Justify all your answers.

1. Let $C_1$ and $C_2$ be $q$-ary linear codes with parameters $[n, k_1, d_1]$ and $[n, k_2, d_2]$, respectively.
   (a) Show that $C_1 \ast C_2$ (the Plotkin construction) is a linear code.
   (b) If $G_1$ and $G_2$ are generator matrices for $C_1$ and $C_2$, respectively, write a generator matrix for $C_1 \ast C_2$ in terms of $G_1$ and $G_2$.
   (c) If $H_1$ and $H_2$ are parity-check matrices for $C_1$ and $C_2$, respectively, write a parity-check matrix for $C_1 \ast C_2$ in terms of $H_1$ and $H_2$.

2. Let $C$ be the binary code with the following parity-check matrix
   \[ H = \begin{bmatrix}
   0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
   0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
   1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
   1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
   \end{bmatrix}. \]
   (a) Determine the $[n, k, d]$ parameters of the code $C$.
   (b) Show that $C$ can be used to correct all errors with weight 1 and all errors with weight 2 with a nonzero $n$-th component. Can this code correct simultaneously all these errors plus a few more with weight 2?
   (c) Describe a decoding algorithm that corrects all errors mentioned in part (b), and decode the received vector $y = 10111011$.

3. Let $C$ be a $q$-ary MDS code with parameters $[n, k]$, where $k < n$.
   (a) Show that there is a $q$-ary MDS code with length $n$ and dimension $n - k$.
   (b) Show that there is a $q$-ary MDS code with length $n - 1$ and dimension $k$.

4. Consider the vector space $V = \mathbb{F}_q^3$.
   (a) Show that $V$ contains $\frac{q^3-1}{q-1} = q^2 + q + 1$ 1-dimensional vector subspaces.
   (b) Show that $V$ contains $\frac{q^2-1}{q-1} = q + 1$ 2-dimensional vector subspaces.
   (c) Let $P$ be the set of 1-dimensional vector subspaces and let $B$ be the set of 2-dimensional vector subspaces. Show that $P$ (as the set of points) and $B$ (as the set of blocks), with the relation $P \in P$ belongs to $B \in B$ if $P$ is a subspace of $B$, define a Steiner system $S(2, q+1, q^2 + q + 1)$. Since the number of points and the number of blocks are the same, this Steiner system is called a 2-dimensional projective geometry (or a projective plane) of order $q$, and its denoted by $PG(2, q)$ or $PG_2(q)$.

5. For any code $C$, we define $A_i = \# \{ x \in C : w(x) = i \}$. Determine the numbers $A_i$ for the extended Golay code $G_{24}$. [Suggestion: Show that $\bar{1} \in G_{24}$.]