CDI-I

1^a Ficha-3^a e 4^a Aulas Práticas

II. Símbolo de Somatório

4.(b) Vamos provar, não usando Indução Matemática, que:

$$\sum_{k=1}^{n} \frac{1}{k\,(k+1)} = \frac{n}{n+1}\ .$$

Ora,

$$\sum_{k=1}^{n} \tfrac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\tfrac{1}{k} - \tfrac{1}{k+1}\right) \underset{\substack{\text{(homogeneidade)}\\ \text{(ver o ex. II.2.(b))}}}{=} -\sum_{k=1}^{n} \left(\tfrac{1}{k+1} - \tfrac{1}{k}\right) =$$

$$\underset{\substack{\left(\text{propriedade telescópica, com } a_k = \frac{1}{k+1}\right)\\ \text{(ver o ex. II.2.(c))}}}{=} -\left(\tfrac{1}{n+1} - 1\right) =$$

$$= 1 - \tfrac{1}{n+1} = \tfrac{n+1-1}{n+1} = \tfrac{n}{n+1}$$

5. Para quaisquer $a, b \in \mathbb{R}$, mostremos que:

$$\forall n \in \mathbb{N} : a^n - b^n = (a-b) \sum_{k=1}^{n} a^{n-k} b^{k-1}\ .$$

Ora, para qualquer $n \in \mathbb{N}$:

$$(a-b) \sum_{k=1}^{n} a^{n-k} b^{k-1} \underset{\substack{\text{(homogeneidade)}\\ \text{(ver o ex. II.2.(b))}}}{=} \sum_{k=1}^{n} (a-b)\, a^{n-k} b^{k-1} =$$

$$= \sum_{k=1}^{n} \left(a^{n+1-k} b^{k-1} - a^{n-k} b^{k}\right) \underset{\substack{\text{(homogeneidade)}\\ \text{(ver o ex. II.2.(b))}}}{=} -\sum_{k=1}^{n} \left(a^{n-k} b^{k} - a^{n+1-k} b^{k-1}\right) =$$

$$= -\sum_{k=1}^{n} \left(a^{n-k} b^{k} - a^{n-(k-1)} b^{k-1}\right) \underset{\substack{\left(\text{propriedade telescópica, com } a_k = a^{n-k} b^{k}\right)\\ \text{(ver o ex. II.2.(c))}}}{=}$$

$$= -\left(a^{n-n} b^{n} - a^{n-0} b^{0}\right) = -\left(a^{0} b^{n} - a^{n} b^{0}\right) = a^n - b^n\ .$$

III. Indução e Somatórios

1 Vamos provar, usando Indução Matemática, que:

$$\forall n \in \mathbb{N} : \underbrace{\sum_{k=1}^{n} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!}}_{P(n)} .$$

.

$$n = 1 :$$

$\displaystyle\sum_{k=1}^{n} \frac{k}{(k+1)!} = \sum_{k=1}^{1} \frac{k}{(k+1)!} = \frac{1}{2!} = \frac{1}{2}$,
$1 - \frac{1}{(n+1)!} = 1 - \frac{1}{2!} = 1 - \frac{1}{2} = \frac{1}{2}$.
Logo, $P(n)$ é satisfeita para $n = 1$.

Hipótese de Indução (H.I.): $\displaystyle\underbrace{\sum_{k=1}^{n} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!}}_{P(n)}$.

Tese de Indução: $\displaystyle\underbrace{\sum_{k=1}^{n+1} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+2)!}}_{P(n+1)}$.

Ora,
$\displaystyle\sum_{k=1}^{n+1} \frac{k}{(k+1)!} = \sum_{k=1}^{n} \frac{k}{(k+1)!} + \frac{n+1}{(n+2)!} \underset{\mathbf{H.I}}{=} 1 - \frac{1}{(n+1)!} + \frac{n+1}{(n+2)!} =$
$= 1 - \frac{n+2-(n+1)}{(n+2)!} = 1 - \frac{1}{(n+2)!}$.

22 Vamos provar, usando Indução Matemática, que:

$$\forall n \in \mathbb{N} : \underbrace{\sum_{k=1}^{n} \frac{(k-2)\, 3^{k-1}}{(k+1)!} = 1 - \frac{3^n}{(n+1)!}}_{P(n)} .$$

.

$$n = 1 :$$

$\displaystyle\sum_{k=1}^{n} \frac{(k-2)3^{k-1}}{(k+1)!} = \sum_{k=1}^{1} \frac{(k-2)3^{k-1}}{(k+1)!} = \frac{(1-2)3^{1-1}}{(1+1)!} = -\frac{1}{2}$,
$1 - \frac{3^n}{(n+1)!} = 1 - \frac{3}{2!} = 1 - \frac{3}{2!} = -\frac{1}{2}$.

