Escola de Verão de Matemática, Estatística e Computação

# A Matemática dos Labirintos

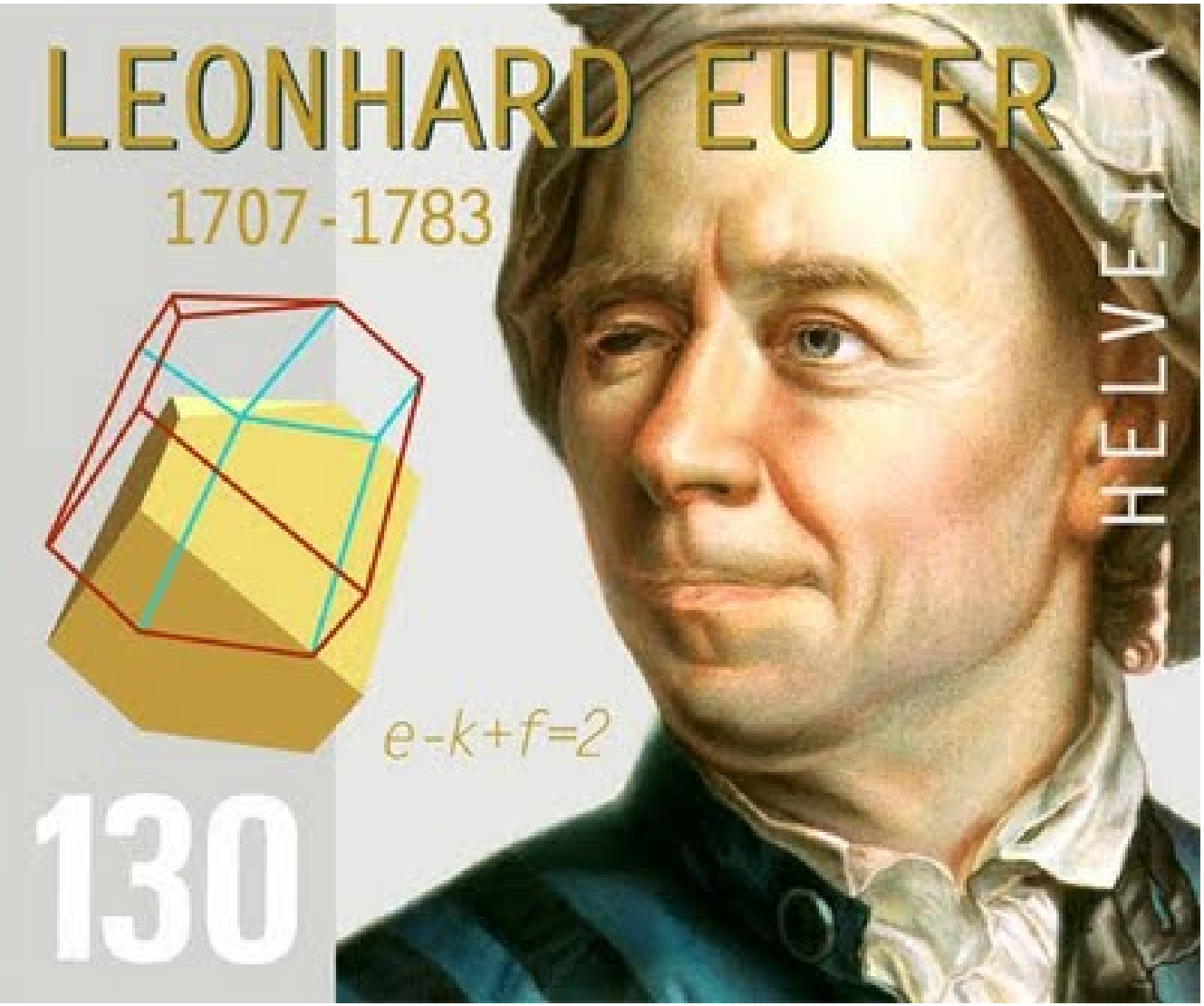
**José Félix Costa**[1,2]

[1] Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa
fgc@math.tecnico.ulisboa.pt

14 – 16 de julho

# 1. As Pontes de Königsberg

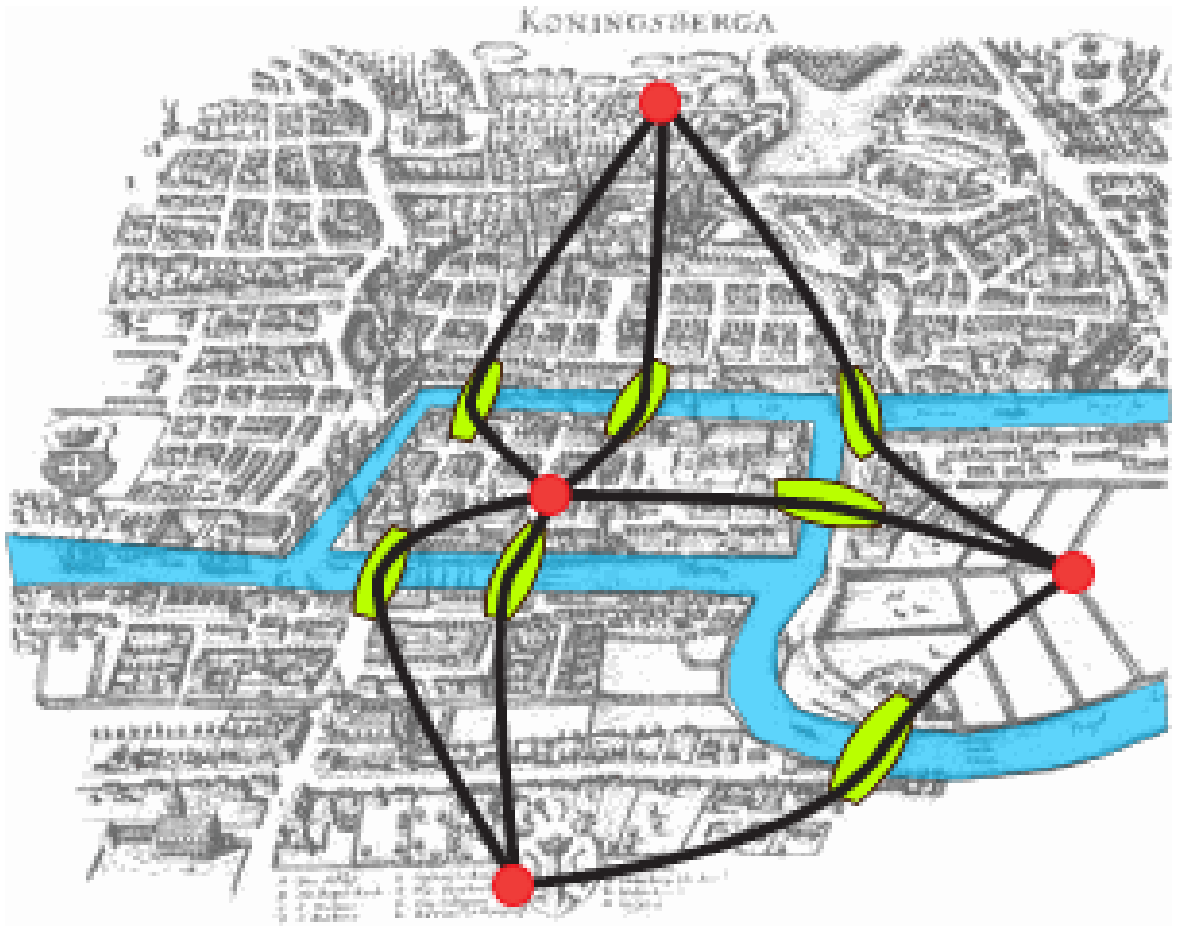
# Leonhard Euler



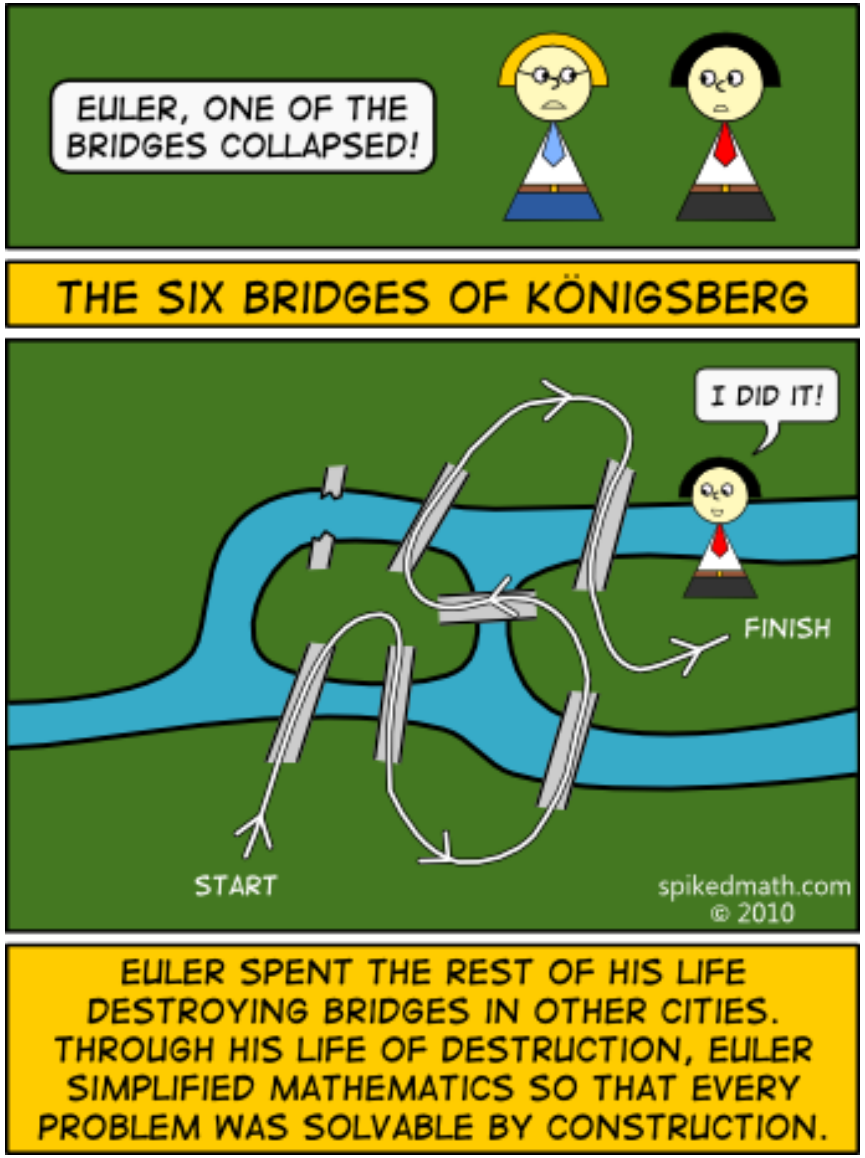
# Leonhard Euler

# Leonhard Euler

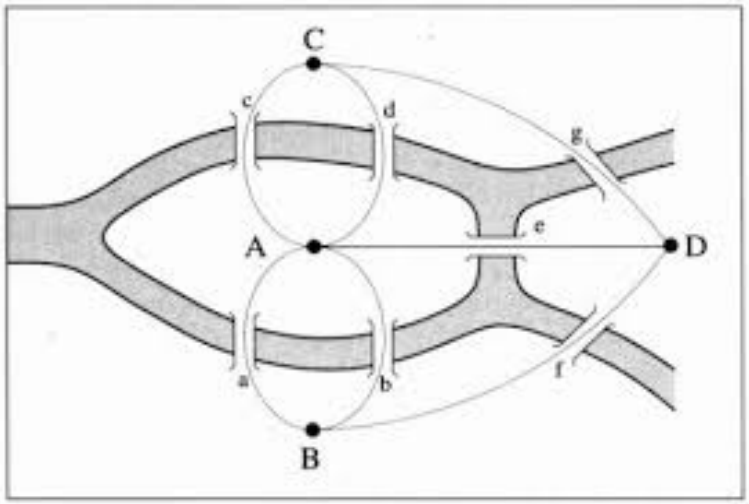
# Problema das pontes de Königsberg



# Problema das pontes de Königsberg



# Problema das pontes de Königsberg



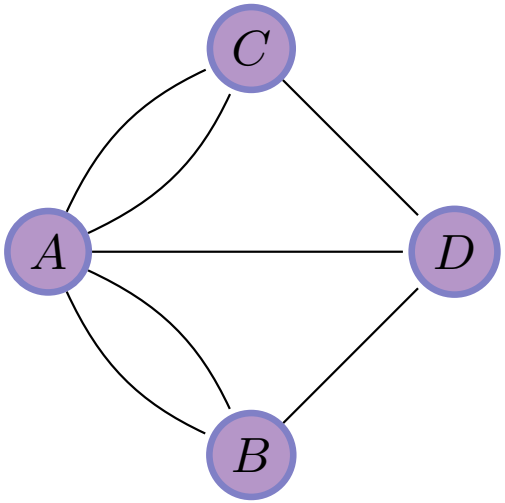




Figura: Problema das pontes de Königsberg. O problema concreto (à esquerda) é resolvido depois de interpretado por grafo modelo (à direita).

