, P]$;

(b) $[P, Q \wedge R] = [P, Q] \wedge R + (-1)^{p(q+1)} Q \wedge [P, R]$;

Verify that this bracket satisfies the following Jacobi type identity:

$$(-1)^{pr}[P, [Q, R]] + (-1)^{qp}[Q, [R, P]] + (-1)^{rq}[R, [P, Q]] = 0.$$

In all these identities, $P \in \mathfrak{X}^{p+1}(M)$, $Q \in \mathfrak{X}^{q+1}(M)$ and $R \in \mathfrak{X}^{r+1}(M)$.

NOTE: This operation is known as the **Schouten bracket** and is the counterpart for multivector fields of the exterior differential for forms. It is an example of a graded Lie bracket.

Ultimately, our interest on differential forms of degree d lies in the fact that they can be integrated over oriented d-manifolds, as we now explain.

Let us start with the case where $M = \mathbb{R}^d$, with the usual orientation. If $U \subset \mathbb{R}^d$ is open, then every differential form $\omega \in \Omega^d(U)$ can be written as:

$$\omega = f \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^d, \quad (f \in C^\infty(U)).$$

We say that ω is **integrable in** U and we define its integral by:

$$\int_U \omega = \int_U f(x^1, \ldots, x^d) \mathrm{d}x^1 \cdots \mathrm{d}x^d,$$

provided the integral in the right hand side exists and is finite.

The usual change of variable formula for the integral in $\mathbb{R}^d$ yields the following result:

Lemma 19.1. *Let $\Phi : U \to \mathbb{R}^d$ be a diffeomorphism defined in an open connected set $U \subset \mathbb{R}^d$. If ω is a differential form integrable in $\Phi(U)$, then $\Phi^*\omega$ is integrable in U and*

$$\int_{\Phi(U)} \omega = \pm \int_U \Phi^* \omega,$$

where $\pm$ is the sign of $\det(\Phi'(p))$ for any $p \in U$.

Therefore, as long as we consider only orientation preserving diffeomorphisms, the integral is invariant under diffeomorphisms. For this reason, we will only consider the integral of differential forms over *oriented* manifolds. It is possible to define the integral over non-oriented manifolds, but this requires introducing **densities**, which generalize the notion of volume form.

We will also assume, in order to avoid convergence issues, that the differential forms $\omega \in \Omega^k(M)$ to be integrated have support

$$\operatorname{supp} \omega = \overline{\{p \in M : \omega_p \neq 0\}},$$

a compact set. We will denote by $\Omega^k_c(M)$ the smooth differential forms of degree k with compact support.

Definition 19.2. *If M is an oriented d-manifold and $\omega \in \Omega^d_c(M)$ has compact support, we define its* ***integral*** *over M as follows:*

- *If $\operatorname{supp} \omega \subset U$, where (U, ϕ) is a positive coordinate chart, then:*

$$\int_M \omega := \int_{\phi(U)} (\phi^{-1})^* \omega.$$

- *More generally, we consider an open cover of M by positive charts (U_α, ϕ_α) and a partition of unity $\{\rho_\alpha\}$ subordinated to this cover, and we define:*

$$\int_M \omega = \sum_\alpha \int_M \rho_\alpha \omega.$$

We remark that the sum in this definition is finite, since we assume that $\operatorname{supp} \omega$ is compact. It is easy to check that the definition is independent of the choices of covering by positive charts and of partition of unity. We leave it to the exercises the check of all these details.

It is also easy to check, that the integral satisfies the following basic properties:

(a) *Linearity:* If $\omega, \eta \in \Omega_c^d(M)$ and $a, b \in \mathbb{R}$, then:

$$\int_M (a\omega + b\eta) = a \int_M \omega + b \int_M \eta.$$

(b) *Additivity:* If $M = M_1 \cup M_2$ and $\omega \in \Omega_c^d(M)$, then:

$$\int_M \omega = \int_{M_1} \omega + \int_{M_2} \omega,$$

provided that $M_1 \cap M_2$ has zero measure.

Moreover, we have:

Theorem 19.3 (Change of Variables Formula). *Let M and N be oriented manifolds of dimension d and let $\Phi : M \to N$ be an orientation preserving diffeomorphism. Then, for every differential form $\omega \in \Omega_c^d(N)$, one has:*

$$\int_N \omega = \int_M \Phi^* \omega.$$

Proof. Since Φ is a diffeomorphism and preserves orientations, we can find an open cover of M by positive charts (U_α, ϕ_α), such that the open sets $\Phi(U_\alpha)$ are domains of positive charts $\psi_\alpha : \Phi(U_\alpha) \to \mathbb{R}^d$ for N. Let $\{\rho_\alpha\}$ be a partition of unity for N subordinated to the cover $\{\Phi(U_\alpha)\}$, so that $\{\rho_\alpha \circ \Phi\}$ is a partition of unity for M subordinated to the cover $\{U_\alpha\}$. By Lemma 19.1, we find:

$$\int_{\Phi(U_\alpha)} \rho_\alpha \omega = \int_{U_\alpha} \Phi^*(\rho_\alpha \omega) = \int_{U_\alpha} (\rho_\alpha \circ \Phi) \Phi^* \omega.$$

Hence, we obtain:

$$\begin{aligned}
\int_N \omega &= \sum_\alpha \int_N \rho_\alpha \omega \\
&= \sum_\alpha \int_{\Phi(U_\alpha)} \rho_\alpha \omega \\
&= \sum_\alpha \int_{U_\alpha} (\rho_\alpha \circ \Phi) \Phi^* \omega \\
&= \sum_\alpha \int_M (\rho_\alpha \circ \Phi) \Phi^* \omega = \int_M \Phi^* \omega.
\end{aligned}$$

∎

The computation of the integral of differential forms from the definition is not practical since it uses a partition of unity . The following result can often be applied to avoid the use of partitions of unity :

Proposition 19.4. *Let M be an oriented manifold of dimension d and let $C \subset M$ be a closed subset of zero measure. For any differential form $\omega \in \Omega_c^d(N)$, we have:*

$$\int_M \omega = \int_{M-C} \omega.$$

Proof. Using a partition of unity we can reduce the result to the case where M is an open subset of $\mathbb{R}^d$. For an open set $U \subset \mathbb{R}^d$, the result reduces to the equality:

$$\int_U f \mathrm{d}x^1 \dots \mathrm{d}x^d = \int_{U-C} f \mathrm{d}x^1 \dots \mathrm{d}x^d,$$

where $f : U \to \mathbb{R}$ is smooth and bounded. This result holds, since C has zero measure. □

Example 19.5.

Given a volume form μ is a compact manifold M, we can define the ***volume of*** *M relative to μ to be the positive number:*

$$\mathrm{vol}_\mu(M) := \int_M \mu,$$

where the integral is relative to the orientation $[\mu]$.

For example, consider on the sphere $\mathbb{S}^2$ the standard orientation defined by the volume form $\mu \in \Omega^2(\mathbb{S}^2)$:

$$\mu = i^*(x\mathrm{d}y \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y).$$

By the proposition above, we have that:

$$\mathrm{vol}_\mu(\mathbb{S}^2) = \int_{\mathbb{S}^2} \mu = \int_{\mathbb{S}^2 - p} \mu,$$

for any $p \in \mathbb{S}^2$. Let us take the north pole $p = p_N$. Then the stereographic projection $\pi_N : \mathbb{S}^2 - \{p_N\} \to \mathbb{R}^2$ defines a global chart for $\mathbb{S}^2 - \{p_N\}$ whose inverse is the parameterization:

$$\pi_N^{-1}(u, v) = \frac{1}{u^2 + v^2 + 1}(2u, 2v, u^2 + v^2 - 1).$$

We then compute:

$$\begin{aligned}(\pi_N^{-1})^*\mu &= (i \circ \pi_N^{-1})^*(x\mathrm{d}y \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y) \\ &= -\frac{4}{(u^2 + v^2 + 1)^2}\mathrm{d}u \wedge \mathrm{d}v.\end{aligned}$$

Which shows that π_N is a negative chart. Therefore:

$$\int_{\mathbb{S}^2} \mu = \int_{\mathbb{R}^2} \frac{4}{(u^2 + v^2 + 1)^2}\mathrm{d}u \wedge \mathrm{d}v.$$

The integral on the right can be computed using polar coordinates, and the final result is:

$$\mathrm{vol}_\mu(\mathbb{S}^2) = \int_{\mathbb{S}^2} \mu = \int_0^{+\infty} \int_0^{2\pi} \frac{4r}{(r^2+1)^2} \mathrm{d}\theta \mathrm{d}r = 4\pi$$

Our next aim is to generalize Stokes Theorem to differential forms.

Let M be a manifold with boundary and $p \in \partial M$. In a boundary chart $(U, x^1, \ldots, x^d)$ centered at p, a tangent vector $\mathbf{v} \in T_pM$ can be written in the form:

$$\mathbf{v} = \sum_{i=1}^d v^i \left. \frac{\partial}{\partial x^i} \right|_p .$$

and the tangent vectors in $T_p(\partial M)$ are exactly the tangent vectors whose last component vanishes:

$$T_p(\partial M) = \{\mathbf{v} \in T_pM : v^d = 0\}.$$

We will say that a tangent vector is **exterior** to ∂M if $v^d < 0$. It is easy to see that this condition is independent of the choice of boundary chart.

We can use this remark to construct the **induced orientation** on ∂M, whenever $(M, [\mu])$ is an oriented manifold with boundary: if $p \in \partial M$, the orientation of $T_p(\partial M)$ is, by definition, $[i_{\mathbf{v}}\mu_p]$ where $\mathbf{v} \in T_pM$ is any exterior tangent vector to ∂M. Is easy to see that this definition is independent of choice of exterior tangent vector so we have a well defined orientation $[\partial\mu]$ for ∂M. Henceforth, whenever M is an oriented manifold with boundary, we will always consider the induced orientation on ∂M.

Theorem 19.6 (Stokes Formula)**.** *Let M be an oriented manifold with boundary of dimension d. If $\omega \in \Omega_c^{d-1}(M)$ then:*

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega.$$

Proof. We divide the proof into several cases.

The case $M = \mathbb{R}^d$: In this case, we can write:

$$\omega = \sum_{i=1}^d f_i \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^d,$$

where f_i are compactly supported functions. We find its differential to be:

$$\mathrm{d}\omega = \sum_{i=1}^d (-1)^{i-1} \frac{\partial f_i}{\partial x^i} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^d.$$

By Fubini's Theorem:

$$\int_{\mathbb{R}^d} \mathrm{d}\omega = \sum_{i=1}^d (-1)^{i-1} \int_{\mathbb{R}^{d-1}} \left(\int_{-\infty}^{+\infty} \frac{\partial f_i}{\partial x^i} \mathrm{d}x^i \right) \mathrm{d}x^1 \cdots \widehat{\mathrm{d}x^i} \cdots \mathrm{d}x^d = 0.$$

where we used that f_i has compact support. Since $\partial\mathbb{R}^d = \emptyset$, Stokes Formula for $\mathbb{R}^d$ follows.

The case $M = \mathbb{H}^d$: We proceed as in the case of $\mathbb{R}^n$, but this time we obtain

$$\begin{aligned}
\int_{\mathbb{H}^d} \mathrm{d}\omega &= \sum_{i=1}^{d} (-1)^{i-1} \int_{\mathbb{H}^d} \frac{\partial f_i}{\partial x^i} \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^d \\
&= \sum_{i=1}^{d-1} (-1)^{i-1} \int_{\mathbb{H}^{d-1}} \left(\int_{-\infty}^{+\infty} \frac{\partial f_i}{\partial x^i} \mathrm{d}x^i \right) \mathrm{d}x^1 \cdots \widehat{\mathrm{d}x^i} \cdots \mathrm{d}x^d + \\
&\qquad\qquad + (-1)^{d-1} \int_{\mathbb{R}^{d-1}} \left(\int_{0}^{+\infty} \frac{\partial f_d}{\partial x^d} \mathrm{d}x^d \right) \mathrm{d}x^1 \cdots \mathrm{d}x^{d-1} \\
&= (-1)^d \int_{\mathbb{R}^{d-1}} f_d(x^1, \dots, x^{d-1}, 0) \mathrm{d}x^1 \cdots \mathrm{d}x^{d-1}.
\end{aligned}$$

On the other hand, $\partial\mathbb{H}^d = \{(x^1, \dots, x^d) : x^d = 0\}$, hence

$$\int_{\partial\mathbb{H}^d} \omega = \int_{\partial\mathbb{H}^d} f_d(x^1, \dots, x^{d-1}, 0) \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{d-1}.$$

In $\partial\mathbb{H}^d = \mathbb{R}^{d-1}$ we must take the induced orientation from the canonical orientation $[\mu] = [\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^d]$ in $\mathbb{H}^d$. The induced orientation is given by: $[(-1)^d \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{d-1}]$ so we conclude that:

$$\int_{\partial\mathbb{H}^d} \omega = (-1)^d \int_{\partial\mathbb{R}^{d-1}} f_d(x^1, \dots, x^{d-1}, 0) \mathrm{d}x^1 \cdots \mathrm{d}x^{d-1}.$$

Therefore, Stokes Formula also holds for the half space $\mathbb{H}^d$.

The case of general M: We fix an open cover of M by positive charts (U_α, ϕ_α) and we choose a partition of unity $\{\rho_\alpha\}$ subordinated to this cover. We can also assume that the charts have been chosen so that $\phi_\alpha(U_\alpha)$ is either $\mathbb{R}^d$ or $\mathbb{H}^d$. The forms $\rho_\alpha\omega$ have compact support contained in U_α:

$$\operatorname{supp} \rho_\alpha\omega \subset \operatorname{supp} \rho_\alpha \cap \operatorname{supp} \omega \subset U_\alpha.$$

and since each U_α is diffeomorphic to either $\mathbb{R}^d$ or to $\mathbb{H}^d$, by the change of variable formula, we already know that:

$$\int_{U_\alpha} \mathrm{d}(\rho_\alpha\omega) = \int_{\partial U_\alpha} \rho_\alpha\omega.$$

Notice that $\partial U_\alpha = U_\alpha \cap \partial M$, and so by the linearity and the additivity of the integral, we obtain:

$$\begin{aligned}
\int_M \mathrm{d}\omega &= \sum_\alpha \int_M \mathrm{d}(\rho_\alpha\omega) = \sum_\alpha \int_{U_\alpha} \mathrm{d}(\rho_\alpha\omega) \\
&= \sum_\alpha \int_{U_\alpha \cap \partial M} \rho_\alpha\omega = \int_{\partial M} \sum_\alpha \rho_\alpha\omega = \int_{\partial M} \omega.
\end{aligned}$$

$\square$

Corollary 19.7. *Let M be an oriented, manifold without boundary of dimension d. For any $\omega \in \Omega_c^{d-1}(M)$:*

$$\int_M \mathrm{d}\omega = 0.$$

Homework.

1. Show that the integral of differential forms is linear and additive relative to the region of integration.

2. In $\mathbb{H}^d$ consider the standard orientation $[\mu] = [\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^d]$. Show that the induced orientation in $\partial\mathbb{H}^d = \mathbb{R}^{d-1}$ is given by $[\partial\mu] = [(-1)^d \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{d-1}]$.

3. Consider the n-torus $\mathbb{T}^n$ as an embedded submanifold of $\mathbb{R}^{2n}$:
$$\mathbb{T}^n = \{(x^1, \dots, x^n, y^1, \dots, y^n) \in \mathbb{R}^{2n} : (x^i)^2 + (y^i)^2 = 1, i = 1, \dots, n\},$$
and let $\omega \in \Omega^n(\mathbb{T}^n)$ be the form
$$\omega = \left(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n\right)|_{\mathbb{T}^n}.$$
Compute the integral $\int_{\mathbb{T}^n} \omega$ for an orientation of your choice, in the following ways:
(a) using the definition, and
(b) using Stokes formula.

4. Find the volume of $\mathbb{S}^d$ for the standard volume form on the sphere:
$$\mu = \sum_{i=1}^{d+1} (-1)^{i+1} x^i \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^{d+1}\bigg|_{\mathbb{S}^d}.$$

5. Let $(M, g, [\mu])$ is an oriented Riemannian manifold with boundary, and with associated volume form μ and Hodge-star operator $*$. If $f : M \to \mathbb{R}$ is a smooth, compactly supported function, define the integral of f over M by:
$$\int_M f \equiv \int_M *f.$$
If X is any vector field, prove the classical **Divergence Theorem**:
$$\int_M \mathrm{div}_\mu X = \int_{\partial M} X \cdot n,$$
where $n : \partial M \to T_{\partial M} M$ is the unit exterior normal vector field along ∂M.

6. Let M be an oriented Riemannian manifold with boundary. For any smooth function $f : M \to \mathbb{R}$ denote by $\frac{\partial f}{\partial n}$ the function $n(f) : \partial M \to \mathbb{R}$, where n is the unit exterior normal vector field along ∂M. Verify the following Green identities:
$$\int_{\partial M} f \frac{\partial g}{\partial n} = \int_M \langle \mathrm{grad}\, f, \mathrm{grad}\, g \rangle - \int_M f \Delta g,$$
$$\int_{\partial M} \left(f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n} \right) = \int_M (g \Delta f - f \Delta g),$$
where $f, g \in C^\infty(M)$.

7. Let G be a Lie group of dimension d.
   (a) Show that if $\omega, \omega' \in \Omega^d(G)$ are left invariant and $[\omega] = [\omega']$, then
$$\int_G f\omega = a \int_G f\omega', \ \forall f \in C^\infty(G),$$
   for some real number $a > 0$.

   Fix an orientation $\mu = [\omega]$ for G defined by a left invariant form $\omega \in \Omega^d(G)$. Define the integral of $f : G \to \mathbb{R}$ relative to this orientation by:
$$\int_G f \equiv \int_G f\omega.$$
   (b) Show that the integral is left invariant, i.e., for every $g \in G$ is valid the identity:
$$\int_G f \circ L_g = \int_G f.$$
   (c) Give an example of a Lie group where the integral is not right invariant.

   For each $g \in G$, the differential form $R_g^*\omega$ is left invariant, hence
$$R_g^*\omega = \tilde{\lambda}(g)\omega,$$
   for some smooth function $\tilde{\lambda} : G \to \mathbb{R}$. The **modular function** $\lambda : G \to \mathbb{R}_+$ is defined to be $\lambda(g) = |\tilde{\lambda}(g)|$.
   (d) Show that the integral is right invariant if and only if G is **unimodular**, i.e., $\lambda \equiv 1$.
   (e) Show that a compact Lie group is unimodular.

8. Let G be a compact Lie group and let $\Phi : G \to GL(V)$ be a representation of G. Show that there exists an inner product $\langle \ , \ \rangle$ in V such that this representation is by orthogonal transformations:
$$\langle \Phi(g) \cdot \mathbf{v}, \Phi(g) \cdot \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle, \quad \forall g \in G.$$
   (Hint: Use the fact that a compact Lie group is unimodular.)

9. Let G be a compact Lie group. Show that G has a bi-invariant Riemannian metric, i.e., a Riemannian metric which is both right and left invariant. (Hint: A left invariant Riemannian metric in G is also right invariant if and only if the inner product $\langle \ , \ \rangle$ induced in $\mathfrak{g} \simeq T_eG$ satisfies:
$$\langle \mathrm{Ad}(g) \cdot X, \mathrm{Ad}(g) \cdot Y \rangle = \langle X, Y \rangle, \quad \forall g \in G, X, Y \in \mathfrak{g}.$$

20. de Rham Cohomology

The equation $\mathrm{d}^2 = 0$, which so far we have made little use, has in fact some deep consequences, as we shall see in the next few sections.

Definition 20.1. *Let $\omega \in \Omega^k(M)$.*

*(i) ω is called a **closed form** if $\mathrm{d}\omega = 0$.*
*(ii) ω is called an **exact form** if $\omega = \mathrm{d}\eta$, for some $\eta \in \Omega^{k-1}(M)$.*

We will denote by $Z^k(M)$, respectively $B^k(M)$, the subspaces of closed, respectively exact, differential forms of degree k.

In other words, the closed forms form the kernel of d, while the exact forms form the image of d. The pair $(\Omega(M), \mathrm{d})$ is called the *de Rham complex* of M and we will often represent it as:

$$\cdots \longrightarrow \Omega^{k-1}(M) \xrightarrow{\mathrm{d}} \Omega^k(M) \xrightarrow{\mathrm{d}} \Omega^{k+1}(M) \longrightarrow \cdots$$

The fact that $\mathrm{d}^2 = 0$ means that every exact form is closed:

$$B^k(M) \subset Z^k(M).$$

One should think of $(\Omega(M), \mathrm{d})$ as a set of differential equations associated with the manifold M. Finding the closed forms, means to solve the differential equation:

$$\mathrm{d}\omega = 0.$$

On the other hand, the exact forms can be thought of as the trivial solutions of this equation. We are interested in the space of all solutions modulus the trivial solutions, and this is called the de Rham cohomology of M:

Definition 20.2. *The **de Rham cohomology space of degree** k is the vector space:*

$$H^k(M) \equiv Z^k(M)/B^k(M).$$

In general, the computation of the cohomology spaces $H^k(M)$ directly from the definition is very hard. In the next sections we will study several properties of the de Rham cohomology spaces which can be used to compute them. For now we list some easy consequences of the definition and we give a very simple example.

Proposition 20.3. *Let M be a smooth manifold. Then:*

(i) $H^0(M) = \mathbb{R}^l$, *where l is the number of connected components of M;*
(ii) $H^k(M) = \{0\}$, *if $k < 0$ or $k > \dim M$.*

Proof. We have $\Omega^0(M) = C^\infty(M)$ and if $f \in C^\infty(M)$ satisfies $\mathrm{d}f = 0$, then f is locally constant. Hence:

$$Z^0(M) = \mathbb{R}^l,$$

where l is the number of connected components of M. Since $B^0(M) = \{0\}$, we have that $H^0(M) = \mathbb{R}^l$. On the other hand, since $\Omega^k(M) = \{0\}$ if $k > \dim M$, the result follows. □

Example 20.4.
Let $M = \mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Since $\mathbb{S}^1$ is connected, it follows that:

$$H^0(\mathbb{S}^1) = \mathbb{R}.$$

Now to compute $H^1(\mathbb{S}^1)$, we consider the 1-form $-y\mathrm{d}x + x\mathrm{d}y \in \Omega^1(\mathbb{R}^2)$. This form restricts to a 1-form in $\mathbb{S}^1$ which we will denote by ω. Since $\dim \mathbb{S}^1 = 1$,

ω is closed. On the other hand, consider the parameterization $\sigma :]0, 2\pi[\to \mathbb{S}^1 - \{(1,0)\}$, given by $\sigma(t) = (\cos t, \operatorname{sen} t)$. Then:

$$\begin{aligned}\int_{\mathbb{S}^1} \omega &= \int_{]0,2\pi[} \sigma^*\omega \\ &= \int_{]0,2\pi[} (-\sin t)\mathrm{d}\cos t + \cos t \mathrm{d}\sin t = \int_0^{2\pi} \mathrm{d}t = 2\pi.\end{aligned}$$

By the corollary to Stokes Formula, we see that ω is not exact, so it represents a non-trivial cohomology class $[\omega] \in H^1(\mathbb{S}^1)$.

The form ω has a simple geometric meaning: since $\sigma^*\omega = \mathrm{d}t$, we have that $\omega = \mathrm{d}\theta$ in $\mathbb{S}^1 - \{(1,0)\}$, where $\theta : \mathbb{S}^1 - \{(1,0)\} \to \mathbb{R}$ is the angle coordinate (the inverse of the parameterization σ). Sometimes one denotes ω by $\mathrm{d}\theta$, in spite of the fact that this is not an exact form.

We claim that $[\omega]$ is a basis for $H^1(\mathbb{S}^1)$. Given a form $\alpha \in \Omega^1(\mathbb{S}^1)$ we have that $\alpha = f\omega$, for some function $f : \mathbb{S}^1 \to \mathbb{R}$. Let

$$c = \frac{1}{2\pi}\int_{\mathbb{S}^1} \alpha = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\mathrm{d}\theta,$$

and define $g : \mathbb{R} \to \mathbb{R}$ by:

$$g(t) = \int_0^t (\alpha - c\omega) = \int_0^t (f(\theta) - c)\mathrm{d}\theta.$$

Since:

$$\begin{aligned}g(t + 2\pi) &= \int_0^{t+2\pi} (f(\theta) - c)\mathrm{d}\theta \\ &= \int_0^t (f(\theta) - c)\mathrm{d}\theta + \int_t^{t+2\pi} (f(\theta) - c)\mathrm{d}\theta \\ &= g(t) + \int_0^{2\pi} (f(\theta) - c)\mathrm{d}\theta = g(t),\end{aligned}$$

we obtain a C^∞ function $g : \mathbb{S}^1 \to \mathbb{R}$. In $\mathbb{S}^1 - \{(1,0)\}$, we have that

$$\mathrm{d}g = f(\theta)\mathrm{d}\theta - c\mathrm{d}\theta = \alpha - c\omega.$$

Hence, we must have $\mathrm{d}g = \alpha - c\omega$ in $\mathbb{S}^1$ so that $[\alpha] = c[\omega]$. This shows that $[\omega]$ generates $H^1(\mathbb{S}^1)$ so we conclude that:

$$H^1(\mathbb{S}^1) \simeq \mathbb{R}.$$

The wedge product $\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$ induces the **cup product** in the de Rham cohomology of M which is defined by setting:

$$[\alpha] \cup [\beta] := [\alpha \wedge \beta].$$

We leave it as an exercise to check that this definition is independent of the choice of representatives of the cohomology classes. With this product the space:

$$H^\bullet(M) = \bigoplus_{k \in \mathbb{Z}} H^k(M)$$

becomes a $\mathbb{Z}$-graded ring (in fact, a $\mathbb{Z}$-graded algebra).

If $\Phi : M \to N$ is a smooth map, then pull-back map gives a linear map $\Phi^* : \Omega^\bullet(N) \to \Omega^\bullet(M)$ which commutes with the differentials:

$$\Phi^* \mathrm{d}\omega = \mathrm{d}(\Phi^*\omega).$$

Therefore, Φ^* takes closed (respectively, exact) forms to closed (respectively, exact) forms, and we have an induced map in cohomology:

$$\Phi^* : H^\bullet(N) \to H^\bullet(M),\ [\omega] \longmapsto [\Phi^*\omega].$$

The induced map $\Phi^* : H^\bullet(N) \to H^\bullet(M)$ is a ring homomorphism:

$$\Phi^*([\alpha] + [\beta]) = \Phi^*[\alpha] + \Phi^*[\beta]. \qquad \Phi^*([\alpha] \cup [\beta]) = \Phi^*[\alpha] \cup (\Phi^*[\beta].$$

Moreover, we have:

(i) If $\Phi : M \to N$ and $\Psi : N \to Q$ are smooth maps, then the composition $(\Psi \circ \Phi)^* : H^\bullet(Q) \to H^\bullet(M)$ satisfies $(\Psi \circ \Phi)^* = \Phi^* \circ \Psi^*$;

(ii) The identity map $M \to M$ induces the identity linear transformation $H^\bullet(M) \to H^\bullet(M)$.

One can summarize (i) and (ii) as follows: the assignment which associates to a differential manifold M its de Rham complex $(\Omega(M), \mathrm{d})$ and to each smooth map $\Phi : M \to N$ the pull-back $\Phi^* : \Omega^\bullet(N) \to \Omega^\bullet(M)$ is a *contravariant functor* from the category of differential manifolds to the category of cochain complexes. In particular, when $\Phi : M \to N$ is a diffeomorphism, the induced linear map $\Phi^* : H^\bullet(N) \to H^\bullet(M)$ is an isomorphism in cohomology. Hence, we have:

Corollary 20.5. *The de Rham cohomology ring is an* invariant *of differentiable manifolds: if M and N are diffeomorphic, then $H^\bullet(M)$ and $H^\bullet(N)$ are isomorphic rings.*

Remark 20.6 (*A Crash Course in Homological Algebra - part I*)**.** The de Rham complex $(\Omega^\bullet(M), \mathrm{d})$ and the compactly supported de Rham complex $(\Omega^\bullet_c(M), \mathrm{d})$ are examples of *cochain complexes.* In general, one calls a **cochain complex** a pair (C, d) where:

(a) C is a $\mathbb{Z}$-graded vector space, i.e., $C = \oplus_{k\in\mathbb{Z}} C^k$ is the direct sum of vector spaces [3];

(b) $\mathrm{d} : C \to C$ is a linear transformation of degree 1, i.e., $\mathrm{d}(C^k) \subset C^{k+1}$, such that $\mathrm{d}^2 = 0$.

One represents a complex by the diagram:

$$\cdots \longrightarrow C^{k-1} \xrightarrow{\mathrm{d}} C^k \xrightarrow{\mathrm{d}} C^{k+1} \longrightarrow \cdots$$

The transformation d is called the **differential** of the complex.

For any cochain complex, (C, d) one defines the subspace of all **cocycles**:

$$Z^k(C) \equiv \{z \in C^k : \mathrm{d}z = 0\},$$

[3]More generally, one can consider complexes formed by $\mathbb{Z}$-graded modules over commutative rings with unit (e.g., abelian groups). Most of the following statements are valid for the category of modules over commutative rings with unit, with obvious modifications.

and the subspace of all **coboundaries**

$$B^k(C) \equiv \{\mathrm{d}z : z \in C^{k-1}\}.$$

Since $\mathrm{d}^2 = 0$, we have that $B^k(C) \subset Z^k(C)$. The **cohomology of** (C, d) is the direct sum $H^\bullet(C) = \oplus_{k\in\mathbb{Z}} H^k(C)$ of all the cohomology spaces of degree k, which are defined by:

$$H^k(C) = \frac{Z^k(C)}{B^k(C)}.$$

Given two cochain complexes (A, d_A) and (B, d_B), a **cochain map** of degree d is a linear map $f : A \to B$ such that:

(a) f shifts the grading by d, i.e., $f(A^k) \subset B^{k+d}$;

(b) f commutes with the differentials, i.e., $f\mathrm{d}_A = \mathrm{d}_B f$.

One represents a cochain map by a commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & A^{k-1} \arrow[r, "\mathrm{d}_A"] \arrow[d, "f"'] & A^k \arrow[r, "\mathrm{d}_A"] \arrow[d, "f"'] & A^{k+1} \arrow[r] \arrow[d, "f"'] & \cdots \\
\cdots \arrow[r] & B^{k+d-1} \arrow[r, "\mathrm{d}_B"'] & B^{k+d} \arrow[r, "\mathrm{d}_B"'] & B^{k+d+1} \arrow[r] & \cdots
\end{tikzcd}$$

It should be clear that a cochain map $f : A \to B$ takes cocycles to cocycles and coboundaries to coboundaries. Hence, f induces a linear map in cohomology, denoted by the same letter: $f : H^\bullet(A) \to H^{\bullet+d}(B)$. Most often we consider cochain maps of degree 0, so we will omit mentioning the degree.

The cochain complexes and cochain maps form a category, and their study is one of the central themes of *Homological Algebra*.

Notice that the differential d takes a compactly supported form to a compactly supported form, so we have another complex $(\Omega_c(M), \mathrm{d})$.

Definition 20.7. *The* ***compactly supported de Rham cohomology space of degree*** k *is the vector space:*

$$H^k_c(M) \equiv Z^k_c(M)/B^k_c(M),$$

where $Z^k_c(M) \subset \Omega^k_c(M)$*, respectively* $B^k_c(M) \subset \Omega^k_c(M)$*, denotes the subspaces of closed, respectively exact, compactly supported forms of degree* k*.*

Obviously, $H^k(M) = H^k_c(M)$ if M is compact. In general, the inclusion $\Omega^k_c(M) \subset \Omega^k(M)$ gives a linear map in cohomology:

$$H^k_c(M) \to H^k(M).$$

Notice that this map, in general, is neither injective nor surjective:

(i) given a closed form $\omega \in \Omega^k(M)$ one may not be able to find a cohomologous form $\omega + \mathrm{d}\eta$ with compact support, and

(ii) given an exact form $\omega = \mathrm{d}\eta \in \Omega^k_c(M)$ one may not be able to find another primitive η' with compact support.

Hence, $H_c^k(M)$ and $H^k(M)$ can be very different. This can be seen already in degree 0:

Proposition 20.8. *Let M be a smooth manifold. Then:*

(i) $H_c^0(M) = \mathbb{R}^s$*, where s is the number of compact connected components of M;*

(ii) $H_c^k(M) = \{0\}$*, if $k < 0$ or $k > \dim M$.*

Proof. If $f \in C_c^\infty(M)$ satisfies $\mathrm{d}f = 0$, then f is constant in the compact connected components of M and is zero in the non-compact connected components. Since $B_c^0(M) = \{0\}$, we conclude that

$$H_c^0(M) = \mathbb{R}^s,$$

where s is the number of compact connected components of M. $\square$

Similar to what we saw before, the wedge product of forms induces a cup product:

$$\cup : H_c^k(M) \times H_c^l(M) \to H_c^{k+l}(M), \qquad [\alpha] \cup [\beta] := [\alpha \wedge \beta],$$

so we have a $\mathbb{Z}$-graded ring (in fact, a $\mathbb{Z}$-graded algebra):

$$H_c^\bullet(M) = \bigoplus_{k \in \mathbb{Z}} H_c^k(M).$$

The pullback by a smooth map $\Phi : M \to N$ of a form ω with compact support is a form $\Phi^*\omega$ that may fail to have compact support. However, for a *proper map* Φ, we do have an induced map in cohomology:

$$\Phi^* : H_c^k(N) \to H_c^*(M).$$

This defines a functor from the category of differential manifolds and smooth proper maps to the category of cochain complexes, which assigns to a differentiable manifold M its compactly supported de Rham complex.

Homework.

1. Show that $H^1(\mathbb{T}^d) = \mathbb{R}^d$, using the definition of the de Rham cohomology. (Hint: Show that a basis for $H^1(\mathbb{T}^d)$ is given by $\{[\mathrm{d}\theta_1], \ldots, [\mathrm{d}\theta_d]\}$, where $(\theta_1, \ldots, \theta_d)$ are the angles on each $\mathbb{S}^1$ factor.)

2. Consider the 2-sphere $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$.
   (a) Show that every closed 1-form $\omega \in \Omega^1(\mathbb{S}^2)$ is exact.
   (b) Show that the 2-form in $\mathbb{R}^3$ given by

   $$\omega = x\mathrm{d}y \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y.$$

   induces by restriction to $\mathbb{S}^2$ a non-trivial cohomology class $[\omega] \in H^2(\mathbb{S}^2)$.
   (Hint: For (a), you can use the fact that a closed 1-form in $\mathbb{R}^2$ is always exact.)

3. Use de Rham cohomology to prove that $\mathbb{T}^2$ and $\mathbb{S}^2$ are not diffeomorphic manifolds.

4. Show that if M is a compact, orientable, d-manifold, then $H^d(M) \neq 0$.

5. Show that the wedge product $\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$ induces a well-defined product $\cup$ in the de Rham cohomology of M, which makes $H(M) = \oplus_k H^k(M)$ into a ring (actually, an algebra over $\mathbb{R}$).

6. A symplectic form on a manifold M of dimension $2n$ is a 2-form $\omega \in \Omega^2(M)$ such that $d\omega = 0$ and $\wedge^n \omega$ is a volume form. Show that if M is compact and admits some symplectic form, then $H^{2k}(M) \neq 0$ for $k = 0, \ldots, n$.
(Hint: Use the ring structure of $H^\bullet(M)$.)

21. The de Rham Theorem

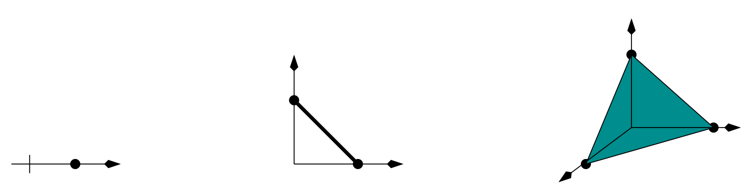
We saw in the previous section that de Rham cohomology is an invariant of differential manifolds. Actually, de Rham cohomology is a topological invariant. This is a consequence of the famous de Rham Theorem, which shows that for any smooth manifold its singular cohomology with real coefficients is isomorphic with its de Rham cohomology. Our aim in this section is to present the ingredients and the statement of this result. Several proofs will be left open, since they go beyond the scope of this notes and require more advanced material.

Singular Homology. We recall the definition of the singular homology of a topological space M. Although we will continue to use the letter M, the following discussion only uses the topology of M.

We denote by $\Delta^k \subset \mathbb{R}^{k+1}$ the **standard k-simplex**:

$$\Delta^k = \{(t_0, \ldots, t_k) \in \mathbb{R}^{k+1} : \sum_{i=0}^{k} t_i = 1, t_i \geq 0\}.$$

Note that $\Delta^0 = \{1\}$ has only one element.



Δ^0 Δ^1 Δ^2

Definition 21.1. *A* ***singular k-simplex*** *in M is a continuous map $\sigma : \Delta^k \to M$. A* ***singular k-chain*** *is a formal linear combination*

$$c = \sum_{i=1}^{p} a_i \sigma_i,$$

where $a_i \in \mathbb{R}$ and the σ_i are singular k-simplices.

We will denote by $S_k(M;\mathbb{R})$ the set of all singular k-chains. It is a real vector space. In fact, formally, $S_k(M;\mathbb{R})$ is the free vector space generated by the set of all singular k-simplices. One can also consider other abelian rings as coefficients besides $\mathbb{R}$, but here we will consider only real coefficients, since this is the case of interest to relate to differential forms.

We define the ***i*-face map** of the standard k-simplex, where $0 \leq i \leq k$, to be the map $\varepsilon^i : \Delta^{k-1} \to \Delta^k$ defined by:

$$\varepsilon^i(t_0, \ldots, t_{k-1}) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{k-1}).$$

These face maps of the standard k-simplex induce face maps ε_i of any singular k-simplex $\sigma : \Delta^k \to M$ by setting:

$$\varepsilon_i(\sigma) = \sigma \circ \varepsilon^i.$$

These clearly extend by linearity to any k-chain, yielding linear maps

$$\varepsilon_i : S_k(M;\mathbb{R}) \to S_{k-1}(M;\mathbb{R}),$$

and these lead to the following definition:

Definition 21.2. *The* ***boundary of a*** $\boldsymbol{k}$***-chain*** *c is the $(k-1)$-chain ∂c defined by*

$$\partial c = \sum_{i=0}^{k} (-1)^i \varepsilon_i(c).$$

The geometric meaning of this definition is that we consider the faces of each simplex with a certain choice of signs, which one should view as some kind of orientations of the faces. We illustrate this choice in the next example.

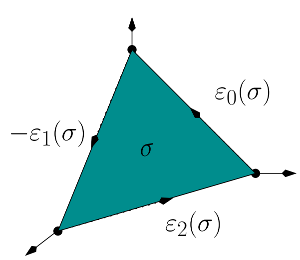
Example 21.3.
The boundary of the standard 2-simplex $\sigma =$id: $\Delta^2 \to \mathbb{R}^3$ is the chain:

$$\partial\sigma = \varepsilon_0(\sigma) - \varepsilon_1(\sigma) + \varepsilon_2(\sigma),$$

where ε_0, ε_1 and ε_2 are the 1-simplices (faces) given by:

$$\varepsilon_0(\sigma)(t_0, t_1) = (0, t_0, t_1),$$
$$\varepsilon_1(\sigma)(t_0, t_1) = (t_0, 0, t_1),$$
$$\varepsilon_2(\sigma)(t_0, t_1) = (t_0, t_1, 0).$$

We can represent this choice of signs by including orientations on the faces of the simplex, as shown schematically by the following figure:



Also, the 1-simplices ε_0, ε_1 and ε_2 have boundaries the 0-chains:

$$\begin{aligned}\partial\varepsilon_0(\sigma)(1) &= \varepsilon_0(\sigma)(0,1) - \varepsilon_0(\sigma)(1,0) = (0,0,1) - (0,1,0),\\ \partial\varepsilon_1(\sigma)(1) &= \varepsilon_1(\sigma)(0,1) - \varepsilon_1(\sigma)(1,0) = (0,0,1) - (1,0,0),\\ \partial\varepsilon_2(\sigma)(1) &= \varepsilon_2(\sigma)(0,1) - \varepsilon_2(\sigma)(1,0) = (0,1,0) - (1,0,0).\end{aligned}$$

Note that:

$$\partial^2\sigma = \partial(\partial\sigma) = \partial\varepsilon_0(\sigma) - \partial\varepsilon_1(\sigma) + \partial\varepsilon_2(\sigma) = 0.$$

We noticed in this example that $\partial^2\sigma = 0$. This is actually a general fact which is a consequence of the judicious choice of signs and parameterizations of the faces. We leave its proof as an exercise:

Lemma 21.4. *For every singular chain c:*

$$\partial(\partial c) = 0.$$

In this way we obtain a complex $S(M;\mathbb{R}) = \oplus_{k\in\mathbb{Z}} S_k(M;\mathbb{R})$:

$$\cdots \longleftarrow S_{k-1}(M;\mathbb{R}) \xleftarrow{\partial} S_k(M;\mathbb{R}) \xleftarrow{\partial} S_{k+1}(M;\mathbb{R}) \longleftarrow \cdots$$

One calls $(S(M;\mathbb{R}),\partial)$ the **complex of singular chains** in M. The homology of the complex $(S(M;\mathbb{R}),\partial)$ is called the **singular homology** of M with real coefficients, and is denoted

$$H_k(M;\mathbb{R}) = \frac{Z_k(M;\mathbb{R})}{B_k(M;\mathbb{R})}.$$

Remark 21.5 (*A Crash Course in Homological Algebra - part II*)**.** In the cochain complexes that we studied related to de Rham cohomology the differentials *increase* the degree, while for the singular chains the differential *decreases* the degree.

We call a complex $C = \oplus_{k\in\mathbb{Z}} C_k$ where the differential decreases the degree

$$\cdots \longleftarrow C_{k-1} \xleftarrow{\partial} C_k \xleftarrow{\partial} C_{k+1} \longleftarrow \cdots$$

a **chain complex**. We say that $z \in C_k$ is a **cycle** if $\partial z = 0$ and we say that z is a **boundary** if $z = \partial b$ ([4]). In this case, one defines the **homology** of the complex C to be the direct sum $H(C) = \oplus_{k\in\mathbb{Z}} H_k(C)$ of the vector spaces:

$$H_k(C) = \frac{Z_k(C)}{B_k(C)},$$

where $Z_k(C)$ is the subspace of all cycles and $B_k(C)$ is the subspace of all boundaries. Note also the position of the indices.

[4]Notice that given a complex (C,∂) where the differential decreases one can define a new complex $(\bar{C},\mathrm{d})$ setting $\bar{C}^k \equiv C_{-k}$ and $\mathrm{d} = \partial$, obtaining a complex where the differential increases the degree. Therefore, these conventions are somewhat arbitrary.

If $\Phi : M \to N$ continuous map, then for any singular simplex $\sigma : \Delta^k \to M$, we have that $\Phi_*(\sigma) \equiv \Phi \circ \sigma : \Delta^k \to N$ is a singular simplex in N. We extend this map to any chain $c = \sum_j a_j \sigma_j$ requiring linearity to hold:

$$\Phi_*(c) \equiv \sum_j a_j (\Phi \circ \sigma_j).$$

It follows that $\Phi_* : S(M;\mathbb{R}) \to S(N;\mathbb{R})$ is a chain map:

$$\begin{tikzcd}
\cdots & S_{k-1}(M;\mathbb{R}) \arrow[l] \arrow[d, "\Phi_*"'] & S_k(M;\mathbb{R}) \arrow[l, "\partial"'] \arrow[d, "\Phi_*"'] & S_{k+1}(M;\mathbb{R}) \arrow[l, "\partial"'] \arrow[d, "\Phi_*"'] & \cdots \arrow[l] \\
\cdots & S_{k-1}(N;\mathbb{R}) \arrow[l] & S_k(N;\mathbb{R}) \arrow[l, "\partial"'] & S_{k+1}(N;\mathbb{R}) \arrow[l, "\partial"'] & \cdots \arrow[l]
\end{tikzcd}$$

Therefore, Φ_* induces a linear map in singular homology:

$$\Phi_* : H_\bullet(M;\mathbb{R}) \to H_\bullet(N;\mathbb{R}).$$

One checks easily that this assignment has the following properties:

(i) If $\Phi : M \to N$ and $\Psi : N \to Q$ are continuous maps, then:

$$(\Psi \circ \Phi)_* = \Psi_* \circ \Phi_*;$$

(ii) The identity map id: $M \to M$ induces the identity map in homology:

$$\mathrm{id}_* = \mathrm{id} : H_\bullet(M;\mathbb{R}) \to H_\bullet(M;\mathbb{R}).$$

It follows that singular homology is a *topological invariant*:

Proposition 21.6. *If M and N are are homeomorphic spaces then $H_\bullet(M,\mathbb{R}) \simeq H_\bullet(N,\mathbb{R})$.*

Smooth Singular Homology. Assume now that M is a smooth manifold. The chain complex $(S_\bullet(M;\mathbb{R}),\partial)$ has a subcomplex $(S^\infty_\bullet(M;\mathbb{R}),\partial)$ formed by the **smooth singular k-chains**:

$$S^\infty_k(M;\mathbb{R}) = \{\sum_{i=1}^p a_i\sigma_i : \sigma_i : \Delta^k \to M \text{ is smooth}\}$$

This is a sub complex because if $c \in S^\infty_k(M;\mathbb{R})$ is a smooth k-chain, then so is $\partial c \in S^\infty_k(M;\mathbb{R})$.

