Secção de Matemática Aplicada e Análise Numérica
Departamento de Matemática/Instituto Superior Técnico

Matemática Computacional (Mestrado em Engenharia Física Tecnológica)

Formulário – 2º Teste

i. Resolução numérica de sistemas não-lineares

Método do ponto fixo ($x = G(x)$)

$$x^{(m+1)} = G(x^{(m)}), \quad m = 0, 1, \ldots$$

$$\|z - x^{(m+1)}\| \leq L\,\|z - x^{(m)}\|, \qquad \|z - x^{(m+1)}\| \leq \frac{L}{1-L}\,\|x^{(m+1)} - x^{(m)}\|, \qquad m = 0, 1, \ldots$$

Método de Newton ($F(x) = 0$)

$$x^{(m+1)} = x^{(m)} + \Delta x^{(m)}, \qquad [J_F(x^{(m)})]\,\Delta x^{(m)} = -F(x^{(m)}), \quad m = 0, 1, \ldots$$

ii. Interpolação polinomial

Fórmula interpoladora de Lagrange:

$$p_n(x) = \sum_{i=0}^{n} f_i\, l_i(x), \qquad l_i(x) = \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}, \quad i = 0, 1, \ldots, n$$

Fórmula interpoladora de Newton:

$$p_n(x) = f[x_0] + \sum_{i=1}^{n} f[x_0, \ldots, x_i]\,(x - x_0)\cdots(x - x_{i-1})$$

Fórmula do erro:

$$e_n(x) = f(x) - p_n(x) = f[x_0, \ldots, x_n, x]\prod_{i=0}^{n}(x - x_i) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{i=0}^{n}(x - x_i), \qquad \xi \in\,]\,x_0; \ldots; x_n; x\,[$$

iii. Melhor aproximação mínimos quadrados discreta

$$\sum_{k=0}^{N} a_k\,\langle \phi_k, \phi_j\rangle \;=\; \langle f, \phi_j\rangle\,, \quad j = 0, \ldots, N\,, \qquad g(x) = \sum_{k=0}^{N} a_k\,\phi_k(x)\,, \qquad \langle \phi, \psi\rangle = \sum_{k=0}^{M} \phi(x_k)\,\psi(x_k)$$

iv. Polinómios ortogonais com respeito ao produto interno $(f,g) = \int_a^b f(x)g(x)\,dx$

$$[a,b] \ni t \longrightarrow x \in [-1,1] : \quad t = \frac{a+b+(b-a)\,x}{2}$$

Polinómios de Legendre: $[a,b]=[-1,1], \quad (f,g) = \int_{-1}^1 f(x)g(x)\,dx$

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}\left(3\,x^2 - 1\right), \quad P_3(x) = \frac{1}{2}\left(5\,x^3 - 3\,x\right)$$

$$P_4(x) = \frac{1}{8}\left(35\,x^4 - 30\,x^2 + 3\right), \quad P_5(x) = \frac{1}{8}\left(63\,x^5 - 70\,x^3 + 15\,x\right)$$

$$(P_n, P_n) = \frac{2}{2n+1}$$

v. Melhor aproximação mínimos quadrados – caso contínuo

$$\Phi^*(x) = \sum_{k=0}^{n} b_k^*\,\varphi_k(x), \qquad \sum_{k=0}^{n} b_k^*\,(\varphi_k, \varphi_j) = (f, \varphi_j), \quad j = 0, \ldots, n\,.$$

vi. Fórmulas de quadratura $Q_n(f) = \sum_{j=0}^n A_j f(x_j) \approx I(f) = \int_a^b f(x)\,dx$

Regra do ponto médio

$$I_0(f) = (b-a)\,f(\frac{a+b}{2})\,, \qquad E_0(f) = \frac{(b-a)^3}{24} f''(\xi), \qquad \xi\, \in]a,b[$$

$$I_0^n(f) = h\sum_{i=1}^{n} f_i, \qquad E_0^n(f) = \frac{h^2(b-a)}{24} f''(\xi), \qquad \xi\, \in]a,b[$$

Regra dos trapézios

$$I_1(f) = (\frac{b-a}{2})[f(a)+f(b)], \qquad E_1(f) = -\frac{(b-a)^3}{12} f''(\xi), \qquad \xi\, \in]a,b[$$

$$I_1^n(f) = h\left[(f_0+f_n)/2 + \sum_{i=1}^{n-1} f_i\right], \qquad E_1^n(f) = -\frac{h^2(b-a)}{12} f''(\xi), \qquad \xi \in (a,b)$$