# Poema de William Thomas Tutte (1917 – 2002)

## Poema de homenagem a Euler

*Some citizens of Koenigsberg*
*Were walking on the strand*
*Beside the river Pregel*
*With its seven bridges spanned.*

*"O, Euler come and walk with us"*
*Those burghers did beseech*
*"We'll walk the seven bridges o'er*
*And pass but once by each."*

*"It can't be done" then Euler cried*
*"Here comes the Q. E. D.*
*Your islands are but vertices,*
*And all of odd degree."*

# Poema de William Thomas Tutte (1917 – 2002)

## Poema de homenagem a Euler

*Some citizens of Koenigsberg*
*Were walking on the strand*
*Beside the river Pregel*
*With its seven bridges spanned.*

*"O, Euler come and walk with us"*
*Those burghers did beseech*
*"We'll walk the seven bridges o'er*
*And pass but once by each."*

*"It can't be done" then Euler cried*
*"Here comes the Q. E. D.*
*Your islands are but vertices,*
*And all of odd degree."*

# Poema de William Thomas Tutte (1917 – 2002)

## Poema de homenagem a Euler

*Some citizens of Koenigsberg*
*Were walking on the strand*
*Beside the river Pregel*
*With its seven bridges spanned.*

*"O, Euler come and walk with us"*
*Those burghers did beseech*
*"We'll walk the seven bridges o'er*
*And pass but once by each."*

*"It can't be done" then Euler cried*
*"Here comes the Q. E. D.*
*Your islands are but vertices,*
*And all of odd degree."*

# Poema de William Thomas Tutte (1917 – 2002)

## Poema de homenagem a Euler

*Some citizens of Koenigsberg*
*Were walking on the strand*
*Beside the river Pregel*
*With its seven bridges spanned.*

*"O, Euler come and walk with us"*
*Those burghers did beseech*
*"We'll walk the seven bridges o'er*
*And pass but once by each."*

*"It can't be done" then Euler cried*
*"Here comes the Q. E. D.*
*Your islands are but vertices,*
*And all of odd degree."*

# Os teoremas de Euler

**Leonhard Euler, letter of April 1736**

"... this type of solution bears little relationship to mathematics, and I do not understand why you expect a mathematician to produce it, rather than anyone else, for the solution is based on reason alone, and its discovery does not depend on any mathematical principle."

**Leonhard Euler, letter of March 1736**

"This question is so banal, but seemed to me worthy of attention in that neither geometry, nor algebra, nor even the art of counting was sufficient to solve it."

# Os teoremas de Euler

**Leonhard Euler, letter of April 1736**

"... this type of solution bears little relationship to mathematics, and I do not understand why you expect a mathematician to produce it, rather than anyone else, for the solution is based on reason alone, and its discovery does not depend on any mathematical principle."

**Leonhard Euler, letter of March 1736**

"This question is so banal, but seemed to me worthy of attention in that neither geometry, nor algebra, nor even the art of counting was sufficient to solve it."

# Os teoremas de Euler

### Leonhard Euler, letter of April 1736

"... this type of solution bears little relationship to mathematics, and I do not understand why you expect a mathematician to produce it, rather than anyone else, for the solution is based on reason alone, and its discovery does not depend on any mathematical principle."

### Leonhard Euler, letter of March 1736

"This question is so banal, but seemed to me worthy of attention in that neither geometry, nor algebra, nor even the art of counting was sufficient to solve it."

# Os Teoremas de Euler

**Teorema (Teorema de Euler-Hierholzer)**

*Um multigrafo $\mathcal{G}$ é euleriano se e só se $\mathcal{G}$ é conexo e todo o vértice de $\mathcal{G}$ é par.*

**Teorema (Teorema de Euler-Hierholzer)**

*Um multigrafo $\mathcal{G}$ é atravessável se e só se $\mathcal{G}$ é conexo e tem exatamente dois vértices ímpares. Nomeadamente, todo o atalho euleriano de $\mathcal{G}$ começa num dos vértices ímpares e termina no outro dos vértices ímpares.*

**Teorema (Teorema de Euler-Hierholzer)**

*Um multigrafo $\mathcal{G}$ com $2n$ vértices ímpares pode ser descrito por $n$ atalhos disjuntos. (Note-se que o número de vértices ímpares é necessariamente par.)*

# Os Teoremas de Euler

**Teorema (Teorema de Euler-Hierholzer)**

*Um multigrafo $\mathcal{G}$ é euleriano se e só se $\mathcal{G}$ é conexo e todo o vértice de $\mathcal{G}$ é par.*

**Teorema (Teorema de Euler-Hierholzer)**

*Um multigrafo $\mathcal{G}$ é atravessável se e só se $\mathcal{G}$ é conexo e tem exatamente dois vértices ímpares. Nomeadamente, todo o atalho euleriano de $\mathcal{G}$ começa num dos vértices ímpares e termina no outro dos vértices ímpares.*

**Teorema (Teorema de Euler-Hierholzer)**

*Um multigrafo $\mathcal{G}$ com $2n$ vértices ímpares pode ser descrito por $n$ atalhos disjuntos. (Note-se que o número de vértices ímpares é necessariamente par.)*

# Os Teoremas de Euler

**Teorema (Teorema de Euler-Hierholzer)**

*Um multigrafo $\mathcal{G}$ é euleriano se e só se $\mathcal{G}$ é conexo e todo o vértice de $\mathcal{G}$ é par.*

**Teorema (Teorema de Euler-Hierholzer)**

*Um multigrafo $\mathcal{G}$ é atravessável se e só se $\mathcal{G}$ é conexo e tem exatamente dois vértices ímpares. Nomeadamente, todo o atalho euleriano de $\mathcal{G}$ começa num dos vértices ímpares e termina no outro dos vértices ímpares.*

**Teorema (Teorema de Euler-Hierholzer)**

*Um multigrafo $\mathcal{G}$ com $2n$ vértices ímpares pode ser descrito por $n$ atalhos disjuntos. (Note-se que o número de vértices ímpares é necessariamente par.)*

# Os Teoremas de Euler

**Teorema (Teorema de Euler-Hierholzer)**

*Um multigrafo $\mathcal{G}$ é euleriano se e só se $\mathcal{G}$ é conexo e todo o vértice de $\mathcal{G}$ é par.*

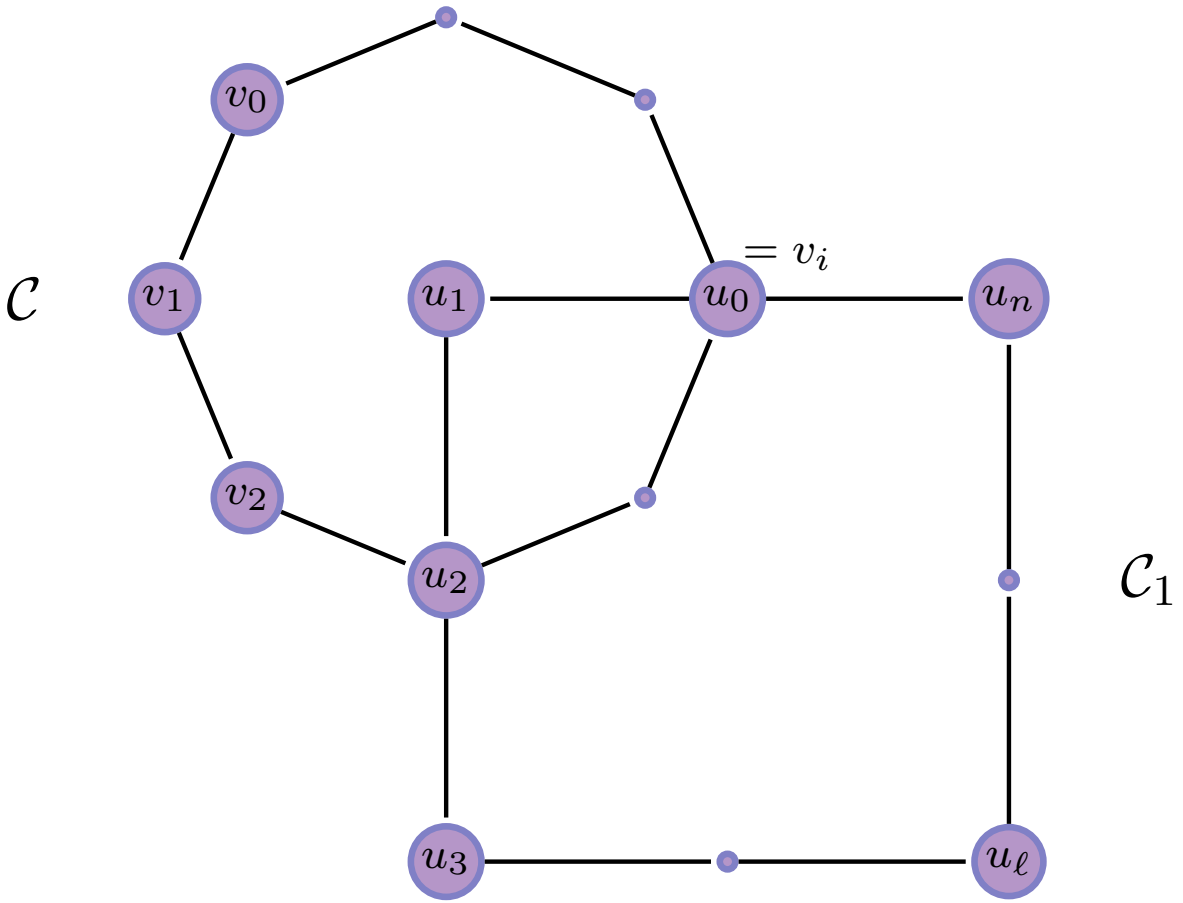
**Teorema (Teorema de Euler-Hierholzer)**

*Um multigrafo $\mathcal{G}$ é atravessável se e só se $\mathcal{G}$ é conexo e tem exatamente dois vértices ímpares. Nomeadamente, todo o atalho euleriano de $\mathcal{G}$ começa num dos vértices ímpares e termina no outro dos vértices ímpares.*

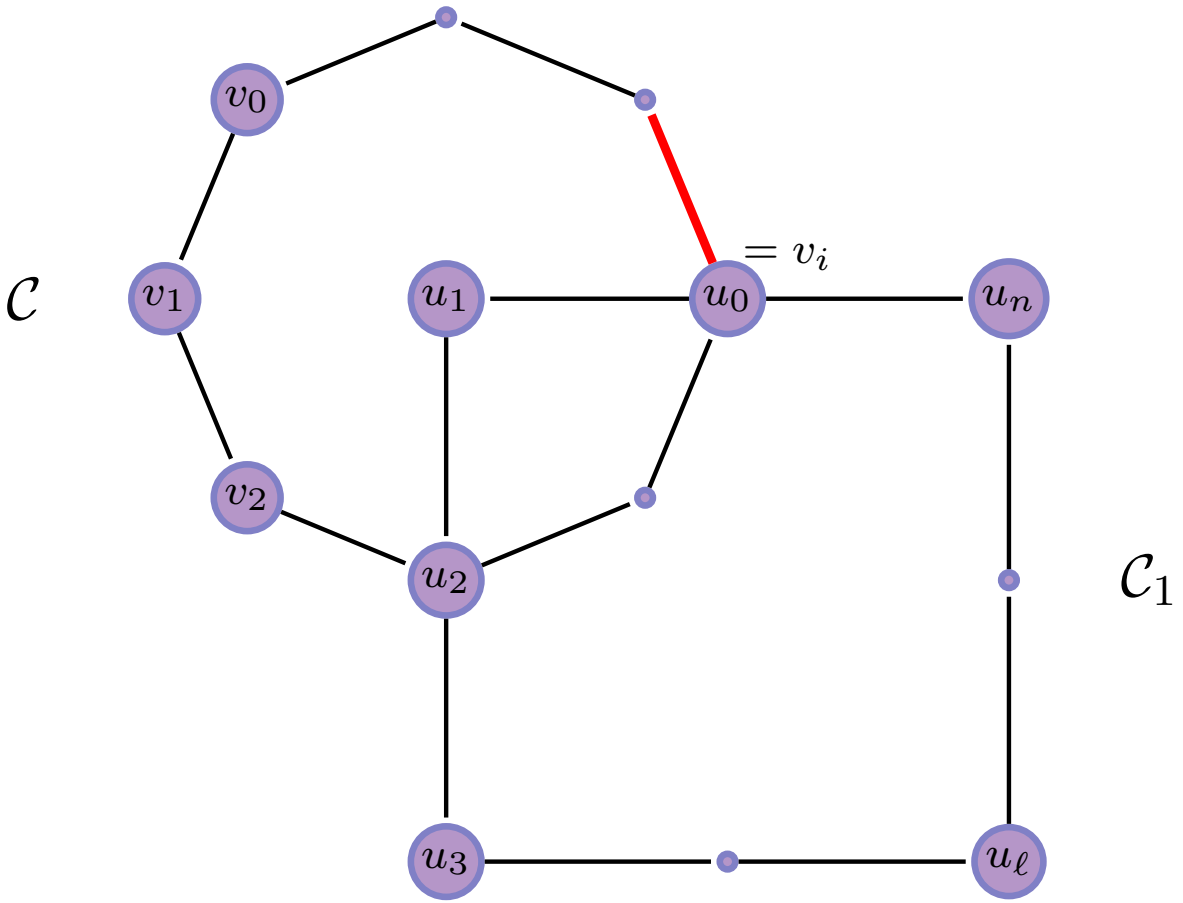
**Teorema (Teorema de Euler-Hierholzer)**