Remark 21.7. Even when c is smooth, the use of the term "singular" is justified by the absence of any assumption on the differentials of the maps σ_i: in general, a smooth k-simplex *does not* parameterize any submanifold and its image may be a rather pathological subset of M.

One has the following important fact:

Proposition 21.8. *The inclusion $S^\infty_\bullet(M,\mathbb{R}) \hookrightarrow S_\bullet(M,\mathbb{R})$ induces an isomorphism in homology:*

$$H(S^\infty_\bullet(M,\mathbb{R})) \simeq H(S_\bullet(M,\mathbb{R})).$$

This proposition says that:

(i) every homology class in $H_\bullet(M;\mathbb{R})$ has a representative c which is a C^∞ cycle, and
(ii) if two C^∞ cycles c and c' differ by a continuous boundary ($c-c'=\partial b$), then they also differ by a C^∞ boundary b' ($c-c'=\partial b'$).

Hence, smooth singular homology and singular homology coincide. The proof of the previous proposition is beyond the scope of these notes.

Singular Cohomology. Dually, one defines the singular cohomology of M as follows. First, one defines the space of **singular k-cochains** with real coefficients to be the vector space dual to $S_k(M.\mathbb{R})$

$$S^k(M;\mathbb{R}) := \mathrm{Hom}(S_k(M;\mathbb{R}),\mathbb{R}).$$

We have a **singular differential** obtained by transposing the singular boundary operator:

$$\mathrm{d} : S^k(M;\mathbb{R}) \to S^{k+1}(M;\mathbb{R}), \quad (\mathrm{d}l)(c) = l(\partial c), \quad \forall c \in S_k(M;\mathbb{R}).$$

It follows that $\mathrm{d}^2 = 0$, so we have a cochain complex $(S^\bullet(M;\mathbb{R}),\mathrm{d})$. The corresponding cohomology is called the **singular cohomology** of M with real coefficients and is denoted by $H^\bullet(M;\mathbb{R})$.

Remark 21.9. A more explicit form of the singular differential is as follows. Since the k-simplices form a basis for the vector space $S_k(M,\mathbb{R})$ a linear map $l : S_k(M,\mathbb{R}) \to \mathbb{R}$ amounts to a collection of real numbers $l = (l_\sigma)$, indexed by all singular simplices (so $l_\sigma = l(\sigma)$). Then the singular differential $\mathrm{d}l \in S^{k+1}(M;\mathbb{R})$ is given by the collection $((\mathrm{d}l)_\sigma)$ indexed by $k+1$-simplices defined by:

$$(\mathrm{d}l)_\sigma = \sum_{i=0}^{k+1} (-1)^i l_{\varepsilon_i(\sigma)}.$$

If $\Phi : M \to N$ we can transpose the map $\Phi_* : S_k(M;\mathbb{R}) \to S_k(N;\mathbb{R})$, obtaining a linear map $\Phi^* : S^k(N;\mathbb{R}) \to S^k(M;\mathbb{R})$ which is a cochain map:

$$\Phi^*\mathrm{d} = \mathrm{d}\Phi^*.$$

Therefore, we have an induced linear map in singular cohomology $\Phi^* : H^\bullet(N;\mathbb{R}) \to H^\bullet(M;\mathbb{R})$, which satisfies the obvious functorial properties, and hence we also have:

Proposition 21.10. *If M and N are homeomorphic spaces then $H^\bullet(M,\mathbb{R}) \simeq H^\bullet(N,\mathbb{R})$.*

Of course, one can also consider smooth singular k-cochains:

$$S^k_\infty(M;\mathbb{R}) := \mathrm{Hom}(S^\infty_k(M;\mathbb{R}),\mathbb{R}).$$

which form a complex $(S^\bullet_\infty(M;\mathbb{R}),\mathrm{d})$. There is an obvious restriction map

$$S^k(M;\mathbb{R}) \to S^k_\infty(M;\mathbb{R}), \quad l \mapsto l|_{S^k_\infty(M;\mathbb{R})},$$

which is easily checked to be a cochain map. So we have an induced map in cohomology and one can prove:

Proposition 21.11. *The restriction map $S^k(M;\mathbb{R}) \to S^k_\infty(M;\mathbb{R})$ yields an isomorphism in cohomology:*

$$H(S^\bullet(M;\mathbb{R}), \mathrm{d}) \simeq H(S^\bullet_\infty(M;\mathbb{R}), \mathrm{d}).$$

For this reason, in the sequel we will not distinguish between these cohomologies.

Singular Cohomology vs. de Rham Cohomology. We now take advantage of the fact that singular cohomology and differentiable singular cohomology coincide to relate it with the de Rham cohomology. For that, we start by explaining that one can integrate differential forms over singular chains.

First, we observe that we can parameterize the standard k-simplex Δ^k by the map $\phi : \Delta^k_0 \to \Delta^k$, where:

$$\Delta^k_0 := \{(x^1, \dots, x^k) : x^i \geq 0, \sum_{i=1}^{k} x^i \leq 1\}$$

$$\phi(x^1, \dots, x^k) = (1 - \sum_{i=1}^{k} x^i, x^1, \dots, x^k),$$

Hence, if $\omega \in \Omega^k(U)$ is a k-form which is defined in some open set $U \subset \mathbb{R}^{k+1}$ containing the standard k-simplex Δ^k, we can write:

$$\phi^*\omega = f(x^1, \dots, x^k)\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^k,$$

and define:

$$\int_{\Delta^k} \omega := \int_{\Delta^k_0} f \mathrm{d}x^1 \cdots \mathrm{d}x^k.$$

Next, given any differential form $\omega \in \Omega^k(M)$ in a smooth manifold M, we define the **integral of ω over a smooth simplex** $\sigma : \Delta^k \to M$ to be the real number:

$$\int_\sigma \omega := \int_{\Delta^k} \sigma^*\omega.$$

We extend this definition to any smooth singular k-chain $c = \sum_{j=1}^{p} a_j \sigma_j$ by linearity:

$$\int_c \omega := \sum_{j=1}^{p} a_j \int_{\sigma_j} \omega.$$

Notice that, unlike the case of integration over manifolds, there is now no assumption about orientation and M may very well be non-oriented.

We leave it to the exercises the proof of the following version of Stokes formula for chains:

Theorem 21.12 (Stokes II)**.** *Let M be a smooth manifold, $\omega \in \Omega^{k-1}(M)$ a $(k-1)$-differential form, and c a smooth singular k-chain. Then:*

$$\int_c \mathrm{d}\omega = \int_{\partial c} \omega.$$

Now we can define an *integration map* $I : \Omega^\bullet(M) \to S^\bullet_\infty(M;\mathbb{R})$:

$$I(\omega)(\sigma) = \int_\sigma \omega, \quad \omega \in \Omega^k(M), \sigma \in S^\infty_k(M;\mathbb{R}).$$

and we have:

Proposition 21.13. *The integration map $I : (\Omega^\bullet(M), \mathrm{d}) \to (S^\bullet_\infty(M;\mathbb{R}), \mathrm{d})$ is a chain map:*

$$I(\mathrm{d}\omega) = \mathrm{d}I(\omega).$$

Proof. This follows from the following computation:

$$\begin{aligned}(I(\mathrm{d}\omega))(\sigma) = \int_\sigma \mathrm{d}\omega &= \int_{\partial\sigma} \omega \\ &= I(\omega)(\partial\sigma) = (\mathrm{d}I(\omega))(\sigma),\end{aligned}$$

where we used Stokes formula for chains and the fact that the singular differential is the transpose of the singular coboundary operator. $\square$

It follows that we have an induced linear map in cohomology:

$$I : H^k(M) \to H^k(M;\mathbb{R}).$$

Theorem 21.14 (de Rham)**.** *For any smooth manifold the integration map $I : H^\bullet(M) \to H^\bullet(M;\mathbb{R})$ is an isomorphism.*

There is also cup product in singular cohomology and one can show that the integration map is actually a ring isomorphism (see the exercises). The proof of this result is beyond the scope of these notes.

An important consequence of the de Rham Theorem is that the de Rham cohomology is actually a topological invariant of smooth manifolds, i.e., if M and N are homeomorphic smooth manifolds then their de Rham cohomologies are isomorphic. For example, the different exotic smooth structures on the spheres all have the same de Rham cohomology!

HOMEWORK.

1. Show that for every singular chain c one has $\partial(\partial c) = 0$.

2. Give a proof of Stokes Formula for singular chains, by showing the following:
   (a) It is enough to prove the formula for chains consisting of a singular simplex.
   (b) It is enough to prove the formula for the standard k-simplex $\Delta^k_0 \subset \mathbb{R}^k$.
   (c) It is enough to prove the formula for $(k-1)$-differential forms in $\mathbb{R}^k$ of the type:
   $$\omega = f\mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^k.$$
   (d) Show that
   $$\int_{\Delta^k_0} \mathrm{d}\omega = \int_{\partial\Delta^k_0} \omega,$$
   where ω is a differential form of the type (c).

3. In the torus $\mathbb{T}^d = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ consider the 1-chains $c_1, \ldots, c_d : [0,1] \to \mathbb{T}^d$ defined by:

$$c_j(t) \equiv (1, \ldots, e^{2\pi i t}, \ldots, 1) \quad (j = 1, \ldots, d).$$

Show that:
(a) The c_j's are 1-cycles: $\partial c_j = 0$;
(b) The c_j's are not 1-boundaries;
(c) The classes $\{[c_1], \ldots, [c_d]\} \subset H_1(\mathbb{T}^d, \mathbb{R})$ form a linearly independent set.

HINT: Use Stokes formula.

4. The de Rham Theorem, shows that the exterior product induces a product

$$\cup : H^k(M;\mathbb{R}) \times H^l(M:\mathbb{R}) \to H^{k+l}(M;\mathbb{R}),$$

so that $H^\bullet(M;\mathbb{R})$ becomes a ring. This product is called the **cup product**. Here is one way of constructing it directly:
(a) Show that for $l < k$ and $0 \le i_0 < \cdots < i_l \le k$ one has maps $\varepsilon_{i_0,\ldots,i_l} : \Delta^l \to \Delta^k$, defined by:

$$\varepsilon_{i_0,\ldots,i_l}(t_0, \ldots, t_l) = (s_0, \ldots, s_k), \quad \text{where } \begin{cases} s_l = 0, \text{ if } l \notin \{i_0, \ldots, i_l\} \\ s_{i_j} = t_j, \text{ otherwise.} \end{cases}$$

(b) Show that if $c_1 \in S^k(M;\mathbb{R})$ and $c_2 \in S^l(M;\mathbb{R})$ the formula:

$$(c_1 \cup c_2)(\sigma) := c_1(\sigma \circ \varepsilon_{1,\ldots,k}) c_2(\sigma \circ \varepsilon_{k+1,\ldots,k+l}),$$

defines an element $c_1 \cup c_2 \in S^{k+l}(M;\mathbb{R})$.
(c) Show that for any chains $c_1 \in S^k(M;\mathbb{R})$ and $c_2 \in S^l(M;\mathbb{R})$ one has:

$$\mathrm{d}(c_1 \cup c_2) = (\mathrm{d}c_1) \cup c_2 + (-1)^k c_1 \cup (\mathrm{d}c_2).$$

It follows that one can define $\cup : H^k(M;\mathbb{R}) \times H^l(M:\mathbb{R}) \to H^{k+l}(M;\mathbb{R})$ by

$$[c_1] \cup [c_2] := [c_1 \cup c_2].$$

Note that for the integration map $I : \Omega^k(M) \to S^k_\infty(M)$, in general, $I(\omega \wedge \eta) \neq I(\omega) \cup I(\eta)$. However, show that this equality holds in cohomology:

$$I([\omega] \wedge [\eta]) = I([\omega]) \cup I([\eta]), \quad [\omega] \in H^k_{dR}(M), [\eta] \in H^l_{dR}(M).$$

5. Let $S_k(M,\mathbb{Z}) \subset S_k(M,\mathbb{R})$ be the abelian group consisting of all **integral singular k-simplex**, i,e, all formal linear combinations

$$c = \sum_{i=1}^p n_i \sigma_i,$$

where $n_i \in \mathbb{Z}$ and the σ_i are singular k-simplices (so $S_k(M,\mathbb{Z})$ be the free abelian group generated by the set of all singular k-simplices).
(a) Show that $S_k(M,\mathbb{Z}) \subset (S_k(M,\mathbb{R}), \partial)$ is a subcomplex of abelian groups. The corresponding homology groups are denoted $H_k(M,\mathbb{Z})$ and are called integral singular homology groups of M.
(b) Dually, define the complex of singular integral cochains $(S^k(M,\mathbb{Z}), \mathrm{d})$ by:

$$S^k(M,\mathbb{Z}) := \mathrm{Hom}(S_k(M,\mathbb{Z}), \mathbb{Z}), \quad \mathrm{d}c(\sigma) := c(\partial\sigma).$$

Denoting by $H_k(M,\mathbb{Z})$ corresponding **integral singular cohomology groups**, show that there is a group homomorphism:

$$i : H^k(M,\mathbb{Z}) \to H^k(M,\mathbb{R}).$$

6. Let $(S_k^\infty(M,\mathbb{Z}),\partial)$ be the complex of smooth integral singular chains and let $(S^k_\infty(M,\mathbb{Z}),\mathrm{d})$ be the complex of smooth integral singular cochains (see previous problem). It is a fact that these complexes still yield the integral singular homology and cohomology:

$$H_k(S_\bullet^\infty(M,\mathbb{Z}),\partial) = H_k(M,\mathbb{Z}), \quad H^k(S^\bullet_\infty(M,\mathbb{Z}),\mathrm{d}) = H^k(M,\mathbb{Z}).$$

Assuming this, show that:

(a) There is a homomorphism of abelian groups:

$$I : H^k(M,\mathbb{Z}) \to H^k(M).$$

(b) For a closed form $\omega \in \Omega^k(M)$ the set:

$$\mathrm{Per}(\omega) := \left\{ \int_c \omega : [c] \in H_k(S_\bullet^\infty(M,\mathbb{Z}),\partial) \right\} \subset \mathbb{R},$$

is an additive subgroup. It is called the **group of periods** of ω,

(c) A cohomology class $[\omega] \in H^k(M)$ belongs to the image of the homomorphism $I : H^k(M,\mathbb{Z}) \to H^k(M)$ if and only if $\mathrm{Per}(\omega) \subset \mathbb{Z}$.

22. Homotopy Invariance and Mayer-Vietoris Sequence

We shall now study some properties of de Rham cohomology which are very useful in the computation of these rings in specific examples.

The Poincaré Lemma. We start with the simplest example of manifold, namely $M = \mathbb{R}^d$. In order to compute its cohomology we will compare the cohomology of M and the cohomology of $M \times \mathbb{R}$, for an arbitrary smooth manifold M.

Proposition 22.1. *If M is a smooth manifold, consider the projection map $\pi : M \times \mathbb{R} \to M$ and the inclusion map $i : M \to M \times \mathbb{R}$ given by:*

$$\begin{tikzcd} M \times \mathbb{R} \arrow[d, shift left, "\pi"] \\ M \arrow[u, shift left, "i"] \end{tikzcd} \qquad \begin{aligned} \pi(p,t) &= p, \\ i(p) &= (p,0). \end{aligned}$$

The induced maps $i^* : H^\bullet(M\times\mathbb{R}) \to H^\bullet(M)$ and $\pi^* : H^\bullet(M) \to H^\bullet(M\times\mathbb{R})$ are inverse to each other.

Since $H^0(\mathbb{R}^0) = \mathbb{R}$ and $H^k(\mathbb{R}^0) = 0$ if $k \neq 0$, repeated use of the proposition gives the cohomology of euclidean space:

Corollary 22.2 (Poincaré Lemma)**.**

$$H^k(\mathbb{R}^d) = H^k(\mathbb{R}^0) = \begin{cases} \mathbb{R} & \textit{if } k = 0, \\ \\ 0 & \textit{if } k \neq 0. \end{cases}$$

In other words, in $\mathbb{R}^d$ every closed form of positive degree is exact.

We now turn to the proof of Proposition 22.1. For that it is useful to recall a bit more of homological algebra.

Remark 22.3 (*A Crash Course in Homological Algebra - part III*)**.** In order to prove this proposition we will use the notion of homotopy operator. Given two cochain complexes (A, d) and (B, d) and cochain maps $f, g : A \to B$ a **homotopy operator** is a linear map $h : A \to B$ of degree -1, such that

$$f - g = \pm(\mathrm{d}h \pm h\mathrm{d})$$

(the choice of signs is irrelevant). In this case, we say that f and g are **homotopic cochain maps** and we express it by the diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & A^{k-1} \arrow[r, "\mathrm{d}"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \arrow[dl] & A^k \arrow[r, "\mathrm{d}"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \arrow[dl, "h"'] & A^{k+1} \arrow[r] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \arrow[dl, "h"'] & \cdots \arrow[dl] \\
\cdots \arrow[r] & B^{k-1} \arrow[r, "\mathrm{d}"'] & B^k \arrow[r, "\mathrm{d}"'] & B^{k+1} \arrow[r] & \cdots
\end{tikzcd}$$

Since $\pm(\mathrm{d}h \pm h\mathrm{d})$ maps closed forms to exact forms, it is induces the zero map in cohomology. Hence. if f and g are homotopic chain maps, they induce the same map in cohomology:

$$f_* = g_* : H^\bullet(A) \to H^\bullet(B).$$

Proof of Proposition 22.1. Note that $\pi \circ i = \mathrm{id}$, hence $i^* \circ \pi^* = \mathrm{id}$. To complete the proof we need to check that $\pi^* \circ i^* = \mathrm{id}$. For this we construct a homotopy operator $h : \Omega^\bullet(M \times \mathbb{R}) \to \Omega^{\bullet-1}(M \times \mathbb{R})$ such that:

$$\mathrm{id} - \pi^* \circ i^* = \mathrm{d}h + h\mathrm{d}.$$

To construct h, we leave as an exercise to check that any differential k-form $\theta \in \Omega^k(M \times \mathbb{R})$ can be expressed as locally finite sum:

$$\theta = \sum_i \theta_i,$$

where each $\theta_i \in \Omega^k(M \times \mathbb{R})$ is of one of the following two kinds:

$$f_1(x, t)\pi^*\omega_1,$$
$$f_2(x, t)\mathrm{d}t \wedge \pi^*\omega_2,$$

with ω_1 and ω_2 differential forms in M of degree k and $k - 1$, respectively, and $f, g : M \times \mathbb{R} \to \mathbb{R}$ smooth functions. So we define the homotopy operator in each of these kinds of forms by:

$$h : \begin{cases} f_1(x,t)\pi^*\omega_1 \longmapsto 0, \\ f_2(x,t)\mathrm{d}t \wedge \pi^*\omega_2 \longmapsto \int_0^t f_2(x,s)\mathrm{d}s\ \pi^*\omega_2, \end{cases}$$

and then we extend it by linearity to all forms. We now check that h is indeed a homotopy operator, i.e., that we have:

$$(\mathrm{id} - \pi^* \circ i^*)\theta = (\mathrm{d}h + \mathrm{d}h)\theta. \tag{22.1}$$

Let $\theta_1 = f_1(x,t)\pi^*\omega_1 \in \Omega^k(M \times \mathbb{R})$ be a form of the first kind. Then:

$$(\mathrm{id} - \pi^* \circ i^*)\theta_1 = \theta_1 - \pi^*(f_1(x,0)\omega_1) = (f_1(x,t) - f_1(x,0))\pi^*\omega_1.$$

On the other hand,

$$\begin{aligned}
(\mathrm{d}h + h\mathrm{d})\theta_1 &= h\mathrm{d}\theta_1 \\
&= h\left((\mathrm{d}f_1 \wedge \pi^*\omega_1 + f_1\pi^*\mathrm{d}\omega_1\right) \\
&= h\left(\frac{\partial f_1}{\partial t}\mathrm{d}t_1 \wedge \pi^*\omega_1\right) \\
&= \int_0^t \frac{\partial f_1}{\partial t}(x,s)\mathrm{d}s\ \pi^*\omega_1 = (f_1(x,t) - f_1(x,0))\pi^*\omega_1.
\end{aligned}$$

Hence, for any form θ_1 formula 22.1 holds.

Let now $\theta_2 = f_2(x,t)\mathrm{d}t \wedge \pi^*\omega_2$ be a differential form of the second kind. On the one hand,

$$(\mathrm{id} - \pi^* \circ i^*)\theta_2 = \theta_2.$$

On the other hand, in any local coordinates (U, x^i) for M, we find:

$$\begin{aligned}
(\mathrm{d}h + h\mathrm{d})\theta_2 &= \mathrm{d}(\int_0^t f_2(x,s)\mathrm{d}s\ \pi^*\omega_2) + h\left(\sum_i \frac{\partial f_2}{\partial x^i}\mathrm{d}x^i \wedge \mathrm{d}t \wedge \pi^*\omega_2 - f_2\mathrm{d}t \wedge \pi^*\mathrm{d}\omega_2\right) \\
&= f_2(x,t)\mathrm{d}t \wedge \pi^*\omega_2 + \sum_i \int_0^t \frac{\partial f_2}{\partial x^i}\mathrm{d}s\ \mathrm{d}x^i \wedge \pi^*\omega_2 + \int_0^t f_2(x,s)\mathrm{d}s\ \mathrm{d}\pi^*\omega_2 \\
&\quad - \sum_i \int_0^t \frac{\partial f_2}{\partial x^i}\mathrm{d}s\ \mathrm{d}x^i \wedge \pi^*\omega_2 - \int_0^t f_2(x,s)\mathrm{d}s\ \pi^*\mathrm{d}\omega_2 \\
&= f_2(x,t)\mathrm{d}t \wedge \pi^*\omega_2 = \theta_2.
\end{aligned}$$

Therefore, for any form θ_2 of the second kind formula 22.1 also holds. □

Homotopy Invariance. Proposition 22.1 is actually very special case of a general property of cohomology: if a manifold can be continuously deformed into another manifold then their cohomologies are isomorphic. In order to formulate a precise statement, we make the following definition.

Definition 22.4. *Let $\Phi, \Psi : M \to N$ be smooth maps. A **smooth homotopy** between Φ and Ψ is a smooth map $H : M \times \mathbb{R} \to N$ such that*

$$H(p,t) = \begin{cases} \Phi(p) & \text{if } t \leq 0, \\ \\ \Psi(p) & \text{if } t \geq 1. \end{cases}$$

Often one defines a smooth homotopy between Φ and Ψ to be a smooth map $H : M \times [0,1] \to N$ such that:

$$H(p,0) = \Phi(p), \quad H(1,p) = \Psi(p), \quad p \in M.$$

It is easy to see that two definitions are equivalent. Less obvious, one can show that:

(i) two smooth maps are smooth homotopic iff they are C^0-homotopic;
(ii) any continuous map between two smooth manifolds is C^0-homotopic to a smooth map.

Theorem 22.5 (Homotopy Invariance). *If $\Phi, \Psi : M \to N$ are smooth homotopic maps then $\Phi^* = \Psi^* : H^\bullet(N) \to H^\bullet(M)$.*

Proof. Denote by $\pi : M \times \mathbb{R} \to M$ the projection and $i_0, i_1 : M \to M \times \mathbb{R}$ the sections:

$$i_0(p) = (p,0) \text{ and } i_1(p) = (p,1).$$

By Proposition 22.1, i_0^* and i_1^* are linear maps which both invert π^*, so they must coincide: $i_0^* = i_1^*$.

Now let $H : M \times \mathbb{R} \to N$ be a homotopy between Φ and Ψ. Then $\Phi = H \circ i_0$ and $\Psi = H \circ i_1$. At the level of cohomology we find:

$$\begin{aligned} \Phi^* &= (H \circ i_0)^* = i_0^* \, H^*, \\ \Psi^* &= (H \circ i_1)^* = i_1^* \, H^*. \end{aligned}$$

Since $i_0^* = i_1^*$, we conclude that $\Phi^* = \Psi^*$.

∎

We say that two manifolds M and N have the **homotopy type** if there exist smooth maps $\Phi : M \to N$ and $\Psi : N \to M$ such that $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are homotopic to id_M and id_N, respectively. A manifold is said to be **contractible** if it has the same homotopy type as a point (i.e., $\mathbb{R}^0$).

Corollary 22.6. *If M and N have the same homotopy type then $H^\bullet(M) \simeq H^\bullet(N)$. In particular, if M is a contractible manifold then:*

$$H^k(M) = \begin{cases} \mathbb{R} & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases}$$

Examples 22.7.

1. *An open set $U \subset \mathbb{R}^d$ is called **star shaped** if there exists some $x_0 \in U$ such that for any $x \in U$, the segment $tx + (1-t)x_0$ lies in U. We leave it as exercise to show that a star shaped open set U is contractible, so that*

$$H^k(U) = \begin{cases} \mathbb{R} & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases}$$

2. The manifold $M = \mathbb{R}^{d+1} - 0$ has the same homotopy type as $\mathbb{S}^d$: the inclusion $i : \mathbb{S}^d \hookrightarrow \mathbb{R}^{d+1} - 0$ and the projection $\pi : \mathbb{R}^{d+1} - 0 \to \mathbb{S}^d$, $x \mapsto x/||x||$, are homotopic inverses to each other. Hence:

$$H^\bullet(\mathbb{S}^d) = H^\bullet(\mathbb{R}^{d+1} - 0).$$

Notice that we don't know yet how to compute $H^\bullet(\mathbb{R}^{d+1} - 0)$!

Mayer-Vietoris Sequence. Let us discuss now another important property of cohomology, which allows to compute the cohomology of a manifold M from a decomposition of M into more elementary pieces of which we already know the cohomology.

Theorem 22.8 (Mayer-Vietoris Sequence)**.** *Let M be a smooth manifold and let $U, V \subset M$ be open subsets such that $M = U \cup V$. There exists a long exact sequence:*

$$\longrightarrow H^k(M) \longrightarrow H^k(U) \oplus H^k(V) \longrightarrow H^k(U \cap V) \xrightarrow{\delta} H^{k+1}(M) \longrightarrow$$

Remark 22.9 (*A Crash Course in Homological Algebra - part IV*)**.** A sequence of vector spaces and linear maps

$$\cdots \longrightarrow C^{k-1} \xrightarrow{f_{k-1}} C^k \xrightarrow{f_k} C^{k+1} \longrightarrow \cdots$$

is called **exact** if $\operatorname{Im} f_{k-1} = \operatorname{Ker} f_k$. An exact sequence of the form:

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is called **a short exact sequence**. This means that:

(a) f is injective,
(b) $\operatorname{Im} f = \operatorname{Ker} g$, and
(c) g is surjective.

A basic property of exact sequences is the following: given any exact sequence ending in trivial vector spaces

$$0 \longrightarrow C^0 \longrightarrow \cdots \longrightarrow C^k \longrightarrow \cdots \longrightarrow C^d \longrightarrow 0$$

the alternating sum of the dimensions is zero:

$$\sum_{i=0}^{d} (-1)^i \dim C^i = 0.$$

We leave the (easy) proof for the exercises.

Note that a short exact sequence of complexes:

$$0 \longrightarrow (A^\bullet, \mathrm{d}) \xrightarrow{f} (B^\bullet, \mathrm{d}) \xrightarrow{g} (C^\bullet, \mathrm{d}) \longrightarrow 0$$

can be represented by a large commutative diagram where all rows are exact:

$$
\begin{tikzcd}
 & & & & \\
0 \arrow[r] & A^{k+1} \arrow[u] \arrow[r, "f"] & B^{k+1} \arrow[u] \arrow[r, "g"] & C^{k+1} \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & A^{k} \arrow[u, "\mathrm{d}"] \arrow[r, "f"] & B^{k} \arrow[u, "\mathrm{d}"] \arrow[r, "g"] & C^{k} \arrow[u, "\mathrm{d}"] \arrow[r] & 0 \\
0 \arrow[r] & A^{k-1} \arrow[u, "\mathrm{d}"] \arrow[r, "f"] & B^{k-1} \arrow[u, "\mathrm{d}"] \arrow[r, "g"] & C^{k-1} \arrow[u, "\mathrm{d}"] \arrow[r] & 0 \\
 & \arrow[u] & \arrow[u] & \arrow[u] &
\end{tikzcd}
$$

We have the following basic fact: given a short exact sequence of complexes as above there exists an *associated long exact sequence* in cohomology

$$\cdots \longrightarrow H^k(A) \xrightarrow{f} H^k(B) \xrightarrow{g} H^k(C) \xrightarrow{\delta} H^{k+1}(A) \longrightarrow \cdots$$

where $\delta : H^k(C) \to H^{k+1}(A)$ is called the *connecting homomorphism*. The fact that $\operatorname{Im} f = \operatorname{Ker} g$ follows immediately from the definition of short exact sequence. On the other hand, the identities $\operatorname{Im} g = \operatorname{Ker} \delta$ and $\operatorname{Im} \delta = \operatorname{Ker} f$ follow from the way δ is constructed, and which we now describe.

For the construction of δ one should keep in mind the large commutative diagram above. Given a cocycle $c \in C^k$ so that $\mathrm{d}c = 0$, it follows from the fact that the rows are exact that there exists $b \in B^k$ such that $g(b) = c$. Since the diagram commutes, we have

$$g(\mathrm{d}b) = \mathrm{d}g(b) = \mathrm{d}c = 0.$$

Using again that the rows are exact, we conclude that there exists a unique $a \in A^{k+1}$ such that $f(a) = \mathrm{d}b$. Note that:

$$f(\mathrm{d}a) = \mathrm{d}f(a) = \mathrm{d}^2 b = 0,$$

and since f is injective, we have $\mathrm{d}a = 0$, i.e., a is cocycle. In this way, we have associated to a cocycle $c \in C^k$ a cocycle $a \in A^{k+1}$.

This association depends on a choice of an intermediate element $b \in C^k$. If we choose a different $b' \in C^k$ such $g(b') = c$, we obtain a different element $a' \in A^{k+1}$. However, note that

$$g(b - b') = g(b') - g(b) = c - c = 0,$$

so there exist $\bar{a} \in A^k$ such that $f(\bar{a}) = b - b'$. Hence, we find

$$f(a - a') = f(a) - f(a') = \mathrm{d}b - \mathrm{d}b' = \mathrm{d}f(\bar{a}) = f(\mathrm{d}\bar{a}).$$

Since f is injective, we conclude that $a - a' = \mathrm{d}\bar{a}$. This shows that different intermediate choices lead to elements in the same cohomology class.

Finally, note that this assignment associates a coboundary to a coboundary. In fact, if $c \in C^k$ is a coboundary, i.e., $c = \mathrm{d}c'$, then there exists $b' \in C^{k-1}$ such that $g(b') = c'$. Moreover,

$$g(b - \mathrm{d}b') = g(b) - \mathrm{d}g(b') = c - \mathrm{d}c' = 0.$$

Therefore, there exists $a' \in A^k$ such that $f(a') = b - \mathrm{d}b'$, and:

$$f(a - \mathrm{d}a') = f(a) - \mathrm{d}f(a') = \mathrm{d}b - \mathrm{d}b + \mathrm{d}^2 b' = 0.$$

Since f is injective, we conclude that $a = \mathrm{d}a'$ is a coboundary, as claimed.

This discussion shows that we have a well-defined linear map

$$\delta : H^k(C) \to H^{k+1}(A), [c] \mapsto [a].$$

We leave it as an exercise to check that this definition leads to $\operatorname{Im} g = \operatorname{Ker} \delta$ and $\operatorname{Im} \delta = \ker f$.

Proof of Theorem 22.8. We claim that we have a short exact sequence:

$$0 \longrightarrow \Omega^\bullet(M) \longrightarrow \Omega^\bullet(U) \oplus \Omega^\bullet(V) \longrightarrow \Omega^\bullet(U \cap V) \longrightarrow 0$$

where the first map is given by:

$$\omega \mapsto (\omega|_U, \omega|_V),$$

while the second map is defined by:

$$(\theta, \eta) \mapsto \theta|_{U\cap V} - \eta|_{U\cap V}.$$

Since $M = U \cup V$, the first map is injective. Also, it is clear that the image of the first map is contained in the kernel of the second map. On the other hand, if $(\theta, \eta) \in \Omega^\bullet(U) \oplus \Omega^\bullet(V)$ belongs to the kernel of the second map, then

$$\theta|_{U\cap V} = \eta|_{U\cap V}.$$

Hence, we can define a smooth differential form in M by:

$$\omega_p = \begin{cases} \theta_p & \text{if } p \in U, \\ \\ \eta_p & \text{if } p \in V. \end{cases}$$

Therefore the image of the first map coincides with the kernel of the second map. Finally, let $\alpha \in \Omega^\bullet(U \cap V)$ and choose a partition of unity $\{\rho_U, \rho_V\}$ subordinated to the cover $\{U, V\}$. Then $\rho_V \alpha \in \Omega^\bullet(U)$ and $\rho_U \alpha \in \Omega^\bullet(V)$ and this pair of forms is transformed by the second map to

$$(\rho_V \alpha, -\rho_U \alpha) \mapsto \rho_V \alpha + \rho_U \alpha = \alpha.$$

Therefore, the second map is surjective and we have a short exact sequence as claimed. The corresponding long exact sequence in cohomology yields the statement of the theorem. $\square$

Example 22.10.
Let us use the Mayer-Vietoris sequence to compute the cohomology of $\mathbb{S}^d$ *for* $d \geq 2$ *(we already know the cohomology* $H^\bullet(\mathbb{S}^1)$*; see in Example 20.4).*

Let $p_N \in \mathbb{S}^d$ *be the north pole and let* $U = \mathbb{S}^d - p_N$*. The stereographic projection* $\pi_N : U \to \mathbb{R}^d$ *is a diffeomorphism, so* U *is contractible. Similarly if* $p_S \in \mathbb{S}^d$ *is the south pole, the open set* $V = \mathbb{S}^d - p_S$ *is contractible. On the other hand, we have that* $M = U \cap V$ *and the intersection* $U \cap V$ *is diffeomorphic to* $\mathbb{R}^d - 0$ *(via any of the stereographic projections). We saw in Example 22.7 that* $\mathbb{R}^d - 0$ *as the same homotopy type as* $\mathbb{S}^{d-1}$*.*

We have all the ingredients to compute the Mayer-Vietoris sequence:

- *if* $k \geq 1$*, the sequence gives:*

$$\cdots \longrightarrow 0 \oplus 0 \longrightarrow H^k(\mathbb{S}^{d-1}) \xrightarrow{\mathrm{d}^*} H^{k+1}(\mathbb{S}^d) \longrightarrow 0 \oplus 0 \longrightarrow \cdots$$

  Hence, $H^{k+1}(\mathbb{S}^d) \simeq H^k(\mathbb{S}^{d-1})$*. By induction, we conclude that:*

$$H^k(\mathbb{S}^d) \simeq H^{k-1}(\mathbb{S}^{d-1}) \simeq \cdots \simeq H^1(\mathbb{S}^{d-k+1}).$$

- *On the other hand, since* U*,* V *and* $U \cap V$ *are connected, the first terms of the sequence are*

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R} \xrightarrow{\delta} H^1(\mathbb{S}^d) \longrightarrow 0 \longrightarrow \cdots$$

  It follows that $\dim H^1(\mathbb{S}^d) = 0$ *if* $d \geq 2$*, since the alternating sum of the dimensions must be zero.*

Since $H^1(\mathbb{S}^1) = \mathbb{R}$*, we conclude that:*

$$H^k(\mathbb{S}^d) = \begin{cases} \mathbb{R} & \text{if } k = 0, d, \\ 0 & \text{otherwise.} \end{cases}$$

Compactly supported cohomology. As we saw in the previous section, compactly supported cohomology does not behave functorialy under smooth maps. Still this cohomology behaves functorialy under proper maps and, because of this, compactly supported cohomology still satisfies properties analogous, but distinct, to the properties we have studied for de Rham cohomology.

Proposition 22.11. *Let* M *be a smooth manifold. Then:*

$$H_c^\bullet(M \times \mathbb{R}) \simeq H_c^{\bullet-1}(M).$$

Proof. Note that if $\pi : M \times \mathbb{R} \to M$ is the projection and $\omega \neq 0$ then $\pi^*\omega$ does not have compact support. Instead, one has "push-forward" maps

$$\pi_* : \Omega_c^{\bullet+1}(M \times \mathbb{R}) \to \Omega_c^\bullet(M),$$
$$e_* : \Omega_c^\bullet(M) \to \Omega_c^{\bullet+1}(M \times \mathbb{R}).$$

which are cochains maps, homotopic inverse to each other.

We start by constructing π_*. Note that every compactly supported k-form in $M \times \mathbb{R}$ is a locally finite sum of forms of two kinds:

$$f_1(x,t)\pi^*\omega_1,$$
$$f_2(x,t)\pi^*\omega_2 \wedge \mathrm{d}t,$$

where $\omega_1 \in \Omega_c^k(M)$, $\omega_1 \in \Omega_c^{k-1}(M)$ and $f_1, f_2 : M \times \mathbb{R} \to \mathbb{R}$ are compactly supported smooth functions. The map π_* is given by:

$$f_1(x,t)(\pi^*\omega_1) \longmapsto 0,$$
$$f_2(x,t)\pi^*\omega_2 \wedge \mathrm{d}t \longmapsto \int_{-\infty}^{+\infty} f_2(x,t)\mathrm{d}t\ \omega_2.$$

and it is known as *integration along the fibers*.

On the other hand, in order to construct e_* one chooses some 1-form $\theta = g(t)\mathrm{d}t \in \Omega_c^1(\mathbb{R})$ with $\int_{\mathbb{R}} \theta = 1$ and sets:

$$e_* :\ \omega \to \pi^*\omega \wedge \theta.$$

It follows from these definitions that:

$$\pi_* \circ e_* = \mathrm{id}, \quad \mathrm{d}\pi_* = \pi_*\mathrm{d}, \quad e_*\mathrm{d} = \mathrm{d}e_*.$$

To finish the proof, we check that $e_* \circ \pi_*$ is homotopic to the identity. We leave it as an exercise to check that the map $h : \Omega_c^\bullet(M \times \mathbb{R}) \to \Omega_c^{\bullet-1}(M \times \mathbb{R})$ defined by:

$$f_1(x,t)(\pi^*\omega_1) \longmapsto 0,$$
$$f_2(x,t)\pi^*\omega_2 \wedge \mathrm{d}t \longmapsto \left(\int_{-\infty}^{t} f_2(x,s)\mathrm{d}s - \int_{-\infty}^{+\infty} f_2(x,s)\mathrm{d}s\ \int_{-\infty}^{t} g(s)\mathrm{d}s\right)\pi^*\omega_2,$$

is indeed a homotopy from $e_* \circ \pi_*$ to the identity. □

The proposition shows that compactly supported cohomology *is not* invariant under homotopy. On the other hand, the proposition shows that the Poincaré Lemma must be modified as follows:

Corollary 22.12 (Poincaré Lemma for compactly supported cohomology)**.**

$$H_c^k(\mathbb{R}^d) = \begin{cases} \mathbb{R} & \textit{if } k = d, \\ 0 & \textit{if } k \neq d. \end{cases}$$

Next we construct the Mayer-Vietoris sequence for compactly supported cohomology. Notice that if $U, V \subset M$ are open sets with $U \cup V = M$, the inclusions $U, V \hookrightarrow M$, $U \cap V \hookrightarrow U$ and $U \cap V \hookrightarrow V$ give a short exact sequence

$$0 \longleftarrow \Omega_c^\bullet(M) \longleftarrow \Omega_c^\bullet(U) \oplus \Omega_c^\bullet(V) \longleftarrow \Omega_c^\bullet(U \cap V) \longleftarrow 0$$

where the first map is:

$$(\theta, \eta) \mapsto \theta + \eta,$$

while the second map is:

$$\omega \mapsto (-\omega, \omega).$$

Hence, it follows that

Theorem 22.13 (Mayer-Vietoris sequence for compactly supported cohomology)**.** *Let M be a smooth manifold and $U, V \hookrightarrow M$ open subsets such that $M = U \cup V$. There exists a long exact sequence*

$$\longleftarrow H^k_c(M) \longleftarrow H^k_c(U) \oplus H^k_c(V) \longleftarrow H^k_c(U \cap V) \xleftarrow{\delta} H^{k-1}_c(M) \longleftarrow$$

We leave the details of the argument for the exercises.

Notice that in the Mayer-Vietoris sequence for compact supported cohomology the inclusions $U, V \hookrightarrow M$, $U \cap V \hookrightarrow U$ and $U \cap V \hookrightarrow V$ induce maps in the same direction, while for the ordinary de Rham cohomology the inclusions are reversed in the sequence. In the next section we will relate these two cohomology theories, and this will explain all the differences of behavior that we have just discussed.

Homework.

1. Let $h : \Omega^k(M \times \mathbb{R}) \to \Omega^{k-1}(M \times \mathbb{R})$ be the homotopy operator used in the proof of Proposition 22.1. Show that h can be written in either of the following more invariant forms:
   (a) If $E = t\frac{\partial}{\partial t} \in \mathfrak{X}(M \times \mathbb{R})$ is the Euler vector field and $\psi^s : M \times \mathbb{R} \to M \times \mathbb{R}$ the family of maps $\psi^s(x,t) = (x, st)$, then:
   $$h(\theta) = \int_0^1 \frac{1}{s}(\psi^s)^* i_E \theta \, \mathrm{d}s,$$
   (b) If E is the Euler vector field and $\phi^s_E(x,t) = (x, e^s t)$ its flow, then:
   $$h(\theta) = \int_{-\infty}^0 (\phi^s_E)^* i_E \theta \, \mathrm{d}s,$$
   (c) Use the second expression and Cartan Calculus to prove that h verifies (22.1).
   (Hint: The flow ϕ^s_X of a vector field X satisfies: $\frac{\mathrm{d}}{\mathrm{d}s}(\phi^s_X)^*\omega = (\phi^s_X)^* \mathcal{L}_X \omega$.)
2. Show that a star shaped open set is contractible.
3. Let $i : N \hookrightarrow M$ be a submanifold. We say that a map $r : M \to N$ is a **retraction of** M **in** N if $r \circ i = \mathrm{id}_N$ and that N is a **deformation retract** of M if there exists a retraction $r : M \to N$ such that $i \circ r$ is homotopic to id_M. Show that:
   (a) If N is a deformation retract of M, then $H^\bullet(N) \simeq H^\bullet(M)$.
   (b) Show that $\mathbb{S}^2$ is a deformation retract of $\mathbb{R}^3 - 0$.
   (c) Show that $\mathbb{T}^2$, viewed as a submanifold of $\mathbb{R}^3$ as in Example 7.8.2, is a deformation retract of $\mathbb{R}^3 - \{L \cup S\}$ where L is the z-axis and S is the circle in the xy-plane of radius R and center the origin.
4. In Remark 22.9, show that the connecting homomorphism in the long exact sequence satisfies $\operatorname{Im} g = \operatorname{Ker} \mathrm{d}^*$ and $\operatorname{Im} \mathrm{d}^* = \ker f$.

5. Given a long exact sequence of vector spaces

$$0 \longrightarrow C^0 \longrightarrow \cdots \longrightarrow C^{k-1} \longrightarrow C^k \longrightarrow \cdots \longrightarrow C^d \longrightarrow 0$$

show that:

$$\sum_{i=0}^{d} (-1)^i \dim C^i = 0.$$

6. Compute the cohomology of $\mathbb{T}^2$ and $\mathbb{RP}^2$.

7. Complete the construction of the Mayer-Vietoris sequence for compactly supported cohomology, by showing that:

$$0 \longleftarrow \Omega_c^\bullet(M) \longleftarrow \Omega_c^\bullet(U) \oplus \Omega_c^\bullet(V) \longleftarrow \Omega_c^\bullet(U \cap V) \longleftarrow 0$$

is a short exact sequence of complexes.

8. Compute the compactly supported cohomology of $\mathbb{R}^d - 0$.
(Hint: Apply the Mayer-Vietoris sequence to $M = \mathbb{S}^d$, $U = \mathbb{S}^d - p_N$ and $V = \mathbb{S}^d - p_S$.)

23. Computations in Cohomology

In the previous section we constructed the Mayer-Vietoris sequence relating the cohomology of the union of open sets with the cohomology of its factors. This sequence leads to a very useful technique to compute cohomology by induction, which also allows to extract many properties of cohomology. In order to apply it, we need to cover M by open sets whose intersections have trivial cohomology.

Definition 23.1. *An open cover $\{U_\alpha\}$ of a smooth manifold M is called a* ***good cover*** *if all finite intersections $U_{\alpha_1} \cap \cdots \cap U_{\alpha_k}$ are diffeomorphic to $\mathbb{R}^d$. We say that M is a* ***manifold of finite type*** *if it admits a finite good cover.*

Proposition 23.2. *Every smooth manifold M admits a good cover. If M is compact then it admits a finite good cover.*

Sketch of proof. [5] Let g be a Riemannian metric for M. In Riemannian geometry one shows that each point $p \in M$ has a strong geodesically convex neighborhood U_p, i.e., a neighborhood such that for any two points $q, q' \in U_p$ there exists a unique (length minimizing) geodesic in U_p which connects q and q'. Then one checks that:

(i) a strong geodesically convex open set is diffeomorphic to $\mathbb{R}^d$, and
(ii) the intersection of two strong geodesically convex open sets is a strong geodesically convex open set.

[5]This proof requires some knowledge of Riemannian geometry. If you are not familiar with the notion of geodesics, you may wish to skip the proof and admit the result as valid.

It follows that a cover $\{U_p\}_{p\in M}$ by strong geodesically convex open neighborhoods is a good cover of M.