Logo, $P(n)$ é satisfeita para $n = 1$.

Hipótese de Indução (H.I.): $\underbrace{\sum_{k=1}^{n} \frac{(k-2)\,3^{k-1}}{(k+1)!} = 1 - \frac{3^n}{(n+1)!}}_{P(n)}$.

Tese de Indução: $\underbrace{\sum_{k=1}^{n+1} \frac{(k-2)\,3^{k-1}}{(k+1)!} = 1 - \frac{3^{n+1}}{(n+2)!}}_{P(n+1)}$.

Ora,
$\sum_{k=1}^{n+1} \frac{(k-2)3^{k-1}}{(k+1)!} = \sum_{k=1}^{n} \frac{(k-2)3^{k-1}}{(k+1)!} + \frac{(n-1)3^n}{(n+2)!} \underset{\textbf{H.I}}{=} 1 - \frac{3^n}{(n+1)!} + \frac{(n-1)3^n}{(n+2)!} =$
$= 1 - \frac{3^n(n+2)-(n-1)3^n}{(n+2)!} = 1 - \frac{3^n(n+2-n+1)}{(n+2)!} = 1 - \frac{3^{n+1}}{(n+2)!}$.

IV. Funções Elementares

9).(b) Sendo $f(x) = \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}$, determinemos D_f (**domínio da função** f):

Ora,
$\cos^2 x = 0 \Longleftrightarrow \cos x = 0 \Longleftrightarrow x = k\pi + \frac{\pi}{2};\ k \in \mathbb{Z}$,
$\sin^2 x = 0 \Longleftrightarrow \sin x = 0 \Longleftrightarrow x = k\pi;\ k \in \mathbb{Z}$.
Logo,
$D_f = \mathbb{R} \setminus \left(\left\{k\pi + \frac{\pi}{2} : k \in \mathbb{Z}\right\} \cup \{k\pi : k \in \mathbb{Z}\}\right) \underset{\text{consulte-se o } \textbf{círculo trigonométrico}}{=}$
$= \left\{\frac{k\pi}{2} : k \in \mathbb{Z}\right\}$.

(d) Sendo $f(x) = \log(\log x)$, determinemos D_f (**domínio da função** f):

Ora, levando em conta que o domínio da **função logarítmica** é $]0, +\infty[$, deverá ter-se:
$\log x > 0 \Longleftrightarrow \log x > \log 1 \underset{\text{a função } g(x)=\log x \text{ é estritamente crescente}}{\Longleftrightarrow} x > 1 \Longleftrightarrow$
$\Longleftrightarrow x \in]1, +\infty[$.
Logo,
$D_f =]1, +\infty[$.

V. Limites Elementares

1).(b) Calculemos $\lim\limits_{x \longrightarrow 1} \frac{2x^2-3x+1}{x-1}$.
A substituição directa de x por 1, conduz a uma indeterminação do tipo $\frac{0}{0}$.
Ora,
$2x^2 - 3x + 1 = 0 \Longleftrightarrow x = \frac{3-\sqrt{9-8}}{4} \vee x = \frac{3+\sqrt{9-8}}{4} \Longleftrightarrow x = \frac{1}{2} \vee x = 1$.

Sendo assim (ver o **Apêndice V)**,
$2x^2 - 3x + 1 = 0 \Longleftrightarrow 2\left(x - \frac{1}{2}\right)(x-1)$.
Logo,
$\lim\limits_{x \longrightarrow 1} \frac{2x^2-3x+1}{x-1} = \lim\limits_{x \longrightarrow 1} \frac{2\left(x-\frac{1}{2}\right)(x-1)}{x-1} = \lim\limits_{x \longrightarrow 1} 2\left(x - \frac{1}{2}\right) = 1$.