*Um multigrafo $\mathcal{G}$ com $2n$ vértices ímpares pode ser descrito por $n$ atalhos disjuntos. (Note-se que o número de vértices ímpares é necessariamente par.)*

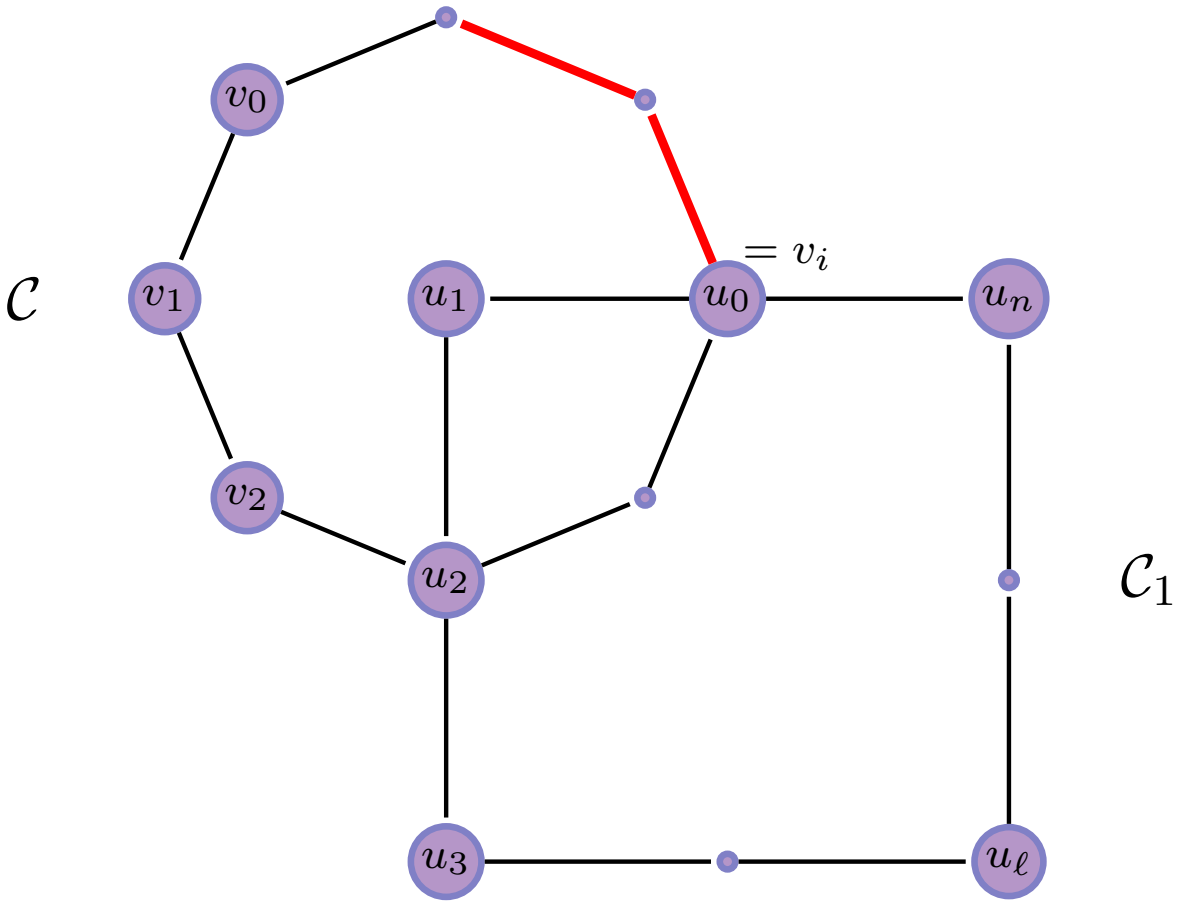
# Os teoremas de Euler



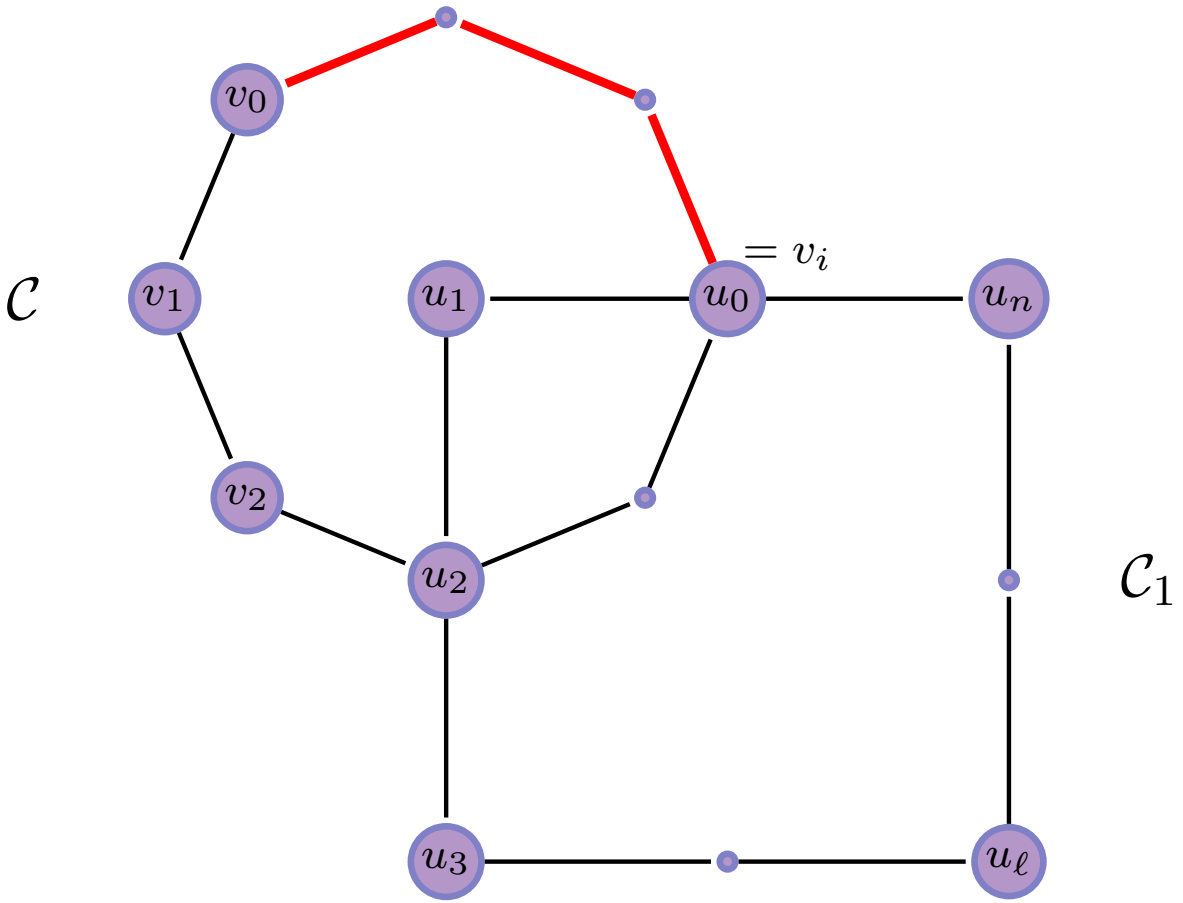
# Os teoremas de Euler



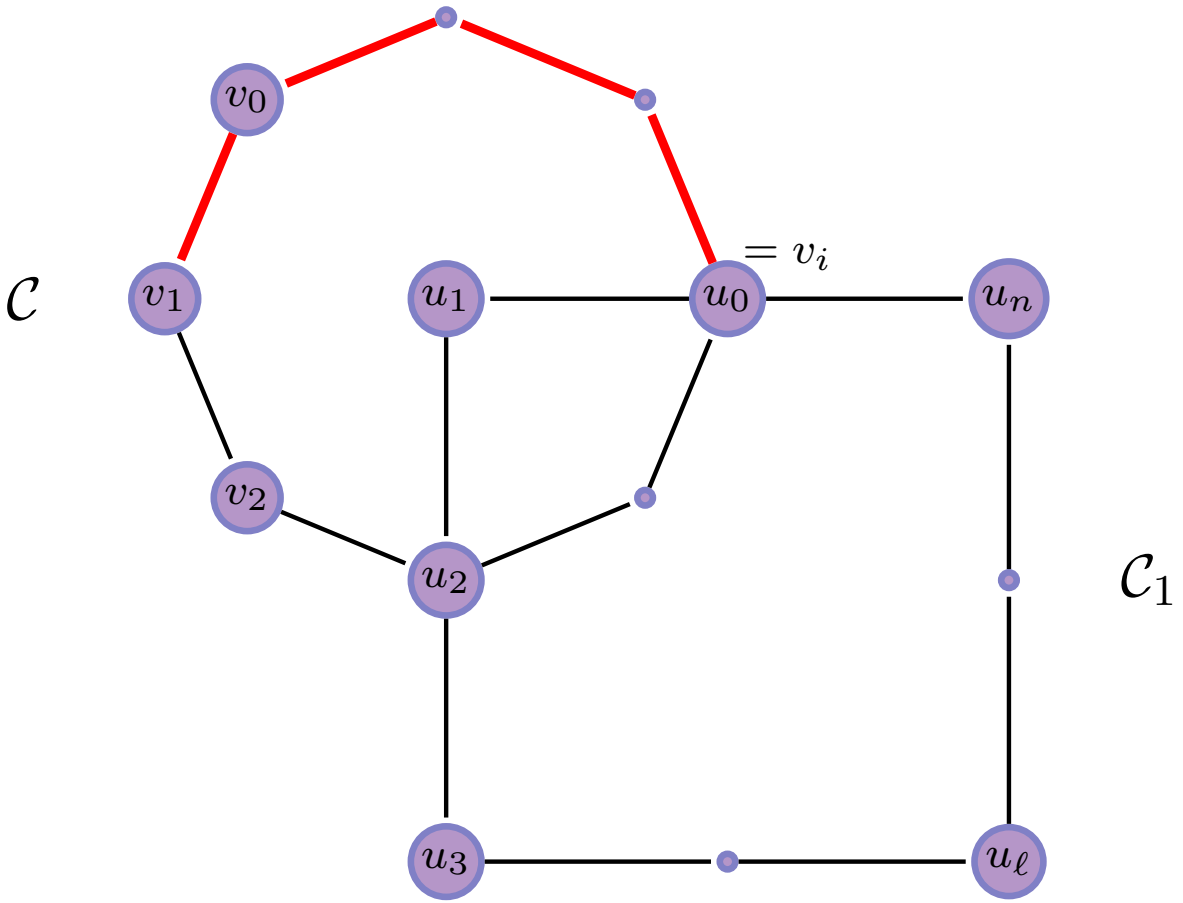