If M is compact, then any good cover has a finite subcover which is also good. $\square$

Finite dimensional cohomology. We can use good covers and the Mayer-Vietoris sequence to show that the cohomology is often finite dimensional:

Theorem 23.3. *If M is a manifold of finite type then the cohomology spaces $H^k(M)$ and $H^k_c(M)$ have finite dimension.*

Proof. For any two open sets U and V, the Mayer-Vietoris sequence:

$$\cdots \longrightarrow H^{k-1}(U\cap V) \xrightarrow{\delta} H^k(U\cup V) \xrightarrow{r} H^k(U)\oplus H^k(V) \longrightarrow \dots$$

shows that:

$$H^k(U\cup V) \simeq \operatorname{Im}\delta \oplus \operatorname{Im} r.$$

Hence, if the cohomologies of U, V and $U\cap V$ are finite dimensional, then so is the cohomology of $U\cup V$.

Now we can use induction on the number of open sets in a cover, to show that manifolds which admit a finite good cover have finite dimensional cohomology:

- If M is diffeomorphic to $\mathbb{R}^d$ the Poincaré Lemma shows that M has finite dimensional cohomology.
- Now assume that all manifolds admitting a good cover with at most n open sets have finite dimensional cohomology. Let M be manifold which admits a good cover with $n+1$ open sets $\{U_1,\dots,U_{n+1}\}$. We observe that the open sets:

  $$\begin{aligned}&U_{n+1},\\ &U_1\cup\cdots\cup U_n, \text{ and}\\ &(U_1\cup\cdots\cup U_n)\cap U_{n+1} = (U_1\cap U_{n+1})\cup\cdots\cup(U_n\cap U_{n+1}),\end{aligned}$$

  all have finite dimensional cohomology, since they all admit a good cover with at most n open sets. Hence, the cohomology of $M = U_1\cup\cdots\cup U_{n+1}$ is also finite dimensional.

The proof for compactly supported cohomology is similar. $\square$

Poincaré duality. Recall (see the exercises in Section 20) that the exterior product induces a ring structure in cohomology:

$$\cup : H^k(M)\times H^l(M) \to H^{k+l}(M),\ [\omega]\cup[\eta] \equiv [\omega\wedge\eta].$$

Obviously, if η has compact support then $\omega\wedge\eta$ also has compact support, hence we obtain also a "product":

$$\cup : H^k(M)\times H^l_c(M) \to H^{k+l}_c(M).$$

Stokes formula shows that the integral of differential forms descends to the level of cohomology. Hence, if M is an oriented manifold of dimension d we obtain a bilinear form

$$H^k(M) \times H_c^{d-k}(M) \to \mathbb{R},\ ([\omega],[\eta]) \mapsto \int_M \omega \wedge \eta. \tag{23.1}$$

Theorem 23.4 (Poincaré duality)**.** *If M is an oriented manifold of finite type the bilinear form (23.1) is non-degenerate. In particular:*

$$H^k(M) \simeq H_c^{d-k}(M)^*.$$

Remark 23.5 (*A Crash Course in Homological Algebra - part V*)**.** For the proof of Poincaré duality we turn once more to Homological Algebra.

Lemma 23.6 (Five Lemma)**.** *Consider a commutative diagram of homomorphisms of vector spaces:*

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "\alpha"'] & B \arrow[r, "f_2"] \arrow[d, "\beta"'] & C \arrow[r, "f_3"] \arrow[d, "\gamma"'] & D \arrow[r, "f_4"] \arrow[d, "\delta"'] & E \arrow[d, "\varepsilon"'] \\
A' \arrow[r, "f_1'"] & B' \arrow[r, "f_2'"] & C' \arrow[r, "f_3'"] & D' \arrow[r, "f_4'"] & E'
\end{tikzcd}$$

where the rows are exact. If α, β, δ and ε are isomorphisms, then γ is also an isomorphism.

The proof of this lemma is by diagram chasing and is left as an easy exercise.

Proof of Theorem 23.4. Let us start by observing that the bilinear form (23.1) gives always a linear map $H^k(M) \to H_c^{d-k}(M)^*$. If U and V are open sets, one checks easily that the Mayer-Vietoris sequence for $\Omega^\bullet$ and $\Omega_c^\bullet$, give a diagram of exact sequences:

$$\begin{tikzcd}
{} \arrow[r] & H^k(U\cup V) \arrow[r] \arrow[d] & H^k(U)\oplus H^k(V) \arrow[r] \arrow[d] & H^k(U\cap V) \arrow[r, "\delta"] \arrow[d] & H^{k+1}(U\cup V) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & H_c^{d-k}(U\cup V)^* \arrow[r] & H_c^{d-k}(U)^*\oplus H_c^{d-k}(V)^* \arrow[r] & H_c^{d-k}(U\cap V)^* \arrow[r, "\delta^*"] & H_c^{d-k-1}(U\cup V)^* \arrow[r] & {}
\end{tikzcd}$$

which commutes up to signs: for example, we have

$$\int_{U\cap V} \omega \wedge \delta\theta = \pm \int_{U\cup V} \delta\omega \wedge \tau.$$

If we apply the Five Lemma to this diagram, we conclude that if Poincaré duality holds for U, V and $U \cap V$, then it also holds for $U \cup V$.

Now let M be a manifold with a finite good cover. We show that Poincaré duality holds using induction on the cardinality of the cover:

- If $M \simeq \mathbb{R}^d$, the Poincaré Lemmas give:

$$H^k(\mathbb{R}^d) = \begin{cases} \mathbb{R} & \text{if } k=0, \\ \\ 0 & \text{if } k \neq 0. \end{cases} \qquad H_c^k(\mathbb{R}^d) = \begin{cases} \mathbb{R} & \text{if } k=d, \\ \\ 0 & \text{if } k \neq d. \end{cases}$$

Also, $(\cdot,\cdot) : H^0(\mathbb{R}^d) \times H_c^d(\mathbb{R}^d) \to \mathbb{R}$ is non-zero, so the bilinear form is non-degenerate in this case.

- Now assume that Poincaré duality holds for any manifold admitting a good cover with at most n open sets. If M is a manifold which admits an open cover $\{U_1, \ldots, U_{n+1}\}$ with $n+1$ open sets, we note that the open sets:

$$\begin{aligned}&U_{n+1},\ U_1 \cup \cdots \cup U_n, \text{ and}\\ &(U_1 \cup \cdots \cup U_n) \cap U_{n+1} = (U_1 \cap U_{n+1}) \cup \cdots \cup (U_n \cap U_{n+1}),\end{aligned}$$

all satisfy Poincaré duality, since they all admit a good cover with at most n open sets. It follows that $M = U_1 \cup \cdots \cup U_{n+1}$ also satisfies Poincaré duality.

□

If M is a compact manifold, we have $H_c^\bullet(M) = H^\bullet(M)$. Hence:

Corollary 23.7. *If M is a compact oriented manifold then:*

$$H^k(M) \simeq H^{d-k}(M).$$

Remark 23.8. One can show that Poincaré duality still holds for oriented manifolds which do not admit a finite good cover: when the cohomology of M is not finite dimensional, one still has an isomorphism:

$$H^k(M) \simeq (H_c^{d-k}(M))^*.$$

However, in general, *one does not have* a dual isomorphism $H_c^{d-k}(M) \simeq H^k(M)^*$. The reason is that while the dual of direct product is a direct sum, the dual of an infinite direct sum is not a direct product. We discuss an example in the exercises.

Because of the previous remark, in the next corollary we omit the assumption that M has a finite good cover.

Corollary 23.9. *Let M be a connected manifold of dimension d. Then:*

$$H_c^d(M) \simeq \begin{cases} \mathbb{R} & \text{if } M \text{ is orientable,} \\ \\ 0 & \text{if } M \text{ is not orientable.} \end{cases}$$

In particular, if M is compact and connected of dimension d, then M is orientable if and only if $H^d(M) \simeq \mathbb{R}$.

Proof. By Poincaré duality, if M is a connected orientable manifold of dimension d, then $(H_c^d(M))^* \simeq H^0(M) \simeq \mathbb{R}$. We leave the proof of the converse to the exercises. □

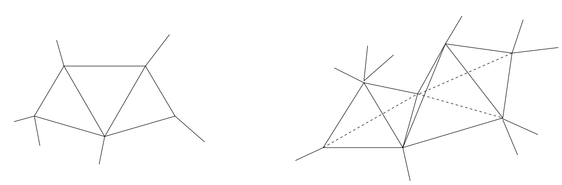
Triangulations and Euler's formula. As another application of the Mayer-Vietoris sequence, we show how the familiar Euler's formula for regular polygons can be extended to any compact manifold M admitting a *triangulation*, i.e., a nice decomposition of M into regular simplices as we now explain([6]).

A **regular simplex** is a simplex $\sigma : \Delta^d \to M$ which can be extended to a diffeomorphism $\tilde{\sigma} : U \to \tilde{\sigma}(U) \subset M$, where U is some open neighborhood of Δ^d. We have defined before the $(d-1)$-dimensional faces of a simplex $\sigma : \Delta^d \to M$. For a regular simplex, these are regular $(d-1)$-simplices $\varepsilon_i(\sigma) : \Delta^{d-1} \to M$ of dimension $(d-1)$. By iterating this construction we obtain the $d-k$**-dimensional faces** of a simplex, which are regular $(d-k)$-simplices $\varepsilon_{i_1,i_2,\dots,i_{d-k}}(\sigma) : \Delta^{d-k} \to M$.

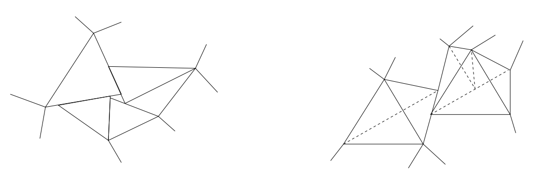
Definition 23.10. *A* ***triangulation*** *of a compact manifold M of dimension d is a finite collection $\{\sigma_i\}$ of regular d-simplices such that:*

(i) the collection $\{\sigma_i\}$ covers M, and
(ii) if two simplices in $\{\sigma_i\}$ have non-empty intersection, then there intersection $\sigma_i \cap \sigma_j$ is a face of both simplices σ_i and σ_j.

The next figure illustrates condition (ii) in this definition for dimensions 2 and 3.



$d = 2$ $\qquad\qquad$ $d = 3$



Notice that on the top the condition is satisfied while on the bottom the condition fails.

If M is a manifold with finite dimensional cohomology (e.g., if M is compact) one defines the **Euler characteristic** of M to be the integer $\chi(M)$ given by:

$$\chi(M) = \dim H^0(M) - \dim H^1(M) + \cdots + (-1)^d \dim H^d(M).$$

Applying Poincaré duality, we have that:

[6]Actually, one can show that every smooth compact manifold can be triangulated. This result is very technical and we will not discuss it in these notes.

Corollary 23.11. *If M is a compact oriented odd dimensional manifold then $\chi(M) = 0$.*

For other manifolds the Euler characteristic, in general, is non-zero and can be computed using a triangulation:

Theorem 23.12 (Euler's Formula)**.** *If M is a compact manifold of dimension d, for any triangulation we have:*

$$(-1)^d \chi(M) = r_0 - r_1 + \cdots + (-1)^d r_d,$$

where r_i denotes the number of faces of dimension i of the triangulation.

Proof. Let us fix a triangulation $\{\sigma_1, \sigma_2, \ldots, \sigma_{r_d}\}$ of M and define open sets:

$$V_k := M - \{k\text{-faces of the triangulation}\}.$$

We claim that for $0 \le k \le d-1$ we have:

$$\chi(M) = \chi(V_k) + (-1)^d (r_0 - r_1 + \cdots + (-1)^k r_k). \tag{23.2}$$

Assuming this claim, since

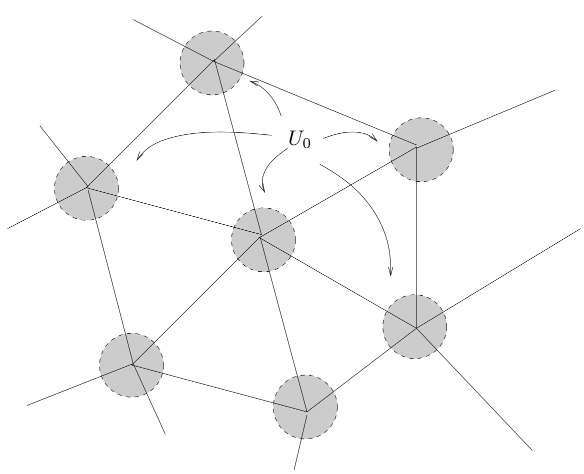
$$V_{d-1} = \bigcup_{j=1}^{r_d} \operatorname{int}(\sigma_j),$$

and each open set $\operatorname{int}(\sigma_j)$ is contractible, we have $H^k(V_{d-1}) = 0$, for $k > 0$. Hence:

$$\chi(V_{d-1}) = \dim H^0(V_{d-1}) = r_d.$$

This identity, together with (23.2), show that Euler's formula holds.

Lets us start by verifying (23.2) for $k = 0$. For each 0-dimensional face we can choose disjoint open neighborhoods $U_{0,1}, \ldots, U_{0,r_0}$, each diffeomorphic to the open ball $B^d_1 = \{x \in \mathbb{R}^d : ||x|| < 1\}$.



We set

$$U_0 := \bigcup_{i=0}^{r_0} U_{0,i}.$$

Notice that $V_0 \cup U_0 = M$. Since each $U_{0,i}$ is contractible, we have:

$$\dim H^k(U_0) = \begin{cases} r_0, & \text{if } k = 0, \\ 0, & \text{if } k \neq 0. \end{cases}$$

On the other hand, the intersection $V_0 \cap U_{0,i}$ deformation retracts in $\mathbb{S}^{d-1}$, hence

$$\dim H^k(V_0 \cap U_0) = \begin{cases} r_0, & \text{if } k = 0, d-1, \\ 0, & \text{if } k \neq 0, d-1. \end{cases}$$

We can apply the Mayer-Vietoris argument to the pair (U_0, V_0): if $d > 2$, this sequence gives the following information:

(i) The lowest degree terms in the sequence are:

$$0 \longrightarrow H^0(M) \longrightarrow H^0(U_0) \oplus H^0(V_0) \longrightarrow H^0(U_0 \cap V_0) \longrightarrow$$
$$\longrightarrow H^1(M) \longrightarrow 0 \oplus H^1(V_0) \longrightarrow 0$$

so it follows that:

$$\dim H^0(M) - \dim H^0(U_0) - \dim H^0(V_0) + \\ + \dim H^0(U_0 \cap V_0) - \dim H^1(M) + \dim H^1(V_0) = 0.$$

Since M and V_0 have the same number of connected components we find

$$\dim H^0(M) = \dim H^0(V_0).$$

On the other hand, the number of connected components of U_0 and $V_0 \cap U_0$ are also the same, hence we conclude that:

$$\dim H^1(M) = \dim H^1(V_0).$$

(ii) For $1 < k < d-1$, the Mayer-Vietoris sequence gives:

$$0 \longrightarrow H^k(M) \longrightarrow 0 \oplus H^k(V_0) \longrightarrow 0$$

Hence:

$$\dim H^k(M) = \dim H^k(V_0).$$

(iii) Finally, the last terms in the sequence give:

$$0 \longrightarrow H^{d-1}(M) \longrightarrow 0 \oplus H^{d-1}(V_0) \longrightarrow H^{d-1}(U_0 \cap V_0) \longrightarrow$$
$$\longrightarrow H^d(M) \longrightarrow 0 \oplus H^d(V_0) \longrightarrow 0$$

Since $\dim H^{d-1}(U_0 \cap V_0) = r_0$, we conclude that:

$$\dim H^{d-1}(M) - \dim H^{d-1}(V_0) + \dim H^{d-1}(V_0) - \dim H^d(M) = -r_0.$$

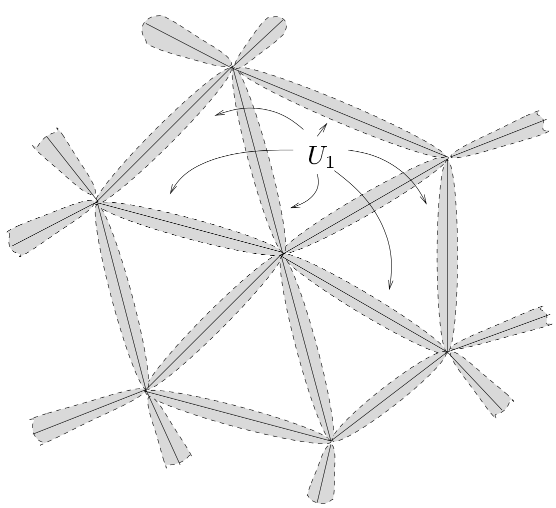
When $d = 2$, we obtain exactly the same results except that we can consider the whole sequence at once. In an any case, we conclude that:

$$\begin{aligned}\chi(M) &= \sum_{i=0}^{d} (-1)^i \dim H^i(M) \\ &= \sum_{i=0}^{d} (-1)^i \dim H^i(V_0) + (-1)^d r_0 = \chi(V_0) + (-1)^d r_0.\end{aligned}$$

which yields (23.2) if $k = 0$.

In order to prove (23.2) when $k = 1$, we can proceed as follows: for each 1-face we choose open disjoint neighborhoods $U_{1,1}, \ldots, U_{1,r_1}$ of the (1-faces)-(0-faces), diffeomorphic to $(\text{int } \Delta^1) \times B_1^{d-1}$, and we define the open set:

$$U_1 = \bigcup_{i=0}^{r_1} U_{1,i}.$$



We have that $V_0 = U_1 \cup V_1$. Moreover, U_1 is a disjoint union of r_1 contractible open sets, while $U_1 \cap V_1$ as the same homotopy type as the disjoint union of $(d-2)$-spheres This allows one to show, exactly like in the case $k = 0$, that the Mayer-Vietoris sequence yields:

$$\chi(V_0) = \chi(V_1) + (-1)^{d-1} r_1.$$

In general, for each k, we choose open disjoint neighborhoods $U_{k,1}, \ldots, U_{k,r_k}$ of $\{k\text{-faces}\} - \{(k-1)\text{-faces}\}$, diffeomorphic to $(\text{int } \Delta^k) \times B_1^{d-k}$, and we define the open set:

$$U_k = \bigcup_{i=0}^{r_k} U_{k,i}.$$

We have that $V_{k-1} = U_k \cup V_k$, where U_k is a union of r_k contractible open sets, while $U_k \cap V_k$ as the same homotopy type as the disjoint union of $(d-k-1)$-spheres. The Mayer-Vietoris sequence then shows that:

$$\chi(V_{k-1}) = \chi(V_k) + (-1)^{d-k} r_k.$$

This proves (23.2) and finishes the proof of Euler's formula. $\square$

Homework.

1. Given an example of a connected manifold which is not of finite type.

2. Prove the Five Lemma and find weaker conditions on the maps α, β, ε and δ, so that the conclusion still holds.

3. Check the commutativity, up to signs, of the diagram of long exact sequences that appears in the proof of Poincaré duality.

4. Let M be a connected manifold of dimension d, which is not orientable. Prove that

$$H^d_c(M) = 0,$$

by proceeding as follows. Let $\widetilde{M}$ denote the set of orientations for all the tangent spaces T_pM:

$$\widetilde{M} = \{(p, [\mu_p]) : [\mu_p] \text{ is an orientation for } T_pM\}.$$

One calls $\widetilde{M}$ the **orientation cover of** M. Show that:
   (a) $\widetilde{M}$ is a connected orientable manifold of dimension d;
   (b) The map:
   $$\pi : \widetilde{M} \to M, \quad (p, [\mu_p]) \mapsto p,$$
   is a double cover, i.e., each $p \in M$ has a neighborhood U such that $\pi^{-1}(U) = V_1 \cup V_2$ (disjoint) and $\pi|_{V_i} : V_i \to U$ is a diffeomorphism;
   (c) The map $\Phi : \widetilde{M} \to \widetilde{M}$, $(p, [\mu_p]) \mapsto (p, -[\mu_p])$ is a diffeomorphism that changes orientation and satisfies:
   $$\pi = \pi \circ \Phi, \quad \Phi \circ \Phi = \mathrm{id};$$
   (d) Given $\widetilde{\omega} \in \Omega^k(\widetilde{M})$, there exists $\omega \in \Omega^k(M)$ such that $\widetilde{\omega} = \pi^*\omega$ if and only if $\Phi^*\widetilde{\omega} = \widetilde{\omega}$;
   (e) Conclude that one must have $H^d_c(M) = 0$.

5. Let $M_1, M_2, \ldots,$ be orientable manifolds of finite type of dimension d and consider the disjoint union of the M_i:

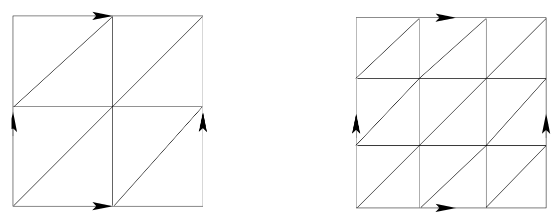
$$M = \bigcup_{i=1}^{+\infty} M_i.$$

Show that:
   (a) $H^k(M) = \prod_{i=1}^{+\infty} H^k(M_i)$;

(b) $H_c^k(M) = \bigoplus_{i=1}^{+\infty} H_c^k(M_i)$;
(c) Conclude that there exists an isomorphism: $H^k(M) \simeq (H_c^{d-k}(M))^*$;
(d) Give an example of an orientable M with $H_c^{d-k}(M)$ not isomorphic to $H^k(M)^*$.

6. Compute $H^k(M)$ and $H_c^k(M)$ for the following manifolds:
(a) Möbius band;
(b) Klein bottle;
(c) The d-torus;
(ANSWER: $\dim H^k(\mathbb{T}^d) = \binom{d}{k}$.)
(d) Complex projective space;
(ANSWER: $\dim H^{2k}(\mathbb{CP}^d) = 1$ if $2k \leq d$, and 0 otherwise.)

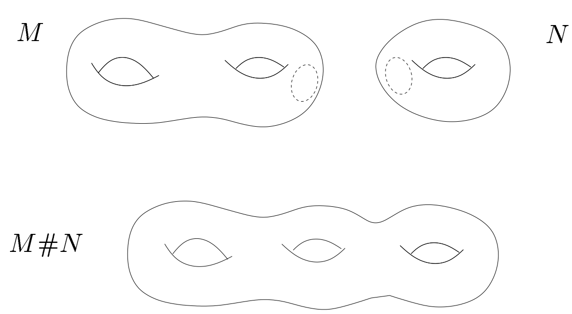
7. Consider the following two subdivisions of the square $[0,1] \times [0,1]$:



(a) Verify that only one of these subdivisions induces a triangulation of the 2-torus $\mathbb{T}^2$;
(b) Compute r_0, r_1 and r_2 for this triangulation.

8. Let M and N be connected compact manifolds of dimension d. Let $M\#N$ be the **connected sum** of M and N, i.e., the manifold obtained by gluing M and N along the boundary of open sets $U \subset M$ and $V \subset N$ both diffeomorphic to the ball $\{x \in \mathbb{R}^d : ||x|| < 1\}$. Show that the Euler characteristics satisfy:

$$\chi(M\#N) = \chi(M) + \chi(N) - \chi(\mathbb{S}^d).$$



Conclude that the Euler characteristic of a compact, oriented, surface of genus g (i.e., with g holes) is $2 - 2g$.

24. The Degree and the Index

We saw in the previous section that a connected manifold M of dimension d is orientable if and only if $H_c^d(M) \simeq \mathbb{R}$. Notice that a choice of orientation for M determines a generator of $H_c^d(M)$. In fact, in this case, integration gives an isomorphism $H_c^d(M) \simeq \mathbb{R}$ by:

$$H_c^d(M) \to \mathbb{R}, \ [\omega] \mapsto \int_M \omega.$$

By the way, this isomorphism is just Poincaré duality, since M being connected $H^0(M)$ is the space of constant functions in M. In the sequel, we will often use the same symbol μ_M to denote the orientation of M and the generator $\mu_M \in H_c^d(M)$ that corresponds to the constant function 1.

Let $\Phi : M \to N$ be a proper map between connected, oriented manifolds of the same dimension: $\dim M = \dim N = d$. The canonical isomorphisms $H_c^d(M) \simeq \mathbb{R}$ and $H_c^d(N) \simeq \mathbb{R}$ give a representation of the induced map in cohomology

$$\Phi^* : H_c^d(N) \to H_c^d(M)$$

as a real number which one calls the *degree of the map*. In other words:

Definition 24.1. *Let $\Phi : M \to N$ be a proper map between connected, oriented manifolds of the same dimension d. The **degree** of Φ is the unique real number* $\deg \Phi$ *such that:*

$$\int_M \Phi^* \omega = \deg \Phi \int_N \omega,$$

for every differential form $\omega \in \Omega_c^d(N)$.

Our aim is to give a geometric characterization of the degree of map, which allows also for its computation. For simplicity, we will consider only the case where both manifolds are compact. You may wish to try to extend these results to any proper map. We start with the following property:

Proposition 24.2. *Let $\Phi : M \to N$ be a smooth map between compact, connected, oriented manifolds of the same dimension d. If Φ is not surjective then* $\deg \Phi = 0$.

Proof. Let $q_0 \in N - \Phi(M)$. Since $\Phi(M)$ is closed, there is an open neighborhood of q_0 such that $U \subset N - \Phi(M)$. Let $\omega \in \Omega^d(N)$ have its support in U be such that $\int_N \omega \neq 0$. Then:

$$0 = \int_M \Phi^* \omega = \deg \Phi \int_N \omega,$$

hence $\deg \Phi = 0$. □

We can now give a geometric interpretation of the degree of a map. This interpretation also shows that the degree is always an integer, something which is not obvious from our definition of the degree.

Theorem 24.3. *Let $\Phi : M \to N$ be a smooth map between compact, connected, oriented manifolds of the same dimension d. Let $q \in N$ be a regular value of Φ and for each $p \in \Phi^{-1}(q)$ define*

$$\operatorname{sgn}_p \Phi \equiv \begin{cases} 1 & \text{if } \mathrm{d}_p\Phi : T_pM \to T_qN \text{ preserves orientations,} \\ \\ -1 & \text{if } \mathrm{d}_p\Phi : T_pM \to T_qN \text{ switches orientations.} \end{cases}$$

Then [7]*:*

$$\deg \Phi = \sum_{p\in\Phi^{-1}(q)} \operatorname{sgn}_p \Phi.$$

In particular, the degree is an integer.

Proof. Let q be a regular value of Φ. If $\Phi^{-1}(q)$ is empty, then Φ is not surjective and the result follows from the previous proposition. On the other hand, if $\Phi^{-1}(q)$ is non-empty then it is a discrete subset of M which, by compactness, must be finite: $\Phi^{-1}(q) = \{p_1, \dots, p_N\}$. We need the following lemma:

Lemma 24.4. *There exists a neighborhood V of q and disjoint neighborhoods $U_1, \dots, U_N$ of $p_1, \dots, p_N$ such that*

$$\Phi^{-1}(V) = U_1 \cup \cdots \cup U_N.$$

Assuming that this lemma holds, let V and $U_1, \dots, U_N$ be as in its statement. Since each p_i is a regular point, we can assume, additionally that V is the domain of a chart $(y^1, \dots, y^d)$ in N and that the restrictions $\Phi|_{U_i} : U_i \to V$ are diffeomorphisms.

Let:

$$\omega = f \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^d \in \Omega^d(N),$$

where $f \geq 0$ has $\operatorname{supp} f \subset V$. Obviously, we have

$$\operatorname{supp} \Phi^*\omega \subset U_1 \cup \cdots \cup U_N,$$

so we find:

$$\int_M \Phi^*\omega = \sum_{i=1}^N \int_{U_i} \Phi^*\omega.$$

Since each $\Phi|_{U_i}$ is a diffeomorphism, the change of variables formula gives:

$$\int_{U_i} \Phi^*\omega = \pm \int_V \omega = \pm \int_N \omega,$$

where the sign is positive if $\Phi|_{U_i}$ preserves orientations and negative otherwise. Since $\Phi|_{U_i}$ preserves orientations if $\operatorname{sgn}_{p_i} \Phi > 0$ and switches orientations if $\operatorname{sgn}_{p_i} \Phi < 0$, we conclude that

$$\int_M \Phi^*\omega = \sum_{i=1}^N \operatorname{sgn}_{p_i} \Phi \int_N \omega.$$

[7] If $\Phi^{-1}(q)$ is empty then q is a regular value and we convention that the sum is zero.

To finish the proof it remains to prove the lemma. Let $O_1, \dots, O_N$ be any disjoint open neighborhoods of $p_1, \dots, p_N$, and W a compact neighborhood of q. The set $\widetilde{W} \subset M$ defined by:

$$\widetilde{W} = \Phi^{-1}(W) - (O_1 \cup \dots \cup O_N),$$

is compact. Hence, $\Phi(\widetilde{W})$ is a closed set which does not contain q. Therefore, there exists an open set $V \subset W - \Phi(\widetilde{W})$, containing q, and we have $\Phi^{-1}(V) \subset O_1 \cup \dots \cup O_N$. If we let $U_i = O_i \cap \Phi^{-1}(V)$, we see that the lemma holds. $\square$

The degrees of two homotopic maps coincide, since homotopic maps induce the same map in cohomology. This is a very useful fact in computing degrees, and can be explored to deduce global properties of manifolds. A classic illustration of this is given in the next example.

EXAMPLE 24.5.
Consider the antipodal map $\Phi : \mathbb{S}^d \to \mathbb{S}^d$, $p \mapsto -p$. For the canonical orientation of the sphere $\mathbb{S}^d$ defined by the form

$$\omega = \sum_{i=1}^{d+1} (-1)^{i+1} x^i \mathrm{d}x^1 \wedge \dots \wedge \widehat{\mathrm{d}x^i} \wedge \dots \wedge \mathrm{d}x^{d+1}.$$

we see that Φ preserves or switches orientations if d is odd or even, respectively. Since $\Phi^{-1}(q)$ contains only one point, we conclude that

$$\deg \Phi = (-1)^{d-1}.$$

By the way, we could also compute the degree directly from the definition, since we have

$$\int_{\mathbb{S}^d} \Phi^* \omega = (-1)^{d-1} \int_{\mathbb{S}^d} \omega.$$

We claim that we can use this fact to show that every vector field on a even dimensional sphere vanishes at some point. In fact, let $X \in \mathfrak{X}(\mathbb{S}^{2d})$ be a nowhere vanishing vector field. Then for each $p \in \mathbb{S}^{2d}$ there exists a unique semi-circle γ_p joining p to $-p$ with $\gamma_p'(0) = X|_p$. It follows that the map $H : \mathbb{S}^{2d} \times [0,1] \to \mathbb{S}^{2d}$ given by

$$H(p,t) = \gamma_p(t),$$

is a homotopy between Φ and the identity map. Hence,

$$-1 = \deg \Phi = \deg(\mathrm{id}) = 1,$$

a contradiction.

You should notice that, in contrast, any odd degree $\mathbb{S}^{2d-1} \subset \mathbb{R}^{2d}$ admits the vector field:

$$X = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2} + \dots + x^{2d} \frac{\partial}{\partial x^{2d-1}} - x^{2d-1} \frac{\partial}{\partial x^{2d}},$$

which is a nowhere vanishing vector field.

As another application of degree theory, we will introduce now the index of a vector field, which will eventually lead to a geometric formula for the Euler characteristic of a manifold, known as the Poincaré-Hopf Theorem.

Consider first a vector field X defined in an open set $U \subset \mathbb{R}^d$ which has an isolated zero at $x_0 \in U$. We can view it as a map $X : U \to \mathbb{R}^d$ which vanishes at x_0 and is non-zero in a deleted neighborhood $V - \{x_0\}$. Let $D_\varepsilon(x_0) \subset U$ be a closed disk of radius ε centered at x_0 which does not contain any other zero of X and let S_ε be the sphere of radius ε centered at x_0:

$$S_\varepsilon := \partial D_\varepsilon(x_0).$$

We define the **Gauss map** $G_\varepsilon : S_\varepsilon \to \mathbb{S}^{d-1}$ by:

$$G_\varepsilon(x) = \frac{X(x)}{||X(x)||}.$$

The **index** of X at x_0 is the degree of the Gauss map:

$$\operatorname{ind}_{x_0} X \equiv \deg G_\varepsilon,$$

where on both spheres we consider the induced orientation from $\mathbb{R}^d$.

Our next result states that the degree is independent of ε and is a diffeomorphism invariant:

Proposition 24.6. *Let $U \subset \mathbb{R}^d$ be open and let $X \in \mathfrak{X}(U)$ a vector field with an isolated zero at x_0.*

(i) Any two Gauss maps G_{ε_0} and G_{ε_1} have the same degree.

(ii) If $\Phi : U \to U'$ a diffeomorphism and $X' = \Phi_* X$ then

$$\operatorname{ind}_{x_0} X = \operatorname{ind}_{\Phi(x_0)} X'.$$

Proof. We leave (i) as an exercise. To prove (ii), we can assume that $\Phi(x_0) = x_0 = 0$ and that U is star shaped with center 0.

Assume first that Φ preserves orientations. Then the map

$$H(t,x) = \begin{cases} \frac{1}{t}\Phi(tx), & \text{if } t > 0, \\ \\ \mathrm{d}_0\Phi(x), & \text{if } t = 0. \end{cases}$$

is a homotopy between Φ and $\mathrm{d}_0\Phi$, consisting of diffeomorphisms that fix the origin. Since $\mathrm{d}_0\Phi$ is homotopic to the identity, via diffeomorphisms that fix the origin, we see that there exists a homotopy, via diffeomorphisms that fix the origin, between Φ and the identity. Hence, the Gauss maps of X and X' are homotopic, so that the indices of X and X' coincide.

Now, the case where Φ switches orientations follows if we can prove the case where Φ is a reflection. In this case Φ is a linear map, so:

$$X' = \Phi_* X = \Phi \circ X \circ \Phi^{-1}.$$

The corresponding Gauss maps are then related by:

$$G'_\varepsilon = \Phi \circ G_\varepsilon \circ \Phi^{-1},$$

and, hence, their degrees coincide. $\square$

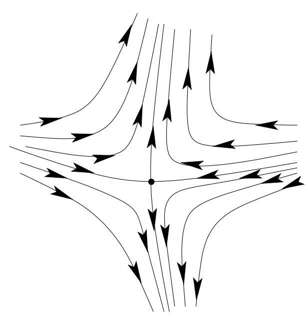
The proposition allows us to define the index for any vector field:

Definition 24.7. *If $X \in \mathfrak{X}(M)$ is a vector field with an isolated zero p_0, the* ***index of*** *X at $p_0 \in M$, is the number*

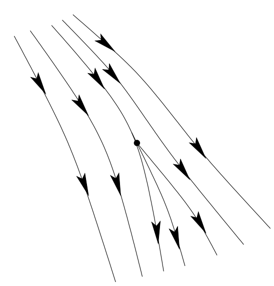
$$\operatorname{ind}_{p_0} X \equiv \operatorname{ind}_0 \phi_*(X|_U),$$

where (U, ϕ) is any coordinate system centered at p_0.

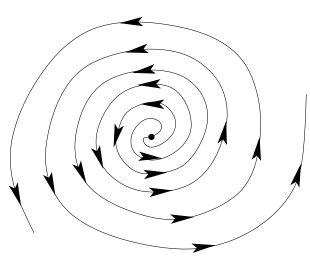
In some simple cases it is possible to determine the index of a vector field from its phase portrait, even if the zeros are degenerate. The pictures in the next page illustrate some examples of planar vector fields with a zero and the value of its index. You should try to check that the degree of the corresponding Gauss maps is indeed the integer in each figure.



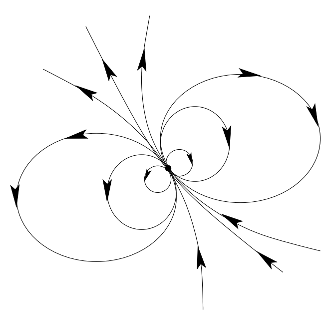
$\operatorname{ind}_{p_0} X = -1$



$\operatorname{ind}_{p_0} X = 0$



$\operatorname{ind}_{p_0} X = 1$



$\operatorname{ind}_{p_0} X = 2$

In general, it maybe hard to compute the index, but there is a method for so-called *non-degenerate* zeros of vector fields, as we now explain.

Let X be a vector field in a manifold M and let $p_0 \in M$ be a zero of X. The zero section $Z \subset TM$ and the fiber $T_{p_0}M \subset TM$ intersect transversely at $0 \in T_{p_0}M$, i.e., we have:

$$T_0(TM) = T_{p_0}Z \oplus T_{p_0}(T_{p_0}M) \simeq T_{p_0}M \oplus T_{p_0}M,$$

where the isomorphism arises from the differential $\mathrm{d}_{p_0}\pi$ of the projection $\pi : TM \to M$. Under this decomposition, the differential of the vector field:

$$\mathrm{d}_{p_0}X : T_{p_0}M \to T_0(TM).$$

has first component the identity (since $\pi \circ X = \mathrm{id}_M$), while the second component is a linear map $T_{p_0}M \to T_{p_0}M$. This linear map will be denoted also by $\mathrm{d}_{p_0}X$, and is called the **linear approximation** to X at the zero p_0. Notice that we can also view $\mathrm{d}_{p_0}X$ as a linear vector field on the tangent space $T_{p_0}M$.

In a chart $(U, x^1, \dots, x^d)$ centered at p_0 the vector field has a representation $X = \sum_{i=1}^d X^i \frac{\partial}{\partial x^i}$ where $X^i(0) = 0$. The linear approximation is the linear map $\mathrm{d}_{p_0}X$ whose matrix relative to the basis $\{ \frac{\partial}{\partial x^1}\big|_{p_0}, \dots, \frac{\partial}{\partial x^d}\big|_{p_0}\}$ is:

$$\mathrm{d}_{p_0}X = \left[\frac{\partial X^i}{\partial x^j}(0)\right]_{i,j=1}^d .$$

If we prefer to view $\mathrm{d}_{p_0}X$ as a linear vector field in $T_{p_0}M$, it has the expression:

$$\mathrm{d}_{p_0}X = \sum_{i,j=1}^d \frac{\partial X^i}{\partial x^j}(0)x^j \left.\frac{\partial}{\partial x^i}\right|_{p_0} .$$

Definition 24.8. *A zero p_0 of $X \in \mathfrak{X}(M)$ is called* ***non-degenerate*** *if the linear approximation $\mathrm{d}_{p_0}X : T_{p_0}M \to T_{p_0}M$ is invertible.*

Non-degenerate zeros are always isolated and their indices can be computed easily:

Proposition 24.9. *Let $p_0 \in M$ be a non-degenerate zero of a vector field $X \in \mathfrak{X}(M)$. Then p_0 is an isolated zero and:*

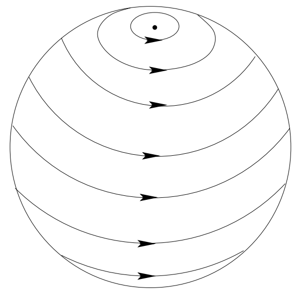
$$\operatorname{ind}_{p_0} X = \begin{cases} +1, & \text{if } \det \mathrm{d}_{p_0}X > 0, \\ \\ -1, & \text{if } \det \mathrm{d}_{p_0}X < 0. \end{cases}$$

Proof. Choose local coordinates (U, ϕ) centered at p_0. The vector field $\phi_*(X|_U)$ has an associated Gauss map $G : S_\varepsilon \to \mathbb{S}^{d-1}$ which is a diffeomorphism. Moreover, this diffeomorphism preserves (switches) orientations if and only if $\det \mathrm{d}_{p_0}X > 0$ (respectively, < 0). Hence the result follows from Theorem 24.3. $\square$

Example 24.10.
Consider $\mathbb{R}^3$ with coordinates (x, y, z). The vector field

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^3),$$

is tangent to the sphere $\mathbb{S}^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ and hence defines a vector field $X \in \mathfrak{X}(\mathbb{S}^2)$, with exactly two zeros: the north pole p_N and the south pole p_S.



The projection $\phi = (u, v) : (x, y, z) \mapsto (x, y)$ restricts on the upper and lower hemispheres to system of coordinates on $\mathbb{S}^2$ centered at p_N and p_S. We have:

$$\phi_* X = v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}.$$

The matrix representation of the linear approximation to X at p_N and p_S relative to the basis $\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\}$ is then given by:

$$\mathrm{d}_{p_N} X = \mathrm{d}_{p_S} X = \left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right].$$

We conclude that p_N and p_S are non-degenerate zeros and:

$$\operatorname{ind}_{p_N} X = \operatorname{ind}_{p_S} X = 1.$$

In the previous example, the sum of the indices of the zeros the vector field $X \in \mathfrak{X}(\mathbb{S}^2)$ equals 2, so it coincides with the value of the Euler characteristic of $\mathbb{S}^2$. This is an illustration of the following famous result:

Theorem 24.11 (Poincaré-Hopf)**.** *Let $X \in \mathfrak{X}(M)$ is a vector field on a compact manifold with a finite number of zeros $\{p_1, \ldots, p_N\}$. Then:*

$$\chi(M) = \sum_{i=1}^{N} \operatorname{ind}_{p_i} X.$$

This beautiful theorem connects the topology of M with its smooth structure, i.e., its tangent bundle. The proof will be given in the next chapter where we will study bundle theory.

Homework.

1. Let $\Phi : \mathbb{C} \to \mathbb{C}$ be a polynomial map of degree d. Find $\deg \Phi$.

2. Show that for a smooth manifold M of dimension $d > 0$ the identity map $M \to M$ is never homotopic to a constant map. Use this fact to prove that there is no retraction of the closed unit disk $D^d \subset \mathbb{R}^d$ on its boundary $\mathbb{S}^{d-1} = \partial D^d$.

   Hint: If there was such a retraction $r : D^d \to \mathbb{S}^{d-1}$ consider the map $H(x,t) = r(rx)$.

3. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix with integer entries. Identifying $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, consider the map $\Phi : \mathbb{T}^2 \to \mathbb{T}^2$ defined by:

$$\Phi([x, y]) = [ax + by, cx + dy].$$

Determine $\deg \Phi$.

4. Let $X : \mathbb{R}^d \to \mathbb{R}^d$ be a vector field with an isolated zero at $x = 0$ and associated Gauss map G_ε. Define $\widetilde{G}_\varepsilon : \mathbb{S}^{d-1} \to \mathbb{S}^{d-1}$ to be the composition of $G_\varepsilon : S_\varepsilon \to \mathbb{S}^{d-1}$ with the map $\mathbb{S}^{d-1} \to S_\varepsilon$, $x \mapsto \varepsilon x$.
   (a) Show that $\deg \widetilde{G}_\varepsilon = \deg G_\varepsilon$;
   (b) Show that for any $\varepsilon_0, \varepsilon_1 > 0$ the maps $\widetilde{G}_{\varepsilon_0}$ and $\widetilde{G}_{\varepsilon_1}$ are homotopic;
   (c) Conclude that the degree of the Gauss map G_ε is independent of ε.

5. Identify $M = \mathbb{R}^2$ with the field of complex numbers $\mathbb{C}$. Show that the polynomial map $z \mapsto z^k$ defines a vector field in $\mathbb{R}^2$ which has a zero at the origin of index k. How would you change $z \mapsto z^k$ to obtain a vector field with a zero of index $-k$?

6. Find the index of the zeros of the following vector fields in $\mathbb{R}^2$:
   (a) $x\frac{\partial}{\partial x} \pm y\frac{\partial}{\partial y}$;
   (b) $(x^2y + y^3)\frac{\partial}{\partial x} - (x^3 + xy^2)\frac{\partial}{\partial y}$;

7. Show that a vector field on a compact, oriented, surface of genus g must have at least one zero if $g \neq 1$.

8. Consider the vector field $X \in \mathfrak{X}(\mathbb{S}^{2d})$ obtained by restriction of the vector field in $\mathbb{R}^{2d+1}$:

$$X = x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2} + \cdots + x^{2d}\frac{\partial}{\partial x^{2d-1}} - x^{2d-1}\frac{\partial}{\partial x^{2d}}.$$

Show that there is a vector field $\overline{X}$ in $\mathbb{RP}^{2d}$ such that $\pi_* X = \overline{X}$ and apply the Poincaré-Hopf theorem to compute the Euler characteristic of $\mathbb{RP}^{2d}$. What can you say about the Euler characteristic of $\mathbb{RP}^{2d+1}$?

Part 4. Fiber Bundles

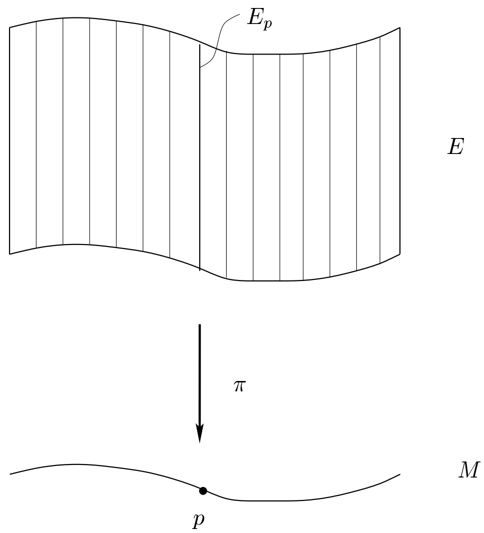
We have seen already several examples of fiber bundles, such as the tangent bundle, the cotangent bundle or the exterior bundles. So far, we have used the concept of a bundle in a more or less informal way. We will see now that one can understand many global properties of manifolds by studying more systematically fibre bundles and their properties.

