A. Prove the universal property of the ring $S^{-1}A$: Let $A$ be a commutative ring and $S \subset A$ be a multiplicative subset. Given a commutative ring $B$ and a ring homomorphism $f : A \to B$ such that $f(S) \subset B^\times$, then there is a unique homomorphism $\tilde{f} : S^{-1}A \to B$ such that $\tilde{f} \circ \varphi_S = f$, i.e., the following diagram commutes

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{\varphi_S} & & \downarrow{\exists!\tilde{f}} \\
S^{-1}A & \end{array}
\]

where $\varphi_S : A \to S^{-1}A$, $\varphi_S(a) = \frac{a}{1}$.

B. Let $A$ be a commutative ring and let $S$ and $T$ be multiplicative subsets of $A$ such that $S \subset T$.

(a) Show that $\psi : S^{-1}A \to T^{-1}A$, given by $\psi\left(\frac{a}{s}\right) = \frac{a}{t}$, defines a ring homomorphism.

(b) If $T \setminus S \subset A^\times$, show that $\psi$ is an isomorphism.