# Os teoremas de Euler



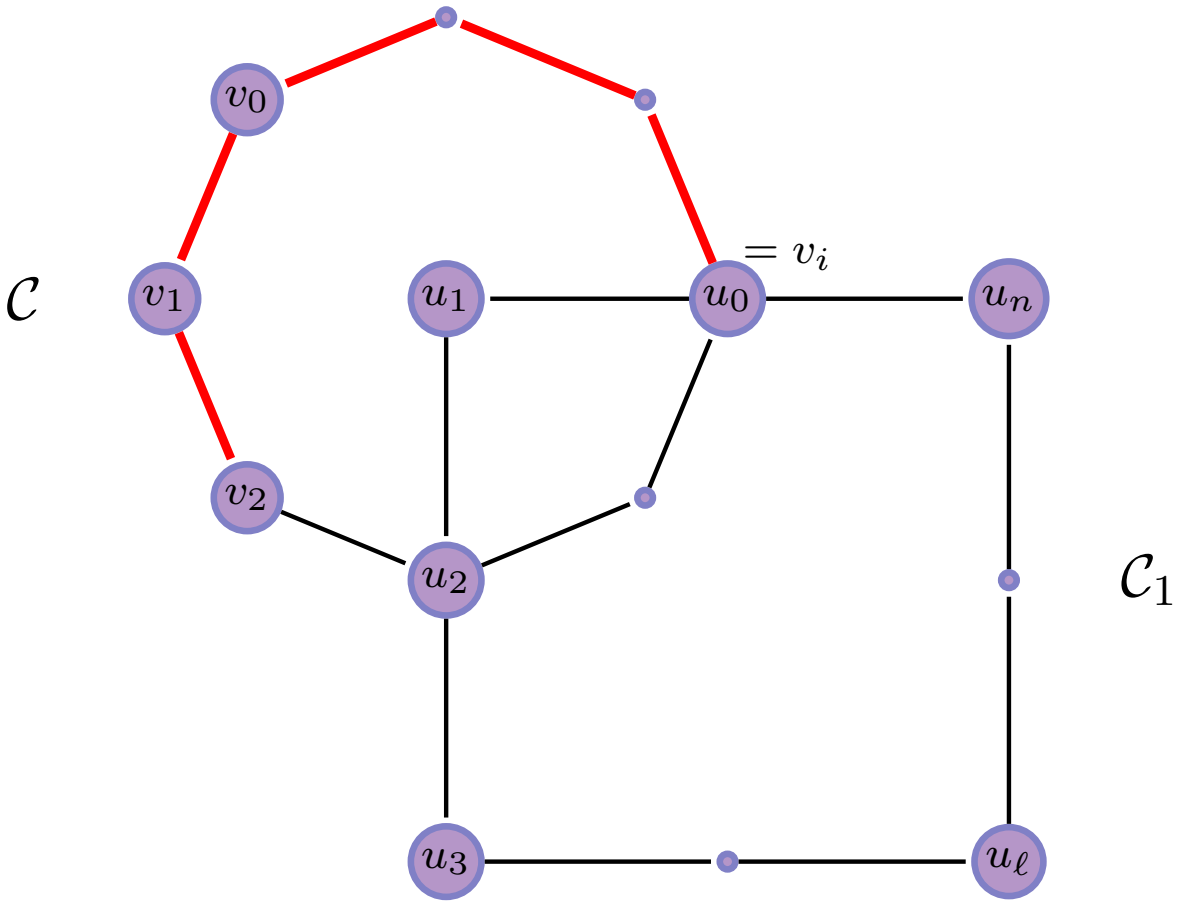
# Os teoremas de Euler



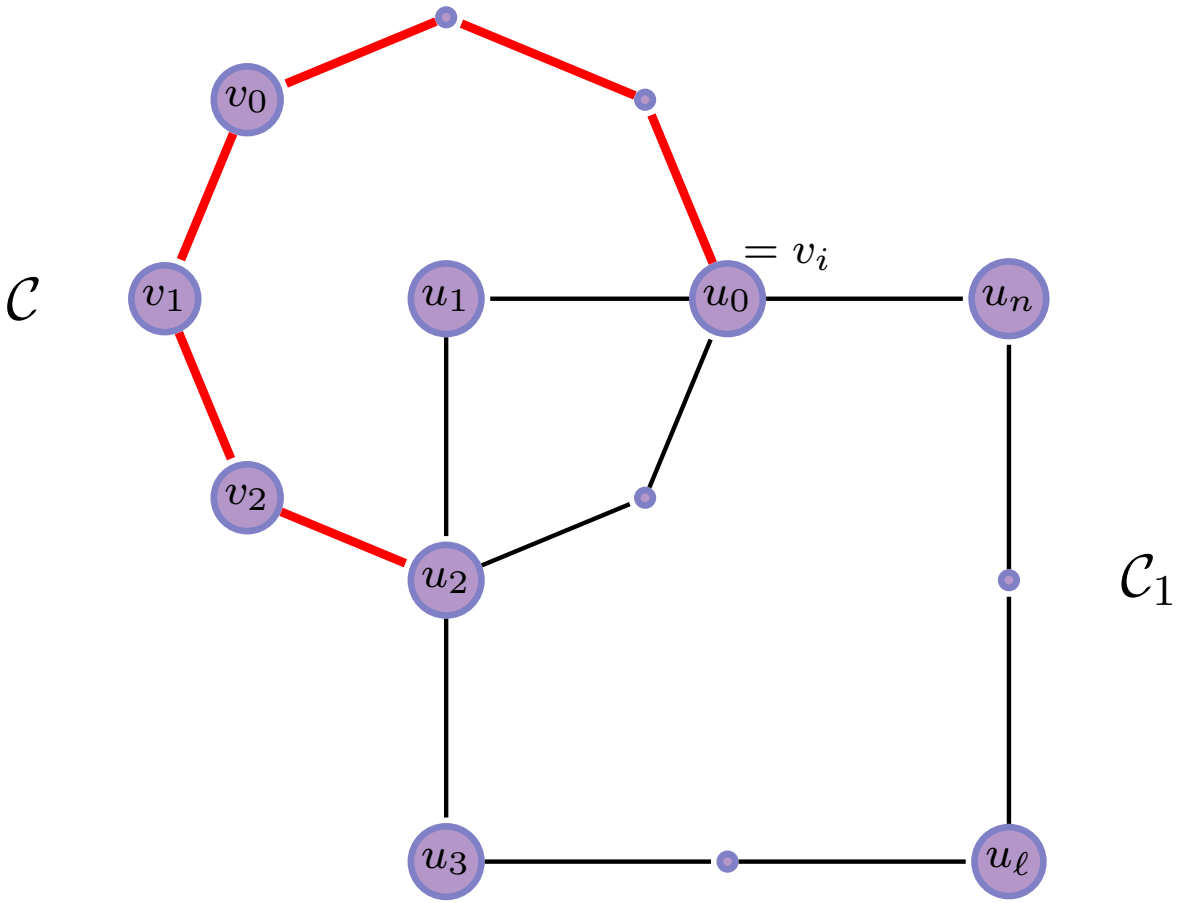
# Os teoremas de Euler



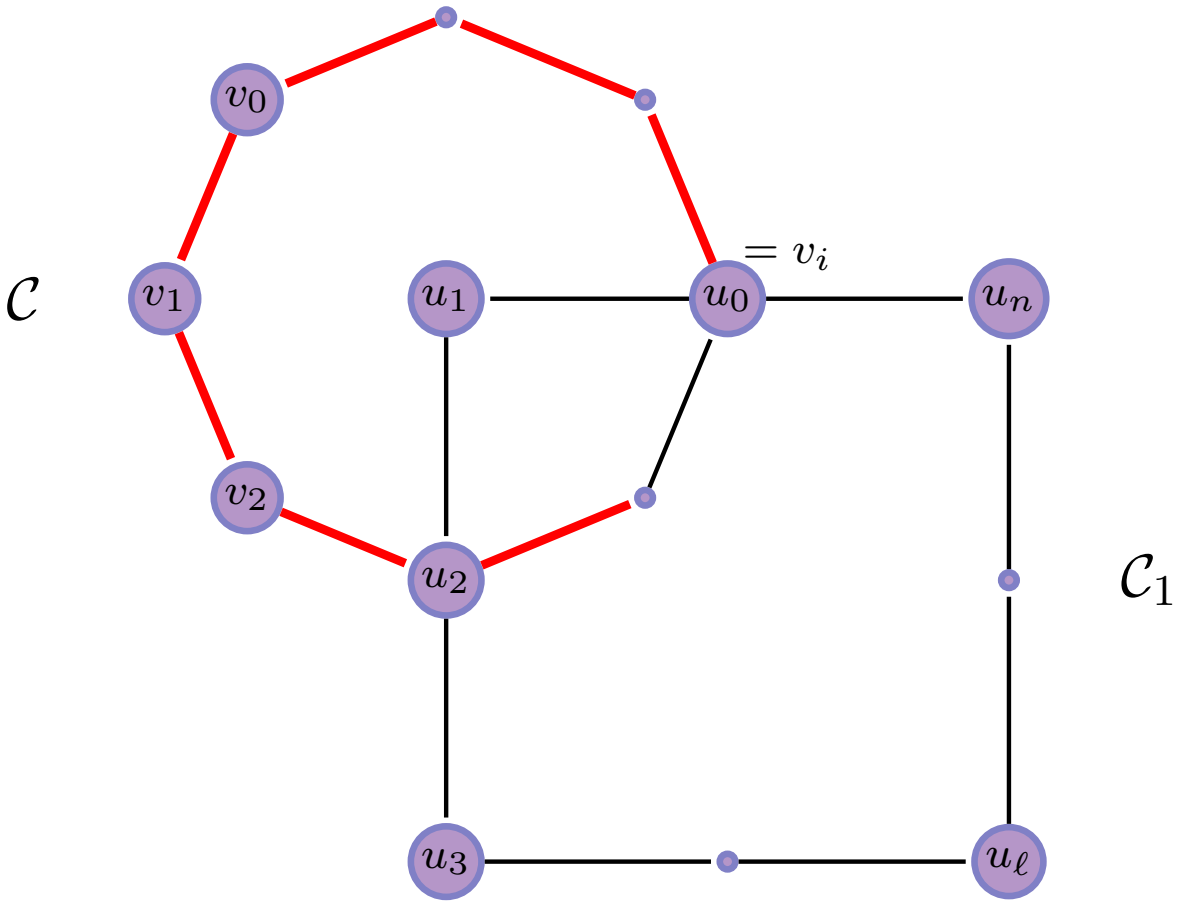
# Os teoremas de Euler



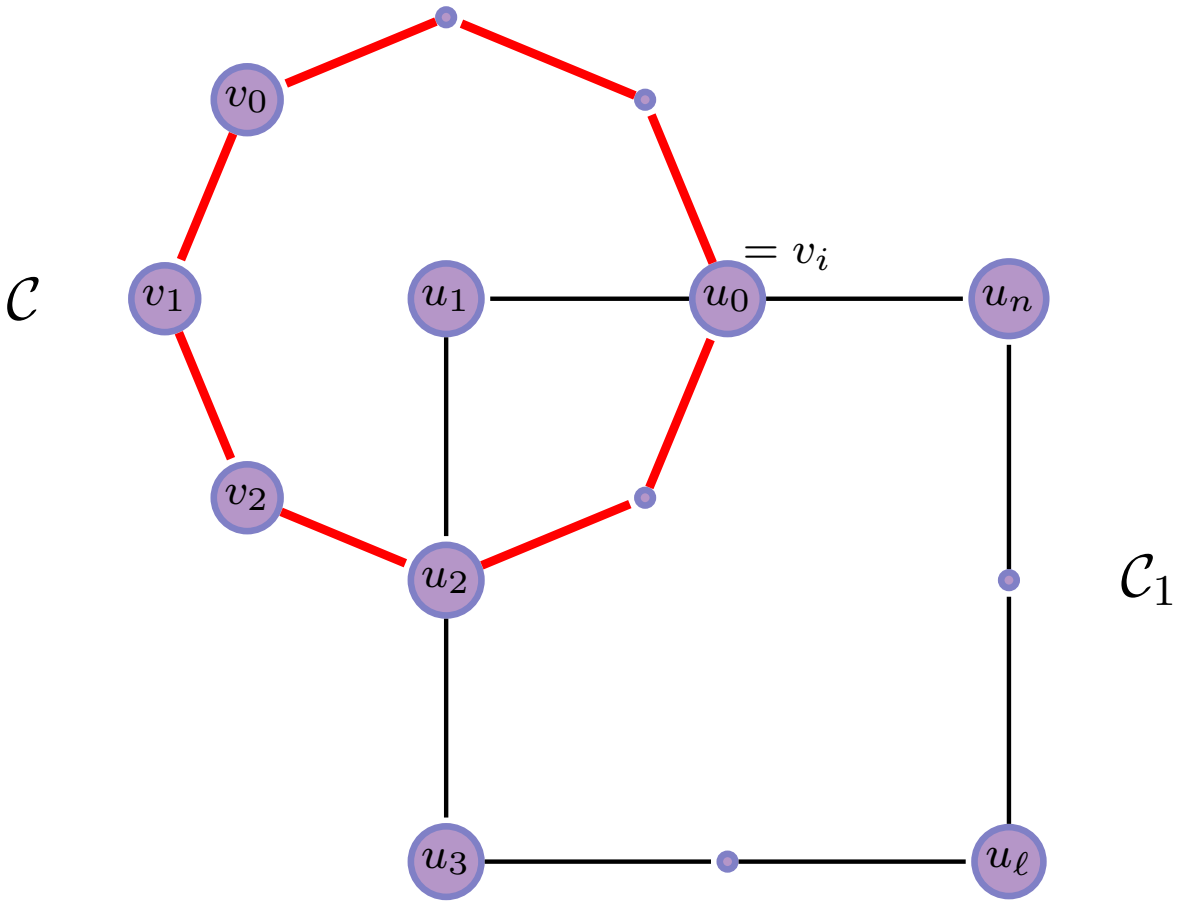