The main notions and concepts to retain from the next series of sections are the following:

- Section 25: The notion of a *vector bundle* and the basic constructions with these bundles, such as the sum, tensor product and exterior product, etc.
- Section 26: Two import invariants of vector bundles that measure how twisted they are: the *Thom class* and the *Euler class*. The relationship between the Euler class of the tangent bundle and the Euler characteristic and, as a consequence, the *Poincaré-Hopf Theorem*.
- Section 27: A fundamental construction with vector bundles, which allows to move between different base manifolds: the *pullback* of vector bundles. The homotopy invariance of pullbacks.
- Section 28: The *classification of vector bundles*, which shows that every vector bundle is the pullback of a *universal vector bundle*.
- Section 29: The concept of a *connection* in a vector bundle, which allows one to differentiate sections of the vector bundle along vector fields in the basis and hence compare different fibers.
- Section 30: The *curvature* of a connection and the *holonomy* of a connection, which allows to characterize the global structure of flat vector bundles.
- Section 31: The *Chern-Weil homomorphism* associating to a vector bundle cohomology classes in a functorial way.
- Section 32: The theory of characteristics classes of real vector bundles (*Pontrjagin classes*) and complex vector bundles (*Chern classes*).
- Section 33: The abstract notion of a *fibre bundle* and of a *principal fibre bundle*. The constructions of the *associated bundles*.
- Section 34: The *classification of principal bundles*, *connections in principal bundles* and *characteristic classes of principal bundles*.

25. Vector Bundles

A vector bundle is a collection $\{E_p\}_{p \in M}$ of vector spaces parameterized by a manifold M. The union of these vector spaces is a manifold E and the map $\pi : E \to M$, $\pi(E_p) = p$ must satisfy a local trivialization condition. You should be able to recognize all these properties in the tangent bundle or in the cotangent bundle of a manifold.



In order to formalize this concept, let $\pi : E \to M$ be a smooth map between differentiable manifolds. A **trivializing chart** of dimension r for π is a pair (U, ϕ), where $U \subset M$ is open and $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^r$ is a diffeomorphism, such that we have a commutative diagram:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^r \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

In this diagram, $\pi_1 : U \times \mathbb{R}^r \to U$ denotes the projection in the first factor.

Let $E_p = \pi^{-1}(p)$ be the fiber over $p \in U$. We define a diffeomorphism $\phi^p : E_p \to \mathbb{R}^r$ as the composition:

$$\phi^p : E_p \xrightarrow{\phi} \{p\} \times \mathbb{R}^r \longrightarrow \mathbb{R}^r .$$

Hence, if $\mathbf{v} \in E_p$, we have

$$\phi(\mathbf{v}) = (p, \phi^p(\mathbf{v})).$$

Notice that since each ϕ^p is a diffeomorphism, we can use ϕ^p to transport the vector space structure of $\mathbb{R}^r$ to E_p. Given two trivializing charts whose domains intersect we would like that the induced vector space structures on the fibers coincide. This leads to the following definition:

Definition 25.1. *A **vector bundle structure** of rank r over a manifold M is a triple $\xi = (\pi, E, M)$, where $\pi : E \to M$ is a smooth map admitting a collection of trivializing charts $\mathcal{C} = \{(U_\alpha, \phi_\alpha) : \alpha \in A\}$ of dimension r, satisfying the following properties:*

(i) $\{U_\alpha : \alpha \in A\}$ is an open cover of M: $\bigcup_{\alpha \in A} U_\alpha = M$;

*(ii) The charts are compatible: for any $\alpha, \beta \in A$ and every $p \in U_\alpha \cap U_\beta$, the **transition functions** $g_{\alpha\beta}(p) \equiv \phi^p_\alpha \circ (\phi^p_\beta)^{-1} : \mathbb{R}^r \to \mathbb{R}^r$ are linear isomorphisms;*

(iii) The collection $\mathcal{C}$ is maximal: if (U, ϕ) is a trivializing chart of dimension r with the property that for every $\alpha \in A$, the maps $\phi^p \circ (\phi^p_\alpha)^{-1}$ and $\phi^p_\alpha \circ (\phi^p)^{-1}$ are linear isomorphisms, then $(U, \phi) \in \mathcal{C}$.

*We call $\xi = (\pi, E, M)$ a **vector bundle** of rank r.*

For a vector bundle $\xi = (\pi, E, M)$ we will use the following notations:

- E is call the **total space**, M is called the **basis space**, and π the **projection** of ξ.
- A collection of charts satisfying (i) and (ii) is called an **atlas of the vector bundle** or a **trivialization** of ξ.

An atlas of a vector bundle defines a vector bundle, since every atlas is contain in a unique maximal atlas. As we have already remarked, (ii) implies that the fiber E_p has a vector space structure such that for any trivializing chart (U, ϕ) the map $\phi^p : E_p \to \mathbb{R}^r$ is a linear isomorphism.

In the definition above of a vector bundle all maps are C^∞. Of course, one can also define C^k-vector bundles over C^k-manifold or even topological manifolds. Also, one can define complex vector bundles over smooth manifolds by replacing $\mathbb{R}^r$ by $\mathbb{C}^r$ and where the the base is still a *real* smooth manifold. In these notes, we will consider mainly real C^∞ vector bundles, but we will see that complex vector bundles will also be important.

Let $\xi = (\pi, E, M)$ be a vector bundle and $U \subset M$ an open set. A map $s : U \to E$ is called a **section** over U if $\pi \circ s = \mathrm{Id}_U$. The sections over U form a real vector space which we denote by $\Gamma_U(E)$. When $U = M$ we call a section over M a **global section** of E and we write $\Gamma(E)$ instead of $\Gamma_M(E)$. If $\operatorname{rank} \xi = r$ a collection $s_1, \dots, s_r$ of sections over U is called a **frame** over U if, for every $p \in U$, the sections $\{s_1(p), \dots, s_r(p)\}$ form a basis for E_p.

Definition 25.2. *Let $\xi_1 = (\pi_1, E_1, M_1)$ and $\xi_2 = (\pi_2, E_2, M_2)$ be two vector bundles. A **morphism of vector bundles** is a smooth map $\Psi : E_1 \to E_2$ which maps the fibers of ξ_1 linearly in the fibers of ξ_2, i.e., Ψ covers a smooth map $\psi : M_1 \to M_2$:*

$$\begin{tikzcd} E_1 \arrow[r, "\Psi"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"'] \\ M_1 \arrow[r, "\psi"] & M_2 \end{tikzcd}$$

and the map of the fibers $\Psi^p \equiv \Psi|_{(E_1)_p} : (E_1)_p \to (E_2)_{\psi(p)}$ is a linear transformation for each $p \in M_1$.

In this way we have the **category of all vector bundles**. Often we will be interested in vector bundles over a *fixed* base manifold M and morphisms over the identity $\psi = \mathrm{Id}_M : M \to M$. These form the **category of vector bundles over** M.

Two vector bundles $\xi_1 = (\pi_1, E_1, M_1)$ and $\xi_2 = (\pi_2, E_2, M_2)$ are called:

- **equivalent** if there exist morphisms $\Psi : \xi_1 \to \xi_2$ and $\Psi' : \xi_2 \to \xi_1$ which are inverse to each other. This means that Ψ is an isomorphism in the category of vector bundles: it covers a diffeomorphism $\psi : M_1 \to M_2$ and each fiber map $\Psi^p : (E_1)_p \to (E_2)_{\psi(p)}$ is a linear isomorphism.
- **isomorphic** if $M_1 = M_2 = M$ and there exist morphisms $\Psi : \xi_1 \to \xi_2$ and $\Psi' : \xi_2 \to \xi_1$, covering the identity which are inverse to each other. This means that Ψ is an isomorphism in the category of vector bundles over M: it covers the identity $\psi = \mathrm{Id}_M$ and each fiber map $\Psi^p : (E_1)_p \to (E_2)_p$ is a linear isomorphism.

EXAMPLES 25.3.

1. *Obviously, for any smooth manifold M, we have the associated vector bundles TM, T^*M, $\wedge^k T^*M$, $\otimes^r TM \otimes^s T^*M$, etc. The sections of these bundles are the vector fields, the differential forms and general tensor fields, that we have studied before. If $\Psi : M \to N$ is a smooth map, its differential $\mathrm{d}\Psi : TM \to TN$ is a morphism of vector bundles (note, however, that the transpose $(\mathrm{d}_x\Psi)^*$, in general,* is not *a vector bundle morphism).*

2. *The **trivial vector bundle** of rank r over M is the vector bundle $\varepsilon^r_M = (\pi, M \times \mathbb{R}^r, M)$, where $\pi : M \times \mathbb{R}^r \to M$ is the projection in the first factor. The global sections of ε^r_M can be identified with $C^\infty(M; \mathbb{R}^r)$. In general, a vector bundle ξ over M of rank r is said to be **trivial** if it is isomorphic to ε^r_M. It is easy to see that a vector bundle is trivial if and only if it admits a global frame.*

   *A **parallelizable manifold** is a manifold M for which TM is a trivial vector bundle. For example, any Lie group G is parallelizable, but $\mathbb{S}^2$ is not parallelizable (actually, one can show that $\mathbb{S}^d$ is parallelizable if and only if $d = 0, 1, 3$ and 7).*

3. *A r-dimensional distribution D in a manifold M, defines a vector bundle over M of rank r. The fibers are the subspaces $D_p \subset T_pM$. A section of this vector bundle is simply a vector field tangent to the distribution.*

4. *A vector bundle of rank 1 is usually refer to as a **line bundle**. For example, any non-vanishing vector field $X \in \mathfrak{X}(M)$ defines a line bundle which is always trivial. More generally, a rank 1 distribution defines a line bundle which is trivial if and only if the distribution is generated by a single vector field.*

5. *Consider the manifold formed by pairs $([x], \mathbf{v})$, where $[x]$ is a line through the origin in $\mathbb{R}^{d+1}$ and $\mathbf{v}$ is a point in this line:*

$$E = \{([x], \mathbf{v}) \in \mathbb{RP}^d \times \mathbb{R}^{d+1} : \mathbf{v} = \lambda x, \text{ for some } \lambda \in \mathbb{R}\}.$$

The map $\pi : E \to \mathbb{RP}^d$ given by $\pi([x], \mathbf{v}) = [x]$ satisfies the the local triviality condition. To see this, given an open set $V \subset \mathbb{S}^d$ such that if $x \in V$ then $-x \notin V$, denoted by $U = \{[x] : x \in V\} \subset \mathbb{RP}^d$ the corresponding open set in real projective space. Then the map defined by:

$$\psi : U \times \mathbb{R} \to \pi^{-1}(U), \quad \psi([x], t) = ([x], tx), \forall x \in V,$$

is a diffeomorphism, and its inverse $\phi = \psi^{-1}$ defines a trivializing chart over U. The family of all such charts (U, ϕ) is an atlas of a vector bundle over $\mathbb{RP}^d$. This vector bundle is called the ***canonical line bundle*** *over $\mathbb{RP}^d$ and is denoted γ_d^1.*

One way of describing vector bundles is through transition functions. Let $\xi = (\pi, E, M)$ be a rank r vector bundle. If (U_α, ϕ_α) and (U_β, ϕ_β) are trivializing charts, the corresponding **transition function** is the map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r)$ given by

$$p \mapsto g_{\alpha\beta}(p) \equiv \phi_\alpha^p \circ (\phi_\beta^p)^{-1},$$

so that:

$$\phi_\alpha \circ (\phi_\beta)^{-1}(p, \mathbf{v}) = (p, g_{\alpha\beta}(p) \cdot \mathbf{v}).$$

The transition functions satisfies the following fundamental identity:

$$g_{\alpha\beta}(p) g_{\beta\gamma}(p) = g_{\alpha\gamma}(p), \quad (p \in U_\alpha \cap U_\beta \cap U_\gamma). \tag{25.1}$$

If $\alpha = \beta = \gamma$, this condition reduces to:

$$g_{\alpha\alpha}(p) = I, \quad (p \in U_\alpha),$$

and when $\gamma = \alpha$ we obtain:

$$g_{\beta\alpha}(p) = g_{\alpha\beta}(p)^{-1}, \quad (p \in U_\alpha \cap U_\beta).$$

The family $\{g_{\alpha\beta}\}$ depends on the choice of trivializing charts. However, we have:

Lemma 25.4. *Let ξ and η be vector bundles over M with trivializations $\{\phi_\alpha\}$ and $\{\phi'_\alpha\}$ subordinated to the same open cover $\{U_\alpha\}$. Denote by $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ the corresponding collections of transition functions. If ξ is isomorphic to η, then there exist smooth maps $\lambda_\alpha : U_\alpha \to GL(r)$ such that:*

$$g'_{\alpha\beta}(p) = \lambda_\alpha(p) \cdot g_{\alpha\beta}(p) \cdot \lambda_\beta^{-1}(p), \quad (p \in U_\alpha \cap U_\beta). \tag{25.2}$$

Proof. Let $\Psi : \xi \to \eta$ be an isomorphism. For each U_α we define smooth maps $\lambda_\alpha : U_\alpha \to GL(r)$ by:

$$\lambda_\alpha(p) = \phi'^p_\alpha \circ \Psi \circ (\phi^p_\alpha)^{-1}.$$

If $p \in U_\alpha \cap U_\beta$, we have:

$$\begin{aligned} g'_{\alpha\beta}(p) &= \phi'^p_\alpha \circ (\phi'^p_\beta)^{-1} = \lambda_\alpha(p) \circ \phi^p_\alpha \circ (\phi^p_\beta)^{-1} \circ (\lambda_\beta(p))^{-1} \\ &= \lambda_\alpha(p) \circ g_{\alpha\beta}(p) \circ \lambda_\beta(p)^{-1}. \end{aligned}$$

□

Given a manifold M and an open cover $\{U_\alpha\}_{\alpha\in A}$ we call a family of maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r)$ satisfying (25.1) a **cocycle** subordinated to the cover. Two cocycles $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ subordinated to the same cover are said to be **equivalent** if they are related by (25.2) for some family of smooth maps $\lambda_\alpha : U_\alpha \to GL(r)$.

We saw above that (i) a trivialization of a vector bundle determines a cocycle and that (ii) two trivializations of isomorphic vector bundles subordinated to the *same* cover determine equivalent cocycles. If two cocycles are subordinated to *different* covers we can refine the covers and obtain cocycles subordinated to the same cover. Moreover, we have the following converse:

Proposition 25.5. *Let $\{g_{\alpha\beta}\}$ be a cocycle subordinated to an open cover $\{U_\alpha\}$ of M. There exists a vector bundle $\xi = (\pi, E, M)$, that admits a trivialization $\{\phi_\alpha\}$ for which the transition functions are the $\{g_{\alpha\beta}\}$. Two equivalent cocycles $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ determine isomorphic vector bundles so there is a 1:1 correspondence:*

$$\left\{\begin{array}{c}\textit{vector bundles } \xi = (\pi, E, M)\\ \textit{up to isomorphism}\end{array}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{array}{c}\textit{cocycles } \{g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r)\}\\ \textit{up to equivalence and refinement}\end{array}\right\}$$

Proof. Given a cocycle $\{g_{\alpha\beta}\}$, subordinated to the cover $\{U_\alpha\}$ of M, we construct the manifold E as the quotient:

$$E = \coprod_{\alpha\in A} (U_\alpha \times \mathbb{R}^r) \Big/ \sim$$

where $\sim$ is the equivalence relation defined by:

$$(p, \mathbf{v}) \sim (q, \mathbf{w}) \text{ iff } \begin{cases} p = q \text{ and} \\ \exists\, \alpha, \beta \in A : \; g_{\alpha\beta}(p) \cdot \mathbf{v} = \mathbf{w}. \end{cases}$$

The quotient topology on E turns this space into a 2nd countable, Hausdorff, topological space. We also have the obvious projection $\pi : E \to M$:

$$\pi([p, \mathbf{v}]) = p.$$

For each chart (V, ψ) for M and each α such that $U\alpha \cap V \not\emptyset$ we construct a chart for E:

$$\pi^{-1}(V \cap U_\alpha \to \mathbb{R}^{d+r}, [(p, \mathbf{v})] \mapsto (\psi(p), \mathbf{v}).$$

This make E into a local euclidean topological space, and it is easy to see that the corresponding change of charts are smooth, so E is a smooth manifold such that π is a smooth map.

The maps $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$ defined by:

$$\phi_\alpha([p, \mathbf{v}]) = (p, \mathbf{v}).$$

give trivializing charts for $\pi : E \to M$ and the corresponding transition functions are exactly the $\{g_{\alpha\beta}\}$. Denote this vector bundle by $\xi = (\pi, E, M)$

If $\{g'_{\alpha\beta}\}$ is another cocycle equivalent to $\{g_{\alpha\beta}\}$ through the family $\{\lambda_\alpha\}$ and $\xi' = (\pi', E', M)$ denotes the vector bundle associated with $\{g'_{\alpha\beta}\}$, we have a vector bundle isomorphism $\Psi : \xi \to \xi'$ defined on each open set $\pi^{-1}(U_\alpha)$ by:

$$\Psi([p, \mathbf{v}]) = [p, \lambda_\alpha(p) \cdot \mathbf{v}].$$

□

Let us now turn to constructions with vector bundles. We have the following general principle:

- *For every functorial construction with vector spaces there is a similar construction with vector bundles.*

This principle can actually be made precise. However, instead of following the abstract route we will describe explicitly the constructions that are most relevant for us.

Subbundles and quotients. Every vector bundle $\xi = (\pi, E, M)$ can be restricted to a submanifold $N \subset M$. The **restriction** ξ_N is the vector bundle with total space:

$$E_N = \{E_p : p \in N\},$$

and projection $\pi_N : E_N \to N$ the restriction of π to E_N. The restriction is an example of a vector subbundle:

Definition 25.6. *A vector bundle $\eta = (\tau, F, N)$ is called a* ***vector subbundle*** *of a vector bundle $\xi = (\pi, E, M)$ if F is a submanifold of E, and the inclusion $F \hookrightarrow E$ is a morphism of vector bundles.*

If $\Psi : \eta \to \xi$ is a morphism of vector bundles covering the identity, in general, its image and its kernel *are not* vector subbundles: these are made of vector spaces of varying dimension. This can be fixed if we assume that Ψ has **constant rank** k, i.e., if all linear maps $\Psi^p : E_p \to F_p$ have the same rank k. For a constant rank morphism we can define the following vector subbundles over M:

- The **kernel** of Ψ is the vector subbundle $\operatorname{Ker}\Psi \subset E$ whose total space is $\{\mathbf{v} \in E : \Psi(\mathbf{v}) = 0\}$;
- The **image** of Ψ is the vector subbundle $\operatorname{Im}\Psi \subset F$ whose total space is $\{\Psi(\mathbf{v}) \in F : \mathbf{v} \in E\}$;
- The **co-kernel** of Ψ is the vector bundle $\operatorname{coKer}\Psi$ whose total space is the quotient $F/\sim$, where $\sim$ the equivalence relation $\mathbf{w}_1 \sim \mathbf{w}_2$ if and only if $\mathbf{w}_1 - \mathbf{w}_2 = \Psi(\mathbf{v})$, for some $\mathbf{v} \in E$.

Note that if Ψ is a *monomorphism* (i.e., each Ψ^p is injective) or if Ψ is an *epimorphism* (i.e., each Ψ^p is surjective) then Ψ has constant rank. Therefore, the kernel, image and cokernel of monomorphisms and epimorphisms are vector subbundles.

The notions associated with exact sequences can be easily extended to vector bundles and morphisms of constant rank. For example, a short exact sequence of vector bundles is a sequence of vector bundle morphisms

$$0 \longrightarrow \xi \xrightarrow{\Phi} \eta \xrightarrow{\Psi} \theta \longrightarrow 0$$

where Φ is a monomorphism, Ψ is an epimorphism and $\operatorname{Im}\Phi = \operatorname{Ker}\Psi$. In this case, we have vector bundle isomorphisms $\xi \simeq \operatorname{Ker}\Psi$ and $\theta \simeq \operatorname{coKer}\Psi$. We say that θ is the **quotient vector bundle** of the monomorphism Φ.

For a concrete example, consider a vector subbundle $\xi = (\tau, F, M) \subset \eta = (\pi, E, M)$. The inclusion is a monomorphism of vector bundles, hence we can form its quotient, which we denote by η/ξ. Notice that the fibers of η/ξ are the quotient vector spaces E_p/F_p.

Example 25.7.

*Let M be a manifold and $N \subset M$ a submanifold. The tangent bundle TN is a vector subbundle of $T_N M$. The quotient bundle $\nu(N) \equiv T_N M/TN$ is usually called the **normal bundle to** N in M.*

*More generally, let $\mathcal{F}$ be a foliation of M. Then $\mathcal{F}$ gives rise to the vector subbundle $T\mathcal{F} \subset TM$. The quotient bundle $\nu(\mathcal{F}) \equiv TM/T\mathcal{F}$ is usually called the **normal bundle of** $\mathcal{F}$ in M. If L is a leaf of $\mathcal{F}$ the restriction of $\nu(\mathcal{F})$ to L is the normal bundle $\nu(L)$.*

Direct sums and tensor products. Let $\xi = (\pi, E, M)$ and $\eta = (\tau, F, M)$ be vector bundles over the same manifold M. The **Whitney sum** or **direct sum** of ξ and η is the vector bundle $\xi \oplus \eta$ whose total space is:

$$E \oplus F := E \times_M F = \{(\mathbf{v}, \mathbf{w}) \in E \times F : \pi(\mathbf{v}) = \tau(\mathbf{w})\},$$

and whose projection is:

$$E \oplus F \to M, \ (\mathbf{v}, \mathbf{w}) \mapsto \pi(\mathbf{v}) = \tau(\mathbf{w}).$$

Note that the fiber of $\xi \oplus \eta$ over $p \in M$ is the direct sum $E_p \oplus F_p$. The local triviality condition is easily verified: if $\{\phi_\alpha\}$ and $\{\psi_\alpha\}$ are trivializations of ξ and η, subordinated to the same covering, with corresponding cocycles $\{g_{\alpha\beta}\}$ and $\{h_{\alpha\beta}\}$, then we have the trivialization of $\xi \oplus \eta$ given by $\{(\phi_\alpha \times \psi_\alpha)|_{E\oplus F}\}$, to which corresponds the cocycle defined by:

$$g_{\alpha\beta} \oplus h_{\alpha\beta} = \begin{bmatrix} g_{\alpha\beta} & 0 \\ 0 & h_{\alpha\beta} \end{bmatrix}.$$

Similarly, we can define:

- The **tensor product** $\xi \otimes \eta$: the fibers are the tensor products $E_p \otimes F_p$ and the transition functions are $g_{\alpha\beta} \otimes h_{\alpha\beta}$.
- The **dual vector bundle** ξ^*: the fibers are the dual vector spaces E_p^* and the transition functions are the inverse transpose maps $(g_{\alpha\beta})^{-T}$.
- The **exterior product** $\wedge^k \xi$: the fibers are the exterior products $\wedge^k E_p$ and the transition functions are the exterior powers $\wedge^k g_{\alpha\beta}$.

- The **Hom(ξ,η)-bundles**: the fibers are the space of all linear morphisms $\mathrm{Hom}(E_x, F_x)$. We leave as an exercise to show that there is a natural isomorphism $\mathrm{Hom}(\xi,\eta) \simeq \xi^* \otimes \eta$.

Orientations. A vector bundle $\xi = (\pi, E, M)$ of rank r is called an **orientable vector bundle** if the exterior product $\wedge^r \xi$ has a section which never vanishes. Note that this section corresponds to a smooth choice of an orientation in each vector space E_p. We call an **orientation** for ξ an equivalence class $[s]$, where two non-vanishing sections $s_1, s_2 \in \Gamma(\wedge^r \xi)$ are equivalent if and only if $s_2 = f s_1$ for some smooth positive function $f \in C^\infty(M)$. We leave as an exercise to check that ξ is orientable if and only if it admits a trivialization $\{\phi_\alpha\}$ for which the associated cocycle $\{g_{\alpha\beta}\}$ takes values in $GL^+(r)$, the group of invertible $r \times r$ matrices with positive determinant:

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL^+(r) \subset GL(r).$$

For a manifold M, the notion of orientation that we studied before corresponds to the notion of orientation of the vector bundle TM. For a vector bundle $\xi = (\pi, E, M)$, the possible orientations for ξ, E and M are related as follows:

Lemma 25.8. *Let $\xi = (\pi, E, M)$ be a vector bundle. If two among the vector bundles TM, TE and ξ are orientable so is the third one.*

The proof is left as an exercise.

Riemmanian structures. A **Riemann structure** in a vector bundle $\xi = (\pi, E, M)$ is a choice of an inner product $\langle\ ,\ \rangle : E_p \times E_p \to \mathbb{R}$ in each fiber which varies smoothly, i.e., for any sections $s_1, s_2 \in \Gamma(E)$ the map $p \mapsto \langle s_1(p), s_2(p)\rangle$ is smooth. This condition is equivalent to say that the section of the vector bundle $\otimes^2 \xi^*$ defined by $\langle\ ,\ \rangle$ is smooth.

It is easy to see, using a partition of unity, that a vector bundle always admits a Riemann structure. Given a trivialization $\{(U_\alpha, \phi_\alpha)\}$, one chooses a partition of unity $\{\rho_\alpha\}$ subordinated to the cover $\{U_\alpha\}$ and defines a Riemmanian structure by:

$$\langle \mathbf{v}, \mathbf{w}\rangle := \sum_\alpha \rho_\alpha(p)(\phi^p_\alpha(\mathbf{v}), \phi^p_\alpha(\mathbf{w}))_{\mathbb{R}^r} \quad (\mathbf{v}, \mathbf{w} \in E_p).$$

On the other hand, it is not hard to see that a vector bundle has a Riemann structure if and only it it admits a trivialization $\{\phi_\alpha\}$ whose associated cocycle $\{g_{\alpha\beta}\}$ take values in the orthogonal group $O(r)$:

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to O(r) \subset GL(r).$$

The fact behind the existence of such a trivialization is the polar decomposition:

$$GL(r) = O(r) \times \{\text{positive definite symmetric matrices}\}.$$

If $\xi = (\pi, E, M)$ is a vector bundle and $\langle\ ,\ \rangle$ is a Riemann structure in ξ, then for any vector subbundle $\eta = (\tau, F, N)$ we can define the **orthogonal vector bundle** $\eta^\perp$ over N as the subbundle of ξ with total space $F^\perp$, where

$$F_p^\perp \equiv \{\mathbf{v} \in E_p : \langle \mathbf{v}, \mathbf{w}\rangle = 0, \forall \mathbf{w} \in F_p\}.$$

When $M = N$, we obtain:

$$\xi = \eta \oplus \eta^\perp.$$

in this case $\eta^\perp \simeq \xi/\eta$, since the natural projection $\xi \to \xi/\eta$ restricts to an isomorphism on $\eta^\perp$.

HOMEWORK.

1. Show that a vector bundle is trivial if and only it admits a global frame.

2. Let $G_r(\mathbb{R}^d)$ be the Grassmannian manifold of r-planes in $\mathbb{R}^d$. Consider the submanifold $E \subset G_r(\mathbb{R}^d) \times \mathbb{R}^d$ defined by:

$$E = \{(S, x) : S \text{ is a subspace of } \mathbb{R}^d \text{ and } x \in S\},$$

and the smooth map $\pi : E \to G_r(\mathbb{R}^d)$ given by:

$$\pi(S, x) = S.$$

Show that $\gamma_d^r = (\pi, E, G_r(\mathbb{R}^d))$ is a vector bundle of rank r. It is called the **canonical bundle** over $G_r(\mathbb{R}^d)$.

3. Let $\Psi : \eta \to \xi$ be a morphism of vector bundles which covers the identity. Show that the kernel and the image of Ψ are vector subbundles if the rank of the linear maps Ψ^p is constant. Give counterexamples when the rank is not constant.

4. Let $\xi = (\pi, E, M)$ and $\eta = (\tau, F, M)$ be vector bundles.
   (a) Show that there exists a vector bundle $\mathrm{Hom}(\xi, \eta)$ whose fibers are the vector spaces $\mathrm{Hom}(E_x, F_x)$.
   (b) Find the transition function of $\mathrm{Hom}(\xi, \eta)$ in terms of the transition functions of ξ and η.
   (c) Find an isomorphism $\mathrm{Hom}(\xi, \eta) \simeq \xi^* \otimes \eta$.

5. Given a vector bundle ξ show that there exists a trivialization of ξ for which the transition functions take values in $O(r)$.

6. Consider a short exact sequence of vector bundles

$$0 \longrightarrow \xi_1 \longrightarrow \xi_2 \xrightarrow{\Psi} \xi_3 \longrightarrow 0$$

Show that:
   (a) Such a short exact sequence always *splits*, i.e., there exists a morphism of vector bundles $\Phi : \xi_3 \to \xi_2$ such that $\Psi \circ \Phi = \mathrm{Id}$;
   (b) There is an isomorphism of vector bundles:

$$\xi_2 \simeq \xi_1 \oplus \xi_3.$$

   (c) If two among the vector bundles ξ_1, ξ_2 and ξ_3 are orientable, so is the third one.

7. Let $\xi = (\pi, E, M)$ be a vector bundle. Show that:
   (a) There exists a natural isomorphism of vector bundles
$$T_M E \simeq \xi \oplus TM;$$
   (b) If two among the vector bundles TM, TE and ξ are orientable so is the third one.

8. For a vector bundle ξ show that the following statements are equivalent:
   (a) ξ is orientable;
   (b) There exists a trivialization of ξ for which the transition functions take values in $GL^+(r)$;
   (c) There exists a trivialization of ξ for which the transition functions take values in $SO(r)$.

26. The Thom Class and the Euler Class

The homotopy invariance of de Rham cohomology relied crucially on the isomorphism:
$$H^\bullet(M \times \mathbb{R}^r) \simeq H^\bullet(M).$$
One can interpret this isomorphism as relating the cohomology of the total space of the trivial bundle with the cohomology of its base. More generally, we have:

Proposition 26.1. *For any vector bundle $\xi = (\pi, E, M)$:*
$$H^\bullet(E) \simeq H^\bullet(M).$$

Proof. Let $s : M \to E$ be the zero section. Its image is a deformation retract of E. Therefore, by homotopy invariance we see that $s^* : H^\bullet(E) \to H^\bullet(M)$ is an isomorphism. □

One may guess that the corresponding statement for compactly supported cohomology:
$$H_c^\bullet(M \times \mathbb{R}^r) \simeq H_c^{\bullet - r}(M),$$
can also be generalized to vector bundles. The following example shows that one must be careful.

Example 26.2.
Consider the canonical line bundle γ_1^1 over $\mathbb{RP}^1 = \mathbb{S}^1$. The total space E of this bundle is the Möbius band, a non-oriented manifold of dimension 2, so we have $H_c^2(E) = 0$. On the other hand, for the base:
$$H_c^{2-1}(\mathbb{S}^1) = H^1(\mathbb{S}^1) \simeq \mathbb{R} \neq 0.$$

On other hand, under a orientability assumption we do have:

Proposition 26.3 (Thom Isomorphism – first version)**.** *Let $\xi = (\pi, E, M)$ be a vector bundle of rank r, where E is orientable and M is of finite type. Then:*

$$H_c^\bullet(E) \simeq H^{\bullet - r}(M).$$

Proof. Since M is of finite type, so is E. Hence, E is both orientable and of finite type and we can apply Poincaré duality to conclude:

$$\begin{aligned} H_c^\bullet(E) &\simeq H^{d+r-\bullet}(E) && \text{(by Poincaré duality for } E\text{)}, \\ &\simeq H^{d+r-\bullet}(M) && \text{(by Proposition 26.1)}. \end{aligned}$$

□

The isomorphism behind the Thom isomorphism can be described explicitly. It relies on a push-forward map

$$\pi_* : \Omega_c^\bullet(E) \to \Omega^{\bullet - r}(M)$$

called **integration along the fibers**. For the case of the trivial line bundle this map appeared in the proof of Proposition 22.11. Using a local trivialization, we ca extended the description given in that proof to any vector bundle. We start by covering M by trivializing oriented charts (U_α, ϕ_α) for the vector bundle ξ, where each U_α is the domain of a chart $(x^1, \dots, x^d)$ of the base M. This yields a chart $(x^1, \dots, x^d, t^1, \dots, t^r)$ for the total space E with domain $\pi^{-1}(U_\alpha)$, where $(t^1, \dots, t^r)$ are linear coordinates on the fibers. If $\omega \in \Omega_c^\bullet(E)$, then $\omega_\alpha = \omega|_{\pi^{-1}(U_\alpha)}$ is a linear combination of two kinds of forms:

$$\begin{aligned} &f_1(x,t)\pi^*\theta_1 \wedge \mathrm{d}t^{i_1} \wedge \cdots \wedge \mathrm{d}t^{i_k}, \text{ with } k < r \\ &f_2(x,t)\pi^*\theta_2 \wedge \mathrm{d}t^1 \wedge \cdots \wedge \mathrm{d}t^r \end{aligned}$$

where θ_i are differential forms in M and the functions $f_i(x,t)$ have compact support. Integration along the fibers $\pi_* : \Omega_c^\bullet(E) \to \Omega^{\bullet - r}(M)$ is given by:

$$\begin{aligned} f_1(x,t)\pi^*\theta_1 \wedge \mathrm{d}t^{i_1} \wedge \cdots \wedge \mathrm{d}t^{i_k} &\longmapsto 0, \quad (k < r) \\ f_2(x,t)(\pi^*\theta_2) \wedge \mathrm{d}t^1 \wedge \cdots \wedge \mathrm{d}t^r &\longmapsto \theta_2 \int_{\mathbb{R}^r} f_2(x, t^1, \dots, t^r)\, \mathrm{d}t^1 \cdots \mathrm{d}t^r. \end{aligned}$$

One checks that this definition is independent of the choices made. Using this explicit description one can check that fiber integration satisfies the following properties:

Proposition 26.4. *If $\pi_* : \Omega_c^\bullet(E) \to \Omega^{\bullet - r}(M)$ denotes integration along the fibers, then:*

(i) *π_* is a cochain map:* $\mathrm{d}\pi_*\omega = \pi_*\mathrm{d}\omega$*;*
(ii) ***Projection formula:*** *for any $\theta \in \Omega^*(M)$ and $\omega \in \Omega_c^\bullet(E)$:*

$$\pi_*(\pi^*\theta \wedge \omega) = \theta \wedge \pi_*\omega. \tag{26.1}$$

(iii) If $\Psi : \xi_1 \to \xi_2$ is a vector bundle map covering a map $\psi : M_1 \to M_2$, which is a fiberwise isomorphism and preserves orientations, then for any $\omega \in \Omega^\bullet_c(E)$:

$$(\pi_1)_* \Psi^* = \psi^*(\pi_2)_*. \tag{26.2}$$

Remark 26.5 (Differential forms with compact vertical support)**.** The description above of integration along the fibers shows that:

(i) The definition of fiber integration π_* only requires that the vector bundle ξ is oriented, so the base and/or the total space can be non-orientable;

(ii) Fiber integration π_* can be defined for any differential form ω in E with compact vertical support, i.e., such that $\operatorname{supp}\omega \cap \pi^{-1}(K)$ is compact for every compact set $K \subset M$.

The space $\Omega^*_{\text{cv}}(E)$ of differential forms with compact vertical support is a subcomplex of the de Rham complex and gives rise to a cohomology $H^\bullet_{\text{cv}}$.

The general version of the Thom isomorphism states that:

Proposition 26.6 (Thom Isomorphism)**.** *Let $\xi = (\pi, E, M)$ be an oriented vector bundle of rank r over a manifold of finite type. Then there is an isomorphism:*

$$H^\bullet_{cv}(E) \simeq H^{\bullet - r}(M).$$

Proof. For a trivial vector bundle, the proof is the same as the proof of Proposition 22.11.

Using a partition of unity argument, one sees that the cohomology $H^\bullet_{cv}(E)$ satisfies the Mayer-Vietoris sequence property. Then, given open sets $U, V \subset M$, one obtains a commutative diagram of Mayer-Vietoris sequences:

$$\begin{tikzcd}
\arrow[r] & H^k_{\text{cv}}(E|_{U\cup V}) \arrow[r] \arrow[d, "\pi_*"] & H^k_{\text{cv}}(E|_U) \oplus H^k_{\text{cv}}(E|_V) \arrow[r] \arrow[d, "\pi_*"] & H^k_{\text{cv}}(E|_{U\cap V}) \arrow[r, "\delta"] \arrow[d, "\pi_*"] & H^{k+1}_{\text{cv}}(E|_{U\cup V}) \arrow[r] \arrow[d, "\pi_*"] & {} \\
\arrow[r] & H^{k-r}(U\cup V)^* \arrow[r] & H^{k-r}_c(U)^* \oplus H^{k-r}_c(V)^* \arrow[r] & H^{k-r}_c(U\cap V)^* \arrow[r, "\delta^*"] & H^{k+1-r}_c(U\cup V)^* \arrow[r] & {}
\end{tikzcd}$$

If the vector bundle ξ is trivial over U and V, then in the previous diagram one obtains that π_* is an isomorphism for U, V and $U \cap V$. By the Five Lemma, it follows that π_* is an isomorphism also over $U \cup V$. Then the proof proceeds by an induction argument over the number of elements of a good cover, as in the proof of Poincaré duality. $\square$

We can now introduce a cohomological invariant of a vector bundle:

Definition 26.7. *The* ***Thom class*** *of an oriented vector bundle $\xi = (\pi, E, M)$ of rank r is the image of 1 under the Thom isomorphism $H^0(M) \simeq H^r_{cv}(E)$. We will denote this class by $U \in H^r_{cv}(E)$.*

The Thom class allows one to write, in a more or less explicit way, the inverse to the integration along fibers $\pi_* : H^\bullet_{\text{cv}}(E) \to H^{\bullet - r}(M)$. In fact, since $\pi_* U = 1$, the projection formula (26.1) shows that the linear map $H^\bullet(M) \to H^{\bullet+r}_{\mathrm{cv}}(E)$ defined by:

$$(\pi_*)^{-1}([\omega]) = [\pi^*\omega] \cup U.$$

is an inverse to π_*.

The following result gives an alternative characterization of the Thom class:

Theorem 26.8. *The Thom class of an oriented vector bundle $\xi = (\pi, E, M)$ is the unique class $U \in H^r_{cv}(E)$ whose pullback to each fiber E_p is the canonical generator of $H^r_c(E_p)$, i.e.,*

$$\int_{E_p} i^* U = 1, \quad \forall p \in M,$$

where $i : E_p \hookrightarrow E$ is the inclusion.

Proof. Since $\pi_* U = 1$, we see that the restriction $i^* U$ to each fiber E_p is a compactly supported form with $\int_{E_p} i^* U = 1$.

Conversely, let $U' \in H^r_{\mathrm{cv}}(E)$ be a class such for each $p \in M$ the restriction $i^* U' \in H^r_c(E_p)$ is the canonical generator. By the projection formula (26.1), we obtain

$$\pi_*(\pi^*\theta \wedge U') = \theta \wedge \pi_* U' = \theta, \quad \forall \theta \in H^\bullet(M).$$

Hence, $\theta \mapsto \pi^*\theta \wedge U'$ inverts π_*. The image of 1, which is U', must then coincide with the Thom class. $\square$

From now on, to simplify the presentation, we will assume that M is compact so that it is of finite type. Moreover it follows that for a vector bundle $\xi = (\pi, E, M)$ we have $H^r_{\mathrm{cv}}(E) = H^r_c(E)$.

The Thom class of a vector bundle $\xi = (\pi, E, M)$ is an invariant of the bundle, but it lies in the cohomology of the total space. We can use a global section to obtain an invariant which lies in the cohomology of the base. For that observe that given a section $s : M \to E$ we have the induced map in cohomology

$$s^* : H^\bullet_c(E) \to H \bullet (M).$$

Notice that we can view this map has the composition of two maps:

$$H^\bullet_c(E) \longrightarrow H^\bullet(E) \xrightarrow{s^*} H^\bullet(M)\ .$$

On the other hand, any two sections $s_0, s_1 : M \to E$ are homotopic:

$$H(p,t) = t s_1(p) + (1-t) s_0(p)$$

From the homotopy invariance of cohomology, we conclude that the maps induced in cohomology by any two section are identical:

$$s_0^* = s_1^* : H^\bullet_c(E) \to H \bullet (M).$$

Therefore, the following definition makes sense:

Definition 26.9. *Let $\xi = (\pi, E, M)$ be an oriented vector bundle of rank r over a compact manifold M. The* ***Euler class*** *of ξ is the class $e(\xi) \in H^r(M)$ defined by:*

$$e(\xi) \equiv s^* U,$$

where U is the Thom class of ξ and $s : M \to E$ is any *global section of ξ.*

Note that, in particular, we can define the Euler class by pulling back along the zero section. The following proposition lists some properties of the Euler class. We leave its proof for the exercises:

Proposition 26.10. *Let $\xi = (\pi, E, M)$ be an oriented vector bundle of rank r over a compact manifold M. Then:*

(i) If $\Psi : \eta \to \xi$ is a vector bundle map covering a map $\psi : M_1 \to M_2$, which is a fiberwise isomorphism and preserves orientations, then: $e(\eta) = \psi^* e(\xi)$.
(ii) If $\bar{\xi}$ denotes the vector bundle ξ with the opposite orientation then $e(\bar{\xi}) = -e(\xi)$.
(iii) If rank r is odd, then $e(\xi) = 0$.
(iv) If $\xi' = (\pi', E', M)$ is another oriented vector bundle of rank r' over M, then $e(\xi \oplus \xi') = e(\xi) \cup e(\xi')$, where $\xi \oplus \xi'$ has the direct sum orientation.

The Euler class of a vector bundle is an *obstruction* to the existence of a non-vanishing global section. In fact, we have:

Theorem 26.11. *Let $\xi = (\pi, E, M)$ be an oriented vector bundle over a compact manifold M. If ξ admits a non-vanishing section then $e(\xi) = 0$.*

Proof. Let $s : M \to E$ be a non-vanishing section. If $\omega \in \Omega^r_c(E)$ is a compactly supported form representing the Thom class, then there exists $c \in \mathbb{R}$ such that the image of the section cs does not intersect $\operatorname{supp} \omega$. Hence:

$$e(\xi) = (cs)^* U = [(cs)^* \omega] = 0.$$

□

Note, however, that the converse to this result does not hold: there are examples of vector bundles ξ with $e(\xi) = 0$ and for which every global section has a zero.

Example 26.12.
Consider an oriented vector bundle $\xi = (\pi, E, M)$ of rank 2. We fix some Riemannian metric on ξ and cover M by charts $\{(U_\alpha, x^i_\alpha)\}$ over which we have a positive orthonormal frame $\{s^\alpha_1, s^\alpha_2\}$. These define coordinates $(\pi^* x^i_\alpha, r_\alpha, \theta_\alpha)$ on

$$(E - \{0_M\})|_{U_\alpha} \simeq U_\alpha \times (\mathbb{R}^2 - \{0\})$$

where $(r_\alpha, \theta_\alpha)$ are polar coordinates on $\mathbb{R}^2 - \{0\}$. On an overlap $U_\alpha \cap U_\beta$ the radial functions coincide $r_\alpha = r_\beta$, while the angles differ by a rotation:

$$\theta_\alpha - \theta_\beta = \pi^* \varphi_{\alpha\beta}, \qquad \varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{S}^1.$$

Note that on triple intersections $U_\alpha \cap U_\beta \cap U_\gamma$ we have:

$$\varphi_{\alpha\beta} + \varphi_{\beta\gamma} = \varphi_{\alpha\gamma}.$$

Let $\{\rho_\alpha\}$ be a partition of unity subordinated to the cover $\{U_\alpha\}$. If we let:

$$\varepsilon_\alpha := \sum_\gamma \rho_\gamma \mathrm{d}\varphi_{\alpha\gamma} \in \Omega^1(U_\alpha),$$

we obtain a 1-form on each open U_α whose differentials on a double intersection $U_\alpha \cap U_\beta$ satisfy:

$$\mathrm{d}\varphi_{\alpha\beta} = \sum_\gamma \rho_\gamma \mathrm{d}\varphi_{\alpha\beta} = \sum_\gamma \rho_\gamma (\mathrm{d}\varphi_{\alpha\gamma} - \mathrm{d}\varphi_{\beta\gamma}) = \varepsilon_\alpha - \varepsilon_\beta.$$

Hence, it follows that we have a well-defined 2-form $\varepsilon \in \Omega^2(M)$ such that:

$$\varepsilon|_{U_\alpha} = \mathrm{d}\varepsilon_\alpha.$$

On the the other hand, on $(E - 0_M)|_{U_\alpha \cap U_\beta}$ we have:

$$\mathrm{d}\theta_\alpha - \mathrm{d}\theta_\beta = \pi^* \mathrm{d}\varphi_{\alpha\beta} = \pi^*\varepsilon_\alpha - \pi^*\varepsilon_\beta.$$

Hence, we also have a global "angular form" $\phi \in \Omega^1(E - 0_M)$ such that:

$$\phi = \mathrm{d}\theta_\alpha - \pi^*\varepsilon_\alpha \quad \text{on } (E - 0_M)|_{U_\alpha}.$$

Notice that:

$$\mathrm{d}\phi = -\pi^*\varepsilon.$$

Finally, let $\delta > 0$ and choose a smooth function $\rho : \mathbb{R} \to \mathbb{R}$ which is non-decreasing, $\rho(r) = -\frac{1}{2\pi}$ for $t < \delta$, $\rho(r) = 0$ for $t \geq 1$ and $\int_\mathbb{R} \rho'(r)\mathrm{d}r = \frac{1}{2\pi}$. We can promote it to a function $\rho : E \to \mathbb{R}$ of the radius:

$$\rho(\mathbf{v}) = \rho(||\mathbf{v}||) \quad (\mathbf{v} \in E).$$

Then we define the 2-form:

$$u := \mathrm{d}(\rho\phi) = \mathrm{d}\rho \wedge \phi - \rho\,\pi^*\varepsilon.$$

A priori this form is only defined outside the zero section, but the second expression shows that it extends smoothly to E, since ρ is constant in a neighborhood of 0_M. The restriction of u to a fiber E_p is the compactly supported 2-form $(\mathrm{d}\rho \wedge \phi)|_{E_p}$, which is positively oriented and has integral 1. Hence, $U = [u]$ is the Thom class of the bundle ξ. Moreover, if we pullback u by the zero section $s_0 : M \to E$ we obtain:

$$s_0^* u = -\rho(0) s_0^* \pi^* \varepsilon = \frac{1}{2\pi}\varepsilon.$$

So we conclude also that $e(\xi) = \frac{1}{2\pi}[\varepsilon]$.