1).(g) Calculemos $\lim\limits_{x \longrightarrow 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$.
A substituição directa de x por 0, conduz a uma indeterminação do tipo $\frac{0}{0}$.
Ora,
$\lim\limits_{x \longrightarrow 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \lim\limits_{x \longrightarrow 0} \frac{\left(\sqrt{1+x}-\sqrt{1-x}\right)\left(\sqrt{1+x}+\sqrt{1-x}\right)}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)} = \lim\limits_{x \longrightarrow 0} \frac{\left(\sqrt{1+x}\right)^2-\left(\sqrt{1-x}\right)^2}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)} =$
$= \lim\limits_{x \longrightarrow 0} \frac{1+x-(1-x)}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)} = \lim\limits_{x \longrightarrow 0} \frac{1+x-1+x}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)} = \lim\limits_{x \longrightarrow 0} \frac{2x}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)} =$
$= \lim\limits_{x \longrightarrow 0} \frac{2}{\sqrt{1+x}+\sqrt{1-x}} = \frac{2}{2} = 1$.

2) Os exercícios seguintes usam o resultado fundamental:

$$\lim_{x \longrightarrow 0} \frac{\sin x}{x} = 1 \ .$$

(b) Mostremos que:

$$\lim_{x \longrightarrow 0} \frac{\sin(5x)}{\sin x} = 5 \ .$$

Ora,
$\lim\limits_{x \longrightarrow 0} \frac{\sin(5x)}{\sin x} = 5 \lim\limits_{x \longrightarrow 0} \frac{\frac{\sin(5x)}{5x}}{\frac{\sin x}{x}} = 5\frac{1}{1} = 5$.

3) (a) Calculemos:

$$\lim_{t \longrightarrow 0} \frac{\sin(\tan t)}{\sin t}$$

Ora,
$\lim\limits_{t \longrightarrow 0} \frac{\sin(\tan t)}{\sin t} = \lim\limits_{t \longrightarrow 0} \frac{(\tan t)\frac{\sin(\tan t)}{\tan t}}{t\frac{\sin t}{t}} \underset{\substack{\lim\limits_{t \longrightarrow 0} \tan t = 0, \\ \lim\limits_{x \longrightarrow 0} \frac{\sin x}{x} = 1}}{=} \lim\limits_{t \longrightarrow 0} \frac{(\tan t)}{t} =$
$= \lim\limits_{t \longrightarrow 0} \left(\frac{1}{\cos t} \frac{\sin t}{t}\right) = \left(\lim\limits_{t \longrightarrow 0} \frac{1}{\cos t}\right)\left(\lim\limits_{t \longrightarrow 0} \frac{\sin t}{t}\right) = \frac{1}{\cos 0}.1 = 1$.

5 (a) $\lim\limits_{x \longrightarrow 0^+} e^{\frac{1}{x}} \underset{\lim\limits_{x \longrightarrow 0^+} \frac{1}{x} = +\infty}{=} \lim\limits_{y \longrightarrow +\infty} e^y = +\infty$.

$\lim\limits_{x \longrightarrow 0^-} e^{\frac{1}{x}} \underset{\lim\limits_{x \longrightarrow 0^-} \frac{1}{x} = -\infty}{=} \lim\limits_{y \longrightarrow -\infty} e^y = 0$.

$\lim\limits_{x \longrightarrow +\infty} e^{\frac{1}{x}} \underset{\lim\limits_{x \longrightarrow +\infty} \frac{1}{x} = 0}{=} \lim\limits_{y \longrightarrow 0^+} e^y = e^0 = 1$.

$$\lim_{x\longrightarrow -\infty} e^{\frac{1}{x}} \underset{\lim\limits_{x\longrightarrow -\infty}\frac{1}{x}=0}{=} \lim_{y\longrightarrow 0^-} e^y = e^0 = 1 \ .$$

6 (c) $$\lim_{x\longrightarrow 0}\cos\left(\frac{2x+\pi}{x^2+1}\right) \underset{\substack{\lim\limits_{x\longrightarrow 0}(2x+\pi)=\pi,\\ \lim\limits_{x\longrightarrow 0}\left(x^2+1\right)=1.}}{=} \lim_{y\longrightarrow \pi}\cos y = \cos\pi = -1 \ .$$

$$\lim_{x\longrightarrow +\infty}\cos\left(\frac{2x+\pi}{x^2+1}\right) \underset{\text{para evitar uma indeterminação do tipo } \frac{\infty}{\infty}}{=} \lim_{x\longrightarrow +\infty}\cos\left(\frac{\frac{2}{x}+\frac{\pi}{x^2}}{1+\frac{1}{x^2}}\right) \underset{\substack{\lim\limits_{x\longrightarrow +\infty}\left(\frac{2}{x}+\frac{\pi}{x^2}\right)=0,\\ \lim\limits_{x\longrightarrow +\infty}\left(1+\frac{1}{x^2}\right)=1.}}{=}$$