# Os teoremas de Euler



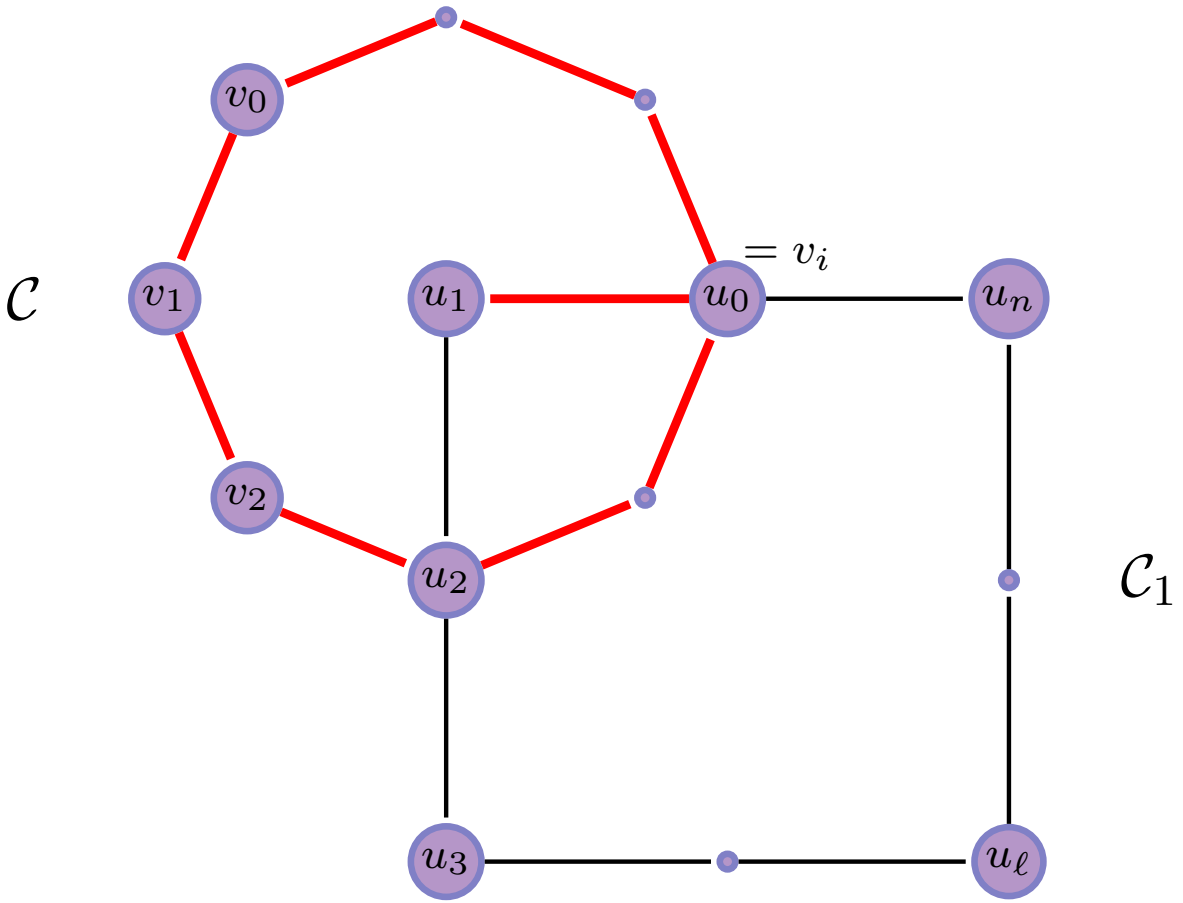
# Os teoremas de Euler



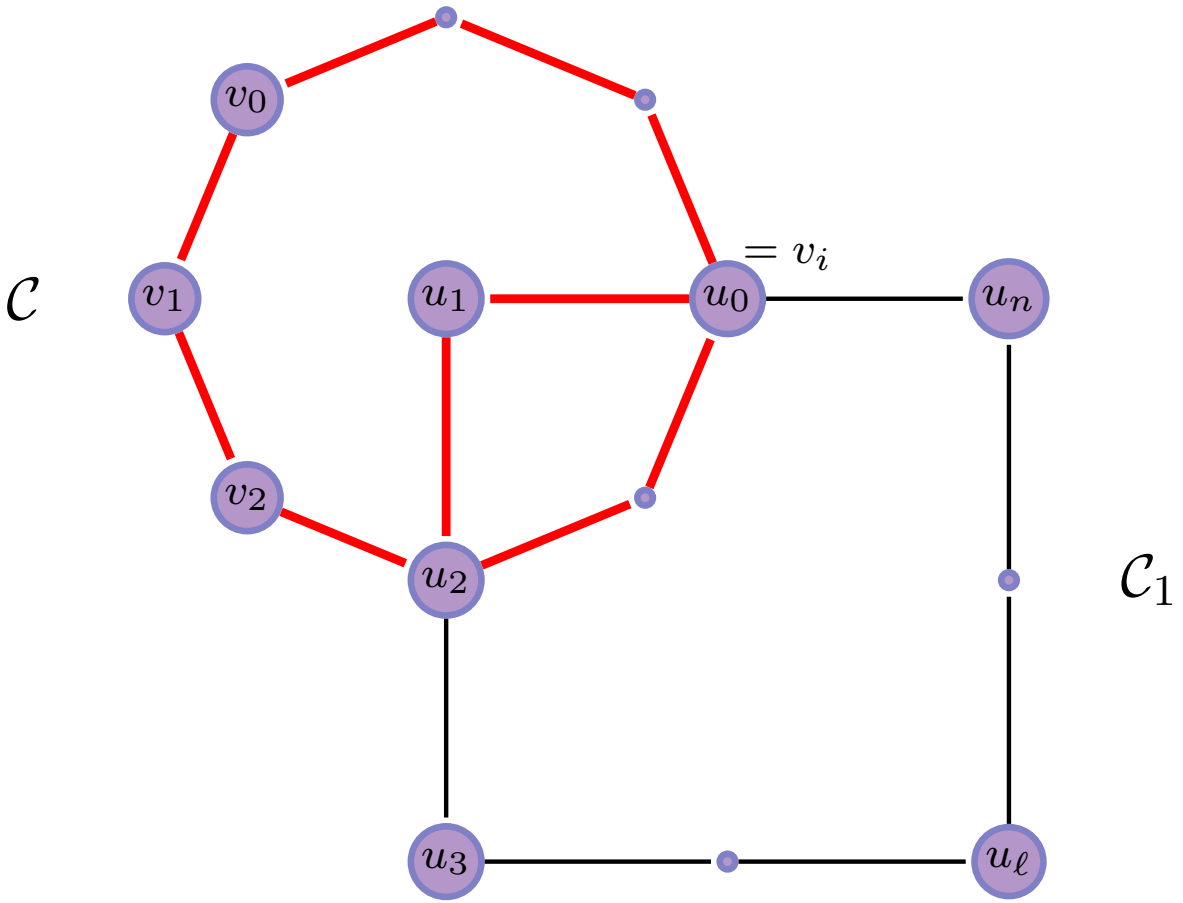
# Os teoremas de Euler



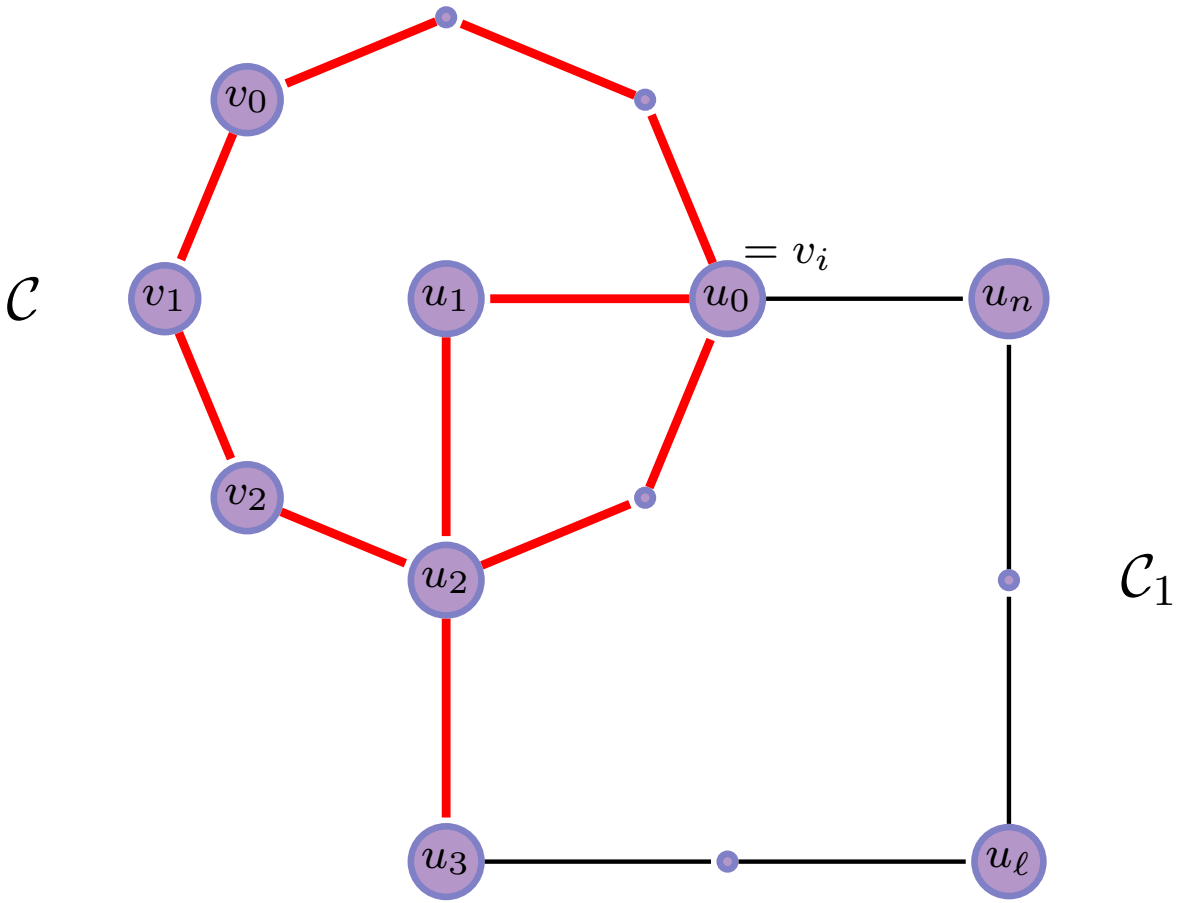
# Os teoremas de Euler



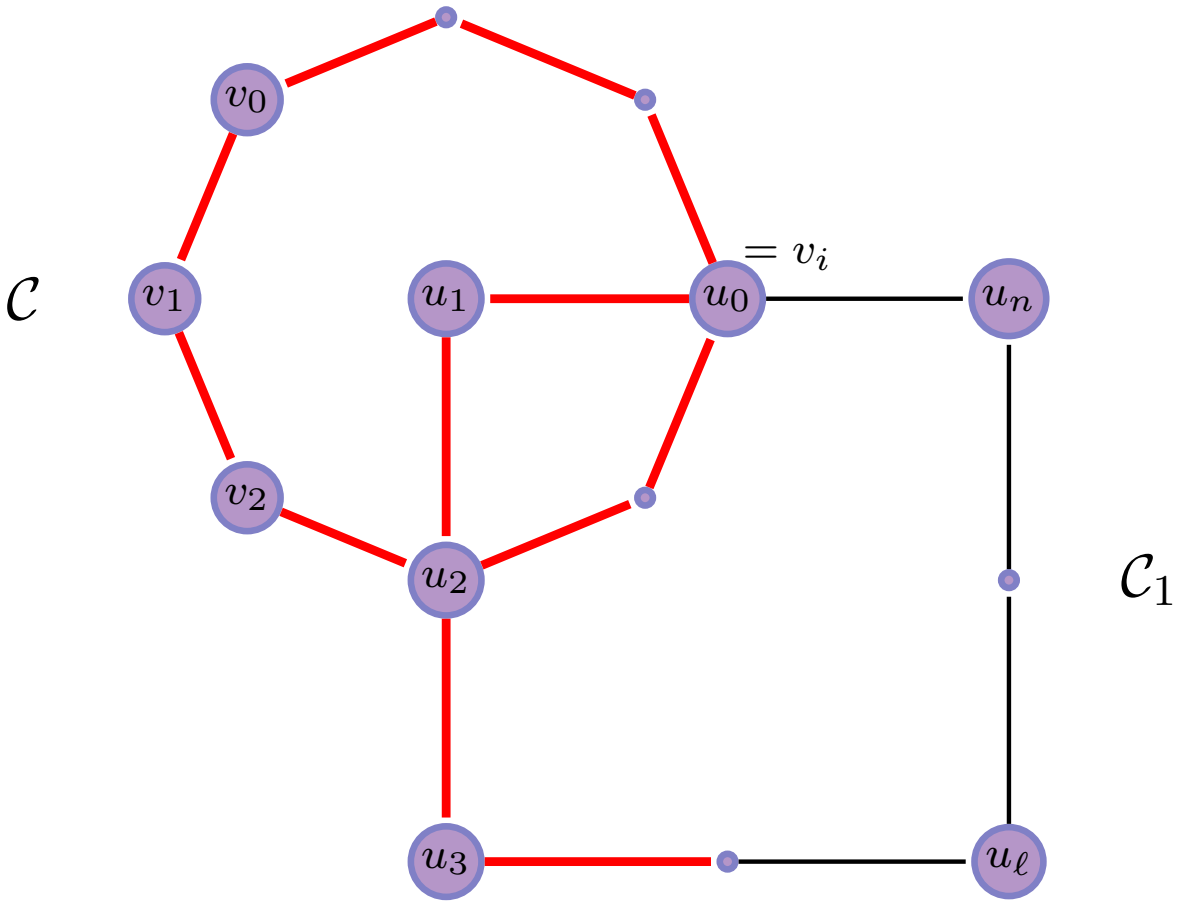
# Os teoremas de Euler



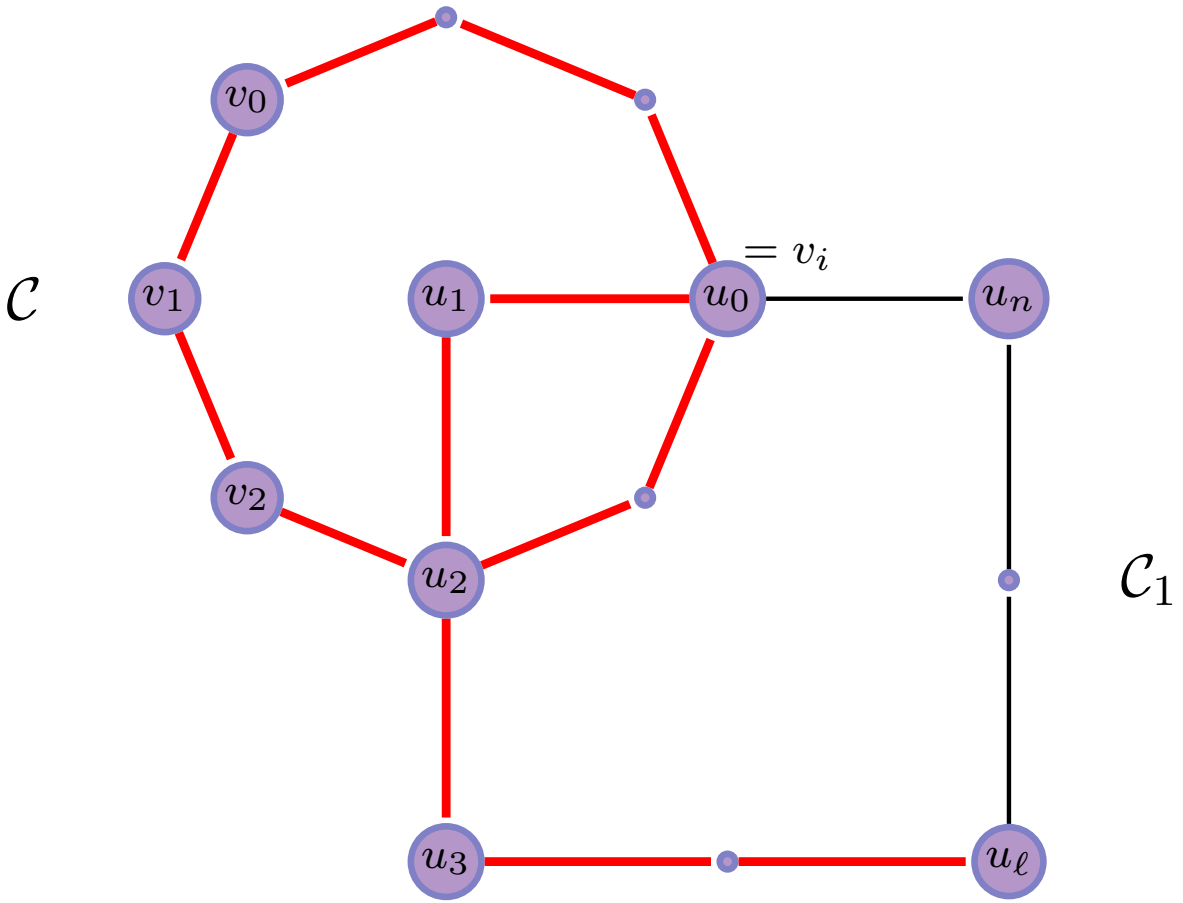