The name *Euler class* is related with the special case where $\xi = TM$. Let M be an oriented, connected, manifold with $\dim M = d$ and denote the orientation by μ. The corresponding canonical generator in cohomology will also be denoted by $\mu \in H^d_c(M)$: it is the class represented by any top degree form $\omega \in \Omega^d_c(M)$ such that:

$$\int_M \omega = 1.$$

If we assume that M is of finite type, then μ is the image of 1 under Poincaré duality $H^0(M) \simeq H^d_c(M)$. Recalling the notion of index of an isolated zero of a vector field from Section 24), we have:

Theorem 26.13. *Let M be an oriented, compact, connected manifold of dimension d. For any vector field $X \in \mathfrak{X}(M)$ with a finite number of zeros $\{p_1, \dots, p_N\}$, one has:*

$$e(TM) = \left(\sum_{i=1}^{N} \operatorname{ind}_{p_i} X \right) \mu \in H^d(M),$$

where $\mu \in H^d(M)$ is the class defined by the orientation of M.

Proof. Let $\omega \in \Omega^d_c(TM)$ be a compactly supported form representing the Thom class. We need to show that:

$$\int_M X^*\omega = \left(\sum_{i=1}^{N} \operatorname{ind}_{p_i} X \right).$$

Choose coordinate systems (U_i, ϕ_i) centered at p_i and denote by D_i the closed balls:

$$D_i = \phi_i^{-1}(\{x \in \mathbb{R}^d : ||x|| \leq 1\}).$$

Consider the identification $TU_i \simeq U_i \times \mathbb{R}^d$ provided by the charts, and denote by $p : TU_i \to \mathbb{R}^d$ the projection on the second factor. Using a partition of unity argument, it follows from Theorem 26.8 that we can choose the representative ω so that on each coordinate system U_i we have:

$$\omega|_{TD_i} = p^* \mathrm{d}\theta \quad \text{where } \int_{\mathbb{S}^{d-1}} \theta = 1.$$

For any $c > 0$ the vector fields X and cX have the same zeros and the same indices. Hence, by choosing c sufficiently large we can assume that:

$$X_p \notin \operatorname{supp} \omega, \quad \forall\, p \notin \bigcup_{i=1}^{N} D_i.$$

Therefore,

$$\int_M X^*\omega = \sum_{i=1}^{N} \int_{D_i} X^*\omega,$$

and so it is enough to verify that:

$$\int_{D_i} X^*\omega = \operatorname{ind}_{p_i} X.$$

Recall that $\operatorname{ind}_{p_i}(X) = \deg G_i$ where the Gauss map G_i is obtained using the identification $TU_i \simeq U_i \times \mathbb{R}^d$ provided by the charts:

$$X|_{U_i} : U_i \to TU_i, \quad p \mapsto (p, X_i(p)), \quad G_i = \frac{X_i}{||X_i||} : \partial D_i \to \mathbb{S}^{d-1} \subset \mathbb{R}^d.$$

Since the maps $X_i : \partial D_i \to \mathbb{R}^d$ and $G_i : \partial D_i \to \mathbb{R}^d$ are homotopic, we find:

$$\begin{aligned}\int_{D_i} X^*\omega &= \int_{D_i} \mathrm{d}(X^*p^*\theta) = \int_{\partial D_i} X_i^*\theta \\ &= \int_{\partial D_i} G_i^*\theta = (\deg G_i) \int_{\mathbb{S}^{d-1}} \theta = \mathrm{ind}_{p_i}(X).\end{aligned}$$

□

An immediate corollary is:

Corollary 26.14. *Let X and Y be vector fields with a finite number of zeros on an oriented, compact, connected manifold M. The sum of the indices of the zeros of X coincides with the sum of the indices of the zeros of Y.*

We must have already guessed that we have:

Theorem 26.15 (Poincaré-Hopf)**.** *Let M be an oriented, compact, connected manifold of dimension d. Then for any vector field $X \in \mathfrak{X}(M)$ with a finite number of zeros $\{p_1, \dots, p_N\}$, we have:*

$$\chi(M) = \sum_{i=1}^{N} \mathrm{ind}_{p_i} X.$$

In particular, $e(TM) = \chi(M)\mu$, where $\mu \in H^d(M)$ is the orientation class.

Remark 26.16. As we remarked before, there exists vector bundles with $e(\xi) = 0$, but where every section has a zero. However, in the case of the tangent bundle one can show that $e(TM) = 0$ if and only if there exists a non-vanishing vector field in M – see the Exercises at the end of this section. This result admits a "dual result" due to Thurston: a compact, oriented manifold admits a codimension 1 foliation if and only if $e(TM) = 0$. Thurston's Theorem is much harder to prove.

Proof. By the corollary above it is enough to construct a vector field X in M, with a finite number of zeros, for which the equality holds. For that, we fix a triangulation $\{\sigma_1, \dots, \sigma_l\}$ of M, and we construct a vector field X with the following properties:

(a) X has exactly one zero p_i in each face of the triangulation.
(b) The zero p_i is non-degenerate and

$$\mathrm{ind}_{p_i} X = (-1)^k,$$

where k is the dimension of the face containing p_i.

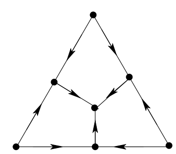
Hence, if r_k is the number of faces of dimension k, we have

$$\sum_{i=1}^{N} \mathrm{ind}_{p_i} X = r_0 - r_1 + \cdots + (-1)^d r_d,$$

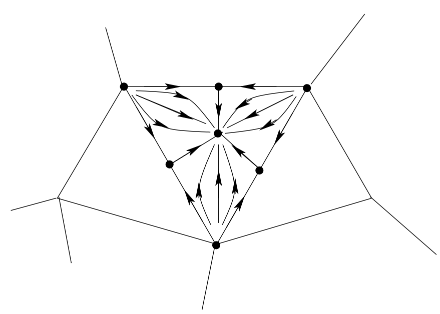
so the result follows from Euler's Formula – see Theorem 23.12. We construct X by describing its phase portrait in each face of the triangulation:

- In each face of dimension 0, the vector field X has a zero.
- In each face of dimension 1, we put a zero in the center of the face and connect it by orbits to the zeros in the vertices, as in the following figure:

- In each face of dimension 2, we put a zero in the center of the face and connect it by heteroclinic orbits to the zeros in the faces of dimension 1, as in the following figure:



  Then we complete the phase portrait of X in the face of dimension 2, so that the zero in its interior becomes an attractor of the vector field restricted to the face:



- In general, once one has constructed the phase portrait in the faces of dimension $k-1$, we construct the phase portrait in a face of dimension k, putting a zero in the center of the face and connecting it by heteroclinic orbits to the zeros in the faces of dimension $k-1$. We then complete the phase portrait so that the new zero is an attractor of the vector field restricted to the face of dimension k.

The vector field one constructs in this way has exactly one zero in each face. Moreover, we can assume that they are non-degenerate zeros. For a zero p_i in the face of dimension k, the linearization of the vector field at p_i is a real matrix with k eingenvalues with negative real part, corresponding to the directions along the face, and $n-k$ eingenvalues with positive real part, corresponding to the directions normal to the face. The sign of the determinant of this matrix is $(-1)^k$. Hence, we have that:

$$\operatorname{ind}_{p_i} X = (-1)^k.$$

This shows that the vector field X satisfies (a) and (b) and completes the proof of the Poincaré-Hopf Theorem. $\square$

Homework.

1. Prove the properties of fiber integration given in Proposition 26.4.

2. Let $E_1 \to M$ and $E_2 \to M$ be oriented vector bundles over a compact manifold M. Consider their Whitney sum with the the direct sum of the orientations. Denoting the projections:

$$\begin{tikzcd} & E_1 \oplus E_2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ E_1 & & E_2 \end{tikzcd}$$

show that the Thom classes of E_1, E_2 and $E_1 \oplus E_2$ are related by:

$$U_{E_1 \oplus E_2} = \pi_1^* U_{E_1} \wedge \pi_2^* U_{E_2}.$$

Use this property to prove that

$$e(\xi \oplus \xi') = e(\xi) \cup e(\xi').$$

3. Let $\xi = (\pi, E, M)$ and $\eta = (\tau, F, N)$ be oriented vector bundles of rank r over compact manifolds M and N. If $\Psi : \eta \to \xi$ is a morphism of vector bundles covering a map $\psi : N \to M$, which preserves orientations and is a fiberwise isomorphism, show that:

$$e(\eta) = \psi^* e(\xi).$$

Use this property to conclude that:
(a) $e(\bar{\xi}) = -e(\xi)$, where $\bar{\xi}$ denotes the vector bundle ξ with the opposite orientation.
(b) $e(\xi) = 0$ whenever $\operatorname{rank} \xi$ is odd.

4. Let $M = \mathbb{CP}^1 \simeq \mathbb{S}^2$ embedded in $\mathbb{CP}^2$ as the submanifold:

$$\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2, \quad [x : y] \mapsto [x : y : 0].$$

Find the Euler class of the normal bundle $\nu(\mathbb{CP}^1)$ and conclude that this vector bundle is non-trivial.

5. Consider the canonical complex line bundle $\gamma_d^1(\mathbb{C})$ over $\mathbb{CP}^d$, defined analogously to the canonical real line bundle γ_d^1 over $\mathbb{RP}^d$. Show that it is orientable and that is Euler class is non-trivial.

6. Let M be a compact manifold of dimension d. One can show that:
(a) If $p_1, \dots, p_N \in M$ there exists an open set $U \subset M$, diffeomorphic to the ball $\{x \in \mathbb{R}^d : ||x|| < 1\}$, such that $p_1, \dots, p_n \in U$.
(b) If $\psi : \mathbb{S}^{d-1} \to \mathbb{S}^{d-1}$ is a map with degree zero, then it is homotopic to the constant map.

Use these facts to show that if $\chi(M) = 0$, then there exists a nowhere vanishing vector field in M.

27. Pullbacks of Vector Bundles

The following *pullback* construction for vector bundles plays a crucial role.

Definition 27.1. *Let $\psi : M \to N$ be a smooth map and $\xi = (\pi, E, N)$ a vector bundle over N of rank r. The* ***pullback*** *of ξ by ψ is the vector bundle $\psi^*\xi = (\hat{\pi}, \psi^*E, M)$ of rank r, with total space given by:*

$$\psi^*E = \{(p, \mathbf{v}) \in M \times E : \psi(p) = \pi(\mathbf{v})\},$$

and projection defined by:

$$\hat{\pi} : \psi^*E \to M, \ (p, \mathbf{v}) \mapsto p.$$

Note that the fiber of $\psi^*\xi$ over p is a copy of the fiber of ξ over $\psi(p)$. Therefore the pullback of ξ by ψ is a vector bundle for which we take a copy of the fiber of ξ over q for each point in the preimage $\psi^{-1}(q)$.

We still need to check that the construction in the definition above does indeed produce a vector bundle. First of all, note that

$$\psi^*E = (\psi \times \pi)(\Delta),$$

where $\Delta \subset N \times N$ is the diagonal. Since $\pi : E \to N$ is a submersion, we have that $(\psi \times \pi) \pitchfork \Delta$, so $\psi^*E \subset M \times E$ is a submanifold. To cheek local triviality of $\psi^*\xi$, let $\{(U_\alpha, \phi_\alpha)\}$ be a trivialization of ξ. Then we obtain a trivialization $\{(\psi^{-1}(U_\alpha), \widetilde{\phi}_\alpha)\}$ for $\psi^*\xi$, where

$$\begin{aligned} \widetilde{\phi}_\alpha : \hat{\pi}^{-1}(\psi^{-1}(U_\alpha)) &\to \psi^{-1}(U_\alpha) \times \mathbb{R}^r \\ (p, \mathbf{v}) &\longmapsto (p, \phi_\alpha^{\psi(p)}(\mathbf{v})). \end{aligned}$$

Moreover, if $\{g_{\alpha\beta}\}$ is the cocycle of ξ associated with the trivialization $\{(U_\alpha, \phi_\alpha)\}$, then $\{\psi^*g_{\alpha\beta}\} = \{g_{\alpha\beta} \circ \psi\}$ is the cocycle of $\psi^*\xi$ associated with the trivialization $\{(\psi^{-1}(U_\alpha), \widetilde{\phi}_\alpha)\}$.

Notice that the map

$$\Psi : \psi^*\xi \to \xi \ (p, \mathbf{v}) \mapsto \mathbf{v},$$

is a morphism of vector bundles covering ψ. Hence, the pullback construction allows to complete the following commutative diagram of morphisms of vector bundles:

$$\begin{tikzcd} \psi^*E \arrow[r, dashed, "\Psi"] \arrow[d, dashed, "\hat{\pi}"'] & E \arrow[d, "\pi"] \\ M \arrow[r, "\psi"] & N \end{tikzcd}$$

In fact, we have the following universal property which characterizes the pullback up to isomorphism:

Proposition 27.2. *Let $\psi : M \to N$ be a smooth map, $\eta = (\tau, F, M)$ and $\xi = (\pi, E, N)$ vector bundles and $\Phi : \eta \to \xi$ a morphism of vector bundles covering ψ. Then there exists a unique morphism of vector bundles*

$\tilde{\Phi} : \eta \to \psi^*\xi$, covering the identity, which makes the following diagram commutative:

$$\begin{tikzcd}
F \arrow[dr, dashed, "\tilde{\Phi}"] \arrow[drr, bend left, "\Phi"] \arrow[ddr, bend right, "\tau"'] & & \\
& \psi^*E \arrow[r, "\Psi"] \arrow[d, "\hat{\pi}"'] & E \arrow[d, "\pi"] \\
& M \arrow[r, "\psi"'] & N
\end{tikzcd}$$

Moreover, $\tilde{\Phi}$ is an isomorphism if an only if $\Phi^p : F_p \to E_{\psi(p)}$ is an isomorphism for all $p \in M$.

Proof. The map $\tilde{\Phi} : \eta \to \psi^*\xi$ is given by:

$$\tilde{\Phi}(\mathbf{w}) = (\tau(\mathbf{w}), \Phi(\mathbf{w})).$$

We leave the details as an (easy) exercise. □

One can also **pullback morphisms** covering the identity: if $\xi = (\pi, E, N)$ and $\eta = (\tau, F, N)$ are vector bundles and $\Phi : \xi \to \eta$ is a morphism covering the identity, then for any smooth map $\psi : M \to N$ we define a morphism of vector bundles $\psi^*(\Phi) : \psi^*\xi \to \psi^*\eta$ by:

$$\psi^*(\Phi)(p, \mathbf{v}) = (p, \Phi(\mathbf{v})).$$

Obviously, this morphism makes the following diagram commute:

$$\begin{tikzcd}
\psi^*E \arrow[rr, "\psi^*(\Phi)"] \arrow[dr] \arrow[dd] & & \psi^*F \arrow[dr] \arrow[dd] & \\
& E \arrow[rr, "\Phi"] \arrow[dd] & & F \arrow[dd] \\
M \arrow[rr] \arrow[dr, "\psi"'] & & M \arrow[dr, "\psi"'] & \\
& N \arrow[rr] & & N
\end{tikzcd}$$

We list some basic properties of the pullback which are immediate from the definitions:

Proposition 27.3. *Let $\psi : M \to N$ and $\phi : Q \to M$ be smooth maps, ξ, η and θ vector bundles over N, and $\Phi : \xi \to \eta$ and $\Psi : \eta \to \theta$ morphisms of vector bundles over the identity. Then:*

(i) $\psi^*(\mathrm{Id}_\xi) = \mathrm{Id}_{\psi^*\xi}$;
(ii) $\psi^*(\Psi \circ \Phi) = \psi^*(\Psi) \circ \psi^*(\Phi)$;
(iii) $\psi^*(\varepsilon^r_N) = \varepsilon^r_M$;
(iv) $(\mathrm{Id})^*\xi = \xi$;
(v) $(\psi \circ \phi)^*\xi = \phi^*(\psi^*\xi)$.

Remark 27.4. Some of the equalities in this proposition are actually isomorphisms. However, they are canonical, i.e., they do not depend on any choices. So we still use the symbol "=" instead of "$\simeq$" to ease the notation. This same remark applies to many of the "equalities" that follow.

The previous result shows that if we fix a smooth map $\psi : M \to N$, then:

- The pullback defines a covariant functor from the category of vector bundles over N to the category of vector bundles over M.

On the other hand, if we denote by $\mathrm{Vect}_r(M)$ the set of isomorphism classes of vector bundles of rank r over a manifold M, there is a distinguish point in $\mathrm{Vect}_r(M)$: the class of the the trivial vector bundles. Given a smooth map $\psi : M \to N$, the pullback $\psi^* : \mathrm{Vect}_r(N) \to \mathrm{Vect}_r(M)$ preserves this distinguished point, so we also have:

- The pullback defines a contravariant functor from the category of smooth manifolds to the category of sets with a distinguished point.

All the functorial constructions with vector bundles are preserved under pullbacks. For example, one finds that:

(i) $\psi^*(\xi \oplus \eta) = \psi^*\xi \oplus \psi^*\eta$;
(ii) $\psi^*(\xi^*) = (\psi^*\xi)^*$;
(iii) $\psi^*(\wedge^k \xi) = \wedge^k \psi^*\xi$.

One can also commute the operation of restriction with pullbacks, provided the map $\psi : M \to N$ is transverse to the submanifold $Q \subset N$ so that $\psi^{-1}(Q) \subset M$ is a submanifold. One then has:

$$\psi^*(\xi|_Q) = \psi^*(\xi)|_{\psi^{-1}(Q)}.$$

There is also an operation of **pullback of sections**, taking sections of a vector bundle $\xi = (\pi, E, N)$ to sections of the pullback $\psi^*\xi = (\hat{\pi}, \psi^*E, M)$:

$$\begin{tikzcd} \psi^*E \arrow[r, "\Psi"] \arrow[d, "\hat{\pi}"] & E \arrow[d, "\pi"'] \\ M \arrow[r, "\psi"'] \arrow[u, bend left, "\psi^*s"] & N \arrow[u, bend right, "s"'] \end{tikzcd} \qquad \psi^*s(p) := (p, s(\phi(p))).$$

In particular, if $\operatorname{rank} \xi = r$, then $\psi^* \wedge^r \xi = \wedge^r \psi^*\xi$, and the pullback of a non-vanishing section of $\wedge^r\xi$ is a non-vanishing section of $\wedge^r\psi^*\xi$. It follows that the pullback $\psi^*\xi$ of an oriented vector bundle ξ has a natural **pullback orientation**. This gives rise to another property of pullbacks: it preserves fiber integration. We state it and leave the proof as an exercise:

Proposition 27.5. *Let $\psi : M \to N$ be a smooth map, let $\xi = (\pi, E, N)$ be an oriented vector bundle and consider $\psi^*\xi = (\hat{\pi}, \psi^*E, M)$ with the pullback orientiation. For any form $\omega \in \Omega^\bullet_{cv}(E)$:*

$$\hat{\pi}_*\Psi^*\omega = \psi^*\pi_*\omega,$$

where $\Psi : \psi^*\xi \to \xi$ is the canonical vector bundle map covering ψ.

Another fundamental property of the pullback of vector bundles is:

Theorem 27.6 (Homotopy invariance)**.** *If ψ and $\phi : M \to N$ are homotopic maps and ξ is a vector bundle over N, then the pullbacks $\psi^*\xi$ and $\phi^*\xi$ are isomorphic vector bundles.*

Proof. Let $H : M \times [0,1] \to N$ be an homotopy between ϕ and ψ. We have:

$$\phi^*\xi = H_0^*\xi = H^*\xi|_{M\times\{0\}},$$
$$\psi^*\xi = H_1^*\xi = H^*\xi|_{M\times\{1\}}.$$

Hence, it is enough to show that for any vector bundle η over $M \times [0,1]$, the restrictions $\eta|_{M\times\{0\}}$ and $\eta|_{M\times\{1\}}$ are isomorphic. Note that H is only C^0, but one can show that:

(a) a vector bundle morphism of class C^0 covering a map of class C^∞ can be approximated by a morphism of classe C^∞ covering the same map.
(b) a vector bundle morphism which is close enough to an isomorphism is also an isomorphism.

Hence, it is enough to proof that for any vector bundle $\eta = (\pi, E, M \times [0,1])$, there exists a C^0-morphism of vector bundles $\Delta : \eta \to \eta$, covering the map

$$\delta : M \times [0,1] \to M \times [0,1],\ (p,t) \mapsto (t,1),$$

and such that the induced maps in the fibers are isomorphisms. In order to construct Δ, we use the following lemma, whose proof is left as an exercise:

Lemma 27.7. *Let η be a vector bundle over $M \times [0,1]$. There exists an open cover $\{U_\alpha\}_{\alpha\in A}$ of M such that the restrictions $\eta|_{U_\alpha\times[0,1]}$ are trivial vector bundles.*

Now choose a locally finite countable open cover $\{U_k\}_{k\in\mathbb{N}}$ of M such that each $\eta|_{U_k\times[0,1]}$ is trivial. Let us denote the trivializing maps ϕ_k by:

$$\begin{tikzcd} E|_{U_k\times[0,1]} \arrow[rr, "\phi_k"] \arrow[dr, "\pi"'] & & (U_k \times [0,1]) \times \mathbb{R}^r \arrow[dl, "\pi_1"] \\ & U_k \times [0,1] & \end{tikzcd}$$

Denote by $\{\rho_k\}_{k\in\mathbb{N}}$ an *envelope of unity* subordinated to the cover $\{U_k\}_{k\in Nn}$, i.e., a collection of continuous maps $\rho_k : M \to \mathbb{R}$ such that $0 \le \rho_k \le 1$, $\operatorname{supp} \rho_k \subset U_k$ and, for all $p \in M$,

$$\max\{\rho_k(p) : k \in \mathbb{N}\} = 1.$$

Such an envelope of unity can be constructed starting with a partition of unity $\{\theta_k\}$ and defining:

$$\rho_k(p) \equiv \frac{\theta_k(p)}{\max\{\theta_k(p) : k \in \mathbb{N}\}}.$$

For each $k \in \mathbb{N}$ we define vector bundle morphisms $\Delta_k : \eta \to \eta$ by:

(a) Δ_k cover the map $\delta_k : M \times [0,1] \to M \times [0,1]$ given by:

$$\delta_k(p,t) = (p, \max(\rho_k(p), t)).$$

(b) In $\pi^{-1}(U_k \times [0,1])$, Δ_k is defined by:

$$\Delta_k(\phi_k^{-1}(p,t,\mathbf{v})) \equiv \phi_k^{-1}(p, \max(\rho_k(p), t), v),$$

and Δ_k is the identity outside $\pi^{-1}(U_k \times [0,1])$.

Finally, one defines $\Delta : \eta \to \eta$ by:

$$\Delta = \cdots \circ \Delta_k \circ \cdots \circ \Delta_1.$$

Since each $p \in M$ has a neighborhood which intersects a finite number of open sets U_k, this is a well-defined vector bundle morphism $\Delta : \eta \to \eta$ which locally is an the composition of vector bundle which isomorphisms on the fibers. Hence, Δ is a vector bundle isomorphism which covers $\delta : M \times [0,1] \to M \times [0,1]$. □

Corollary 27.8. *Any vector bundle over a contractible manifold is trivial.*

Proof. Let $\xi = (\pi, E, M)$ be a vector bundle and let $\phi : M \to \{*\}$ and $\psi : \{*\} \to M$ be smooth maps such that $\psi \circ \phi$ is homotopic to id_M. The theorem above shows that:

$$\xi \simeq (\psi \circ \phi)^* \xi \simeq \phi^*(\psi^* \xi).$$

Since $\psi^*\xi$ is a vector bundle over a set which consist of a single point, it is a trivial vector bundle. Hence $\xi \simeq \phi^*(\psi^*\xi)$ is also a trivial vector bundle. □

Hence, when M is contractible the space $\mathrm{Vect}_r(M)$ consisting of isomorphism classes of vector bundles of rank r over M has only one point.

Example 27.9.
Given a line bundle $\xi = (\pi, E, \mathbb{S}^1)$, we can cover $\mathbb{S}^1$ by the two contractible open sets $U = \mathbb{S}^1 - \{p_N\}$ and $V = \mathbb{S}^1 - \{p_S\}$. By the corollary, over each open set U and V the vector bundle trivializes: $\phi_U : E|_U \simeq U \times \mathbb{R}$ and $\phi_V : E|_V \simeq V \times \mathbb{R}$. Therefore, the line bundle is completely characterized by the transition function $g_{UV} : U \cap V \to \mathbb{R}$, so that:

$$\phi_V \circ \phi_U^{-1} : U \times \mathbb{R} \to V \times \mathbb{R}, \ (p,v) \mapsto (p, g_{UV}(p)v).$$

The intersection $U \cap V$ has two connected components, and we leave it as an exercise to check that if $g_{UV}(x)$ has the same sign in both components, then ξ is trivial, while if $g_{UV}(x)$ has the opposite signs in the two components then the line bundle is isomorphic to the line bundle whose total space is the Möbius band. In other words, the space $\mathrm{Vect}_1(\mathbb{S}^1)$ *consisting of isomorphism classes of line bundles over $\mathbb{S}^1$ has two elements.*

*For a general manifold, up to isomorphism, a line bundle ξ over M is uniquely determined by its **first Stiefel-Whitney class** $w_1(\xi) \in H^1(M, \mathbb{Z}_2)$,* *which can be defined as follows. Define a group homomorphism* $\widetilde{w}_1(\xi) : \pi_1(M) \to \mathbb{Z}_2$ *by setting:*

$$\widetilde{w}_1(\xi)([\gamma]) := \begin{cases} 0 & \text{if } \gamma^*\xi \to \mathbb{S}^1 \text{ is trivial,} \\ \\ 1 & \text{if } \gamma^*\xi \to \mathbb{S}^1 \text{ is not trivial.} \end{cases}$$

Since $\mathbb{Z}_2$ *is abelian, any commutator in* $\pi_1(M)$ *must be in the kernel of this group homomorphism. Hence,* $\widetilde{w}_1(\xi)$ *descends to a group homomorphism:*

$$w_1(\xi) : H_1(M, \mathbb{Z}) \to \mathbb{Z}_2,$$

i.e., we have an element $w_1(\xi) \in H^1(M, \mathbb{Z}_2)$*. We leave as an exercise to check that* w_1 *gives a bijection:*

$$w_1 : \mathrm{Vect}_1(M) \simeq H^1(M, \mathbb{Z}_2).$$

For example, for the real projective space one has $H^1(\mathbb{RP}^d, \mathbb{Z}_2) = \mathbb{Z}_2$*. So* $\mathrm{Vect}_1(\mathbb{RP}^d)$ *has two elements: the class of the trivial bundle and the class of the canonical line bundle* γ^1_d*.*

Homework.

1. Give a proof of the universal property of pullbacks (Proposition 27.2). Show that this property characterizes the pullback of vector bundles up to isomorphism.

2. Verify the properties of the pullback of vector bundles given by Proposition 27.3.

3. Let $\Psi : \eta \to \xi$ be a vector bundle map covering a map $\psi : M \to N$. Show that if Ψ is a fiberwise isomorphism then η is isomorphic to $\phi^*\eta$. Use this to conclude that Proposition 27.5 follows from Proposition 26.4.

4. Let $\phi : M \to N$ be a submersion and denote by $\mathcal{F}$ the foliation of M by the fibers of ϕ. Show that the normal bundle $\nu(\mathcal{F})$ is naturally isomorphic to the pulback bundle ϕ^*TN.

5. Let ξ be a vector bundle over $M \times [0,1]$. Show that there exists an open cover $\{U_\alpha\}_{\alpha \in A}$ of M such that the restrictions $\xi|_{U_\alpha \times [0,1]}$ are trivial.
Hint: Show that if ξ is a vector bundle over $M \times [a,c]$ which is trivial when restricted to both $M \times [a,b]$ and $M \times [b,c]$, for some $a < b < c$, then ξ is a trivial vector bundle.

6. Complete the details of Example 27.9, showing that $\mathrm{Vect}_1(\mathbb{S}^1) \simeq \mathbb{Z}_2$.

7. For a line bundle ξ denote by $w_1(\xi) \in H^1(M, \mathbb{Z}_2)$ its first Stiefel-Whitney class.
   (a) Given a class $c \in H^1(M, \mathbb{Z}_2)$ show that there exists a line bundle ξ whose first Stiefel-Whitney class is $w_1(\xi) = c$;
   (b) Conclude that there is a bijection:

$$w_1 : \mathrm{Vect}_1(M) \simeq H^1(M, \mathbb{Z}_2);$$

(c) Show that the tensor product makes $\mathrm{Vect}_1(M)$ into a group. What is the group structure induced on $H^1(M, \mathbb{Z}_2)$?

8. Determine $\mathrm{Vect}_1(\mathbb{T}^d)$.

9. Denote by $\mathrm{Pic}(M)$ is the space of isomorphism classes of complex line bundles over a manifold M. Show the tensor product turns $\mathrm{Pic}(M)$ into a group, called the **Picard group** of M. Find $\mathrm{Pic}(\mathbb{S}^1)$.

28. The Classification of Vector Bundles

The problem of determining $\mathrm{Vect}_k(M)$ can be reduced to a problem in homotopy theory. We will only sketch this briefly since this topic belongs to the realm of algebraic topology.

Recall that γ_n^r denotes the **canonical bundle** over the Grassmannian $G_r(\mathbb{R}^n)$ (Section 25, Exercise 2): the total space of γ_n^r is defined by:

$$E = \{(S, x) : S \subset \mathbb{R}^n \text{ is } r\text{-dimensional subspace and } x \in S\},$$

and the projection $\pi : E \to G_r(\mathbb{R}^n)$ is given by $\pi(S, x) = S$. The canonical bundle is a subbundle of the trivial vector bundle $\varepsilon^n_{G_r(\mathbb{R}^n)}$.

There is another important vector bundle over the Grassmannian, called the **universal quotient bundle** and denoted η_n^r. We can define it as the vector bundle of rank r over $G_{n-r}(\mathbb{R}^n)$ whose total space is:

$$F = \{(S, x + S) : S \subset \mathbb{R}^n \text{ is } (n-r)\text{-dimensional subspace and } x \in \mathbb{R}^n\},$$

and the projection $\pi : F \to G_r(\mathbb{R}^n)$ is given by $\pi(S, x + S) = S$. In other words, the fiber over S is the normal space to S in $\mathbb{R}^n$.

These vector bundles are related via the short exact sequence of vector bundles:

$$0 \longrightarrow \gamma_n^{n-r} \longrightarrow \varepsilon^n_{G_{n-r}(\mathbb{R}^n)} \longrightarrow \eta_n^r \longrightarrow 0$$

In particular, choosing n global sections of the trivial bundle $\varepsilon^n_{G_{n-r}(\mathbb{R}^n)}$ yields n global sections of η_n^r which at each point S generate the fiber $\mathbb{R}^n/S$.

The reason for the name *universal* is justified by the following proposition:

Proposition 28.1. *Let ξ be a rank r vector bundle over a manifold M. If ξ admits n global sections $s_1, \dots, s_n$ which generate E_p for all $p \in M$, then there exists a smooth map $\psi : M \to G_{n-r}(\mathbb{R}^n)$ such that:*

$$\xi \simeq \psi^*(\eta_n^r).$$

Proof. Let

$$V := \bigoplus_{i=1}^{n} \mathbb{R} s_i,$$

so $V \simeq \mathbb{R}^n$. Since the sections s_i generate E_p, for each $p \in M$, there exists a linear surjective map

$$V \xrightarrow{\mathrm{ev}_p} E_p \longrightarrow 0\ .$$

The kernel $\operatorname{Ker} \mathrm{ev}_p$ of this map is a subspace of V of codimension r.

Define a smooth map:

$$\psi : M \to G_{n-r}(V),\ p \mapsto \operatorname{Ker} \mathrm{ev}_p.$$

Then we have a vector bundle map:

$$\xi \mapsto \psi^*\eta_n^r, \quad \mathbf{v} \mapsto (\pi(\mathbf{v}), \mathrm{ev}_{\pi(\mathbf{v})}^{-1}(\mathbf{v})).$$

This is a fiberwise isomorphism covering the identity, so it is a vector bundle isomorphism. After choosing a basis for V, we obtain the desired classifying map $\psi : M \to G_{n-r}(\mathbb{R}^n)$. □

A map $\psi : M \to G_{n-r}(\mathbb{R}^n)$ such that $\xi \simeq \psi^*\eta_n^r$ is called a **classifying map** for the vector bundle ξ. We leave as an exercise to check that any such classifying map arises from the choice of n global sections $s_1, \dots, s_n \in \Gamma(\xi)$ generating each fiber E_p, as in the previous proof.

The next result shows that given a vector bundle over a manifold of finite type one can always find a classifying map by taking n sufficient large.

Proposition 28.2. *Let ξ be a rank r vector bundle over a manifold M. If M admits a finite good cover with k open sets, then for $n \geq rk$:*

(i) There exist classifying maps $\psi : M \to G_{n-r}(\mathbb{R}^n)$ for ξ;
(ii) Any two classifying maps are homotopic.

Proof. (i) We claim that ξ admits global sections $s_1, \dots, s_n$ which generate E_p, for all $p \in M$, so (i) follows from Proposition 28.1. To see this, let $U_1, \dots, U_k$ be a finite good cover of M. Since each U_i is contractible, the restriction $\xi|_{U_i}$ is trivial. Hence, we can choose a basis of local sections $\{s_1^i, \dots, s_r^i\}$ for $\Gamma(\xi|_{U_i})$. Note that there are open sets $V_1, \dots, V_k$, with $\overline{V}_i \subset U_i$ which still cover M. If we choose smooth functions $f_i : U_i \to \mathbb{R}$ such that $f_i|_{V_i} = 1$ and $f_i = 0$ outside U_i, then $\{f_i s_1^i, \dots, f_i s_r^i : i = 1, \dots, k\}$ are the desired global sections.

(ii) Let $\psi : M \to G_{n-r}(V)$ and $\psi : M \to G_{n-r}(V')$ be two classifying maps constructed from two choices of global sections $\{s_1, \dots, s_n\}$ and $\{s_1', \dots, s_n'\}$, as in the proof of the previous proposition. Then we have a canonical identification between V and V' and also between $G_{n-r}(V)$ and $G_{n-r}(V')$. It follows that the classifying map is well-defined up to a choice of identification $V \simeq \mathbb{R}^n$. If we fix this choice, then we conclude that two classifying maps $\psi : M \to G_{n-r}(\mathbb{R}^n)$ and $\psi : M \to G_{n-r}(\mathbb{R}^n)$ differ by the action of an element $A \in GL(n)$:

$$\psi' = A \circ \psi.$$

Note that A can be chosen to have positive determinant. Since $GL^+(n)$ is connected, we can choose a continuous path $A_t \in GL^+(n)$ with $A_1 = A$ and $A_0 = I$, so that the map:

$$\psi_t := A_t \circ \psi : M \to G_{n-r}(\mathbb{R}^n),$$

is a homotopy between ψ and ψ'. □

Denote by $[M, N]$ the set of homotopy classes of maps $\phi : M \to N$. We obtain:

Theorem 28.3 (Classification of vector bundles). *Let M be a manifold which admits a good open cover with k open sets. For every $n \geq rk$, there exists a bijection:*

$$\mathrm{Vect}_r(M) \simeq [M, G_{n-r}(\mathbb{R}^n)].$$

Proof. We saw above that the homotopy class of a classifying map for ξ is determined by the isomorphism class of ξ, so we have a well-defined map:

$$\mathrm{Vect}_r(M) \to [M : G_{n-r}(\mathbb{R}^n)].$$

On the other hand, by the homotopy invariance of the pullbacks, we conclude that the pullback of the universal bundle induces a map

$$[M : G_{n-r}(\mathbb{R}^n)] \to \mathrm{Vect}_r(M), \ \psi \mapsto \psi^* \eta_n^r.$$

We leave as an exercise to show that these maps are inverse to each other, so the result follows. □

This result reduces the classification of vector bundles to a homotopy problem. We illustrate this in the next example, which assumes some knowledge of homotopy theory.

Example 28.4.
Recall that if X is a path connected topological space then the free homotopies and the homotopies based at $x_0 \in X$ are related by:

$$\pi_k(X, x)/\pi_1(X, x) \simeq [\mathbb{S}^k, X],$$

where the right-hand side is the orbit space for the natural action of $\pi_1(X, x)$ in $\pi_k(X, x)$. Therefore, we have:

$$\mathrm{Vect}_r(\mathbb{S}^k) = [\mathbb{S}^k, G_{n-r}(\mathbb{R}^n)] \simeq \pi_k(G_{n-r}(\mathbb{R}^n))/\pi_1(G_{n-r}(\mathbb{R}^n)),$$

for n large enough. On the other, since the Grassmannian is a homogeneous space:

$$G_{n-r}(\mathbb{R}^n) = O(n)/(O(n-r) \times O(r)),$$

and $\pi_k(O(n)/O(n-r)) = 0$, if n is large enough, the long exact sequence in homotopy yields:

$$\pi_k(G_{n-r}(\mathbb{R}^n)) = \pi_{k-1}(O(r)).$$

Hence, we conclude that:

$$\mathrm{Vect}_r(\mathbb{S}^k) = \pi_{k-1}(O(r))/\pi_0(O(r)) = \pi_{k-1}(O(r))/\mathbb{Z}_2.$$

In order to understand this quotient, one needs to figure out the action of $\pi_0(O(r))$ on $\pi_{k-1}(O(r))$. If $g \in O(r)$, the action by conjugation $i_g : O(r) \to O(r)$, $i_g(h) = ghg^{-1}$, induces an action in homotopy:

$$(i_g)_* : \pi_{k-1}(O(r)) \to \pi_{k-1}(O(r)).$$

If g_1 and g_2 belong to the same connected component, then $(i_{g_1})_* = (i_{g_2})_*$. Hence, we obtain an action of $\pi_0(O(r)) = \mathbb{Z}_2$ on $\pi_{k-1}(O(r))$, which is precisely the action above.

For example, if r is odd then $-I$ represents the non-trivial class in $\pi_0(O_r)$. Since the action by conjugation of $-I$ is trivial, we conclude that

$$\mathrm{Vect}_r(\mathbb{S}^k) = \pi_{k-1}(O(r)), \ \textit{if } r \textit{ is odd.}$$

For instance, we have:

$$\mathrm{Vect}_3(\mathbb{S}^4) = \pi_3(SO(3)) = \pi_3(\mathbb{S}^3) = \mathbb{Z}.$$

On the other hand, when r is even, the action maybe non-trivial. Take for instance $r = 2$, so we have $\pi_1(O(2)) = \mathbb{Z}$. The action of $\pi_0(O_2) = \mathbb{Z}_2$ in $\mathbb{Z}$ is just $\pm 1 \cdot n = \pm n$. Hence, we have

$$\mathrm{Vect}_2(\mathbb{S}^k) = \pi_{k-1}(O(2))/\mathbb{Z}_2 = \pi_{k-1}(\mathbb{S}^1)/\mathbb{Z}_2 = \begin{cases} \mathbb{Z}/\mathbb{Z}_2 & \textit{if } k = 2, \\ 0 & \textit{if } k \geq 3. \end{cases}$$

Remark 28.5. If a manifold is not of finite type, there still exists a classification of vector bundles over M. In this case, we need to consider the the space:

$$\mathbb{R}^\infty = \bigoplus_{d=0}^{\infty} \mathbb{R}^d,$$

which is the direct limit of the increasing sequence of vector spaces:

$$\cdots \subset \mathbb{R}^d \subset \mathbb{R}^{d+1} \subset \mathbb{R}^{d+2} \subset \cdots$$

This is an example of a so-called *profinite manifold*, a class of infinite dimensional manifolds sharing many properties with the class of finite dimensional manifolds.

In $\mathbb{R}^\infty$ we can still consider the Grassmannian:

$$\tilde{G}_r(\mathbb{R}^\infty) = G_{\infty - r}(\mathbb{R}^\infty) = \{S \subset \mathbb{R}^\infty : \text{ linear subspace of codimension } r\}.$$

Over this infinite dimensional Grassmannian there is a tautological vector bundle $\eta^r_\infty = (\pi, E, \tilde{G}_r(\mathbb{R}^\infty))$, called the **universal bundle of rank** r. It has total space:

$$E = \{(S, x) : S \subset \mathbb{R}^\infty \text{ subspace of codimension } r, \, x \in \mathbb{R}^\infty/S\},$$

and projection:

$$\pi : E \to \tilde{G}_r(\mathbb{R}^\infty), \ (S, x) \mapsto S.$$

One can show that every vector bundle of rank r over a manifold M is isomorphic to a pullback $\psi^* \eta^r_\infty$ for some classifying map

$$\psi : M \to \tilde{G}_r(\mathbb{R}^\infty).$$

Has before, any two classifying maps are homotopic, and one obtains for *any* manifold M a bijection:

$$\mathrm{Vect}_r(M) \simeq [M, \tilde{G}_r(\mathbb{R}^\infty)].$$

This approach, via infinite dimensional Grassmanian, has the advantage of avoiding any reference to "large enough n", as we did before in the case of a manifold of finite type. On the other had, it forces one to deal with vector bundles over infinite dimensional manifolds.

HOMEWORK.

1. Let $\xi = (\pi, E, M)$ be a vector bundle and $N \subset M$ a closed submanifold. Show that every section $s : N \to E$ over N, admits an extension to a section $\tilde{s} : U \to E$ definided over an open set $U \supset N$.

2. Let $\psi : M \to G_{n-r}(\mathbb{R}^n)$ be a classifying map for a vector bundle $\xi = (\pi, E, M)$. Show that ψ is obtained from the choice of n global sections $s'_1, \dots, s'_n \in \Gamma(\xi)$ generating each fiber E_p, as in the proof of Proposition 28.1.

3. Let M admit a finite good cover with k open sets and let $n \geq kr$. Show that the map
$$\mathrm{Vect}_r(M) \to [M : G_{n-r}(\mathbb{R}^n)],\ [\xi] \mapsto f,$$
associating to an isomorphism class of a vector bundle ξ the homotopy class of a classifying map f, and the map
$$[M : G_{n-r}(\mathbb{R}^n)] \to \mathrm{Vect}_r(M),\ \psi \mapsto \psi^* \eta_n^r,$$
are inverse to each other.

4. Determine $\mathrm{Vect}_r(\mathbb{S}^1)$, $\mathrm{Vect}_r(\mathbb{S}^2)$ and $\mathrm{Vect}_r(\mathbb{S}^3)$.

29. Connections and Parallel Transport

In general, there is no natural way to differentiate sections of a vector bundle. The reason is that there is no canonical way of comparing fibers of a vector bundle over different points of the base. This can be fixed with the following notion:

Definition 29.1. *A* ***connection on a vector bundle*** $\xi = (\pi, E, M)$ *is a map*

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E),\ (X, s) \mapsto \nabla_X s,$$

which satisfies the following properties:

(i) $\nabla_{X_1+X_2} s = \nabla_{X_1} s + \nabla_{X_2} s$;
(ii) $\nabla_X (s_1 + s_2) = \nabla_X s_1 + \nabla_X s_2$;
(iii) $\nabla_{fX} s = f \nabla_X s$;
(iv) $\nabla_X (fs) = f \nabla_X s + X(f) s$.

Properties (iii) and (iv) show that a connection ∇ can be restrict to any open set $U \subset M$, yielding a connection in $\xi|_U$. On the other hand, the map $X \mapsto \nabla_X$ is $C^\infty(M)$-linear, hence, for any section s definided in a neighborhood U of $p \in M$ and any $\mathbf{v} \in T_pM$, we can define

$$\nabla_{\mathbf{v}} s \equiv \nabla_X s(p) \in E_p,$$

where X is any vector field defined in a neighborhood of p such that $X_p = \mathbf{v}$. Note, however, that $\nabla_{\mathbf{v}} s$ depends on the values of s in a neighborhood of p, not only on $s(p)$ (property (iv) in the definition).