Regra de Simpson

$$I_2(f) = \frac{h}{3}\left[f(a) + 4f(\frac{a+b}{2}) + f(b)\right], \qquad E_2(f) = -\frac{h^5}{90} f^{(4)}(\xi), \qquad \xi \in (a,b)$$

$$I_2^n(f) = \frac{h}{3}\left[(f_0+f_n) + 4\sum_{i=1}^{n/2} f_{2i-1} + 2\sum_{i=1}^{n/2-1} f_{2i}\right], \qquad E_2^n(f) = -\frac{h^4(b-a)}{180} f^{(4)}(\xi), \qquad \xi \in (a,b)$$

Fórmula de erro para as quadraturas de Newton-Côtes:

$$E_n(f) := I(f) - I_n(f) = \begin{cases} \dfrac{E_n(x^{n+1})}{(n+1)!}\, f^{(n+1)}(\eta)\,, & n \text{ impar} \\[2ex] \dfrac{E_n(x^{n+2})}{(n+2)!}\, f^{(n+2)}(\eta)\,, & n \text{ par} \end{cases} \qquad \eta \in\ (a,b)\,.$$

Quadraturas de Gauss-Legendre:

$$I(f) := \int_{-1}^{1} f(x)\,dx \approx Q_n(f) := \sum_{j=0}^{n} A_j f(x_j)\,,$$

$$P_{n+1}(x_j) = 0\,, \quad j = 0, 1, \ldots, n\,.$$

vii. Resolução numérica de EDO's – problemas de valor inicial

$$\begin{cases} y'(t) = f(t, y(t)), & t > t_0 \\ y(t_0) = y_0 \end{cases}$$

Método de Euler (explícito)

$$y_{n+1} = y_n + hf(t_n, y_n)\,, \quad n \geq 0\,, \qquad T_{n+1}(h) = O(h^2)$$

Método de Crank-Nicolson

$$y_{n+1} = y_n + \frac{h}{2}\Big[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\Big]\,, \quad n \geq 0\,, \qquad T_{n+1}(h) = O(h^3)$$

Métodos de Runge-Kutta explícitos de s etapas

$$\begin{cases} y_{n+1} = y_n + h \displaystyle\sum_{i=1}^{s} b_i k_i\,, \\ k_1 = f(t_n, y_n)\,, \\ k_i = f\Big(t_n + c_i h,\ y_n + h \displaystyle\sum_{j=1}^{i-1} a_{ij} k_j\Big)\,, \quad i = 2, \ldots, s\,, \\ c_i = \displaystyle\sum_{j=1}^{i-1} a_{ij}\,, \quad i = 2, \ldots, s\,. \end{cases}$$

Tabela de Butcher:

$$\begin{array}{c|cccccc} 0 & 0 & 0 & \ldots & \ldots & 0 \\ c_2 & a_{21} & 0 & \ldots & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ c_s & a_{s1} & a_{s2} & \ldots & a_{s,s-1} & 0 \\ \hline & b_1 & b_2 & \ldots & b_{s-1} & b_s \end{array} \qquad \begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

Consistência e ordem (≤ 3)

$$\sum_{i=1}^{s} b_i = 1\,, \qquad \sum_{i=2}^{s} b_i c_i = \frac{1}{2}\,, \qquad \sum_{i=2}^{s} b_i c_i^2 = \frac{1}{3}\,, \qquad \sum_{i=3}^{s} \sum_{j=2}^{i-1} b_i a_{ij} c_j = \frac{1}{6}$$

Métodos multipasso lineares a p passos

$$\sum_{j=0}^{p} a_j y_{n+j} = h \sum_{j=0}^{p} b_j f(t_{n+j}, y_{n+j}) \,, \qquad n \geq 0 \,, \quad a_p = 1 \,, \quad a_0^2 + b_0^2 \neq 0$$

Consistência e ordem q

$$\sum_{j=0}^{p} a_j = 0 \,, \qquad\qquad \forall\ k = 1, \ldots, q : \quad \sum_{j=0}^{p} j^k a_j = k \sum_{j=0}^{p} j^{k-1} b_j$$

Polinómio característico

$$\rho(r) = \sum_{j=0}^{p} a_j r^j$$

Métodos de Adams

$$y_{n+p} - y_{n+p-1} = h \sum_{j=0}^{p} b_j f(t_{n+j}, y_{n+j}) \,, \qquad n \geq 0$$