# Os teoremas de Euler



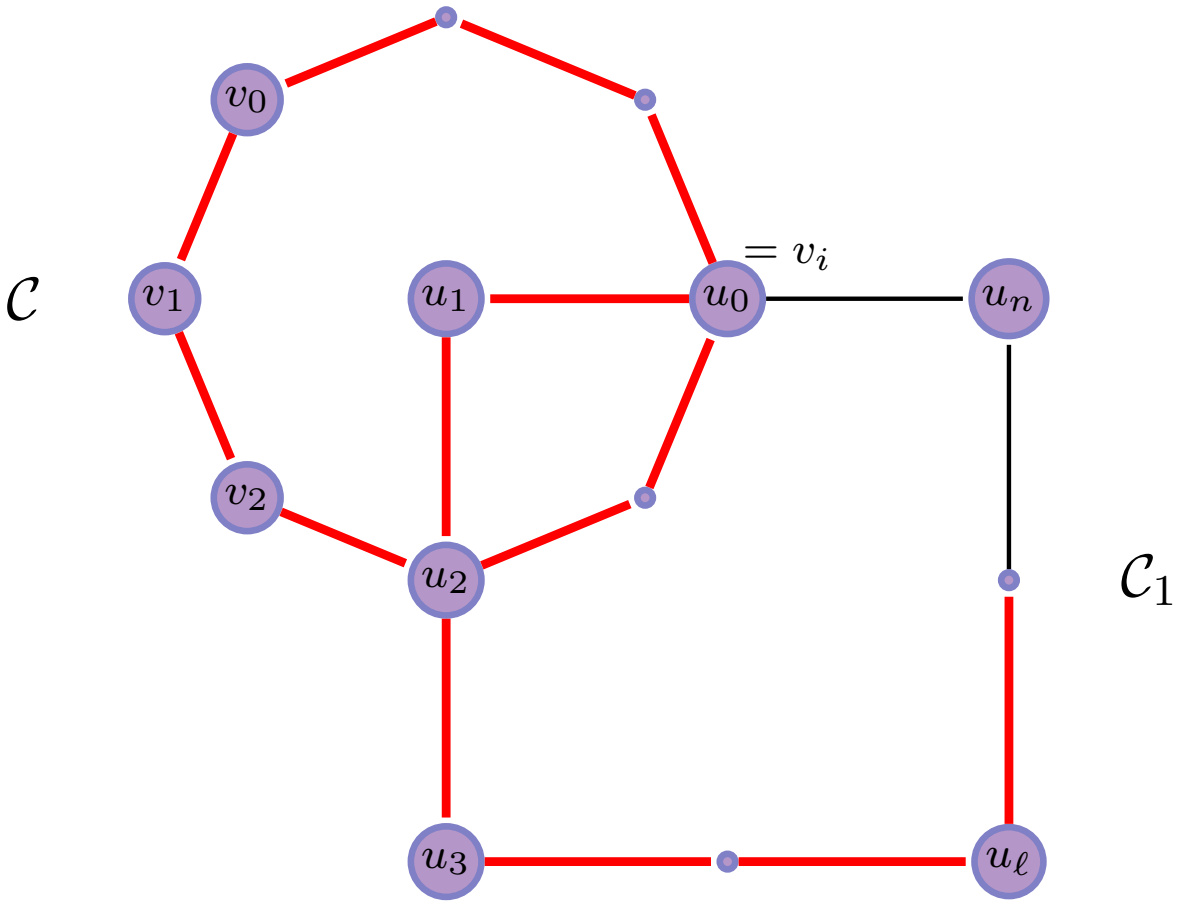
# Os teoremas de Euler



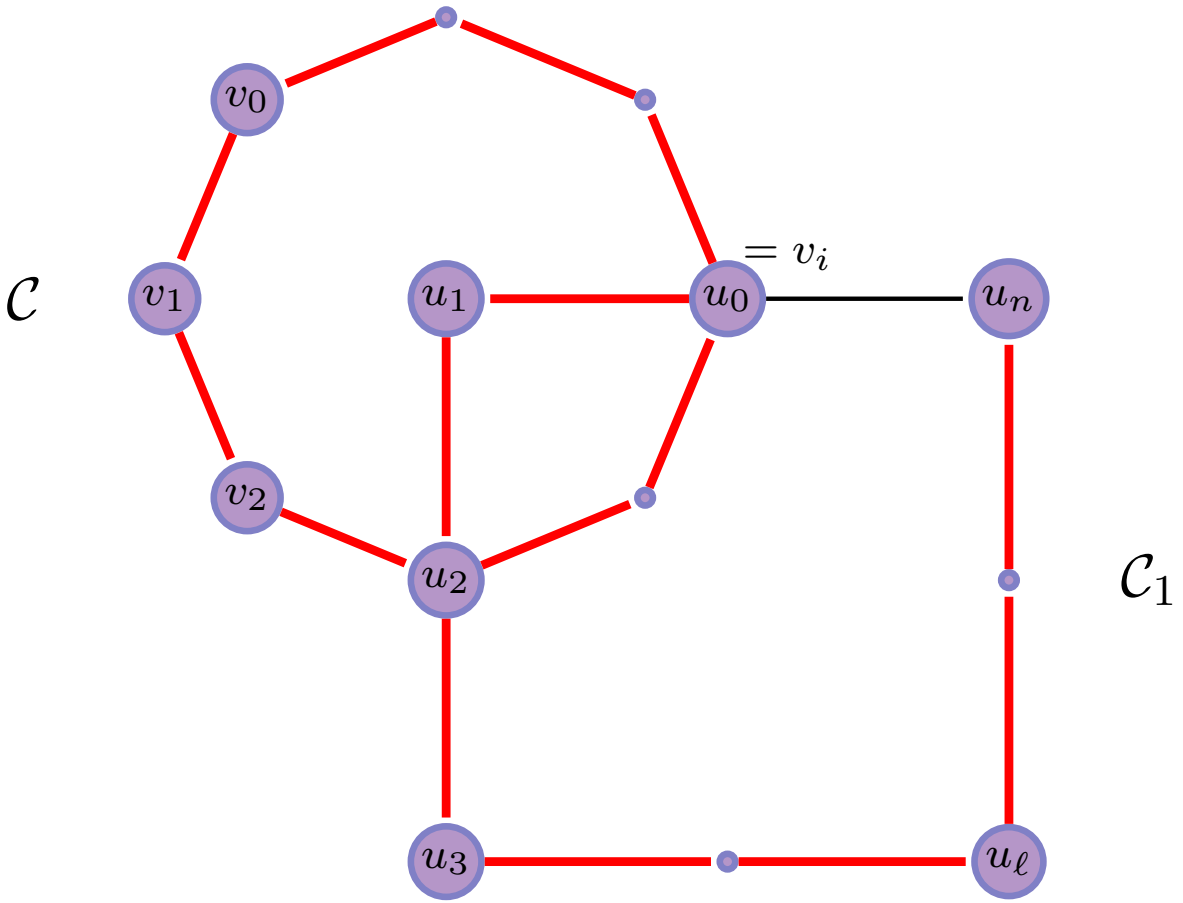
# Os teoremas de Euler



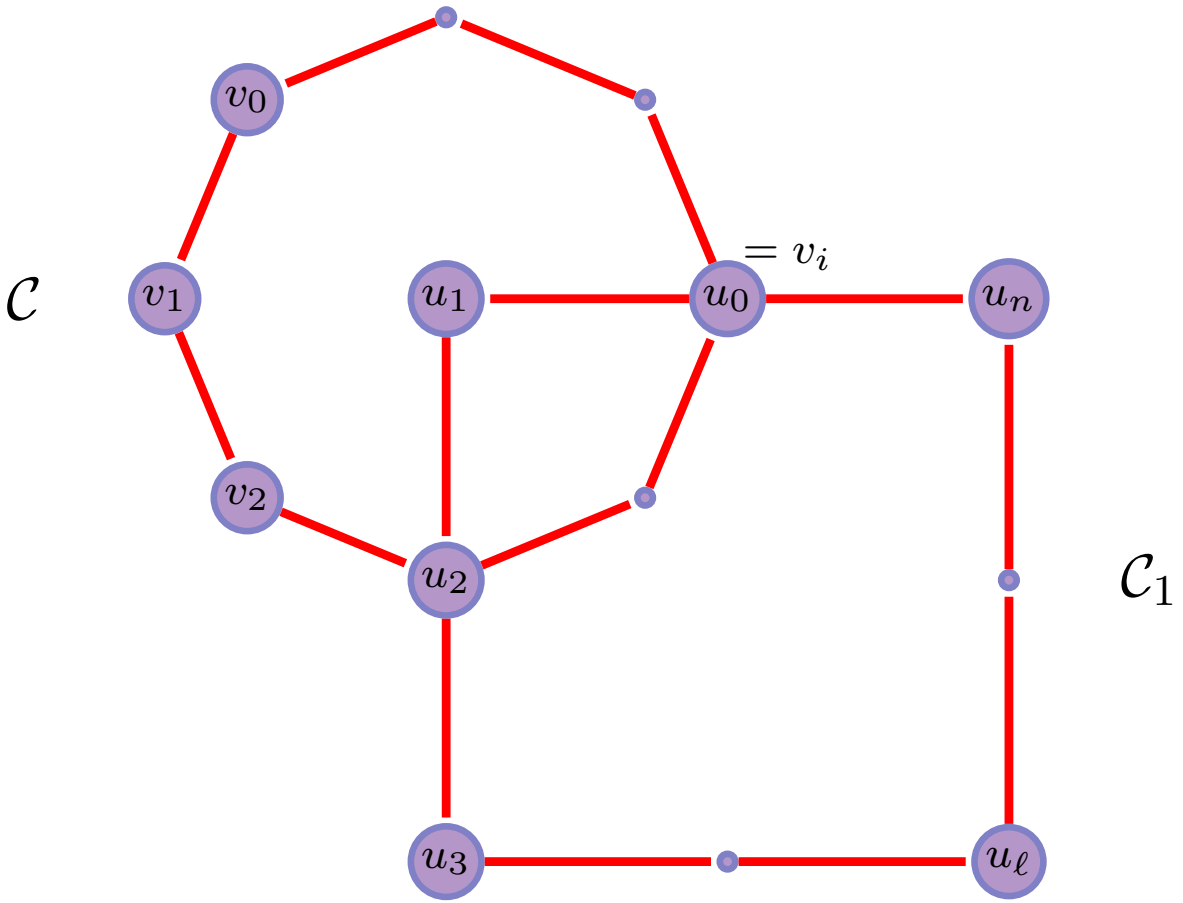
# Os teoremas de Euler



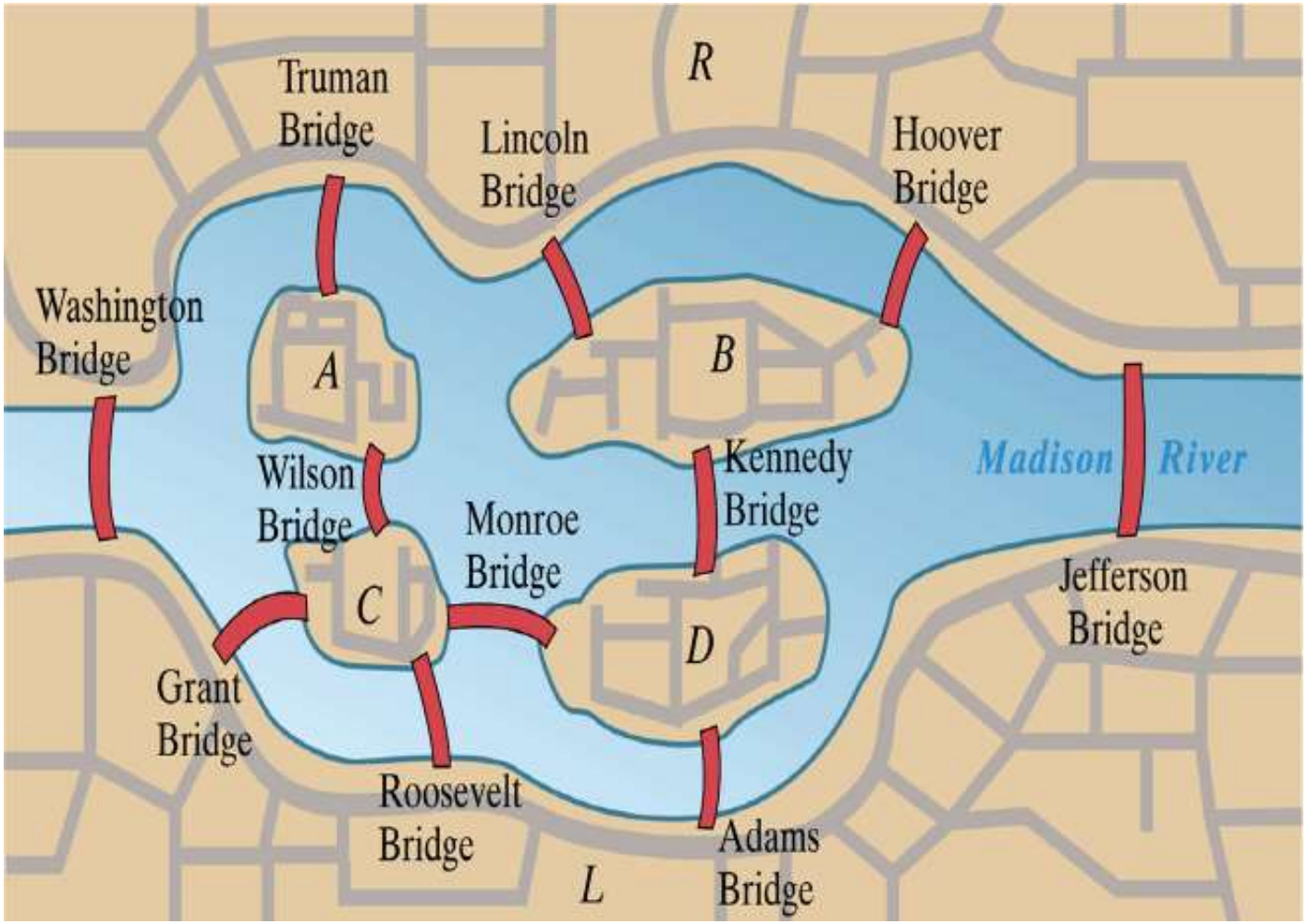