Let $U \subset M$ be an open set where ξ trivializes, so we can choose a basis of sections $\{s_1, \dots, s_r\}$ for $\xi|_U$. Given any section $s \in \Gamma(\xi)$ we have that:

$$s|_U = f^1 s_1 + \cdots + f^r s_r.$$

for unique smooth functions $f^i \in C^\infty(U)$. The connection ∇ on the open set U is then completely determined by its effect on the sections s_i: for any vector field $X \in \mathfrak{X}(M)$, by property (iv), we have:

$$(\nabla_X s)|_U = \sum_{a=1}^{r} \left(f^a \nabla_X s_a + X(f^a) s_a \right).$$

We can write the local section $\nabla_X s_a$ in terms of the local basis as

$$\nabla_X s_a = \sum_{b=1}^{r} \omega_a^b(X) s_b,$$

where, by properties (i) and (iii), $\omega_a^b \in \Omega^1(U)$. One calls the matrix of 1-forms $\omega = [\omega_b^a]$ the **connection 1-form**. It determines completely the connection on U:

$$(\nabla_X s)|_U = \sum_{a=1}^{r} \left(\sum_{b=1}^{r} f^b \omega_b^a(X) + X(f^a) \right) s_a.$$

Exercise 3, in the Homework, discusses how the connection 1-form depends on the choice of trivializing sections.

Assume, additionally, that U is the domain of a chart $(x^1, \dots, x^d)$. Then there exists unique functions $\Gamma_{ia}^b \in C^\infty(U)$ such that:

$$\nabla_{\frac{\partial}{\partial x^i}} s_a = \sum_{b=1}^{r} \Gamma_{ia}^b s_b.$$

The functions Γ_{ia}^b are called the **Christoffel symbols** of the connection relative to the coordinate systems and basis of local sections. They are related to the connection 1-form by:

$$\omega_b^a = \sum_{i=1}^{r} \Gamma_{ib}^a \mathrm{d}x^i.$$

If $X = \sum_{i=1}^d X^i \frac{\partial}{\partial x^i}$, then the local form for the connection becomes:

$$(\nabla_X s)|_U = \sum_{a=1}^r \sum_{i=1}^d \left(\sum_{b=1}^r f^b X^i \Gamma_{ib}^a + X^i \frac{\partial f^a}{\partial x^i} \right) s_a.$$

Example 29.2.

Recall that the vector bundle $\xi = (\pi, E, M)$ of rank r is trivial if and only if it admits a basis of global sections $\{s_1, \dots, s_r\}$. For each such choice of basis, we can define a connection in ξ by setting:

$$\nabla_X s_a := 0, \quad (a = 1, \dots, r).$$

Note that this connection depends on the choice of trivializing sections.

The collection of all connections on a fixed vector bundle ξ has an affine structure: if $\rho \in C^\infty(M)$ is any smooth function, ∇_1 and ∇_2 are connections, then the affine combination

$$\rho \nabla_1 + (1 - \rho) \nabla_2,$$

also defines a connection in ξ. This fact that allows us to show that:

Proposition 29.3. *Every vector bundle $\xi = (\pi, E, M)$ admits a connection.*

Proof. Let $\{U_\alpha\}$ be an open cover of M by trivializing open sets. The previous example shows that in each U_α we can choose a connection ∇^α. We define a connection ∇ in M "gluing" these connections: if $\{\rho_\alpha\}$ is a partition of unity subordinated to the cover $\{U_\alpha\}$, then

$$\nabla \equiv \sum_\alpha \rho_\alpha \nabla^\alpha,$$

defines a connection in ξ. □

If one starts with vector bundles with a connection, the usual constructions lead to vector bundles with connections. The proof is left as an exercise.

Proposition 29.4. *Let ξ and ξ' be vector bundles over M, furnished with connections ∇ and ∇'. Then the associated bundles $\xi \oplus \xi'$, ξ^* and $\wedge^k \xi$, have induced connections satisfying:*

$$\begin{aligned} \nabla_X(s_1 \oplus s_2) &= \nabla_X s_1 \oplus \nabla_X s_2, \\ \nabla_X(s_1 \wedge \cdots \wedge s_k) &= \nabla_X s_1 \wedge \cdots \wedge s_k + \cdots + s_1 \wedge \cdots \wedge \nabla_X s_k \\ X(\langle s, \eta \rangle) &= \langle \nabla_X s, \eta \rangle + \langle s, \nabla_X \eta \rangle. \end{aligned}$$

If $\psi : N \to M$ is a smooth map, then $\psi^* \xi$ has a connection induced from ∇ such that:

$$(\nabla_X \psi^* s)(p) = (p, \nabla_{\mathrm{d}_p \psi(X_p)} s), \quad \forall p \in N, s \in \Gamma(\xi).$$

Connections can be used to compare different fibers of a vector bundle. Let $\xi = (\pi, E, M)$ be a vector bundle with a connection ∇. If $c : [0,1] \to M$ is a smooth curve then the pullback bundle $c^*\xi$ has an induced connection which we still denote by ∇. Notice that a section s of the bundle $c^*\xi$ is just a section of ξ along c, i.e., a smooth map $s : [0,1] \to E$ such that $\pi(s(t)) = c(t)$, for all $t \in [0,1]$.

Definition 29.5. *The* ***covariant derivative*** *of a section along a curve c is the section along c given by:*

$$D_c s \equiv \nabla_{\frac{\mathrm{d}}{\mathrm{d}t}} s.$$

A section along c is called a ***parallel section*** *if it has vanishing covariant derivative: $D_c s = 0$*

The operation of covariant derivative enjoys the following properties:

(i) $D_c(s_1 + s_2) = D_c s_1 + D_c s_2$;
(ii) $D_c(fs) = (f \circ c) D_c s + \mathrm{d}f(\dot{c})s$.

Choose local coordinates $(U, x^1, \dots, x^d)$, admitting trivializing sections $\{s_1, \dots, s_r\}$ over U. Given a curve $c(t)$ in U we set $c^i(t) = x^i(c(t))$. Any section s along c can be expressed as $s(t) = \sum_a v^a(t) s_a(c(t))$, and then the covariant derivative along c has components:

$$(D_c s)^a = \frac{\mathrm{d}v^a}{\mathrm{d}t}(t) + \sum_{ib} \frac{\mathrm{d}c^i}{\mathrm{d}t}(t) \Gamma^a_{ib}(c(t)) v^b(t), \quad (a = 1, \dots, r). \tag{29.1}$$

Remark 29.6. One can define the covariant derivative alternatively as follows. Given a section $s(t)$ along a curve $c(t)$ one chooses a *time-dependent section* $\tilde{s}_t \in \Gamma(E)$ such that:

$$\tilde{s}_t(c(t)) = s(t), \quad \forall t \in I.$$

Then:

$$D_c s(t) := \nabla_{c(t} \tilde{s}_t + \frac{\mathrm{d}}{\mathrm{d}t} \tilde{s}_t(p) \bigg|_{p=c(t)}. \tag{29.2}$$

One can show that this is independent of the choice of extension $\tilde{s}_t$, either by working in a local chart or by showing that it coincides with our first definition.

Notice, in particular, that even for a constant curve $c(t) = p_0$ the covariant derivative along c may not be zero! In fact, in this case, a section along c is just a curve $s : [0,1] \to E_{p_0}$ in the fiber over p_0 and the covariant derivative is the usual derivative of this curve.

Lemma 29.7. *For any curve $c : [0,1] \to M$ and any $v_0 \in E_{c(0)}$, there exists a unique parallel section s along c with initial condition $s(0) = v_0$.*

Proof. Since an interval is contractible, the pullback bundle $c^*\xi$ is trivial. This means that we can find sections $\{s_1, \dots, s_r\}$ along c such that any section s along c can be uniquely written as $s(t) = \sum_{a=1}^{r} v^a(t)s_a(t)$, for some smooth functions $v^a : [0,1] \to \mathbb{R}$. In particular, if we define $\omega_b^a(t)$ by:

$$D_c s_b(t) = \sum_{a=1}^{r} \omega_b^a(t) s_a(t),$$

we find that:

$$D_c s = \sum_{a=1}^{r} \left(\frac{\mathrm{d}v^a}{\mathrm{d}t}(t) + \sum_{b=1}^{r} \omega_b^a(t) v^b(t) \right) s_a(t).$$

Hence, the parallel sections along c are the solutions of the system of ODEs:

$$\begin{cases} \frac{\mathrm{d}v^a}{\mathrm{d}t}(t) = -\sum_{b=1}^{r} \omega_b^a(t) v^b(t), \\ \\ v^a(0) = v_0^a \end{cases} \qquad (a = 1, \dots, r).$$

Hence the lemma follows from the well-known results about existence and uniqueness of solutions of ODEs with time dependent coefficients. □

Under the conditions of this lemma, we say that the vectors $s(t) \in E_{c(t)}$ are obtained by parallel transport along the curve c. We denote the operation of **parallel transport** along c by:

$$\tau_t : E_{c(0)} \to E_{c(t)}, \qquad \tau_t(v_0) := s(t).$$

The next result shows that parallel transport contains all the information about the connections ∇:

Proposition 29.8. *Let $\xi = (\pi, E, M)$ be a vector bundle with a connection ∇ and let $c : [0,1] \to M$ be a smooth curve. Then:*

(i) Parallel transport $\tau_t : E_{c(0)} \to E_{c(t)}$ along c is a linear isomorphism.
(ii) If $\mathbf{v} = c'(0) \in T_{c(0)}M$, then for any section $s \in \Gamma(\xi)$:

$$\nabla_{\mathbf{v}} s = \lim_{t \to 0} \frac{1}{t} \left(\tau_t^{-1}(s(c(t))) - s(c(0)) \right).$$

Proof. Since the differential equation defining parallel transport is linear, it depends linearly on the initial conditions, so τ_t is linear. On the other hand, τ_t is invertible, since its inverse is parallel transport along the curve $\bar{c} : [0,t] \to M$, given by $\bar{c}(\varepsilon) = c(t - \varepsilon)$.

For the proof of (ii), first we use Lemma 29.7 to produce $\{s_1, \dots, s_r\}$ sections along c that at each point $c(t)$ generate the fiber $E_{c(t)}$. Then there are functions $v^a : [0,1] \to \mathbb{R}$ such that

$$s(c(t)) = \sum_{a=1}^{r} v^a(t) s_a(t),$$

and we find that:

$$\begin{aligned}
\lim_{t\to 0}\frac{1}{t}\left(\tau_t^{-1}(s(c(t))) - s(c(0))\right) &= \lim_{t\to 0}\sum_{a=1}^{r}\frac{1}{t}\left(v^a(t)\tau_t^{-1}(s_a(t)) - v^a(0)s_a(0)\right)\\
&= \lim_{t\to 0}\sum_{a=1}^{r}\frac{1}{t}\left(v^a(t) - v^a(0)\right)s_a(0)\\
&= \sum_{a=1}^{r}\frac{\mathrm{d}v^a}{\mathrm{d}t}(0)s_a(0) = D_c(\sum_{a=1}^{r}v^a s_a)(0) = \nabla_v s.
\end{aligned}$$

where in the last line we have used (29.2). □

Consider now the tangent bundle $\xi = TM$ of a manifold M. For a connection ∇ in TM, the notions above have a more geometric meaning. For example, in $M = \mathbb{R}^d$, there is a canonical connection ∇ in $T\mathbb{R}^d = \mathbb{R}^d \times \mathbb{R}^d$, which corresponds to the usual directional derivative. A vector field X (i.e., a section of TM) is parallel for this connection along a curve $c(t)$ if and only if the vectors $X_{c(t)}$ are parallel in the usual sense.

For a connection in the tangent bundle TM there are additional notions that do not make sense for connections on a general vector bundle. This is because a connection in TM differentiates vector fields along vector fields, so we have a more symmetric situation. Here is a first example:

Definition 29.9. *Let ∇ be a connection in TM. A* ***geodesic*** *is a curve $c(t)$ for which its derivative $\dot{c}(t)$ (a vector field along $c(t)$) is parallel, i.e., we have:*

$$D_c\dot{c}(t) = 0.$$

If we choose local coordinates $(U, x^1, \dots, x^d)$, we have trivializing vector fields $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^d}\}$ for $TM|_U$, and we can write:

$$\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x^k}.$$

The equations for the components $c^i(t) = x^i(c(t))$ of a geodesic $c(t)$ in local coordinates are:

$$\frac{\mathrm{d}^2c^k(t)}{\mathrm{d}t^2} = -\sum_{ij}\Gamma_{ij}^k(c(t))\frac{\mathrm{d}c^i(t)}{\mathrm{d}t}\frac{\mathrm{d}c^j(t)}{\mathrm{d}t}, \quad (k = 1, \dots, n).$$

Using these equations, it should be clear that given $p_0 \in M$ and $\mathbf{v} \in T_{p_0}M$, there exists a unique geodesic $c(t)$ such that $c(0) = p_0$ and $\dot{c}(0) = \mathbf{v}$. This geodesic is defined for $0 \le t < \varepsilon$, and if we choose $\mathbf{v}$ sufficiently small we can assume that $\varepsilon > 1$. In this case, we set::

$$\exp_{p_0}(\mathbf{v}) \equiv c(1).$$

In this way, we obtain the **exponential map** $\exp_{p_0} : U \to M$, which is defined in an open neighborhood $U \subset T_{p_0}M$ of the origin.

Another notion which only makes sense for connections ∇ in TM is the **torsion of a connection**: this is the map $T : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by:

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

One checks that T is $C^\infty(M)$-linear in both arguments, so it defines a morphism of vector bundles $T : TM \otimes TM \to TM$. One calls T the **torsion tensor** of the connection. A **symmetric connection** is a connection ∇ whose torsion is zero.

The next proposition gives a characterization of the torsion in terms of the covariant derivative. For that we choose a smooth map $\phi : [0,1] \times [0,1] \to M$ which one can think as a parameterized surface. Denoting the parameters by (x,y) we have maps $[0,1] \times [0,1] \to TM$ covering ϕ defined by:

$$\frac{\partial \phi}{\partial x} \equiv \phi_*(\frac{\partial}{\partial x}), \quad \frac{\partial \phi}{\partial y} \equiv \phi_*(\frac{\partial}{\partial y}).$$

One can think of these as vector fields along ϕ. If one fixes y, they give vector fields along the curve $t \mapsto \phi(t,y)$, and similarly if one fixes x. So we may consider the covariant derivatives:

- $D_x \frac{\partial \phi}{\partial y} \equiv$ covariant derivative along the curve $t \mapsto \phi(t,y)$ at $t = x$;
- $D_y \frac{\partial \phi}{\partial x} \equiv$ covariant derivative along the curve $t \mapsto \phi(x,t)$ at $t = y$;

Proposition 29.10. *Consider a parameterized surface $\phi : [0,1] \times [0,1] \to M$. The torsion of a connection ∇ in TM satisfies:*

$$D_x \frac{\partial \phi}{\partial y} - D_y \frac{\partial \phi}{\partial x} = T\Big(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial x}\Big).$$

Proof. The proof is similar (but simpler!) to the proof of Proposition 30.2 below, and so is left as an exercise. $\square$

The most classical example of a connection is the *Levi-Civita connection* in the tangent bundle of a Riemannian manifold, which we now describe. We start with a definition:

Definition 29.11. *Let ξ be a vector bundle over M with a fiber metric $\langle\ ,\ \rangle$. A connection in ξ is said to be* ***compatible with the metric*** *if*

$$X(\langle s_1, s_2\rangle) = \langle \nabla_X s_1, s_2\rangle + \langle s_1, \nabla_X s_2\rangle,$$

for every vector field $X \in \mathfrak{X}(M)$ and every pair of sections $s_1, s_2 \in \Gamma(\xi)$.

For a Riemannian manifold we have a natural choice of compatible metric:

Proposition 29.12. *Let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold. There exists a unique symmetric connection in TM compatible with the metric.*

Proof. Let $X, Y, Z \in \mathfrak{X}(M)$ be vector fields in M. The compatibility of ∇ with the metric gives:

$$\begin{aligned} X \cdot \langle Y, Z\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \\ Y \cdot \langle Z, X\rangle &= \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle, \\ Z \cdot \langle X, Y\rangle &= \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle. \end{aligned}$$

Adding the first two equations and subtracting the third one, gives:

$$\begin{aligned} X \cdot \langle Y, Z\rangle + Y \cdot \langle Z, X\rangle - Z \cdot \langle X, Y\rangle = 2\langle \nabla_X Y, Z\rangle \\ - \langle X, [Z, Y]\rangle - \langle Y, [Z, X]\rangle - \langle Z, [X, Y]\rangle, \end{aligned}$$

where we have used the symmetry of the connection. This relation shows that the two conditions completely determine the connection by the formula:

$$\begin{aligned} \langle \nabla_X Y, Z\rangle = \frac{1}{2}\left(X \cdot \langle Y, Z\rangle + Y \cdot \langle Z, X\rangle - Z \cdot \langle X, Y\rangle\right) \\ + \frac{1}{2}\left(\langle X, [Z, Y]\rangle + \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle\right). \end{aligned}$$

On the other, one checks easily that this formula does define a connection in TM which is symmetric and compatible with the metric. $\square$

The connection in the proposition is known as the **Levi-Civita connection** of the Riemannian manifold. This allows to define parallel transport, geodesics, exponential map, etc., for a Riemannian manifold. The fact that this connection comes from a metric leads to additional properties of these concepts. We will not go into any deeper discussion of Riemannian geometry and refer the reader to any standard text on the subject.

HOMEWORK.

1. Let ξ and ξ' be vector bundles over M, furnished with connections ∇ and ∇'. Show that the associated bundles $\xi \oplus \xi'$, ξ^* and $\wedge^k \xi$, have induced connections satisfying:

$$\begin{aligned} \nabla_X(s_1 \oplus s_2) &= \nabla_X s_1 \oplus \nabla_X s_2, \\ \nabla_X(s_1 \wedge \cdots \wedge s_k) &= \nabla_X s_1 \wedge \cdots \wedge s_k + \cdots + s_1 \wedge \cdots \nabla_X \wedge s_k \\ X(\langle s, \eta\rangle) &= \langle \nabla_X s, \eta\rangle + \langle s, \nabla_X \eta\rangle. \end{aligned}$$

Determine the connection 1-form of these connections in terms of the original connection 1-forms.

2. Let ξ be a vector bundle over M with a connection ∇. If $\psi : N \to M$ is a smooth map, show that $\psi^*\xi$ has a connection induced from ∇ such that:

$$(\nabla_v \psi^* s) = \psi^*(\nabla_{\mathrm{d}_p\psi(v)} s), \quad \forall v \in T_pN, s \in \Gamma(\xi).$$

Determine the connection 1-form of the pullback connections in terms of the connection 1-form of the original connection.

3. Let $\{s_1, \dots, s_r\}$ and $\{s'_1, \dots, s'_r\}$ be two basis of local sections for a vector bundle $\xi = (\pi, E, M)$ over a common open set $U \subset M$. Denote by $A = (a_i^j) : U \to \mathrm{GL}(r)$ the matrix of change of basis so that $s'_i = \sum_j a_i^j s_j$. Show that the corresponding connection 1-forms ω and ω' are related by:

$$\omega' = A^{-1}\omega A + A^{-1}\mathrm{d}A.$$

4. Deduce formula (29.1) for the local expression of the covariant derivative of a connection.

5. Show that the covariant derivative of a section $s(t)$ along a curve $c(t)$ as given in Definition 29.5 can be computed by choosing a time-dependent section extending s and applying formula (29.2). In particular, conclude that this formula does not depend on the choice of extension.

6. Let ξ be a vector bundle over M with a fiber metric $g := \langle\ ,\ \rangle$. Viewing the metric as a section $g \in \Gamma(\otimes^2 E^*)$, verify that the condition that the connection ∇ is compatible with the metric g is equivalent to:

$$\nabla_X g = 0, \quad \forall X \in \mathfrak{X}(M).$$

Show that one can always find such a compatible connection ∇.

7. Let $\xi = (\pi, E, M)$ be a vector bundle with a fiber metric $\langle\ ,\ \rangle$. For a connection ∇ in ξ, show that the following are equivalent:
   (i) ∇ is compatible with the metric.
   (ii) Parallel transport $\tau_t : E_{c(0)} \to E_{c(t)}$ along any curve c is an isometry.
   (iii) For any basis of orthonormal trivializing sections the connection 1-form $\omega = [\omega_a^b]$ is a skew-symmetric matrix.

8. Let $M \subset \mathbb{R}^n$ be an embedded submanifold so that $T_pM \subset \mathbb{R}^n$ has the inner product induced from the standard inner product on $\mathbb{R}^n$. Show that these yield a Riemannian metric g in M, whose associated Levi-Civita connection is given by:

$$(\nabla_X Y)(p) = \mathrm{pr}_{T_pM}\left(\mathrm{d}_p Y(X_p)\right),$$

where $\mathrm{pr}_{T_pM} : \mathbb{R}^n \to T_pM$ denotes the orthogonal projection and we view $Y \in \mathfrak{X}(M)$ as a map $Y : M \to \mathbb{R}^n$.

30. Curvature and Holonomy

A trivial vector bundle carries natural connections defined in terms of trivializing sections s_i, for which $\nabla s_i = 0$. In general, for an arbitrary connection ∇ on a vector bundle $\xi = (\pi, E, M)$, it is not possible to choose a basis of local sections s_i such that $\nabla s_i = 0$. The obstruction is given by the **curvature** of ∇, which is the map

$$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(\xi) \to \Gamma(\xi)$$

defined by:

$$R(X, Y)s = \nabla_X(\nabla_Y s) - \nabla_Y(\nabla_X s) - \nabla_{[X,Y]}s.$$

A simple computation shows that R is $C^\infty(M)$-linear in all the arguments, so we can think of R as a vector bundle map $R : TM \otimes TM \otimes E \to E$. For this reason one also calls R the **curvature tensor**.

The local expression for the curvature over a chart (U, x^i) where one has a basis of sections $\{s_1, \dots, s_r\}$ for ξ, is:

$$R\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big)s_a = \sum_{b=1}^r R^b_{ija} s_b,$$

where the components R^b_{ija} can be expressed in terms of the Christoffel symbols Γ^b_{ia} by:

$$R^b_{ija} = \frac{\partial \Gamma^b_{ja}}{\partial x^i} - \frac{\partial \Gamma^b_{ia}}{\partial x^j} + \sum_{c=1}^r \left(\Gamma^c_{ia}\Gamma^b_{jc} - \Gamma^c_{ja}\Gamma^b_{ic}\right).$$

We can also codify the curvature in terms of a matrix of differential forms:

$$\Omega^b_a = \sum_{i<j} R^b_{ija} \mathrm{d}x^i \wedge \mathrm{d}x^j,$$

and $\Omega = [\Omega^b_a]$ is called the **curvature 2-form** of the connection. This matrix-valued 2-form is independent of the choice of local coordinates, and it can also be defined from the relation:

$$R(X, Y)s_a = \sum_{b=1}^r \Omega^b_a(X, Y)s_b.$$

The dependence of Ω on the choice of trivializing sections is discussed in the Homework.

Theorem 30.1. *For a connection in a vector bundle ξ, the connection 1-form ω and the curvature 2-form Ω associated with some trivializing sections, are related by the* ***structure equations****:*

$$\Omega^b_a = \mathrm{d}\omega^b_a + \sum_c \omega^c_a \wedge \omega^b_c \quad \Longleftrightarrow \quad \Omega = \mathrm{d}\omega + \omega \wedge \omega,$$

and one has the ***Bianchi's identity****:*

$$\mathrm{d}\Omega^b_a = \sum_c \left(\Omega^c_a \wedge \omega^b_c - \omega^c_a \wedge \Omega^b_c\right) \quad \Longleftrightarrow \quad \mathrm{d}\Omega = \Omega \wedge \omega - \omega \wedge \Omega.$$

Proof. Direct computation. $\square$

Let us turn now to the geometric interpretation of curvature in term of parallel transport. For that we choose a smooth map $\phi : [0, 1] \times [0, 1] \to M$ which one can think as a parameterized surface. Denoting the parameters by (x, y) we have maps $[0, 1] \times [0, 1] \to TM$ covering ϕ defined by:

$$\frac{\partial \phi}{\partial x} \equiv \phi_*(\frac{\partial}{\partial x}), \quad \frac{\partial \phi}{\partial y} \equiv \phi_*(\frac{\partial}{\partial y}).$$

One can think of these as vector fields along ϕ. If one fixes y, they give vector fields along the curve $t \mapsto \phi(t, y)$, and similarly if one fixes x. Given a section s of the vector bundle ξ along ϕ, we can introduce the covariant derivatives:

- $D_x s(x,y) \equiv$ covariant derivative along the curve $t \mapsto \phi(t,y)$ at $t=x$;
- $D_y s(x,y) \equiv$ covariant derivative along the curve $t \mapsto \phi(x,t)$ at $t=y$;

We have:

Proposition 30.2. *For any section s of ξ along a parameterized surface $\phi : [0,1]\times[0,1] \to M$, the curvature of the connection satisfies:*

$$D_x D_y s - D_y D_x s = R\Big(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial x}\Big) s.$$

Proof. Choose (x,y)-dependent vector fields $X_{x,y}, Y_{x,y} \in \mathfrak{X}(M)$ extending $\frac{\partial \phi}{\partial x}$ and $\frac{\partial \phi}{\partial y}$:

$$X_{x,y}(\phi(x,y)) = \frac{\partial \phi}{\partial x}(x,y), \quad Y_{x,y}(\phi(x,y)) = \frac{\partial \phi}{\partial y}(x,y).$$

We will need the following result whose proof we leave as an exercise:

Lemma 30.3.

$$\Big(\frac{\mathrm{d}}{\mathrm{d}y} X_{x,y} - \frac{\mathrm{d}}{\mathrm{d}x} Y_{x,y}\Big)\Big|_{\phi(x,y)} = [X_{x,y}, Y_{x,y}]\Big|_{\phi(x,y)}.$$

We choose also a (x,y)-dependent section $s_{x,y} \in \Gamma(\xi)$ extending s:

$$s_{x,y}(\phi(x,y)) = s(x,y).$$

Using Remark 29.6, we can compute the covariant derivatives:

$$D_x s(x,y) = \Big(\nabla_{X_{x,y}} s_{x,y} + \frac{\mathrm{d}}{\mathrm{d}x} s_{x,y}\Big)\Big|_{\phi(x,y)},$$
$$D_y s(x,y) = \Big(\nabla_{\beta_{x,y}} s_{x,y} + \frac{\mathrm{d}}{\mathrm{d}y} s_{x,y}\Big)\Big|_{\phi(x,y)}.$$

It follows that:

$$D_x D_y s(x,y) = \Big(\nabla_{X_{x,y}} \nabla_{Y_{x,y}} s_{x,y} + \frac{\mathrm{d}}{\mathrm{d}x} \nabla_{Y_{x,y}} s_{x,y} + \nabla_{X_{x,y}} \frac{\mathrm{d}s_{x,y}}{\mathrm{d}y} + \frac{\mathrm{d}^2 s_{x,y}}{\mathrm{d}x\mathrm{d}y}\Big)\Big|_{\phi(x,y)},$$
$$D_y D_x s(x,y) = \Big(\nabla_{\beta_{x,y}} \nabla_{X_{x,y}} s_{x,y} + \frac{\mathrm{d}}{\mathrm{d}y} \nabla_{X_{x,y}} s_{x,y} + \nabla_{Y_{x,y}} \frac{\mathrm{d}s_{x,y}}{\mathrm{d}x} + \frac{\mathrm{d}^2 s_{x,y}}{\mathrm{d}y\mathrm{d}x}\Big)\Big|_{\phi(x,y)}.$$

Taking the difference of these two equations, we obtain:

$$\begin{aligned} D_x D_y s(x,y) - D_y D_x s(x,y) &= \\ &= \Big(\nabla_{X_{x,y}} \nabla_{Y_{x,y}} s_{x,y} - \nabla_{Y_{x,y}} \nabla_{X_{x,y}} s_{x,y} + \nabla_{\frac{\mathrm{d}}{\mathrm{d}x} Y_{x,y} - \frac{\mathrm{d}}{\mathrm{d}y} X_{x,y}} s_{x,y}\Big)\Big|_{\phi(x,y)}. \end{aligned}$$

Using the lemma above, we obtain the result:

$$\begin{aligned} D_x D_y s(x,y) - D_y D_x s(x,y) &= \\ &= \Big(R(X_{x,y}, Y_{x,y}) s_{x,y}\Big)\Big|_{\phi(x,y)} = R\Big(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial x}\Big) s(x,y). \end{aligned}$$

□

A **flat connection** is a connection for which the curvature tensor vanishes. We will often refer to a vector bundle with a flat connection as a **flat bundle**. Clearly, if around each point one can choose coordinates and trivializing sections for which the Christoffel symbols vanish, the connection is flat. The converse is also true, as a consequence of the following local normal form for flat bundles:

Corollary 30.4. *Let $\xi = (\pi, E, M)$ be a vector bundle of rank r with a flat connection ∇. For each $p \in M$, there exists a base of local sections $\{s_1, \ldots, s_r\}$ definided in a neighborhood U of p, such that*

$$\nabla_X s_i = 0, \quad \forall X \in \mathfrak{X}(M).$$

Hence, $\xi|_U$ is isomorphic to the trivial vector bundle ε^r_U with the canonical flat connection.

Proof. See Exercise 3 in the homework at the end of this section. □

In the case of Riemannian manifolds, Corollary 30.4 takes the following more geometric meaning:

Corollary 30.5. *Let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold with vanishing curvature tensor: $R = 0$. For each $p \in M$, there exists a neighborhood U of p which is isometric to an open in $\mathbb{R}^d$ furnished with the Euclidean metric.*

Proof. See Exercise 5 in the homework at the end of this section. □

The previous results describe flat connections locally. To describe what happens with a flat connection globally, we need to introduce the notion of *holonomy* of a connection. Given a vector bundle $\xi = (\pi, E, M)$ of rank r with a connection ∇ fix a base point $p_0 \in M$. For each a closed path $\gamma : [0, 1] \to M$ based at p_0, so $\gamma(0) = \gamma(1) = p_0$, parallel transport along the curve $\gamma(t)$ gives a linear isomorphism called the **holonomy** of γ:

$$H_{p_0}(\gamma) \equiv \tau_1 : E_{p_0} \to E_{p_0}.$$

If we extend this definition, in the obvious way, to closed paths which are piecewise smooth, it is clear that:

$$H_{p_0}(\gamma_1 \cdot \gamma_2) = H_{p_0}(\gamma_1) \circ H_{p_0}(\gamma_2),$$

where $\gamma_1 \cdot \gamma_2$ denotes the concatenation of the two paths:

$$\gamma_1 \cdot \gamma_2(t) := \begin{cases} \gamma_2(2t) & \text{if } 0 \le t \le \frac{1}{2}, \\ \\ \gamma_1(2t - 1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

When the connection is flat we also have:

Proposition 30.6. *Given a flat connection, any two path-homotopic closed curves γ_0 and γ_1 have the same holonomy: $H_{p_0}(\gamma_0) = H_{p_0}(\gamma_1)$.*

Proof. One can show that two smooth curves which are C^0 path-homotopic are also smooth path-homotopic. So denote by $\gamma : [0,1] \times [0,1] \to M$ a path-homotopy between γ_0 and γ_1:

$$\gamma(t,0) = \gamma_0(t), \quad \gamma(t,1) = \gamma_1(t), \quad \gamma(0,\varepsilon) = \gamma(1,\varepsilon) = p_0.$$

Fixing $v_0 \in E_{p_0}$, we define a section $s : [0,1] \times [0,1] \to E$ along $\gamma : [0,1] \times [0,1] \to M$, by:

$$s(t,\varepsilon) := \tau_t^{\gamma(\cdot,\varepsilon)}(v_0) = \begin{cases} \text{parallel transport of } v_0 \text{ along} \\ s \mapsto \gamma(s,\varepsilon) \text{ with } s \in [0,t]. \end{cases}$$

Notice that, by construction, for each fixed ε:

$$D_t s := D_{\gamma(\cdot,\varepsilon)} s = 0.$$

We claim that for each fixed t one also has:

$$D_\varepsilon s := D_{\gamma(t,\cdot)} s = 0.$$

Indeed, since $\gamma(0,\varepsilon) = 0$ and $s(0,\varepsilon) = v_0$, we have:

$$D_\varepsilon s(0,\varepsilon) = D_{\gamma(0,\cdot)} s(0,\varepsilon) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} s(0,\varepsilon) = 0.$$

On the other hand, using Proposition 30.2, we find:

$$D_t D_\varepsilon s = R\Big(\frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial \varepsilon}\Big) + D_\varepsilon D_t s = 0.$$

Hence, $D_\varepsilon s$ is parallel along the curve $t \mapsto \gamma(t,\varepsilon)$ so we must have $D_\varepsilon s(t,\varepsilon) = 0$, as claimed.

Now, applying our claim, and the fact that $\gamma(1,\varepsilon) = p_0$, we conclude that:

$$0 = D_\varepsilon s(1,\varepsilon) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} s(1,\varepsilon).$$

Hence,

$$\tau_1^{\gamma_0}(v_0) = s(1,0) = s(1,1) = \tau_1^{\gamma_1}(v_0).$$

Since $v_0 \in E_{p_0}$ was an arbitrary vector, we conclude that $H_{p_0}(\gamma_0) = H_{p_0}(\gamma_1)$. $\square$

Since every element in $\pi_1(M,p_0)$ has a smooth a representative, we conclude that for a flat connection, one has a group homomorphism

$$H_{p_0} : \pi_1(M,p_0) \to \mathrm{GL}(E_{p_0}),$$

called the **holonomy representation** of ∇ with base point p_0. If $q_0 \in M$ is a different point in the same connected component of M, we can choose a smooth path $c : [0,1] \to M$ with $c(0) = p_0$ and $c(1) = q_0$. Parallel transport along $c(t)$ gives an isomorphism $\tau : E_{p_0} \to E_{q_0}$ and

$$H_{q_0} = \tau \circ H_{p_0} \circ \tau^{-1}.$$

Hence, the holonomy representations of different points in the same component are related by conjugacy.

Theorem 30.7. *Let M be a connected manifold with base point $p_0 \in M$, there is a 1:1 correspondence:*

$$\left\{\begin{array}{c}\textit{isomorphism classes of}\\ \textit{flat vector bundles of rank } r \textit{ over } M\end{array}\right\} \stackrel{\sim}{\longleftrightarrow} \mathrm{Hom}(\pi_1(M,p_0),\mathrm{GL}(r))/\mathrm{GL}(r).$$

where $\mathrm{GL}(r)$ *acts on* $\mathrm{Hom}(\pi_1(M,p_0),\mathrm{GL}(r))$ *by conjugation.*

Proof. We already know that a flat vector bundle (ξ,∇) induces a representation of the fundamental group, namely the holonomy representation:

$$H_{p_0} : \pi_1(M,p_0) \to \mathrm{GL}(E_{p_0}).$$

Fixing a basis for the fiber E_{p_0}, we obtain a group homomorphism:

$$H_{p_0} : \pi_1(M,p_0) \to \mathrm{GL}(r).$$

Two different basis for E_{p_0} are related by conjugation of an element of $\mathrm{GL}(r)$. It follows that isomorphic vector bundles induce homomorphisms which are related by conjugation too, so one can associate to an isomorphism class of vector bundles an element in the quotient

$$\mathrm{Hom}(\pi_1(M,p_0),\mathrm{GL}(r))/\mathrm{GL}(r).$$

Conversely, given a representation $H : \pi_1(M,p_0) \to \mathrm{GL}(r)$ representing some element in this quotient, we construct a flat vector bundle as follows: on the one hand, the representation gives an action of $\pi(M,p_0)$ in $\mathbb{R}^r$. On the other hand, the fundamental group $\pi_1(M,p_0)$ acts in the universal cover $\widetilde{M}$ by deck transformations: identifying $\widetilde{M}$ with the set of homotopy classes of paths $[c]$ with initial point $c(0) = p_0$, the action of $\pi_1(M,p_0)$ in $\widetilde{M}$ is given by concatenation:

$$\pi_1(M,p_0) \times \widetilde{M} \to \widetilde{M},\ ([\gamma],[c]) \mapsto [\gamma \cdot c].$$

Since this action is proper and free, the resulting diagonal action of $\pi_1(M,p_0)$ in $\widetilde{M} \times \mathbb{R}^r$ is also proper and free. Hence, the quotient space $E = (\widetilde{M} \times \mathbb{R}^r)/\pi_1(M,p_0)$ is a manifold, and we have the projection

$$\pi : E \to M, [[c],\mathbf{v}] \mapsto c(1).$$

The triple $\xi = (\pi, E, M)$ is a vector bundle. Moreover, the canonical flat connection in $\widetilde{M} \times \mathbb{R}^r$ induces a connection in ξ for which the holonomy with base point p_0 is precisely $H : \pi_1(M,p_0) \to \mathrm{GL}(r)$. Finally, one checks that given two homomorphisms $H_0, H_1 : \pi_1(M,p_0) \to \mathrm{GL}(r)$ in the same conjugacy class this construction produces isomorphic flat vector bundles. $\square$

Remark 30.8. The space appearing in the previous result is an example of a **character variety**. More general, given a Lie group G and a finitely generated group π, the G-character variety of π is the space of equivalence classes of group homomorphisms:

$$\mathrm{Hom}(\pi, G)/G.$$

HOMEWORK.

1. Show that the connection 1-form and the curvature 2-form of a connection satisfy the structure equations and Bianchi's identity of Theorem 30.1.

2. Prove Lemma 30.3.

3. Let $\{s_1, \ldots, s_r\}$ and $\{s'_1, \ldots, s'_r\}$ be two basis of local sections for a vector bundle $\xi = (\pi, E, M)$ over a common open set $U \subset M$. Denote by $A = (a_i^j) : U \to \mathrm{GL}(r)$ the matrix of change of basis so that $s'_i = \sum_j a_i^j s_j$. Show that the corresponding curvature 2-forms Ω and Ω' are related by:

$$\Omega' = A^{-1}\Omega A.$$

4. Show that if ∇ is a flat connection on a vector bundle $\xi = (\pi, E, M)$, then around every point $p \in M$ one can find a local basis of flat sections for ξ. HINT: Using Exercise 3 in the previous section and the previous exercise, show that the condition $\omega' = 0$ defines an integrable distribution in $U \times \mathrm{GL}(r)$, so one can apply Frobenius.

5. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Show that there exists a unique connection ∇ in TG, which is invariant under left and right translations, and under inversion. Show that ∇ satisfies the following properties:
   (a) For any left invariant vector fields $X, Y \in \mathfrak{g}$:
$$\nabla_X Y = \frac{1}{2}[X, Y].$$
   (b) The torsion of ∇ vanishes and its curvature is given by:
$$R(X, Y) \cdot Z = \frac{1}{4}[[X, Y], Z], \quad \forall X, Y, Z \in \mathfrak{g}).$$
   (c) The exponential map of ∇ at the identity $\exp_e$ coincides with the Lie group exponential map $\exp : \mathfrak{g} \to G$.
   (d) Parallel transport along the curve $c(t) = \exp(tX)$, $X \in \mathfrak{g}$, is given by:
$$\tau_t(\mathbf{v}) = \mathrm{d}L_{\exp(\frac{t}{2}X)} \cdot \mathrm{d}R_{\exp(\frac{t}{2}X)} \cdot \mathbf{v}, \quad \forall \mathbf{v} \in T_e G.$$
   (e) The geodesics are translations of the 1-parameter subgroups of G.

6. Let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold whose curvature tensor vanishes: $R = 0$. Show that for each $p \in M$, there exists a neighborhood U isometric to an open in $\mathbb{R}^d$ with the Euclidean metric.

31. THE CHERN-WEIL HOMOMORPHISM

We saw in the previous section that a flat vector bundle is globally characterized by its holonomy representation. We will now study the non-flat case, a situation that is more complicated but more interesting. Eventually, we will see that one can use a connection on a vector bundle to construct cohomology classes which are invariants of the vector bundle, and which characterize certain properties of the vector bundle up to isomorphism.

Let $\pi : E \to M$ be a vector bundle. We consider differential forms in M with values in E, which we denote by

$$\Omega^\bullet(M;E) := \Gamma(\wedge^k T^*M \otimes E).$$

So a differential form of degree k with values in E is a $C^\infty(M)$-multilinear alternating map:

$$\omega : \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k\text{-times}} \to \Gamma(E).$$

In particular, $\Omega^0(M;E)$ is the space $\Gamma(E)$ of global sections of the vector bundle $\pi : E \to M$. Notice that we also have:

$$\Omega^\bullet(M;E) = \Omega^\bullet(M) \otimes \Gamma(E),$$

where $\otimes$ denotes here the tensor product of $C^\infty(M)$-modules. This last interpretation shows that we have a well-defined wedge product $\omega \wedge \eta \in \Omega^{k+l}(M;E)$, for any $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M;E)$.

A choice of connection ∇ in $\pi : E \to M$ allows us to take the differential of E-valued differential forms as follows. Such a connection determines an operator $\mathrm{d}_\nabla : \Omega^0(M;E) \to \Omega^1(M;E)$ through the formula:

$$(\mathrm{d}_\nabla s)(X) = \nabla_X s.$$

The map d_∇ is $\mathbb{R}$-linear and satisfies the Leibniz identity:

$$\mathrm{d}_\nabla(fs) = \mathrm{d}f \otimes s + f\mathrm{d}_\nabla s.$$

Remark 31.1. Conversely, any $\mathbb{R}$-linear map $\mathrm{d}_\nabla : \Omega^0(M;E) \to \Omega^1(M;E)$ that satisfies the Leibniz identity determines a unique connection ∇. So this gives an alternative approach to the theory of connections E.

One can extend d_∇ to arbitrary forms by requiring that for any form $\omega \in \Omega^\bullet(M)$ and section $s \in \Gamma(E)$ the following general Leibniz identity holds:

$$\mathrm{d}_\nabla(\omega \otimes s) = \mathrm{d}_\nabla(\omega) \otimes s + (-1)^{\deg \omega} \omega \wedge \mathrm{d}_\nabla(s). \tag{31.1}$$

In fact, one has:

Proposition 31.2. *Given a connection ∇ and $\omega \in \Omega^k(M;E)$ define $\mathrm{d}_\nabla\omega \in \Omega^{k+1}(M;E)$ by:*

$$\begin{aligned} \mathrm{d}_\nabla\omega(X_0, \ldots, X_k) = &\sum_{i=0}^{k+1} (-1)^i \nabla_{X_i}(\omega(X_0, \ldots, \widehat{X}_i, \ldots, X_k)) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_k). \end{aligned} \tag{31.2}$$

Then $\mathrm{d}_\nabla : \Omega^\bullet(M;E) \to \Omega^{\bullet+1}(M;E)$ is the unique operator satisfying:

(i) For any 0-form $s \in \Gamma(E)$, one has $(\mathrm{d}_\nabla s)(X) = \nabla_X s$;
(ii) d_∇ is $\mathbb{R}$-linear and satisfies the Leibniz identity (31.1).

Proof. One checks easily that the operator d_∇ defined by (31.2) satisfies (i) and (ii). Since any E-valued k-form η can be written as a linear combination:

$$\eta = \sum_{i=1}^{l} \omega_i \otimes s_i \quad (\omega_i \in \Omega^k(M),\ s_i \in \Gamma(E)),$$

it is clear that (i) and (ii) determined completely d_∇. □

Note that, in general, $\mathrm{d}_\nabla^2 \neq 0$, so d_∇ is not a differential. In fact, the curvature of ∇ can be seen as the failure in d_∇ being a differential.

Proposition 31.3. *Let ∇ be a connection on a vector bundle $\xi = (\pi, E, M)$ with curvature R. Then:*

(i) For any 0-form $s \in \Gamma(E)$

$$\mathrm{d}_\nabla^2 s(X,Y) = R(X,Y)s, \quad (X,Y \in \mathfrak{X}(M));$$

(ii) Viewing the curvature as a 2-form $R \in \Omega^2(M, \operatorname{End} E)$, for the connection on $\operatorname{End}(E)$ induced by ∇:

$$\mathrm{d}_\nabla R = 0. \tag{31.3}$$

Proof. Using the definition of d_∇ that:

$$\begin{aligned}\mathrm{d}_\nabla^2 s(X,Y) &= \nabla_X(\mathrm{d}_\nabla s(Y)) - \nabla_Y(\mathrm{d}_\nabla s(X)) - \mathrm{d}_\nabla s([X,Y])) \\ &= \nabla_X(\nabla_Y s) - \nabla_Y(\nabla_X s) - \nabla_{[X,Y]} s = R(X,Y)s.\end{aligned}$$

The proof of (ii) is left as an exercise.. □

Remark 31.4. The previous result shows that d_∇ is a differential if and only if the connection is flat. In this case, one calls the cohomology of the complex $(\Omega^\bullet(M;E), \mathrm{d}_\nabla)$ the **de Rham cohomology of M with coefficients in** E and denotes it by $H^\bullet(M;E)$. Notice that the usual de Rham cohomology corresponds to the case where $E = M \times \mathbb{R}$ is the trivial flat line bundle.