$$= \lim_{y\longrightarrow 0}\cos y = \cos 0 = 1 \ .$$

$$\lim_{x\longrightarrow -\infty}\cos\left(\frac{2x+\pi}{x^2+1}\right) \underset{\text{para evitar uma indeterminação do tipo } \frac{\infty}{\infty}}{=} \lim_{x\longrightarrow -\infty}\cos\left(\frac{\frac{2}{x}+\frac{\pi}{x^2}}{1+\frac{1}{x^2}}\right) \underset{\substack{\lim\limits_{x\longrightarrow -\infty}\left(\frac{2}{x}+\frac{\pi}{x^2}\right)=0,\\ \lim\limits_{x\longrightarrow -\infty}\left(1+\frac{1}{x^2}\right)=1.}}{=}$$

$$= \lim_{y\longrightarrow 0}\cos y = \cos 0 = 1 \ .$$

7 (d) $$\lim_{x\longrightarrow 0^+}\log\left(\frac{x}{1+\sqrt{x}}\right) \underset{\lim\limits_{x\longrightarrow 0^+}\left(\frac{x}{1+\sqrt{x}}\right)=0,}{=} \lim_{y\longrightarrow 0^+}\log y = -\infty \ .$$

$$\lim_{x\longrightarrow +\infty}\log\left(\frac{x}{1+\sqrt{x}}\right) \underset{\text{para evitar uma indeterminação do tipo } \frac{\infty}{\infty}}{=} \lim_{x\longrightarrow +\infty}\log\left(\frac{\frac{x}{\sqrt{x}}}{\frac{1}{\sqrt{x}}+1}\right) =$$

$$= \lim_{x\longrightarrow +\infty}\log\left(\frac{\sqrt{x}}{\frac{1}{\sqrt{x}}+1}\right) \underset{\substack{\lim\limits_{x\longrightarrow +\infty}\sqrt{x}=+\infty,\\ \lim\limits_{x\longrightarrow +\infty}\left(\frac{1}{\sqrt{x}}+1\right)==1.}}{=} \lim_{y\longrightarrow +\infty}\log y = +\infty \ .$$

(j) $$\lim_{x\longrightarrow 0^+}\log\left(\frac{x}{\sqrt{1+x^2}}\right) \underset{\lim\limits_{x\longrightarrow 0^+}\left(\frac{x}{\sqrt{1+x^2}}\right)=0,}{=} \lim_{y\longrightarrow 0^+}\log y = -\infty \ .$$

$$\lim_{x\longrightarrow +\infty}\log\left(\frac{x}{\sqrt{1+x^2}}\right) \underset{\text{para evitar uma indeterminação do tipo } \frac{\infty}{\infty}}{=} \lim_{x\longrightarrow +\infty}\log\left(\sqrt{\frac{x^2}{1+x^2}}\right) =$$

$$\underset{\text{para evitar uma indeterminação do tipo } \frac{\infty}{\infty}}{=} \lim_{x\longrightarrow +\infty}\log\left(\sqrt{\frac{1}{\frac{1}{x^2}+1}}\right) =$$

$$\underset{\lim\limits_{x\longrightarrow +\infty}\left(\sqrt{\frac{1}{\frac{1}{x^2}+1}}\right)=1}{=} \lim_{y\longrightarrow 1}\log y = \log 1 = 0$$

8) (j) $\lim\limits_{x\longrightarrow 0^+} e^{\frac{x}{\sqrt{1+x^2}}} \underset{\lim\limits_{x\longrightarrow 0^+}\left(\frac{x}{\sqrt{1+x^2}}\right)=0}{=} \lim\limits_{y\longrightarrow 0^+} e^y = e^0 = 1$.

$\lim\limits_{x\longrightarrow +\infty} e^{\frac{x}{\sqrt{1+x^2}}} \underset{\text{para evitar uma indeterminação do tipo } \frac{\infty}{\infty}}{=} \lim\limits_{x\longrightarrow +\infty} e^{\sqrt{\frac{x^2}{1+x^2}}} \underset{\text{para evitar uma indeterminação do tipo } \frac{\infty}{\infty}}{=}$

$= \lim\limits_{x\longrightarrow +\infty} e^{\sqrt{\frac{1}{\frac{1}{x^2}+1}}} \underset{\lim\limits_{x\longrightarrow +\infty}\left(\sqrt{\frac{1}{\frac{1}{x^2}+1}}\right)=1}{=} \lim\limits_{y\longrightarrow 1} e^y = e^1 = e$.