# Os teoremas de Euler



# Os teoremas de Euler

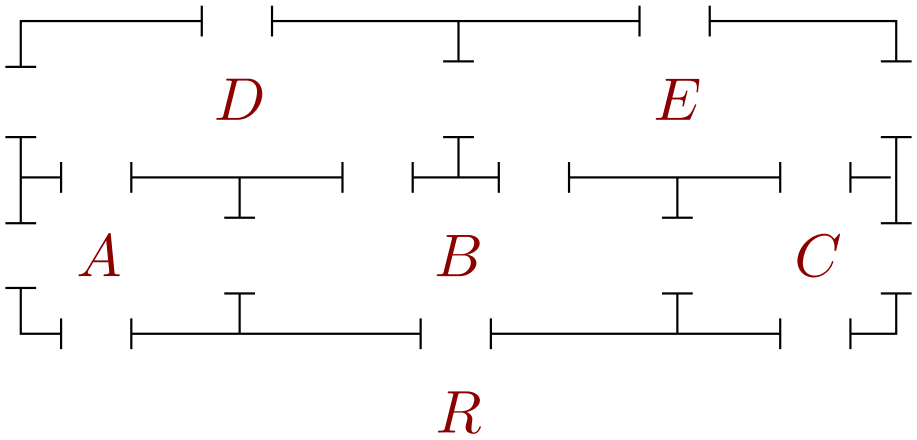


# As pontes do rio Madison

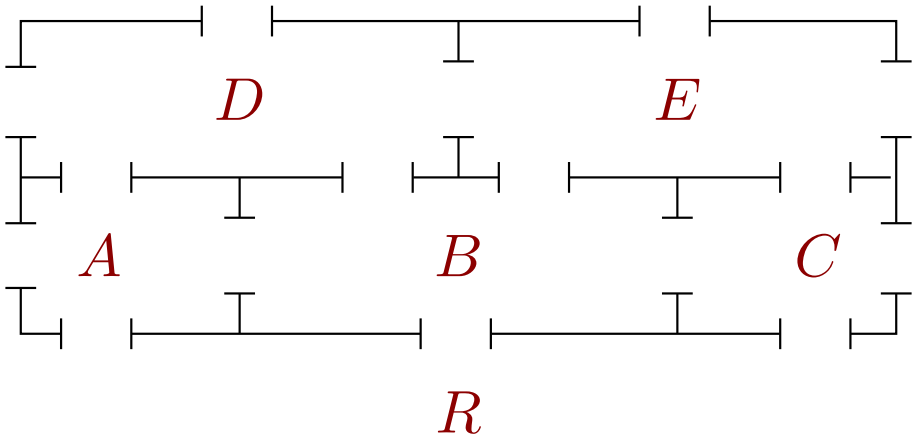


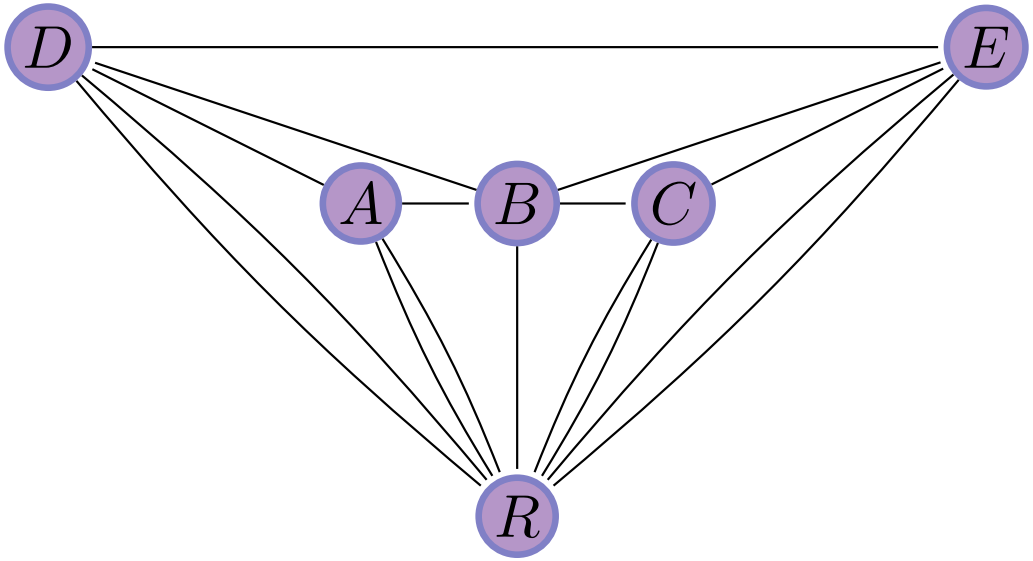
# 2. Plantas e Redes

# Plantas, *vide* [Cha85]

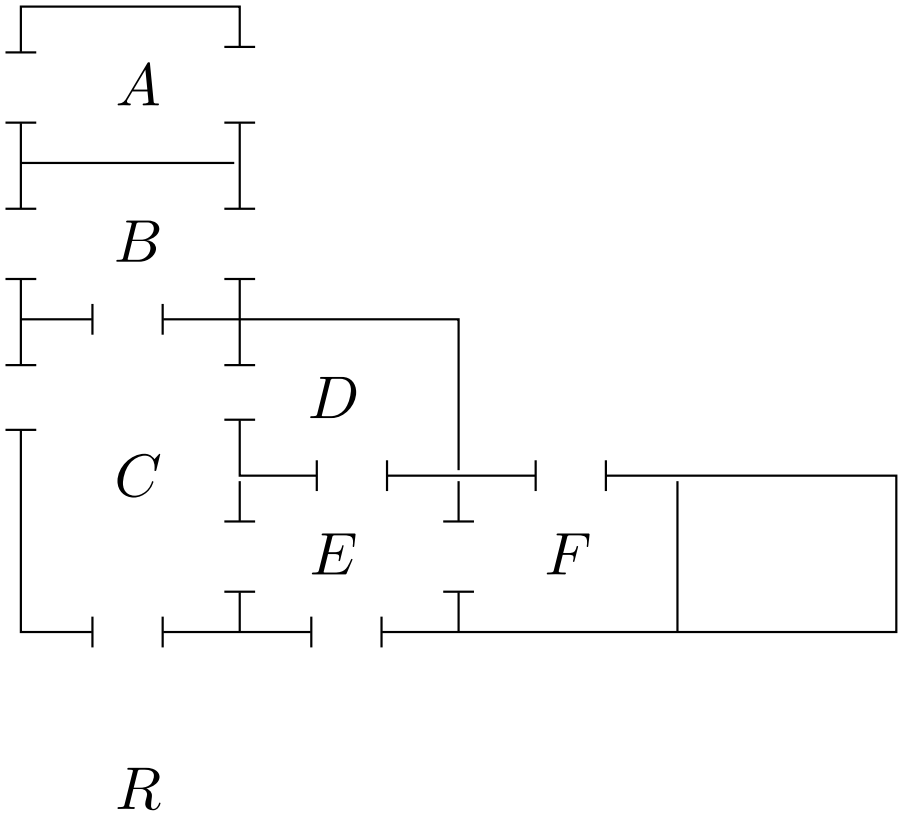


# Plantas, *vide* [Cha85]

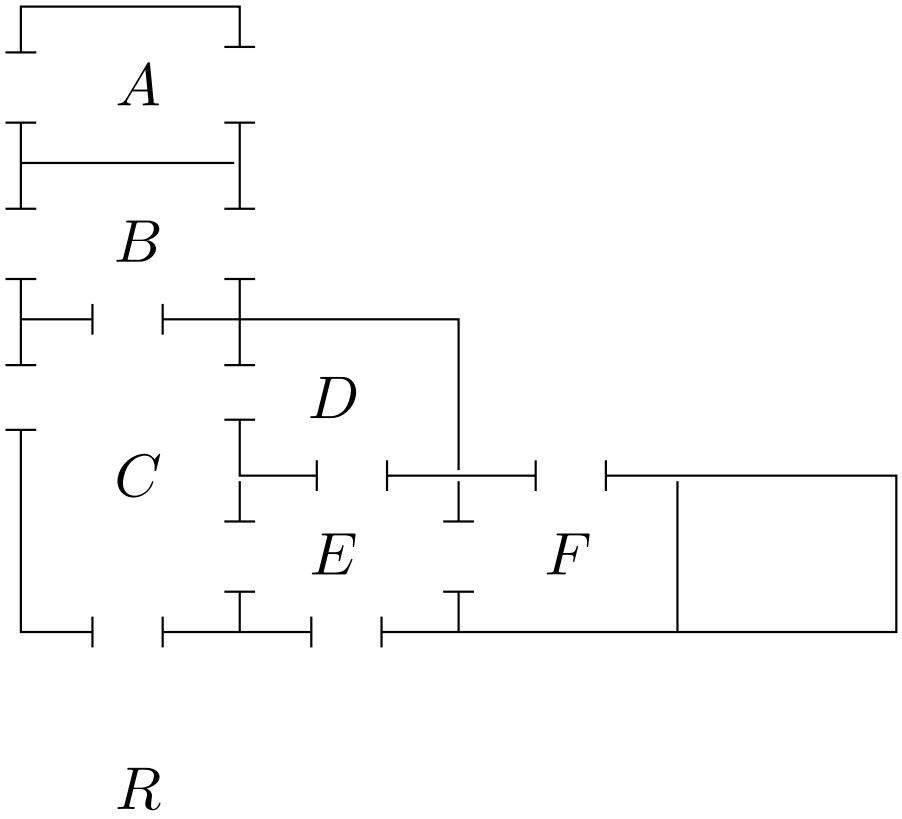