The Bianchi identity can be used to define certain cohomology classes. For that we need first to recall that for a finite dimensional vector space V one has a canonical identification between the homogeneous polynomials and the multilinear symmetric functions:

(i) Every k-multilinear symmetric map $P : V \times \cdots \times V \to \mathbb{R}$ determines a homogeneous polynomial $\widetilde{P} : V \to \mathbb{R}$ of degree k, by the formula:

$$\widetilde{P} : v \mapsto P(v, \dots, v).$$

(ii) Conversely, every homogeneous polynomial $\widetilde{P} : V \to \mathbb{R}$ of degree k determines a k-multilinear symmetric map $P : V \times \cdots \times V \to \mathbb{R}$ by polarization:

$$P(v_1, \dots, v_k) = \frac{1}{k!} \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_k} P(t_1 v_1 + \cdots + t_k v_k)\Big|_{t_1 = \cdots = t_k = 0}.$$

These correspondences are inverse to each other, and the usual product of polynomials corresponds to the product of k-multilinear, symmetric maps, defined by:

$$P_1 \circ P_2(v_1, \dots, v_{k+l}) = \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} P_1(v_{\sigma(1)}, \dots, v_{\sigma(k)}) P_2(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)}).$$

Example 31.5.
If one fixes a base $\xi^1, \dots, \xi^r$ for V^*, then one can think of polarization of the polynomial $\widetilde{P}: V \to \mathbb{R}$ as follows: one can write the polynomial:

$$\widetilde{P}(v) = \sum_{i_1 \cdots i_k = 1}^{r} a_{i_1 \cdots i_k} \xi^{i_1}(v) \cdots \xi^{i_k}(v),$$

where the coefficients $a_{i_1 \cdots i_k}$ are symmetric *in the indices. Then the corresponding k-multilinear, symmetric map $P : V \times \cdots \times V \to \mathbb{R}$ is given by:*

$$P(v_1, \dots, v_k) = \sum_{i_1 \cdots i_k = 1}^{r} a_{i_1 \cdots i_k} \xi^{i_1}(v_1) \cdots \xi^{i_k}(v_k).$$

For example, let $V = \mathbb{R}^3$ with linear coordinates (x, y, z). The homogeneous polynomial of degree 2:

$$\widetilde{P}(x, y, z) = x^2 + xy + z^2 = x^2 + \frac{1}{2}(xy + yx) + z^2,$$

corresponds to the bilinear symmetric map:

$$P(v, w) = v_1 w_1 + \frac{1}{2}(v_1 w_2 + v_2 w_1) + v_3 w_3.$$

We are interested in the case where $V = \mathfrak{g}$ is the Lie algebra of a Lie group G. We will denote by $I^k(G)$ the space of k-multilinear, symmetric maps $P : \mathfrak{g} \times \cdots \times \mathfrak{g} \to \mathbb{R}$ which are invariant under the adjoint action:

$$P(\operatorname{Ad} g \cdot v_1, \dots, \operatorname{Ad} g \cdot v_k) = P(v_1, \dots, v_k), \quad \forall g \in G,\ v_1, \dots, v_k \in \mathfrak{g}.$$

and we let

$$I(G) = \bigoplus_{k=0}^{\infty} I^k(G).$$

Note that $I(G)$ is a ring with the symmetric product. Under the correspondence above, we can identify $I(G)$ with the algebra of polynomials in $\mathfrak{g}$ which are Ad-invariant.

For now, we are only interested in the case where $G = GL(r)$, so that $\mathfrak{g} = \mathfrak{gl}(r)$ is the space of all $r \times r$-matrices. In this case the adjoint action is given by matrix conjugation:

$$\operatorname{Ad} A \cdot X = AXA^{-1}, \quad A \in GL(r),\ X \in \mathfrak{gl}(r).$$

Then the invariance condition is just invariance under conjugation:

$$P(AX_1A^{-1},\dots,AX_kA^{-1})=P(X_1,\dots,X_k)\quad (X_1,\dots,X_k\in\mathfrak{gl}(r)),$$

which must hold for any invertible matrix $A\in GL(r)$.

Example 31.6.

Invariant polynomials on $\mathfrak{gl}(r)$ can be obtained by taking traces of powers:

$$X\mapsto \operatorname{tr}(X^k).$$

Actually, these polynomials generate the ring of $\mathrm{Ad}_{GL(r)}$-invariant polynomials. We will came back to this issue in the next section.

Returning to the discussion of vector bundles with connection, the key remark is now the following:

Proposition 31.7. *Let $\xi=(\pi,E,M)$ be a rank r vector bundle with a connection ∇. Every element $P\in I^k(GL(r))$ determines a map*

$$P:\Omega^\bullet(M;\otimes^k\operatorname{End}(E))\to\Omega^\bullet(M),\quad \omega\mapsto P\circ\omega, \tag{31.4}$$

which satisfies:

$$\mathrm{d}P=P\mathrm{d}_\nabla.$$

Proof. Note that if $s_1,\dots,s_r$ is a base of local of sections of E then for any section $A\in\Gamma(\operatorname{End}(E))$, we have:

$$As_i=\sum_{j=1}^r A_i^j s_j,$$

for some functions A_i^j. Given a $P\in I^k(GL(r))$, we define a map $P:\Gamma(\otimes^k\operatorname{End}(E))\to C^\infty(M)$ by:

$$P(A_1\otimes\cdots\otimes A_k):=P([(A_1)_i^j],\cdots,[(A_k)_i^j]).$$

The invariance condition shows that this expression is independent of the choice of base of local of sections, so this map is well-defined. A degree l form $\omega\in\Omega^l(M;\otimes^k\operatorname{End}(E))$ can be seen as an l-multilinear alternating map

$$\omega:\mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)\to\Gamma(\otimes^k\operatorname{End}(E)),$$

so composing with P determines an l-multilinear alternating map

$$P\circ\omega:\mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)\to C^\infty(M).$$

Hence, $P(\omega)\in\Omega^l(M)$ and an elementary computation using the definitions of d, d_∇ and the fact that P is multilinear, shows that $\mathrm{d}P=P\mathrm{d}_\nabla$. □

Now let R denote the curvature of the connection ∇. The k-symmetric power of R is an element $R^k\in\Omega^{2k}(M;\otimes^k\operatorname{End}(E))$ defined by:

$$R^k(X_1,\dots,X_{2K}):=\frac{1}{(2k)!}\sum_{\sigma\in S_{2k}}(-1)^\sigma R(X_{\sigma(1)},X_{\sigma(2)})\otimes\cdots\otimes R(X_{\sigma(2k-1)},X_{\sigma(2k)}).$$

Therefore, if $P \in I^k(GL(r))$, we obtain a differential form $P(R^k) \in \Omega^{2k}(M)$. If one fixes some local basis of sections $\{s_1, \dots, s_r\}$ and lets $\Omega = [\Omega_a^b]$ denotes the curvature 2-form of the connection relative to this basis, this form is given explicitly by:

$$
\text{(31.5)} \quad P(R^k)(X_1, \dots, X_{2k}) = \frac{1}{(2k)!} \sum_{\sigma \in S_{2k}} (-1)^\sigma P(\Omega(X_{\sigma(1)}, X_{\sigma(2)}), \dots, \Omega(X_{\sigma(2k-1)}, X_{\sigma(2k)})).
$$

Using the previous expression, one checks that if $P_1 \in I^k(GL(r))$ and $P_2 \in I^l(GL(r))$, then:

$$
P_1 \circ P_2(R^{k+l}) = P_1(R^k) \wedge P_2(R^l) \in \Omega^{2(k+l)}(M).
$$

On the other hand, the Bianchi identity (31.3) gives:

$$
\mathrm{d}P(R^k) = P(\mathrm{d}_\nabla R^k) = kP(R^{k-1}\mathrm{d}_\nabla R) = 0,
$$

so $P(R^k) \in \Omega^{2k}(M)$ is a closed form. Now, we have:

Theorem 31.8 (Chern-Weil)**.** *Let ∇ be a connection in a vector bundle $\pi : E \to M$ of rank r, with curvature R. The map $I(GL(r)) \to H(M)$ defined by:*

$$
I^k(GL(r)) \to H^{2k}(M), \ P \longmapsto [P(R^k)],
$$

is a ring homomorphism. This homomorphism is independent of the choice of connection.

Proof. All that it remains to be proved is that the homomorphism is independent of the choice of connection. For that we claim that if ∇^0 and ∇^1 are two connections in $\pi : E \to M$, then for all $P \in I^k(GL(r))$ the differential forms $P(R^k_{\nabla_0})$ and $P(R^k_{\nabla_1})$ differ by an exact form.

To prove the claim, consider the projection $p : M \times [0,1] \to M$. The pullback bundle p^*E carries a connection ∇ defined by requiring that on pullback sections:

$$
\nabla_{\frac{\partial}{\partial t}} p^*s = 0, \qquad \nabla_X p^*s := tp^*(\nabla^1_X s) + (1-t)p^*(\nabla^0_X s), \quad (X \in \mathfrak{X}(M)).
$$

On the other hand, we have *integration along the fibers* of p:

$$
\int_0^1 : \Omega^\bullet(M \times [0,1]) \to \Omega^{\bullet-1}(M),
$$

which is explicitly given by:

$$
\Big(\int_0^1 \omega\Big)(X_1, \dots, X_{l-1}) = \int_0^1 \omega\Big(\frac{\partial}{\partial t}, X_1, \dots, X_{l-1}\Big)\mathrm{d}t.
$$

Then one defines the **Chern-Simons transgression form** by setting

$$
\text{(31.6)} \qquad P(\nabla_0, \nabla_1) \equiv k \int_0^1 P(R^k_\nabla) \in \Omega^{2k-1}(M).
$$

and one checks by direct computation that:

$$\mathrm{d}P(\nabla_0, \nabla_1) = P(R^k_{\nabla_1}) - P(R^k_{\nabla_0}).$$

This proves the claim, showing that $[P(R^k_{\nabla_1})] = [P(R^k_{\nabla_0})] \in H^{2k}(M)$. $\square$

The ring homomorphism given by the previous result:

$$\mathrm{CW}[\xi] : I(GL(r)) \to H^\bullet(M),$$

is called the **Chern-Weil homomorphism** $\xi = (\pi, E, M)$. This homomorphism depends only on the isomorphism class of the vector bundle ξ:

Proposition 31.9. *Let $\psi : N \to M$ be a smooth map and let $\xi = (\pi, E, M)$ be a vector bundle of rank r. For every $P \in I^\bullet(GL(r))$,*

$$\psi^* P(R^k_\nabla) = P(R^k_{\psi^*\nabla}),$$

where ∇ is any connection in ξ. Hence, the Chern-Weil homomorphisms of ξ and $\psi^*\xi$ fit into a commutative diagram:

$$\begin{tikzcd}
 & H^\bullet(M) \arrow[dd, "\phi^*"] \\
I(GL(r)) \arrow[ur, "\mathrm{CW}[\xi]"] \arrow[dr, "\mathrm{CW}[\phi^*\xi]"'] & \\
 & H^\bullet(N)
\end{tikzcd}$$

We leave the proof for the Homework.

Homework.

1. Let ∇ be a connection in a vector bundle $\pi : E \to M$ with curvature R. Prove Bianchi's identity:

$$\mathrm{d}_\nabla R = 0.$$

2. Let $\pi : E \to M$ be a vector bundle of rank r. Given $P \in I^k(GL(r))$, show that the map $P : \Omega^\bullet(M; \otimes^k \operatorname{End}(E)) \to \Omega^\bullet(M)$ given by (31.4) satisfies: $\mathrm{d}P = P\mathrm{d}_\nabla$.

3. Show that the Chern-Simons transgression form (31.6), satisfies:

$$\mathrm{d}P(\nabla_0, \nabla_1) = P(R^k_{\nabla_1}) - P(R^k_{\nabla_0}).$$

4. Let $\psi : N \to M$ be a smooth map and $\xi = (\pi, E, M)$ a vector bundle of rank r with a connection ∇. Show that for all $P \in I^\bullet(GL(r))$,

$$\psi^* P(R^k_\nabla) = P(R^k_{\psi^*\nabla}).$$

A cohomology class in the image of the Chern-Weil homomorphism is called a **characteristics class** of ξ. There are certain canonical characteristic classes that arise from natural choices of elements in the ring of invariant polynomials $I(GL(r))$.

We have already observed that traces of powers yield invariant polynomials in $\mathfrak{gl}(r)$. One can show that any homogeneous polynomial $P \in I^k(GL(r))$ can be written as a $\mathbb{R}$-linear combination of invariant polynomials of the form:

$$X \mapsto \operatorname{tr}(X^{k_1}) \cdots \operatorname{tr}(X^{k_s}), \quad k_1 + \cdots + k_s = k.$$

However, these are not algebraically independent.

Theorem 32.1. *The coefficients of the characteristic polynomial*

$$\det\big(\lambda I + X\big) = \lambda^r + \sigma_1(X)\lambda^{r-1} + \cdots + \sigma_{r-1}(X)\lambda + \sigma_r(X) \quad (X \in \mathfrak{gl}(r)),$$

are algebraically independent and generate the ring $I(GL(r))$.

Remark 32.2. The coefficients $\sigma_k : \mathfrak{gl}(r) \to \mathbb{R}$ can be expressed using the elementary symmetric functions. Recall that if $x_1, \dots, x_r$ denote r indeterminates then for polynomial $p(x) = \prod_{i=1}^r (x + x_i)$ with coefficients in the field of fractions $\mathbb{R}(x_1, \dots, x_n)$, we have:

$$p(x) = \prod_{i=1}^{r} (x + x_i) = x^r + s_1 x^{r-1} + \cdots + s_{r_1} x + s_r,$$

where the coefficients are the elementary symmetric functions:

$$s_1 = \sum_i x_i, \quad s_2 = \sum_{i<j} x_i x_j, \quad \dots \quad s_r = x_1 \cdots x_r.$$

Applying this to the characteristic polynomial, one obtains:

$$\begin{aligned}
\sigma_1(X) &= \operatorname{tr} X, \\
\sigma_2(X) &= \frac{1}{2}\left((\operatorname{tr} X)^2 - \operatorname{tr} X^2\right), \\
&\ \vdots \\
\sigma_r(X) &= \det X.
\end{aligned}$$

One can show that the field $\mathbb{R}(x_1, \dots, x_r)$ is a Galois extension of the field $\mathbb{R}(s_1, \dots, s_r)$ with Galois group the symmetric group S_n. In other words, any symmetric expression in the indeterminates $x_1, \dots, x_r$ is a polynomial in the elementary symmetric functions $s_1, \dots, s_r$. Applying this to the invariant polynomials, one obtains the theorem above. Note that in this discussion one can replace $\mathbb{R}$ by $\mathbb{C}$, or any other field of characteristic zero.

The previous discussion suggest to apply the Chern-Weil homomorphism to the invariant polynomials $\sigma_1, \dots, \sigma_r$. Before we do that, let us recall that one can equip any vector bundle ξ with a fiber metric, and then one can choose a connection ∇ compatible with the metric. We leave as an exercise to check that for such a connection the curvature 2-form relative to an orthonormal frame is always skew-symmetric:

$$\Omega = -\Omega^T.$$

On the other hand, it follows from the above discussion that if X is skew-symmetric then

$$\sigma_{2k+1}(X) = 0.$$

Hence, by (31.5), we have $\sigma_{2k+1}(R^{2k+1}) = 0$ for such a connection. This explains why in the following definition we only consider even dimensional classes.

Definition 32.3. *Let $\xi = (\pi, E, M)$ be a vector bundle of rank r. For $k = 1, 2, \dots$, the* ***Pontrjagin classes*** *of ξ are:*

$$p_k(\xi) = \left[\sigma_{2k}\left(\left(\frac{1}{2\pi}R\right)^{2k}\right)\right] \in H^{4k}(M),$$

where R is the curvature of any connection ∇ in ξ. The ***total Pontrjagin class*** *of the vector bundle ξ is:*

$$p(\xi) = 1 + p_1(\xi) + \cdots + p_{[r/2]}(\xi),$$

where $[r/2]$ denotes the largest integer less or equal to $r/2$.

Remark 32.4. The normalization factor $\frac{1}{2\pi}$ is included so that the Pontrjagin classes belong to the image of the natural homomorphism:

$$H^\bullet(M, \mathbb{Z}) \to H^\bullet(M).$$

The next proposition lists basic properties of the Pontrjagin classes. The proof follows from the construction of these classes and is left as an exercise.

Proposition 32.5. *Let M be a smooth manifold, ξ and η vector bundles over M. The Pontrjagin classes satisfy:*

(i) $p(\xi \oplus \eta) = p(\xi) \cup p(\eta)$*;*
(ii) $p(\psi^*\xi) = \psi^* p(\xi)$*, for any smooth map* $\psi : N \to M$*;*
(iii) $p(\xi) = 1$*, if ξ admits a flat connection.*

The Pontrjagin classes $p_i = p_i(TM)$ of the tangent bundle of a manifold M give an important invariant of a smooth manifold. Although, from its definition it seems that these classes are only invariants of diffeomorphism type, Novikov proved that these classes are in fact topological invariants: two smooth manifolds which are homemorphic have the same Pontrjagin classes p_i. Here it is important that we are dealing with classes in de Rham cohomology. Using classifying bundles, one can also define Pontrjagin classes leaving in integral cohomology and the integral Pontrjagin classes of TM are not topological invariants.

Over a compact oriented manifold of dimension $\dim M = 4m$ one can also define **Pontrjagin numbers of** ξ. One chooses non-negative integers $a_1, \dots, a_{[r/2]}$ such that:

$$4(a_1 + 2a_2 + \cdots + [r/2]a_{[r/2]}) = 4m,$$

and defines a Pontrjagin number:

$$\int_M p_1^{a_1} \wedge p_2^{a_2} \wedge \cdots \wedge p_{[r/2]}^{a_{[r/2]}}.$$

The **Pontrjagin numbers of** M, where M is compact, oriented, of dimension $4m$ are, by definition, the Pontrjagin numbers of its tangent bundle. For example, a compact, oriented manifold of dimension 4 has only one Pontrjagin number $\int_M p_1$ while in dimension 8 there are two Pontrjagin numbers:

$$\int_M p_1^2, \quad \int_M p_2.$$

EXAMPLES 32.6.

1. *Let $M = \mathbb{S}^d \hookrightarrow \mathbb{R}^{d+1}$ and denote by $\nu(\mathbb{S}^d) = T_{\mathbb{S}^d}\mathbb{R}^{d+1}/T\mathbb{S}^d$ the normal bundle of $\mathbb{S}^d$. Notice that the Whitney sum*

$$T\mathbb{S}^d \oplus \nu(\mathbb{S}^d) = T_{\mathbb{S}^d}\mathbb{R}^{d+1},$$

is the trivial vector bundle over $\mathbb{S}^d$. On the other hand, the normal bundle $\nu(\mathbb{S}^d)$ is also trivial, for it is a line bundle which admits a nowhere vanishing section. By properties (i) and (iii) in the Proposition we conclude that $p(T\mathbb{S}^d) = 1$. Note that $\mathbb{S}^d$ has trivial tangent bundle only for $d = 1, 3, 7$.

2. *Let $M = \mathbb{CP}^d$. Recall that we have $\mathbb{CP}^d = \mathbb{S}^{2d+1}/\mathbb{S}^1$, where $\mathbb{S}^{2d+1} \subset \mathbb{C}^{d+1}$ and $\mathbb{S}^1$ acts by complex multiplication: $\theta \cdot z = e^{i\theta}$. The Euclidean metric in $\mathbb{C}^{d+1} = \mathbb{R}^{2d+2}$ induces a Riemannian metric in $\mathbb{S}^{2d+1}$ which is invariant under the $\mathbb{S}^1$-action. Hence, this induces a Riemannian metric in the quotient $\mathbb{CP}^d = \mathbb{S}^{2d+1}/\mathbb{S}^1$, called the **Fubini-Study metric**.*

   One can use the connection associated with the Fubini-Study metric to compute the Pontrjagin classes $p(T\mathbb{CP}^d)$. For example, in the Homework we sketch how in the case of $\mathbb{CP}^2$ one finds that $\mathbb{CP}^2$ with its canonical orientation (the one induced from the standard orientation of $\mathbb{S}^5$) has Pontrjagin number

$$\int_M p_1 = 3.$$

So far, all our vector bundles were *real* vector bundles. One can also consider **complex vector bundles** $\xi = (\pi, E, M)$, where the fibers E_x are now complex vector spaces of complex dimension r and the transition functions are maps:

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{C}).$$

Every complex vector bundle of rank r can be viewed as a real vector bundle of rank $2r$ equipped with a complex structure J, i.e., an endomorphism of (real) vector bundles $J : \xi \to \xi$ such that $J^2 = -\mathrm{id}$. The complex structure J and the complex structure in the fibers are related by:

$$(a + ib)\mathbf{v} = a\mathbf{v} + bJ(\mathbf{v}), \quad \forall \mathbf{v} \in E.$$

On a complex vector bundle ξ one can consider **$\mathbb{C}$-connections**, i.e., connections ∇ such that for each vector field $X \in \mathfrak{X}(M)$ the map $s \mapsto \nabla_X s$ is $\mathbb{C}$-linear:

$$\nabla_X(\lambda s) = \lambda \nabla_X s, \qquad \forall \lambda \in \mathbb{C}, s \in \Gamma(\xi).$$

Using the endomorphism J, this condition can be expressed as

$$\nabla_X(Js) = J\nabla_X s, \quad \forall\, s \in \Gamma(\xi)),\ X \in \mathfrak{X}(M).$$

Hence, a $\mathbb{C}$-connection is an ordinary connection which is compatible with the complex structure J:

$$\nabla J = 0.$$

Any complex vector bundle admits a $\mathbb{C}$-connection.

The connection 1-form ω and the curvature 2-form Ω of a $\mathbb{C}$-connection ∇ relative to any local $\mathbb{C}$-basis of sections defined over an open $U \subset M$ are matrices of complex-valued forms:

$$\omega = [\omega_a^b] \in \Omega^1(U, \mathfrak{gl}(r, \mathbb{C})), \quad \Omega = [\Omega_a^b] \in \Omega^2(U, \mathfrak{gl}(r, \mathbb{C})).$$

Hence, using a $\mathbb{C}$-connection, one defines the **Chern-Weil homomorphism** much the same way as in the real case, obtaining now a ring homomorphism into the *complex* de Rham cohomology:

$$I(GL(r, \mathbb{C})) \to H^\bullet(M, \mathbb{C}).$$

Again, the ring of invariant polynomials $I(GL(r, \mathbb{C}))$ is generated by the elementary invariant polynomials now viewed as polynomials $\sigma_1, \dots, \sigma_r$ in $\mathfrak{gl}(r, \mathbb{C})$:

$$\det(\lambda I + X) = \lambda^r + \sigma_1(X)\lambda^{r-1} + \cdots + \sigma_{r-1}(X)\lambda + \sigma_r(X), \quad X \in \mathfrak{gl}(r, \mathbb{C}).$$

These allow us to define:

Definition 32.7. *Let $\xi = (\pi, E, M)$ be a complex vector bundle of rank r. For $k = 1, \dots, r$ we define the **Chern classes** of ξ by:*

$$c_k(\xi) = \left[\sigma_k\left(\left(\frac{i}{2\pi}R\right)^k\right)\right] \in H^{2k}(M),$$

*where R is the curvature of any $\mathbb{C}$-connection ∇ in ξ. The **total Chern class** of ξ is sum:*

$$c(\xi) = 1 + c_1(\xi) + \cdots + c_r(\xi) \in H(M).$$

Note that, a priori, the Chern classes are cohomology classes lying in *complex* de Rham cohomology $H^\bullet(M, \mathbb{C})$. However, the normalization factor makes them real cohomology classes. To see this, we use the following lemma which is the complex analogue of the fact that real vector bundles admit fiber metrics and compatible connections. The proof is left as an exercise.

Lemma 32.8. *Every complex vector bundle $\xi = (\pi, E, M)$ admits a fiber hermitian metric $h = \langle\cdot,\cdot\rangle$ and a compatible $\mathbb{C}$-connection: $\nabla h = 0$.*

Choosing a connection as in the lemma, for any orthonormal $\mathbb{C}$-basis of local sections $\{s_1, \dots, s_r\}$ of E, the connection 1-form ω and the curvature 2-form Ω take values in:

$$\mathfrak{u}(r) = \{X \in \mathfrak{gl}(r, \mathbb{C}) : X + \overline{X}^T = 0\}.$$

Then the eigenvalues of Ω are purely imaginary, so $i\Omega$ has real eigenvalues. It follows that $\sigma_k((iR/2\pi)^k)$ is a real form, showing that the Chern classes are real cohomology classes, as claimed.

Similar to the real case, the Chern classes enjoy the following properties:

Proposition 32.9. *Let M be a smooth manifold, ξ and η complex vector bundles over M. The Chern classes satisfy:*

(i) $c(\xi \oplus \eta) = c(\xi) \cup c(\eta)$*;*
(ii) $c(\psi^*\xi) = \psi^* c(\xi)$*, for any smooth map* $\psi : N \to M$*;*
(iii) $c(\xi) = 1$*, if* ξ *admits a flat* $\mathbb{C}$*-connection;*
(iv) $c(\gamma_1^1) = 1 - \mu$ *where* μ *denotes the canonical orientation of* $\mathbb{CP}^1$*.*

Remark 32.10. One can show that properties (i)–(iv) above determine completely the Chern class.

Proof. We leave the proof of properties (i)–(iii) to the exercises in the homework. To prove (iv), we define a $\mathbb{C}$-connection ∇ on the canonical (complex) line bundle γ_1^1 over $\mathbb{CP}^1 = \mathbb{S}^2$ as follows. First, γ_1^1 is a subbundle of the trivial bundle:

$$\gamma_1^1 \subset \mathbb{CP}^1 \times \mathbb{C}^2,$$

so a section of γ_1^1 can be viewed as map $s : \mathbb{CP}^1 \to \mathbb{C}^2$. Then we set:

$$(\nabla_X s)(p) = \mathrm{pr}_{E_p}(\mathrm{d}_p s(X)),$$

where $E_p \subset \mathbb{C}^2$ is the fiber over p, and $\mathrm{pr}_{E_p} : \mathbb{C}^2 \to E_p$ denotes the projection relative to the standard hermitian inner product on $\mathbb{C}^2$.

The bundle trivializes on the open set:

$$U_0 := \{[z_0 : z_1] \in \mathbb{CP}^1 : z_0 \neq 0\}.$$

The non-vanishing section $s : U_0 \to \gamma_1^1$ defined by:

$$s([1 : z]) := ([1, z], (1, z)),$$

is a $\mathbb{C}$-basis of sections over U_0. Defining local coordinates (x, y) on U_0 by $z = x + iy$, a straightforward computation gives the corresponding $\mathbb{C}$-valued connection 1-form ω defined by $\nabla_X s = \omega(X)s$ is given by:

$$\omega = \frac{1}{1 + x^2 + y^2}\left((x\mathrm{d}x + y\mathrm{d}y) + i(-y\mathrm{d}x + x\mathrm{d}y)\right).$$

It follows from the structure equations that the curvature 2-form is:

$$\Omega = \mathrm{d}\omega = \frac{2i}{(1 + x^2 + y^2)^2}\,\mathrm{d}x \wedge \mathrm{d}y.$$

The 1st Chern class is then:

$$c_1(\gamma_1^1) = \left[\frac{i}{2\pi}\Omega\right] = -\left[\frac{\mathrm{d}x \wedge \mathrm{d}y}{\pi(1+x^2+y^2)^2}\right] \in H^2(\mathbb{CP}^2).$$

Now (iv) follows by observing that, since U_0 is an open dense set and $(U_0, (x, y))$ is a positive chart, we find:

$$\begin{aligned}\int_M c_1(\gamma_1^1) &= -\int_{\mathbb{R}^2} \frac{1}{\pi(1+x^2+y^2)^2}\mathrm{d}x\,\mathrm{d}y \\ &= -\int_0^{2\pi}\int_0^{+\infty} \frac{r\,\mathrm{d}r}{\pi(1+r^2)^2}\,\mathrm{d}\theta = -\int_0^{+\infty} \frac{2r\,\mathrm{d}r}{(1+r^2)^2} = -1.\end{aligned}$$

□

One natural way of obtaining complex vector bundles is to start with a **complex manifold** M. Such a manifold is specified by an atlas $\{(U_\alpha, \phi_\alpha)\}$, where the charts are homeomorphisms

$$\phi_\alpha : U_\alpha \to \mathbb{C}^d, \quad x \mapsto (z_\alpha^1(x), \dots, z_\alpha^d(x))$$

and the transition functions are holomorphic maps

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta),$$

defined on open subsets of $\mathbb{C}^d$. Such charts are called *holomorphic charts*. Notice that if we write $z_\alpha^k = x_\alpha^k + iy_\alpha^k$, then the charts $(x_\alpha^k, y_\alpha^k) : U_\alpha \to \mathbb{R}^{2d}$ yield a real smooth structure on M of dimension $2d$.

For a complex manifold M the tangent bundle TM is a complex vector bundle over M (viewed as a real manifold). This can be seen either by construction local $\mathbb{C}$-trivializations, using the holomorphic charts, or by observing that there is a well defined endomorphism $J : TM \to TM$ with $J^2 = -\mathrm{Id}$, which in local holomorphic coordinates $z_\alpha^k = x_\alpha^k + iy_\alpha^k$ is given by:

$$J\Big(\frac{\partial}{\partial x^k}\Big) = \frac{\partial}{\partial y^k}, \quad J\Big(\frac{\partial}{\partial y^k}\Big) = -\frac{\partial}{\partial x^k}.$$

Similarly, the cotangent bundle and all the associated bundles are also complex vector bundles over M. Hence, one can define the Chern classes of these bundles. For example, you are asked to show in the homework that for complex projective space the total Chern class is

$$c(T\mathbb{CP}^d) = (1+a)^{d+1},$$

where $a \in H^2(\mathbb{CP}^d)$ is an appropriate generator.

Notice that since a holomorphic map preserves the canonical orientation of $\mathbb{C}^d$, every complex manifold has a canonical orientation. Hence, for a compact complex manifold M of (complex) dimension d, one can define **Chern numbers** by:

$$\int_M c_1^{a_1} \wedge c_2^{a_2} \wedge \cdots \wedge c_d^{a_d},$$

where $c_i = c_i(TM)$ and $a_1, \dots, a_d$ are any non-negative integers such that:

$$2(a_1 + 2a_2 + \cdots + da_d) = 2d.$$

Another class of examples of complex vector bundles arises by complexification of a real vector bundle. If $\xi = (\pi, E, M)$ is a real vector bundle of rank r we can form its tensor product with the trivial *real* rank 2 vector bundle $M \times \mathbb{C} \to M$. The resulting bundle, denoted $\xi \otimes \mathbb{C}$, is a real vector bundle of rank 2r admitting the endomorphism $J : \xi \otimes \mathbb{C} \to \xi \otimes \mathbb{C}$ given by:

$$J(v \otimes \lambda) := v \otimes i\lambda.$$

Since $J^2 = -\mathrm{Id}$, this defines a complex structure in $\xi \otimes \mathbb{C}$. One calls the resulting complex vector bundle $\xi \otimes \mathbb{C}$ the **complexification** of ξ.

Proposition 32.11. *Let ξ be a real vector bundle. Then the Pontrjagin classes of ξ and the Chern classes of $\xi \otimes \mathbb{C}$ are related by:*

$$p_k(\xi) = (-1)^k c_{2k}(\xi \otimes \mathbb{C}).$$

Proof. Immediate from the formulas defining them! □

Our discussion of the Pontrjagin classes suggest that the odd classes $c_{2k+1}(\xi \otimes \mathbb{C})$ vanish. To see this, given a complex vector bundle $\xi = (\pi, E, M)$ its **complex conjugate** is the complex vector bundle $\bar{\xi}$ which, as a real vector bundle, coincides with ξ, but where the complex structure is the opposite: $J_{\bar{\xi}} = -J_\xi$. Notice, e.g., that the identity map id: $\xi \to \bar{\xi}$ satisfies:

$$\mathrm{id}(\lambda \mathbf{v}) = \bar{\lambda}\ \mathrm{id}(\mathbf{v}), \quad \forall \mathbf{v} \in E, \lambda \in \mathbb{C}.$$

Proposition 32.12. *Let $\xi = (\pi, E, M)$ be a complex vector bundle. The Chern classes of ξ and $\bar{\xi}$ are related by $c_k(\bar{\xi}) = (-1)^k c_k(\xi)$ so that:*

$$c(\bar{\xi}) = 1 - c_1(\xi) + c_2(\xi) - \cdots + (-1)^r c_r(\xi).$$

Proof. Let ∇ be a $\mathbb{C}$-connection in ξ. It defines also a $\mathbb{C}$-connection in $\bar{\xi}$ which we denote by $\overline{\nabla}$. If one fixes local trivializing sections $\{s_1, \dots, s_r\}$ for ξ, then we have:

$$\nabla_X s_a = \sum_b \omega_a^b(X) s_b, \quad \overline{\nabla}_X s_a = \sum_b \overline{\omega}_a^b(X) s_b.$$

Hence, the curvature 2-forms of these two connections relative to this basis are related by:

$$\Omega_{\overline{\nabla}}(X, Y) = \overline{\Omega_\nabla(X, Y)},$$

and it follows that:

$$\begin{aligned}
\sigma_k\left(\left(\frac{i}{2\pi} R_{\overline{\nabla}}\right)^k\right) &= \sigma_k\left(\overline{\left(-\frac{i}{2\pi} R_\nabla\right)^k}\right) \\
&= (-1)^k \overline{\sigma_k\left(\left(\frac{i}{2\pi} R_\nabla\right)^k\right)} = (-1)^k \sigma_k\left(\left(\frac{i}{2\pi} R_\nabla\right)^k\right)
\end{aligned}$$

so that $c_k(\bar{\xi}) = (-1)^k c_k(\xi)$. □

Remark 32.13. The complexification $\xi \otimes \mathbb{C}$ and its conjugate complex vector bundle $\overline{\xi \otimes \mathbb{C}}$ are isomorphic complex vector bundles. An explicit isomorphism is given by the complex conjugation map:

$$\xi \otimes \mathbb{C} \to \overline{\xi \otimes \mathbb{C}}, \ v \otimes \lambda \mapsto v \otimes \overline{\lambda}.$$

Hence, by Proposition 32.12, we conclude that

$$c_k(\xi \otimes \mathbb{C}) = 0, \text{ if } k \text{ is odd.}$$

This gives another explanation for why the Pontrjagin classes of a real vector bundle are concentrated in degree $4k$.

Different choices of invariant function lead to other interesting characteristic classes. For example, the invariant function $\chi : \mathfrak{gl}(r, \mathbb{C}) \to \mathbb{C}$ given by:

$$\chi(X) := \operatorname{tr}\big(\exp(X)\big),$$

gives rise to the **Chern character** of the vector bundle:

$$\operatorname{ch}(\xi) = \left[\chi\Big(\frac{i}{2\pi}R\Big)\right] \in H^\bullet(M).$$

The Chern character is a semi-ring homomorphism:

$$\operatorname{ch}(\xi_1 \oplus \xi_1) = \operatorname{ch}(\xi_1) + \operatorname{ch}(\xi_2), \quad \operatorname{ch}(\xi_1 \otimes \xi_1) = \operatorname{ch}(\xi_1) \cup \operatorname{ch}(\xi_2),$$

and for this reason it is important in K-theory. Other examples of characteristic classes include the Todd class of a complex vector bundle, that appears in the Hirzebruch-Riemann-Roch formula in algebraic geometry, or the L-class of a real vector bundle that appears in Hirzebruch's signature formula in differential topology.

The presence of extra data on a vector bundle can also lead to special characteristic classes. For example, the **Euler class** of an oriented vector bundle $\xi = (\pi, E, M)$ can be viewed as a characteristic class. For that, fix a fiberwise metric g and a connection ∇ compatible with the metric g. Then for any local positive orthonormal basis of sections $\{s_1, \dots, s_r\}$ the corresponding connection 1-form ω takes values in the Lie algebra $\mathfrak{so}(r)$ consisting of all skew-symmetric matrices. If we change to a new basis of sections $\{s'_1, \dots, s'_r\}$ the two bases are related by:

$$s'_a = \sum_{b=1}^{r} A_a^b s_b, \qquad A = [A_a^b] : U \to SO(r).$$

Hence, we now look for invariant functions in $I(SO(r))$ to produce characteristic classes.

The restriction of the elementary invariant polynomials σ_k to $\mathfrak{so}(r)$ give obvious elements in $I(SO(r))$. When r is odd, one can show that these generate all invariant polynomials, but when r is even, this is not true anymore and one needs to add an extra polynomial to obtain a set of generators. This can already be seen for $r = 2$.

Example 32.14.
The Lie algebra

$$\mathfrak{so}(2) = \left\{ \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix} : x \in \mathbb{R} \right\} \subset \mathfrak{gl}(2, \mathbb{R})$$

is abelian, so the invariance condition is empty. The elementary polynomial $\sigma_1(X) = \operatorname{tr} X$ *restricts to zero, while* $\sigma_2(X) = \det X$ *restricts to a perfect square:*

$$\det(X) = x^2.$$

We also have the degree 1 invariant polynomial $\mathrm{Pf} : \mathfrak{so}(2) \to \mathbb{R}$ *defined by:*

$$\mathrm{Pf}(X) = x,$$

which is not generated by $\{\sigma_1, \sigma_2\}$.

An analogous invariant polynomial Pf can be defined for any $r \geq 2$ as follows. Any skew-symmetric matrix $X \in \mathfrak{so}(2m)$ is conjugate to a block diagonal matrix:

$$X = ADA^T, \qquad D = \begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & . & & S_m \end{pmatrix}$$

where S_k is a 2x2 matrix of the form:

$$S_k = \begin{pmatrix} 0 & x_k \\ -x_k & 0 \end{pmatrix}.$$

It follows that the determinant is a perfect square:

$$\det(X) = \Big(\det(A) \prod_{i=1}^{m} x_k \Big)^2.$$

and one defines the **Pfaffian** of X to be the function given by:

$$\mathrm{Pf}(X) := \det(A) \prod_{i=1}^{m} x_k.$$

That this is well-defined degree m polynomial follows from the following explicit formula, whose proof is left as an exercise:

$$\mathrm{Pf}(X) = \frac{1}{2^m m!} \sum_{\sigma \in S_{2m}} (-1)^\sigma \prod_{k=1}^{m} X_{\sigma(2k-1)\sigma(2k)}.$$

Also, when $B \in SO(2m)$, so that $B^{-1} = B^T$ and $\det B = 1$, we find that:

$$\mathrm{Pf}(BXB^{-1}) = \det(BA) \prod_{i=1}^{m} x_k = \det(A) \prod_{i=1}^{m} x_k = \mathrm{Pf}(X).$$

Hence, $\mathrm{Pf} \in I^m(SO(2m))$. One can show that the invariant polynomial $\{\sigma_{2k}, \mathrm{Pf}\}$ generate $I(SO(2m))$.

The characteristic class corresponding to the Pfaffian is the Euler class:

Theorem 32.15. *Let $\xi = (\pi, E, M)$ be an oriented vector bundle of rank $r = 2m$. Then its Euler class $e(\xi)$ is represented by the form:*

$$\mathrm{Pf}\left(\left(\frac{1}{2\pi}R\right)^m\right) \in \Omega^{2m}(M),$$

where R the curvature tensor of any connection ∇ compatible with a fiberwise metric.

We will not give a proof of this result. It can be thought of as a somewhat more involved exercise.

Homework.

1. Show that every complex vector bundle $\xi = (\pi, E, M)$ admits a $\mathbb{C}$-connection ∇ compatible with a fiber hermitian metric $h = \langle \cdot, \cdot \rangle$ (i.e., such that $\nabla h = 0$).

2. Prove the properties of the Pontrjagin classes and the Chern classes stated in Propositions 32.5 and 32.9.

3. Let $\xi = (\pi, E, M)$ be a complex vector bundle. Show that its $\mathbb{C}$-dual $\xi^* = \mathrm{Hom}(\xi, \mathbb{C})$ is a complex vector bundle and that their Chern classes are related by:
$$c_k(\xi^*) = (-1)^k c_k(\xi).$$
(Hint: Use a fiber hermitian metric.)

4. Let γ_d^1 be the canonical complex line bundle over $\mathbb{CP}^d$. Show that:
$$c(\gamma_d^1) = 1 - a,$$
where $a \in H^2(\mathbb{CP}^d)$ is an appropriate generator.

5. Denote by $\varepsilon_{\mathbb{CP}^d}^{d+1} = \mathbb{CP}^d \times \mathbb{C}^{d+1} \to \mathbb{CP}^d$ the trivial complex vector bundle equipped with the standard hermitian inner product h on the fibers. Let $(\gamma_d^1)^\perp \subset \varepsilon_{\mathbb{CP}^d}^{d+1}$ denote the h-orthogonal bundle to the canonical complex line bundle γ_d^1, so that:
$$\varepsilon_{\mathbb{CP}^d}^{d+1} = \gamma_d^1 \oplus (\gamma_d^1)^\perp.$$
   (a) Show that there is an isomorphism of complex vector bundles:
   $$T\mathbb{CP}^d \simeq \mathrm{Hom}_{\mathbb{C}}(\gamma_d^1, (\gamma_d^1)^\perp).$$
   (b) Show that there are isomorphisms of complex vector bundles:
   $$T\mathbb{CP}^d \oplus \varepsilon_{\mathbb{CP}^d}^1 \simeq \mathrm{Hom}_{\mathbb{C}}(\gamma_d^1, \varepsilon_{\mathbb{CP}^d}^{d+1}) = \mathrm{Hom}_{\mathbb{C}}(\gamma_d^1, \varepsilon_{\mathbb{CP}^d}^1 \oplus \cdots \oplus \varepsilon_{\mathbb{CP}^d}^1).$$
   (c) Conclude that the total Chern class of the tangent bundle to $\mathbb{CP}^d$ is:
   $$c(T\mathbb{CP}^d) = (1+a)^{d+1},$$
   where $a \in H^2(\mathbb{CP}^d)$ is an appropriate generator.

6. Let $\xi = (\pi, E, M)$ be an oriented vector bundle of rank r. Show that:
$$e(\xi)^2 = p_{[r/2]}(\xi).$$

7. Prove that if a compact, oriented, manifold M of dimension $4m$ can be embedded in $\mathbb{R}^{4m+1}$ then all its Pontrjagin classes must vanish: $p(TM) = 1$. (HINT: The normal bundle $\nu(M)$ is trivial.)

8. Two oriented manifolds M_1 and M_2 are said to be cobordant if $\dim M_1 = \dim M_2$ and there exists an oriented manifold with boundary N such that, as oriented manifolds,
$$\partial N = M_1 - M_2,$$
where $-M_2$ denotes M_2 with the opposite orientation. Show that if M_1 and M_2 are compact oriented cobordant manifolds of dimension $4m$ then they must have the same Pontrjagin numbers.
(HINT: Show first that if $M = \partial N$, where N is compact, oriented, then the Pontrjagin numbers of M must vanish. For this, choose a connection ∇ on N with the property that $\nabla_X Y$ is tangent to ∂N whenever X and Y are tangent to ∂N)

33. FIBER BUNDLES

Bundles with fiber which are not vector spaces also occur frequently in Differential Geometry. We will study them briefly in these last two sectionrs.

Let $\pi : E \to M$ be a surjective submersion. A **trivializing chart** for π with fiber type F is a pair (U, ϕ), where $U \subset M$ is an open set and $\phi : \pi^{-1}(U) \to U \times F$ is a diffeomorphism such that the following diagram commuttes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

where $\pi_1 : U \times F \to U$ denotes the projection in the first factor. If $E_p = \pi^{-1}(p)$ is the fiber over $p \in U$ we obtain a diffeomorphism $\phi^p : E_p \to F$ as the composition of the maps:

$$\phi^p : E_p \xrightarrow{\phi} \{p\} \times F \longrightarrow F \ .$$

Hence, if $\mathbf{v} \in E_p$, we have $\phi(\mathbf{v}) = (p, \phi^p(\mathbf{v}))$.

Given two trivializing charts (U_α, ϕ_α) and (U_β, ϕ_β) have the transition map:

$$\phi_\alpha \circ \phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F, \ (p, f) \mapsto (p, \phi_\alpha^p \circ (\phi_\beta^p)^{-1}(f)).$$

This defines the **transition functions** $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Diff}(F)$, where $g_{\alpha\beta}(p) \equiv \phi_\alpha^p \circ (\phi_\beta^p)^{-1}$.

If one is given a covering of M by trivializing charts $\{(U_\alpha, \phi_\alpha) : \alpha \in A\}$, this leads to a cocycle $\{g_{\alpha\beta}\}$ with values in the group $\mathrm{Diff}(F)$. We would like this to determine the fiber bundle, and recover the bundle from the cocycle. However, $\mathrm{Diff}(F)$ is infinite dimensional, so this may pose some difficulties. For this reason, we will restrict our attention to fibre bundles for which the transition functions take values in a finite dimensional Lie group $G \subset \mathrm{Diff}(F)$:

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to G \subset \mathrm{Diff}(F).$$

Equivalently, we assume that the we have an *effective* action of a Lie group G on F and that the transition functions take the form:

$$\phi^p_\alpha \circ (\phi^p_\beta)^{-1}(f) = g_{\alpha\beta}(p) \cdot f,$$

for a map $g_{\alpha,\beta} : U_\alpha \cap U_\beta \to G$. Our formal definition of a G-fiber bundle is then the following:

Definition 33.1. *Let G be a Lie group and $G \times F \to F$ a smooth, effective, action. A* ***G-fiber bundle*** *over M with fiber type F is a triple $\xi = (\pi, E, M)$, where $\pi : E \to M$ is a smooth map admitting a collection of trivializing charts $\mathcal{C} = \{(U_\alpha, \phi_\alpha) : \alpha \in A\}$ with fiber type F, satisfying the following properties:*

(i) $\{U_\alpha : \alpha \in A\}$ is an open cover of M: $\bigcup_{\alpha \in A} U_\alpha = M$;
(ii) The charts are compatible: for any $\alpha, \beta \in A$ there are smooth maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ such that the transition functions take the form:

$$(p, f) \mapsto (p, g_{\alpha\beta}(p) \cdot f);$$

(iii) The collection $\mathcal{C}$ is maximal: if (U, ϕ) is a trivializing chart of fiber type F with the property that for every $\alpha \in A$, there exist $g_\alpha : U \cap U_\alpha \to G$ such that

$$\phi^p \circ (\phi^p_\alpha)^{-1}(f) = g_\alpha(p) \cdot f, \qquad \forall f \in F,$$

then $(U, \phi) \in \mathcal{C}$.

We shall use the same notation as in the case of vector bundles, so we have the **total space**, the **base space**, and the **projection** of the G-bundle. Also, one calls G the **structure group** of the fiber bundle. Given a G-fiber bundle ξ a subcollection of charts of $\mathcal{C}$ which still covers M is called an **atlas of fiber bundle** or a **trivialization** of ξ. We define a **section** over an open set U in the obvious way and we denote the set of all sections over U by $\Gamma_U(E)$. Although a fiber bundle always has local sections, it may fail to have global sections.

Among the most important classes of G-bundles we have:

- **Vector bundles**: In this case the fiber F is a vector space and the structure group is the group of linear invertible transformations $G = GL(V)$. These are precisely the bundles that we have studied in the previous sections.
- **Principal G-bundles** In this case the fiber F is itself a Lie group G and the structure group is the same Lie group G acting on itself by translations $G \times G \to G$, $(g, h) \mapsto gh$. We shall see that principal bundles play a central role among all G-bundles.

A morphism of G-fiber bundles can be defined in a fashion similar to the definition of a morphism of vector bundles, where we replace $GL(r)$ by the structure group G.

Definition 33.2. *Let $\xi = (\pi, E, M)$ and $\xi' = (\pi', E', M')$ be two G-bundles with the same fiber F and structure group G. A **morphism of G-bundles** is a smooth map $\Psi : E \to E'$ mapping fibers of ξ to fibers of ξ', so Ψ covers a smooth map $\psi : M \to M'$:*

$$\begin{tikzcd} E \arrow[r, "\Psi"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "\psi"] & M' \end{tikzcd}$$

and such that for each $p \in M$, the map on the fibers

$$\Psi^p \equiv \Psi|_{E_p} : E_p \to E'_q, \quad (q = \psi(p))$$

satisfies

$$\phi'^q_\beta \circ \Psi^p \circ (\phi^p_\alpha)^{-1} \in G,$$

for any trivializations $\{\phi_\alpha\}$ of ξ and $\{\phi'_\beta\}$ of ξ'.

In this way, we have the category of fiber bundles with fiber type F structure group G. Just like in the case of vector bundles, we shall also distinguish between *equivalence* and *isomorphism* of G-bundles, according to wether the base map is the identity map or not.

The set of transition functions associated with an atlas of a G-fiber bundle completely determined the bundle. The discussion is entirely analogous to the case of vector bundles. First, if $\xi = (\pi, E, M)$ is a G-bundle the transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$, relative to some trivialization $\{(U_\alpha, \phi_\alpha)\}$, satisfy the cocycle condition:

$$g_{\alpha\beta}(p) g_{\beta\gamma}(p) = g_{\alpha\gamma}(p), \quad (p \in U_\alpha \cap U_\beta \cap U_\gamma).$$

We say that two cocycles $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ are equivalent if there exist smooth maps $\lambda_\alpha : U_\alpha \to G$ such that:

$$g'_{\alpha\beta}(p) = \lambda_\alpha(p) \cdot g_{\alpha\beta}(p) \cdot \lambda_\beta^{-1}(p), \quad (p \in U_\alpha \cap U_\beta).$$

One checks easily the following analogue of Proposition 25.5:

Proposition 33.3. *Let M be a manifold and G a Lie group acting on another smooth manifold F. Given a cocycle $\{g_{\alpha\beta}\}$ with values in G, subordinated to a covering $\{U_\alpha\}$ of M, there exists a G-bundle $\xi = (\pi, E, M)$ with fiber type F which admits an atlas $\{\phi_\alpha\}$, for which the transition functions give the cocycle $\{g_{\alpha\beta}\}$. Two equivalent cocycles determine isomorphic G-bundles.*

Let $\xi = (\pi, E, M)$ be a G-fiber bundle with fiber type F and let $\{g_{\alpha\beta}\}$ be a cocycle associated with a trivialization $\{\phi_\alpha\}$ of ξ. If $H \subset G$ is a Lie subgroup, we say that the **structure group of ξ can be reduced to H**

if the cocycle is equivalent to a cocycle $\{g'_{\alpha\beta}\}$ where the transition functions take values in H:

$$g'_{\alpha\beta}: U_\alpha \cap U_\beta \to H \subset G.$$

We will see later how to describe this notion independently of choice of trivializations. The next examples illustrate how the structure group (and its possible reductions) are intimately related with geometric properties of the bundle.

Examples 33.4.

1. *A G-fiber bundle $\xi = (\pi, E, M)$ with fiber type F and is called* ***trivial*** *if it is isomorphic to the trivial bundle* $\mathrm{pr}: M \times F \to M$*. This is the case if and only if its structure group can be reduced to the trivial group $\{e\}$.*
2. *We saw before that a vector bundle of rank r is orientable if and only if its structure group can be reduced to $GL^+(r)$. Similarly, a vector bundles admits a fiber metric if and only if its structure group can be reduced to $O(r)$ (and by the polar decomposition, this can always be achieved). A further reduction of its structure group to $SO(r)$ amounts to an additional choice of an orientation for the bundle (which may or may not be possible).*

Remark 33.5. The specification of the structure group is crucial. For example, a G-cocycle may take values in a subgroup $H \subset G$ and be trivial as a G-cocycle, but not as an H-cocycle. An example is given in the Homework at the end of this section.

Notice that the cocycles associated with a G-fiber bundle, as well as the notion of equivalence of cocycles, does not make any use of the G-action on F. For this reason principal G-bundles play a fundamental role among all G-bundles. In fact:

- Given a principal G-bundle $\xi = (\pi, P, M)$, a trivialization $\{\phi_\alpha\}$ of ξ determines a cocycle $\{g_{\alpha\beta}\}$ with values in G. If G acts in F we obtain a G-fiber bundle $\xi_F = (\pi, E, M)$ with fibre type F.
- Conversely, given a G-fiber bundle $\xi_F = (\pi, E, M)$ and fixing a trivialization $\{\phi_\alpha\}$ of ξ_F, the associated cocycle $\{g_{\alpha\beta}\}$ takes values in G. Since G acts on itself by translations this cocycle defines a principal G-bundle $\xi = (\pi, P, M)$.

To make this more explicit, we observe that principal G-bundles can also be described more succinctly:

Proposition 33.6. *A fiber bundle $\xi = (\pi, P, M)$ is a principal G-bundle if and only if there exists a right action $P \times G \to P$ satisfying the following properties:*

(i) The action is free and proper;
(ii) M is diffeomorphic to P/G and under this identification $\pi : P \to M \simeq P/G$ is the quotient map;
(iii) The local trivializations (U, ϕ) are G-equivariant: $\phi^p(\mathbf{v} \cdot g) = \phi^p(\mathbf{v})g$.

Proof. Given a principal G-bundle $\xi = (\pi, P, M)$ one constructs a right action $P \times G \to P$ working on trivializing charts (U, ϕ): the action of G on $\pi^{-1}(U)$ is defined by

$$u \cdot g := \phi^{-1}(p, \phi^p(u)g), \quad (p = \pi(u)).$$

One checks easily that this definition is independent of the choice of trivialization. The rest of the statements are left as an exercise. $\square$

Conversely, by an exercise in Section 16, if one is given a free and proper right action $P \times G \to P$, one obtains a principal G-bundle $\xi = (\pi, P, M)$ by setting $M = P/G$ and letting $\pi : P \to M$ be the quotient map. So principal bundles amount simply to free and proper right actions – which will be called **principal actions**.

If one is given a principal action $P \times G \to P$ and an action $G \times F \to F$ one can form the **associated fiber bundle** $\xi_F = (\pi_F, E, M)$ with total space

$$E := P \times_G F,$$

the quotient space for the action of G on $P \times F$ defined by:

$$(u, f) \cdot g \equiv (u \cdot g, g^{-1} \cdot f)$$

(recall that G acts on the right in P and on the left in F). The projection $\pi_F : E \to M$ is given by: $\pi_F([u, f]) = \pi(u)$.

These descriptions of principal G-bundles and the associated bundles allows one to give many examples of G-fiber bundles.

Examples 33.7.

1. *For any Lie group G, we have the trivial principal G-bundle $M \times G \to M$. Sections of this bundle are just smooth maps $M \to G$. Moreover, if G acts on some space F, then the associated bundle is also the trivial bundle $M \times F \to M$.*
2. *For any Lie group G and any closed subgroup $H \subset G$ the right action of H on G is principal, so the quotient $G \to G/H$ is a principal H-bundle. For example, if we let $\mathbb{S}^3$ be the group of unit quaternions and let $\mathbb{S}^1 \subset \mathbb{S}^3$ be the subgroup of unit complex numbers, then we obtain a principal $\mathbb{S}^1$-bundle, which is easily seen to be isomorphic to the Hopf fibration.*
3. *If $\pi : \tilde{M} \to M$ is the universal covering space of a manifold M, the triple $(\pi, \tilde{M}, M)$ is a principal bundle with structure group the fundamental group $\pi_1(M)$ (the topology in $\pi_1(M)$ is the discrete topology). More generally, if $H \subset \pi_1(M)$ is a normal subgroup then the covering space $P := \tilde{M}/H \to M$ with group of deck transformations $G := \pi_1(M)/H$ is also a principal G-bundle.*
4. *Let M be a smooth manifold of dimension d. The **frame bundle** is the principal bundle $\pi : F(M) \to M$ with structure group $GL(d)$ whose fiber over $p \in M$ consists of the set of all ordered basis of T_pM:*

$$F(M)_p = \{(\mathbf{v}_1, \dots, \mathbf{v}_d) : \ \mathbf{v}_1, \dots, \mathbf{v}_d \textit{ is a basis of } T_pM\}.$$

The group $GL(d)$ acts on the right on $F(M)$: if $u = (\mathbf{v}_1, \ldots, \mathbf{v}_d)$ is a frame and $A = (a_i^j)$ is an invertible matrix, then $u \cdot A = (\mathbf{w}_1, \ldots, \mathbf{w}_d)$ is the frame:

$$\mathbf{w}_i = \sum_{j=1}^{d} a_i^j \mathbf{v}_j, \qquad (i = 1, \ldots, d).$$

This is a proper and free action, hence $F(M)$ is a principal bundle with structure group $GL(d)$.

The group $GL(d)$ acts (on the left) in $\mathbb{R}^d$ by matrix multiplication. Hence, $F(M)$ has an associated fibre bundle with fiber $\mathbb{R}^d$, i.e., a vector bundle. We leave it as an exercise to check that this bundle is canonically isomorphic to the tangent bundle $T(M)$. Similarly, one obtains the cotangent bundle, exterior bundles, tensor bundle, etc., if one considers instead the induced actions of $GL(d)$ in $(\mathbb{R}^d)^*$, $\wedge^k\mathbb{R}^d$, $\otimes^k\mathbb{R}^d$, etc.

More generally, for any (real) vector bundle $\pi : E \to M$ of rank r, one can form the frame bundle $F(E)$, a principal bundle with structure group $GL(r)$. For the usual action of $GL(r)$ on $\mathbb{R}^r$ one obtains an associated bundle to $F(E)$ with fiber $\mathbb{R}^r$, which is canonically isomorphic to the original vector bundle $\pi : E \to M$. Similarly, one can obtained as associated bundles E^*, $\wedge^k E$, $\otimes^k E$, etc.

An entirely similar discussion is valid for complex vector bundles and the bundle of complex frames where $GL(d)$ is replaced by $GL(d, \mathbb{C})$.

If $\xi = (\pi, P, M)$ is a principal G-bundle and $G \times F \to F$ is a smooth action, then one should expect that *any functorial construction in the associated bundle $\xi_F = (\pi, E, M)$ should be expressed in terms of ξ and F*. As an example of this principle, for the sections of ξ_F we have

Proposition 33.8. *Let $\xi = (\pi, P, M)$ be a principal G-bundle and $G \times F \to F$ a smooth action. The sections of the associated bundle $\xi_F = (\pi, E, M)$ are in one to one correspondence with the G-equivariant maps $h : P \to F$.*

Proof. The total space of the associated bundle is

$$E = P \times_G F = (P \times F)/G.$$

An element $v \in E_p$ is an equivalence class in $P_p \times_G F$, which can be written as:

$$v = [(u, h_p(u))], \quad \forall u \in P_p,$$

for a unique map $h_p : P_p \to F$ which is G-equivariant:

$$h_p(u \cdot g) = g^{-1} \cdot h_p(u).$$

Hence, a section $s : M \to E$ can be written in the form:

$$s(p) = [(u, h(u))], \quad \forall u \in P \text{ com } \pi(u) = p,$$

where $h : P \to F$ is a G-equivariant map. Conversely, a G-equivariant map $h : P \to F$ determines through this formula a section of ξ_F. $\square$

Note that a G-fiber bundle $\xi_F = (\pi, E, M)$ may not have any sections and even if has a section it may not be trivial (e.g., vector bundles). However, it is not hard to see that a principal G-bundle is trivial if and only if it admits a section. Another important general fact, which we will not prove, is the following:

Theorem 33.9. *Let $\xi_F = (\pi, E, M)$ be a G-fiber bundle with contractible fiber F. Then ξ_F admits a section and any two sections of ξ_F are homotopic.*

In order to understand the issue of reduction of the structure group without referring to cocycles, it is convenient to enlarge the notion of morphism of principal bundles as follows:

Definition 33.10. *Let $\xi' = (\pi', P', M')$ be a principal G'-bundle, $\xi = (\pi, P, M)$ a principal G-bundle and $\phi : G' \to G$ a Lie group homomorphism. A **ϕ-morphism** $\Psi : \xi' \to \xi$ is a map $\Psi : P' \to P$ such that*

$$\Psi(u \cdot g) = \Psi(u)\phi(g), \forall u \in P', g \in G'.$$

A ϕ-morphism of principal bundles $\Psi : \xi' \to \xi$ takes fibers to fibers so it covers a smooth map $\psi : M' \to M$:

$$\begin{tikzcd} P' \arrow[r, "\Psi"] \arrow[d, "\pi'"'] & P \arrow[d, "\pi"] \\ M' \arrow[r, "\psi"'] & M \end{tikzcd}$$

If $\Psi : P' \to P$ and $\phi : G' \to G$ are both embeddings, one can identify P' and G' with its images $\Psi(P') \subset P$ and $H := \Phi(G') \subset G$. We then say that ξ' is a **subbundle** of the principal bundle ξ. When $M' = M$ and ψ =id we say that ξ' is a **reduced subbundle** of ξ. You should check that this matches the notion of reduction of the structure group from G to H that we have introduced before in terms of cocycles.

Example 33.11.
*If M carries a Riemannian structure, then we can consider the **orthogonal frame bundle** whose fiber is:*

$$OF(M)_p = \{(\mathbf{v}_1, \dots, \mathbf{v}_d) : \ \mathbf{v}_1, \dots, \mathbf{v}_d \text{ is an orthonormal basis of } T_pM\}.$$

*This is a principal $O(d)$-bundle, which is a reduced subbundle of $F(M)$, obtained by reduction of the structure group from $GL(d)$ to $O(d)$. In general, a reduction of $F(M)$ to a closed subgroup $G \subset GL(d)$ is called a **G-structure** on M. We leave the details as an exercise.*

Homework.

1. Show that $\xi = (\pi, P, M)$ is a principal G-bundle principal if and only if there exists a right action $P \times G \to P$ satisfying the following properties:
   (i) The action is free and proper;

(ii) The quotient P/G is a manifold, $M \simeq P/G$ and $\pi : P \to P/G \simeq M$ is the quotient map;
(iii) The local trivializations (U, ϕ) are G-equivariant: $\phi^p(g \cdot \mathbf{v}) = g \cdot \phi^p(\mathbf{v})$.

2. Give a proof of Proposition 33.3

3. Consider the covering of $M = \mathbb{S}^1$ by the open sets:

$$U_\pm = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} - \{(\pm 1, 0)\}.$$

Define a cocycle $\{g_{\alpha\beta}\}$ relative to this covering by

$$g_{+-}(x, y) = \begin{cases} I & \text{if } (x, y) \in y > 0, \\ \\ -I & \text{if } (x, y) \in y < 0. \end{cases}$$

where I is the 2×2 identity matrix Show that:
(a) This cocycle defines a G-bundle with fibre type $\mathbb{S}^1$ and structure group $\mathbb{S}^1 = S0(2)$ which is isomorphic (as an $\mathbb{S}^1$-bundle) to the trivial bundle.
(b) This cocycle defines a G-bundle with fibre type $\mathbb{S}^1$ and structure group $\mathbb{Z}_2 = \{I, -I\}$ which is not isomorphic (as a $\mathbb{Z}_2$-bundle) to the trivial bundle.

4. Show that a principal bundle is trivial if and only if it has a global section. (Note: This exercise is a very special case of the next exercise.)

5. Let $\xi = (\pi, P, M)$ be a principal G-bundle and $H \subset G$ a closed subgroup. Note that G acts in the quotient G/H hence there is an associate bundle $\xi_{G/H} = (\pi', P \times_G (G/H), M)$. Show that this bundle can be identified with the quotient $(\pi', P/H, M)$, where $\pi' : P/H \to M$ is the map induced in the quotient by $\pi : P \to M$. Show that the following statements are equivalent:
(a) The structure group of ξ can be reduced to H.
(b) The associated bundle $\xi_{G/H}$ has a section, i.e., there exists a map $s : M \to P/H$ such that $\pi' \circ s =$id.
(c) There exists a G-equivariant map $h : P \to G/H$.

6. Let M be a Riemannian manifold and let $\pi : OF(M) \to M$ be the principal $O(d)$-bundle formed by the orthogonal frames:

$$OF(M)_p = \{(\mathbf{v}_1, \ldots, \mathbf{v}_r) : \ \mathbf{v}_1, \ldots, \mathbf{v}_r \text{ is an orthonormal base of } T_pM\}.$$

Show that $OF(M)$ is the reduced bundle from $F(M)$, which corresponds to the reduction of the structure group from $GL(d)$ to $O(d)$.

34. Principal Fiber Bundles

Let $\psi : N \to M$ be a smooth map and let $\xi = (\pi, P, M)$ be a principal G-bundle. Similar to the case of vector bundles, we have a **pullback** principal bundle $\psi^*\xi$: it is the the principal G-bundle over N with total space:

$$\psi^* P \equiv \{(q, u) \in N \times P : \psi(q) = \pi(u)\}.$$

The G-action on the right of this space is given by $(q,u)\cdot g \equiv (q, u\cdot g)$, which is clearly a principal action.

We also have a morphism of principal bundles $\Psi : \psi^*\xi \to \xi$, given by $\Psi(q,u) = u$, which allows to complete the diagram:

$$\begin{tikzcd} \psi^* P \arrow[r, dashed, "\Psi"] \arrow[d, dashed, "\hat{\pi}"'] & P \arrow[d, "\pi"] \\ M \arrow[r, "\psi"] & N \end{tikzcd}$$

The pullback of principal bundles satisfies properties analogous to those of the pullback of vector bundles, such as the universal property, homotopy invariance, etc., which we leave as an exercise to state (and to prove!).

Let us look briefly at the classification of principal bundles. Let $\xi = (\pi, P, M)$ be a principal G-bundle and let E be a space with a right G-action. If we make the right action into a left action, we can form the associated fiber bundle

$$\xi_E = (\pi, P \times_G E, M).$$

We make the following assumptions on the G-space:

(i) *E is contractible:* By Theorem 33.9 the bundle ξ_E admits sections and any pair of sections s_0, s_1 of ξ_E are homotopic. By Proposition 33.8, a section $s : M \to P \times_G E$ corresponds to a G-equivariant map $\Phi : P \to E$. So we conclude that the there exists G-equivariant maps and any two such maps are G-equivariantly homotopic.

(ii) *Action is free and proper:* We have a principal G-bundle $\eta = (\mathrm{pr}, E, E/G)$ with projection the quotient map

$$\mathrm{pr} : E \to E/G.$$

Also, any G-equivariant map $\Phi : P \to E$ is injective and covers a map $\phi : M \to E/G$:

$$\begin{tikzcd} P \arrow[r, "\Phi"] \arrow[d, "\pi"'] & E \arrow[d, "\mathrm{pr}"] \\ M \arrow[r, "\phi"'] & E/G \end{tikzcd}$$

The map $\Phi : \xi \to \eta$ is a morphism of principal G-bundles and yields an isomorphism of principal G-bundles:

$$\xi \simeq \phi^*\eta, \quad u \mapsto (\pi(u), \Phi(u)).$$

Since the homotopy class of the map $\phi : M \to E/G$ is unique and independent of the choice of section, one is lead to:

Theorem 34.1 (Classification of Principal Bundles)**.** *Let E be a contractible space with a free and proper action of G. The isomorphism classes of principal G-bundles over M are in one to one correspondence with the homotopy classes of maps $[M : E/G]$.*

The space $B_G = E/G$ is usually called the **classifying space** of principal G-bundles, while the principal G-bundle $E_G \to B_G$, where $E_G = E$, is called the **classifying bundle**. The classifying space and bundle are far from being unique, but one can show that the homotopy type of B_G is unique. Also, if G is non-trivial, it is never possible to choose E_G (and hence B_G) to be a finite dimensional manifold. However, B_G can be chosen to be a *simplicial manifold* and this is enough to talk about differential forms, vector fields, etc., on B_G. We will not go into the differential geometry of B_G in these notes.

Examples 34.2.

1. *The infinite sphere $\mathbb{S}^\infty$ is the direct limit of the finite dimensional spheres*
$$\cdots \subset \mathbb{S}^d \subset \mathbb{S}^{d+1} \subset \mathbb{S}^{d+2} \subset \cdots$$
It can be identified with the unit sphere in $\mathbb{R}^\infty$ furnished with the inner product:
$$\langle (x_n), (y_n) \rangle = \sum_{n=1}^{\infty} x_n y_n$$
(note that in this sum only a finite number terms is non-zero) so that:
$$\mathbb{S}^\infty = \{(x_n) \in \mathbb{R}^\infty : \sum_{n=1}^{\infty} x_n = 1.\}$$
We leave as exercise to check that $\mathbb{S}^\infty$ is a contractible space. On the other hand, the antipodal action of the group $\mathbb{Z}_2$ on $\mathbb{S}^\infty$ is free and proper, so we conclude that:
$$E_{\mathbb{Z}_2} = \mathbb{S}^\infty, \qquad B_{\mathbb{Z}_2} = \mathbb{RP}^\infty.$$
2. *Consider the space of r-frames in $\mathbb{R}^\infty$:*
$$F_r(\mathbb{R}^\infty) = \{\phi : \mathbb{R}^r \to \mathbb{R}^\infty : \phi \text{ is linear injective map}\}.$$
This space is contractible and the right action of $GL(r)$ on $F_r(\mathbb{R}^\infty)$ by pre-composition is proper and free. The quotient is the infinite Grassmannian $G_r(\mathbb{R}^\infty)$. We conclude that:
$$E_{GL(r)} = F_r(\mathbb{R}^\infty), \qquad B_{GL(r)} = G_r(\mathbb{R}^\infty).$$
Since vector bundles of rank r can be thought of as the associated bundles of principal bundles with structure group $GL(r)$ – see Example 33.7.4 – it follows that the set $[M : G_r(\mathbb{R}^\infty)]$ is also in bijection with the isomorphism classes of vector bundles of rank r over M.

We studied before connections on vector bundles. By our general principle, there must exist a notion of connection on a principal $GL(r)$-bundle which induces a vector bundle connection on any associated bundle. Actually, one can define a general notion of connection for any principal G-bundle:

Definition 34.3. *Let $\xi = (\pi, P, M)$ be a principal G-bundle. A* ***principal bundle connection*** *in ξ is a distribution $H \subset TP$ such that:*

(i) H is horizontal: for every $u \in P$,

$$T_u P = H_u \oplus \ker \mathrm{d}_u \pi;$$

(ii) H is G-invariant: for all $g \in G$ and $u \in P$,

$$H_{ug} = (R_g)_* H_u,$$

where $R_g : P \to P$ is the translation by g: $R_g(u) = u \cdot g \equiv ug$.

Let $\xi = (\pi, P, M)$ be a principal G-bundle with a connection H. Given $u \in P$, we call $V_u \equiv \ker \mathrm{d}_u \pi$ – the tangent space to the fiber containing u – the **vertical space** and H_u the **horizontal space**. An arbitrary tangent vector $\mathbf{v} \in T_u P$ has a unique decomposition:

$$\mathbf{v} = h(\mathbf{v}) + v(\mathbf{v}), \qquad \text{where } h(\mathbf{v}) \in H_u, v(\mathbf{v}) \in V_u.$$

Any vector field on the total space $X \in \mathfrak{X}(P)$ also splits into an horizontal vector field $h(X)$ and a vertical vector field $v(X)$.

The next example shows how a vector bundle connection ∇ determines a principal bundle connection H.

Example 34.4.

Let M be a smooth manifold and $\xi_F = (\pi, E, M)$ a vector bundle over M furnished with a connection ∇. Denote by $\xi = (\pi, F(E), M)$ the bundle of frames of ξ_F: it is a principal bundle with structure group $GL(r)$, where $r = \operatorname{rank} \xi_F$. If $u = (\mathbf{v}_1, \dots, \mathbf{v}_r) \in F(E)$ is a frame and $c : I \to M$ is a curve with $c(0) = \pi(u)$, then the vector fields $X_1, \dots, X_r$ along $c(t)$ obtained by parallel transport of $\mathbf{v}_1, \dots, \mathbf{v}_r$, determine a curve $u(t) = (X_1(t), \dots, X_r(t))$ in $F(E)$. We consider all the curves $u(t)$ obtained in this way, and we define the subspace:

$$H_u = \{u'(0) \in T_u F(E) : \textit{ for all curves } u(t)\}.$$

It is not hard to see that the distribution $u \mapsto H_u$ is C^∞ and satisfies conditions (i) and (ii) of the definition of a principal bundle connection. Hence, every connection ∇ in a vector bundle determines a principal bundle connection H in the corresponding bundle of frames.

Let $\xi = (\pi, P, M)$ be a principal G-bundle. The G-action on P induces an infinitesimal Lie algebra action $\psi : \mathfrak{g} \to \mathfrak{X}(P)$. We denote by $X^* = \psi(X)$ the vector field in P determined by the element $X \in \mathfrak{g}$. This vector field is vertical, i.e., it is tangent to the fibers of P. Moreover, for each $u \in P$, the map $X \mapsto X^*_u$ gives a linear isomorphism $\mathfrak{g} \simeq V_u$.

Let us fix now a principal bundle connection H in P. Its associated **connection 1-form** is the $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(P;\mathfrak{g})$ defined by:

$$\omega(\mathbf{v}) = X, \quad \text{if } \mathbf{v} \in T_uP \text{ and } X \in \mathfrak{g} \text{ is such that } X^*_u = v(\mathbf{v}).$$

Note that $\omega(\mathbf{v}) = 0$ iff $\mathbf{v}$ is a vertical vector, so ω determines the distribution. Indeed, the connection 1-form completely characterizes the connection, as stated in the following proposition, whose proof is left as an exercise:

Proposition 34.5. *Let $\xi = (\pi, P, M)$ be a principal G-bundle. Given a principal bundle connection H on P its connection 1-form ω satisfies:*

(i) $\omega(X^*) = X$, for all $X \in \mathfrak{g}$;
(ii) $(R_g)_*\omega = \mathrm{Ad}(g^{-1})\omega$, for all $g \in G$.

Conversely, if $\omega \in \Omega^1(P;\mathfrak{g})$ satisfies (i) and (ii), there exists a unique principal bundle connection H in P whose connection 1-form is ω.

The description of principal bundle connections in terms of connections 1-forms also allows to give a simple proof of existence of connections on any principal bundle using a partition of unity.

In order to define the curvature of a connection H on a principal G-bundle $\xi = (\pi, P, M)$, we introduce the *exterior covariant derivative* to be the differential operator $D : \Omega^k(P;\mathfrak{g}) \to \Omega^{k+1}(P;\mathfrak{g})$ by setting:

$$(D\theta)(X_0, \dots, X_k) = (\mathrm{d}\theta)(h(X_0), \dots, h(X_k)), \quad (X_0, \dots, X_k \in \mathfrak{X}(P)).$$

The **curvature 2-form** of H is the $\mathfrak{g}$-valued 2-form $\Omega \in \Omega^2(P,\mathfrak{g})$ given by:

$$\Omega \equiv D\omega.$$

Given a trivialization $\{(U_\alpha, \phi_\alpha)\}$ of the principal G-bundle $\xi = (\pi, P, M)$, we have local sections $s_\alpha : U_\alpha \to P$, $p \mapsto \phi_\alpha^{-1}(p, e)$, where $e \in G$ is the identity element. The connection 1-form ω determines a family of **local connection 1-forms** $\omega_\alpha \in \Omega^1(U_\alpha;\mathfrak{g})$ by:

$$\omega_\alpha = (s_\alpha)^*\omega.$$

On the other hand, the curvature 2-form Ω determines a family of **local curvature 2-forms** $\Omega_\alpha \in \Omega^2(U_\alpha;\mathfrak{g})$ by:

$$\Omega_\alpha = (s_\alpha)^*\Omega.$$

We leave as an exercise to check how the local forms are related on overlaps.

Example 34.6.

We saw in Example 34.4 that a connection ∇ on a vector bundle $\pi : E \to M$ determines a principal bundle connection H on the principal $GL(r)$-bundle of frames $\pi : F(E) \to M$. The associated connection 1-form and curvature 2-form takes values in the Lie algebra $\mathfrak{gl}(r)$.

A trivialization $\{(U_\alpha, \phi_\alpha)\}$ for the vector bundle $\pi : E \to M$ yields also a trivialization for the bundle of frames $\pi : F(E) \to M$. The matrices of connection 1-forms $\omega_\alpha = [\omega_a^b]$ and curvature 2-forms $\Omega_\alpha = [\Omega_a^b]$ associated with ∇ defined in Section 22, are the same as the local connection 1-form and curvature 2-form of the principal connection H.

Just like in the case of connections on vector bundles, we have:

Theorem 34.7. *Let H be a connection in a principal G-bundle $\xi = (\pi, P, M)$, with connection 1-form ω and curvature 2-form Ω. Then the following hold:*

(i) Structure equation: $\Omega = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega]$.
(ii) Bianchi's identity: $D\Omega = 0$.

Note that in item (i) of the statement, we have used the bracket of $\mathfrak{g}$-valued 1 forms $\eta_1, \eta_2 \in \Omega^1(P, \mathfrak{g})$: it is the $\mathfrak{g}$-valued 2-form given by

$$[\eta_1, \eta_2](X, Y) := [\eta_1(X), \eta_2(Y)] - [\eta_1(Y), \eta_2(X)] \qquad (X, Y \in TP).$$

The notion of *parallel transport* can also be introduced for a connection H in a principal G-bundle $\xi = (\pi, P, M)$. First, if $X \in \mathfrak{X}(M)$ is a vector field on the base M, there exists a unique vector field $\widetilde{X}$ in the total space P such that:

(a) $\widetilde{X}$ is horizontal: $\widetilde{X}_u \in H_u$ for all $u \in P$;
(b) $\widetilde{X}$ is π-related to X.

One calls $\widetilde{X} \in \mathfrak{X}(P)$ the **horizontal lift** of $X \in \mathfrak{X}(M)$. The next results states the most important properties of the horizontal lift, and follows immediately from the definitions:

Proposition 34.8. *Let $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$. Then:*

(i) $\widetilde{X} + \widetilde{Y}$ is the horizontal lift of $X + Y$;
(ii) $(\pi^* f)\widetilde{X}$ is the horizontal lift of fX;
(iii) $h([\widetilde{X}, \widetilde{Y}])$ is the horizontal lift of $[X, Y]$.

Notice that, by property (iii), the vector field

$$[\widetilde{X}, \widetilde{Y}] - \widetilde{[X, Y]},$$

is vertical. This leads to a geometric interpretation of curvature, whose proof we leave as an exercise:

Theorem 34.9. *Let H be a connection in a principal G-bundle $\xi = (\pi, P, M)$, with curvature 2-form $\Omega \in \Omega^2(P; \mathfrak{g})$. For any local section $s : U \to P$ and vector fields $X, Y \in \mathfrak{X}(U)$ we have:*

$$(s^*\Omega)(X, Y)^*_p = \left([\widetilde{X}, \widetilde{Y}] - \widetilde{[X, Y]}\right)_{s(p)}.$$

A **flat connection** is a connection whose curvature vanishes identically: $\Omega \equiv 0$. Since the horizontal lifts $\widetilde{X}$ of vector fields $X \in \mathfrak{X}(M)$ generate the horizontal distribution of the connection, we obtain:

Corollary 34.10. *A connection is flat if and only if the horizontal distribution is integrable.*

In order to define parallel transport we need to define the horizontal lift of curves $c : I \to M$: a curve $u : I \to P$ is called a **horizontal lift** of the curve $c(t)$ if $\pi(u(t)) = c(t)$ and $u(t)$ is an horizontal curve (i.e., is tangent to the horizontal distribution).

Proposition 34.11. *Let H be a connection in a principal G-bundle $\xi = (\pi, P, M)$. If $c : I \to M$ is a curve and $u_0 \in \pi^{-1}(c(0))$ there exists a unique horizontal lift $u : I \to P$ of $c(t)$ with $u(0) = u_0$.*

Proof. Local triviality of the bundle, shows that we can always lift $c(t)$ to a curve $v : I \to P$, such that $v(0) = u_0$ and $\pi(v(t)) = c(t)$. The horizontal lift $u : I \to P$ through u_0, if it exists, takes the form:

$$u(t) = v(t)g(t),$$

for some curve $g : I \to G$ with $g(0) = e$. If ω denotes the connection 1-form of H, differentiating this expression we obtain:

$$\omega(\dot{u}(t)) = \mathrm{Ad}(g(t)^{-1})\omega(\dot{v}(t) + g(t)^{-1}\dot{g}(t),$$

where $t \mapsto g(t)^{-1}\dot{g}(t) \equiv \mathrm{d}_{g(t)}L_{g(t)^{-1}}\dot{g}(t)$ is a curve in the Lie algebra $\mathfrak{g}$. The curve $u(t)$ will be horizontal iff $g(t)$ satisfies the equation:

$$g(t)^{-1}\dot{g}(t) = -\,\mathrm{Ad}(g(t)^{-1})\omega(\dot{v}(t).$$

Hence, the proposition follows from the following lemma, whose proof is left as an exercise:

Lemma 34.12. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. If $t \mapsto X(t)$ is a curve in $\mathfrak{g}$, then there exists a unique curve $g : I \to G$, with $g(0) = e$, satisfying:*

$$g(t)^{-1}\dot{g}(t) = X(t), \quad (t \in [0, 1]).$$

$\square$

Therefore, given a curve $c : I \to M$, we can proceed to define **parallel transport** along c to be the map $\tau_t : P_{c(0)} \to P_{c(t)}$ given by

$$\tau_t(u_0) := u(t),$$

where $u(t)$ is the unique horizontal lift $u : I \to P$ of $c(t)$ such that $u(0) = u_0$. Note that we can also define parallel transport along curves which are only piecewise smooth, by making parallel transport successively along its smooth components.

Parallel transport is an isomorphism between fibers, since we have:

Proposition 34.13. *Parallel transport along a piecewise smooth curve $c : I \to M$ commutes with the G-action:*

$$\tau_t \circ R_g = R_g \circ \tau_t, \quad \forall g \in G.$$

Moreover:

(i) τ_1 *is an isomorphism whose inverse is parallel transport along the curve* $\bar{c}(t) \equiv c(1 - t)$.
(ii) *If* c_1 *and* c_2 *are piecewise smooth curves and* $c_1(1) = c_2(0)$, *then parallel transport along the concatenation* $c_1 \cdot c_2$ *coincides with the composition the parallel transports.*

Proof. The first statement follows by observing that R_g takes horizontal curves to horizontal curves. The rest is obvious. $\square$

The **holonomy group** $\Phi(p_0)$ of a connection H on a principal G-bundle $\xi = (\pi, P, M)$ with base point $p_0 \in M$ consists of the set of all isomorphisms $\tau_1 : P_{p_0} \to P_{p_0}$ obtain by performing parallel transport along piecewise smooth curves $c : I \to M$ with $c(0) = c(1) = p_0$.

Now choose $u_0 \in \pi^{-1}(p_0)$. For each $\tau \in \Phi(p_0)$ there is an element $g \in G$ such that $\tau(u) = u_0 g$. This establishes an injective group homomorphism between $\Phi(p_0)$ and a subgroup $\Phi(u_0) \subset G$. Given two points $u_0, u_0' \in \pi^{-1}(p_0)$ there exists an element $g_0 \in G$ such that $u_0' = u_0 g_0$, and we have:

$$\Phi(u_0') = g_0 \Phi(u_0) g_0^{-1}.$$

Hence, the subgroups $\Phi(u)$, for $u \in \pi^{-1}(p_0)$, are all conjugate. Moreover, one can show that the following fundamental result holds:

Theorem 34.14 (Ambrose-Singer)**.** *Let H be a connection in a principal G-bundle $\xi = (\pi, P, M)$ with curvature 2-form Ω. Given $u \in P$ denote by $P(u) \subset P$ the set of all $u' \in P$ which can be connected to u through an horizontal curve. Then $\Phi(u)$ is a Lie subgroup of G with Lie algebra:*

$$\{\Omega_{u'}(\mathbf{v}, \mathbf{w}) : u' \in P(u), \mathbf{v}, \mathbf{w} \in H_{u'}\} \subset \mathfrak{g}.$$

Let $\xi = (\pi, P, M)$ be a principal G-bundle and let $\rho : G \to GL(r)$ be a representation of G. The associated bundle $\xi_{\mathbb{R}^r} = (\pi, E, M)$ is then a vector bundle, and parallel transport in ξ induces a parallel transport operation in $\xi_{\mathbb{R}^r}$ as we explain next.

If $c : I \to M$ is a piecewise smooth curve, the horizontal lift of $c(t)$ in the associated bundle $\xi_{\mathbb{R}^r}$ is, by definition, a curve $\mathbf{v}(t) \in E$ of the form:

$$\mathbf{v}(t) = [(u(t), \mathbf{v})] \in P \times_G \mathbb{R}^r \equiv E,$$

where $u(t)$ is an horizontal lift of $c(t)$ in P. It is easy to see that, for any $\mathbf{v}_0 \in E_{c(0)}$, there exists a unique horizontal lift $\mathbf{v}(t)$ of $c(t)$ such that $\mathbf{v}(0) = \mathbf{v}_0$. As before, we can now define the parallel transport along $c(t)$, in the associated bundle: $\tau_t : E_{c(0)} \to E_{c(1)}$.

Now let s be a section of the associated bundle $\xi_{\mathbb{R}^r}$. Given $\mathbf{v} \in T_pM$ let $c : I \to M$ be a curve such that $c(0) = p$ and $\dot{c}(0) = \mathbf{v}$. The covariant derivative $\nabla_{\mathbf{v}} s$ of s in the direction $\mathbf{v}$ is defined to be:

$$\nabla_{\mathbf{v}} s \equiv \lim_{t \to 0} \frac{1}{t} \left[\tau_t^{-1}(s(c(t)) - s(p) \right] \in E_p.$$

It is easy to check that this definition is independent of the choice of curve c. Moreover, if $X \in \mathfrak{X}(M)$ is a vector field, we define the covariant derivative of a section s along X, to be the section defined by:

$$(\nabla_X s)(p) \equiv \nabla_{X_p} s.$$

We have:

Proposition 34.15. *The covariant derivative $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ associated with a connection H in ξ is a vector bundle connection on $\xi_{\mathbb{R}^r}$. Moreover, every vector bundle connection ∇ on $\xi_{\mathbb{R}^r}$ arises in this way from a unique principal bundle connection H on ξ.*

This bijective correspondence between connections on a principal bundle and vector bundle connections on the associated vector bundle, suggests that the theory of characteristics classes for vector bundles, that we studied before, can be generalized to principal bundles. Indeed, if H is a connection in the principal G-bundle $\xi = (\pi, P, M)$, one defines the **Chern-Weil homomorphism** similarly to the way it was defined for vector bundles:

$$\mathrm{CW}[\xi] : I^k(G) \to H^{2k}(M),\ P \mapsto [P(\Omega^k)],$$

This homomorphism is independent of the choice of connection.

One can use the Chern-Weil homomorphism to construct characteristics classes. For example, if ξ is a principal bundle with structure group $GL(r, \mathbb{R})$ the **Pontrjagin classes** of ξ are obtained by considering the elementary symmetric polynomials:

$$p_k(\xi) \equiv \left[\sigma_{2k}\Big(\frac{1}{2\pi}\Omega\Big)^{2k}\right] \in H^{4k}(M).$$

Similarly, if ξ is a principal bundle with structure group $GL(r, \mathbb{C})$ the **Chern classes** of ξ are given by:

$$c_k(\xi) \equiv \left[\sigma_k\Big(\frac{i}{2\pi}\Omega\Big)^{k}\right] \in H^{2k}(M).$$

An alternative approach is to use the classifying space B_G. Using the fact that $E_G \to B_G$ is a simplicial principal G-bundle, one can introduce connections on this principal bundle.The curvature 2-form of such a connection allows one to define a **universal Chern-Weil homorphism**:

$$\mathrm{CW}_G : I(G) \to H(B_G).$$

The reason for calling this universal is that for any principal G-bundle $\xi = (\pi, P, M)$ the Chern-Weil homomorphism factors through its classifying map $\psi : M \to B_G$, so one obtains a commutative diagram:

$$\begin{tikzcd} & H^\bullet(B_G) \arrow[d, "\psi^*"] \\ I^\bullet(G) \arrow[ur, "\mathrm{CW}_G"] \arrow[r, "\mathrm{CW}[\xi]"'] & H^\bullet(M) \end{tikzcd}$$

One can show that for any compact Lie group G the universal Chern-Weil homorphism is actually an isomorphism. Hence, one can think of elements in $H^\bullet(B_G)$ as **universal characteristic classes** and any characteristic class is a pullback of such a universal characteristic class. One can also consider elements in the integral cohomology $H^\bullet(B_G, \mathbb{Z})$ as universal characteristic classes giving rise to integral characteristic classes, so one has integral Chern classes, integral Pontrjagin classes, etc. Of course, these are related to the real cohomology classes defined using connections and curvature via the natural homomorphism $H^\bullet(M,\mathbb{Z}) \to H^\bullet(M)$.

Homework.

1. Show that a principal G-bundle is trivial if and only if it admits a section. Moreover, given any principal G-bundle $\xi = (\pi, P, M)$:
   (a) Prove that there is an open cover $\{U_\alpha : \alpha \in A\}$ of M over which ξ admits local sections $s_\alpha : U_\alpha \to P$;
   (b) Given an open cover as in (a), show that any two sections s_α and s_β with overlapping domains are related by:
$$s_\beta(p) = s_\alpha(p) g_{\alpha\beta}(p) \qquad (p \in U_\alpha \cap U_\beta),$$
   for unique functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ forming a cocycle.

2. Let $\xi = (\pi, P, M)$ be a principal G-bundle with connection H, and denote by ω its connection 1-form and by Ω its curvature 2-form. Let $\{U_\alpha : \alpha \in A\}$ be an open cover of M over which ξ admits local sections $s_\alpha : U_\alpha \to P$ and denote by $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ the associated cocycle (see the previous exercise). Show that the local connection 1-forms $\omega_\alpha = s_\alpha^*\omega$ and local curvature 2-forms $\Omega_\alpha = s_\alpha^*\Omega$ satisfy:
$$\omega_\beta = \mathrm{Ad}(g_{\alpha\beta})^{-1}\omega_\alpha + g_{\alpha\beta}^*\omega_{MC}, \qquad \Omega_\beta = \mathrm{Ad}(g_{\alpha\beta})^{-1}\Omega_\alpha,$$
where $\omega_{MC} \in \Omega^1(G, \mathfrak{g})$ is the Maurer-Cartan form of G, defined by:
$$\omega_{MC}(v) = \mathrm{d}_g L_{g^{-1}}(v) \qquad (v \in T_g G).$$

3. Show that a principal G-bundle always admits a connection.

4. If H is a connection in a principal G-bundle $\xi = (\pi, P, M)$, with curvature 2-form $\Omega \in \Omega^2(P; \mathfrak{g})$. For any local section $s : U \to P$, show that:
$$(s^*\Omega)(X,Y)^* = [\widetilde{X}, \widetilde{Y}] - \widetilde{[X,Y]}.$$

5. Let G be a Lie group with Lie algebra $\mathfrak{g}$. If $t \mapsto X(t)$ is a curve in $\mathfrak{g}$, show that there exists a unique curve $g : I \to G$, with $g(0) = e$, satisfying:
$$g(t)^{-1}\dot{g}(t) = X(t), \quad (t \in [0,1]).$$

6. Give a proof of the bijective correspondence between vector bundle connections and principal bundle with connections stated in Proposition 34.15.

7. Give the details of the construction of the Chern-Weil homomorphism for the case of principal bundles.

8. Let G be a (non-trivial) compact Lie group. Show that there exists no finite dimensional manifold which is contractible and admits a free G-action.

9. Prove that $\mathbb{S}^\infty$ and $F_r(\mathbb{R}^\infty)$ are both contractible spaces.

10. Let $G = \mathbb{S}^1$. Show that $E_G = \mathbb{S}^\infty$ and $B_G = \mathbb{CP}^\infty$.

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