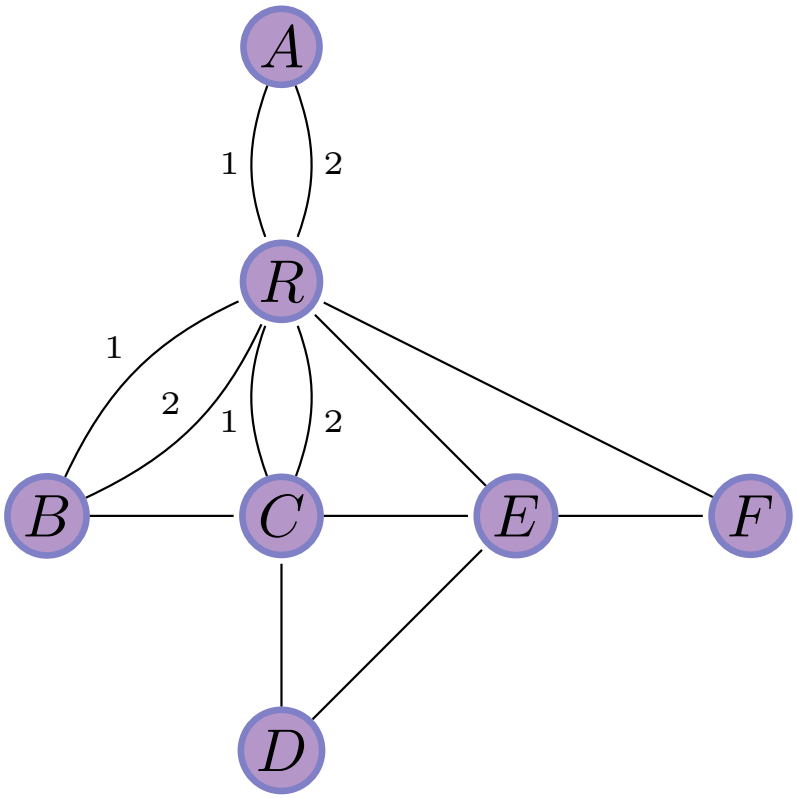



# Plantas, *vide* [Cha85]

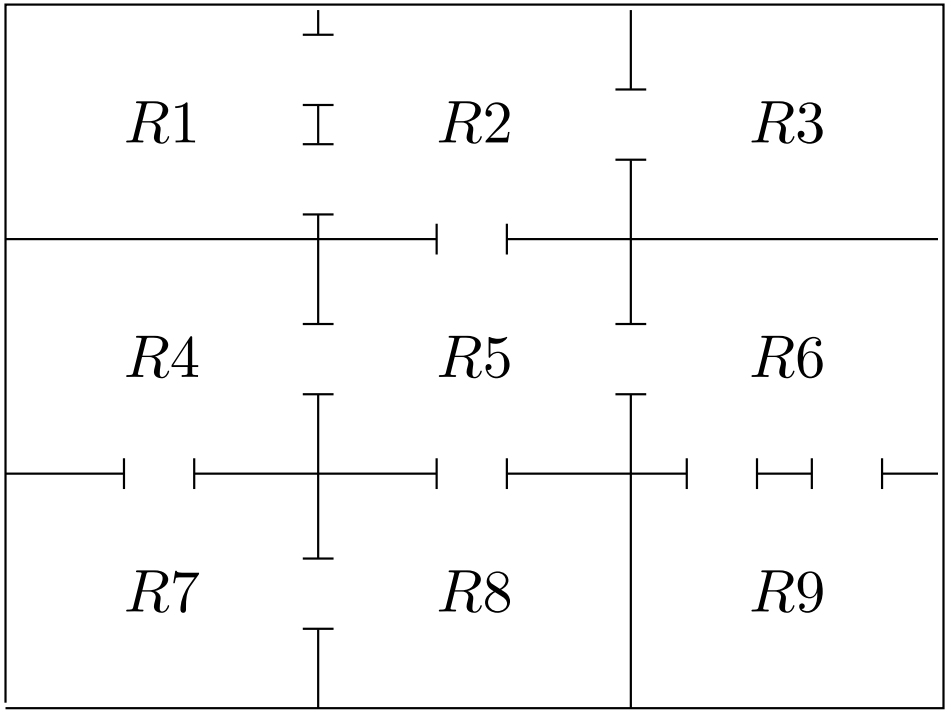


# Plantas, *vide* [Cha85]

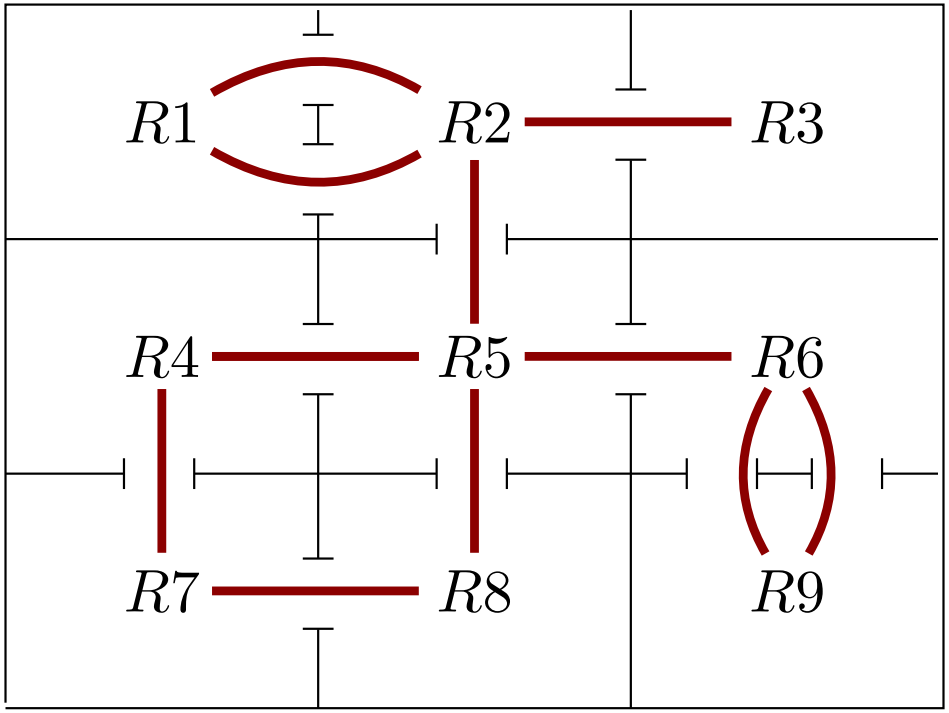


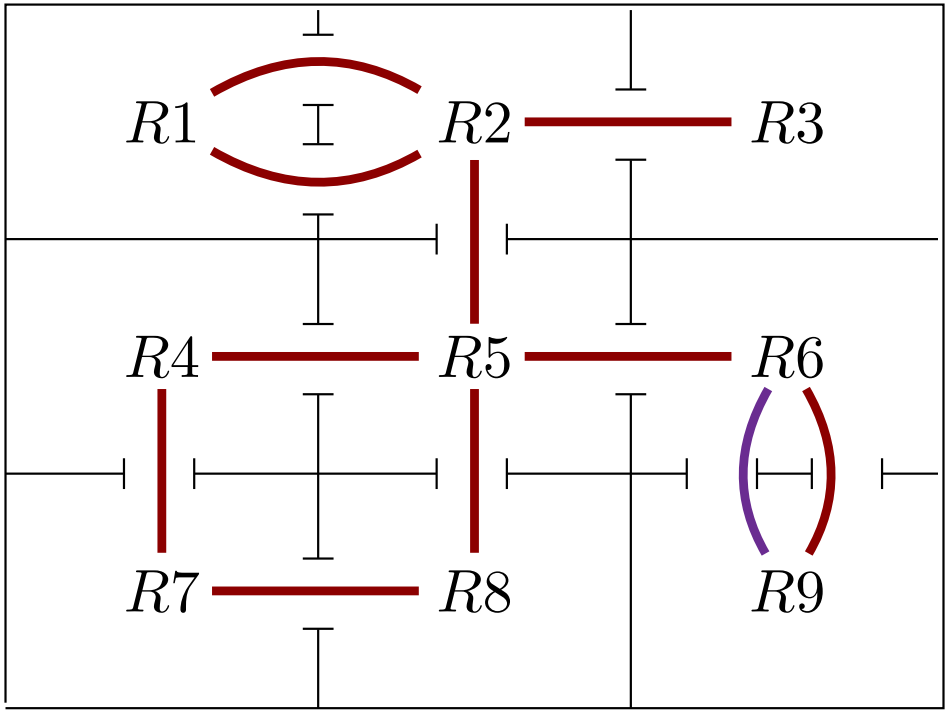
# Plantas



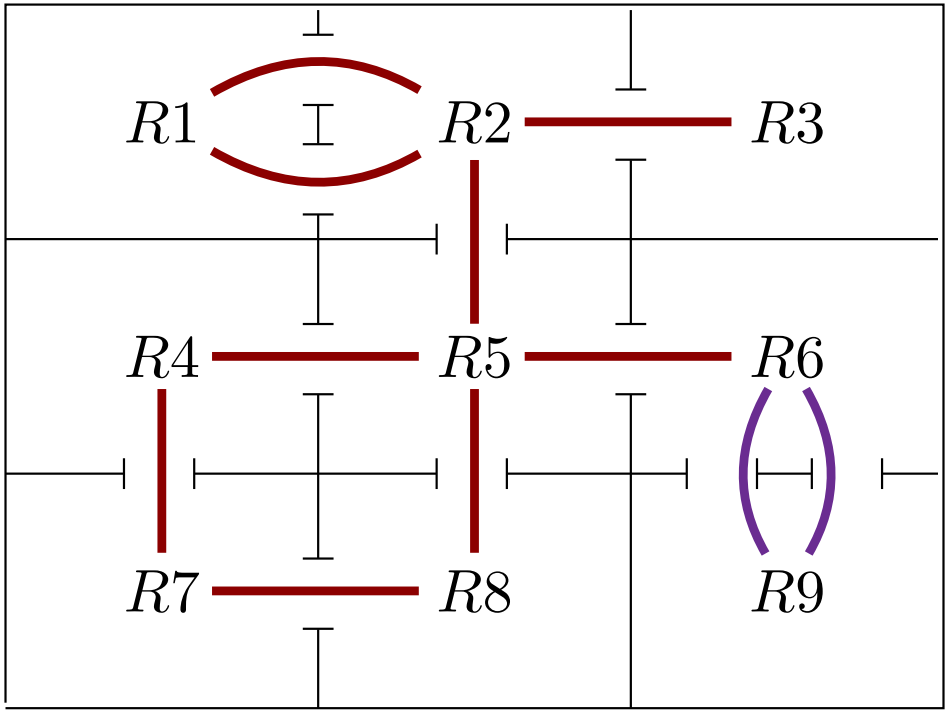
# Plantas



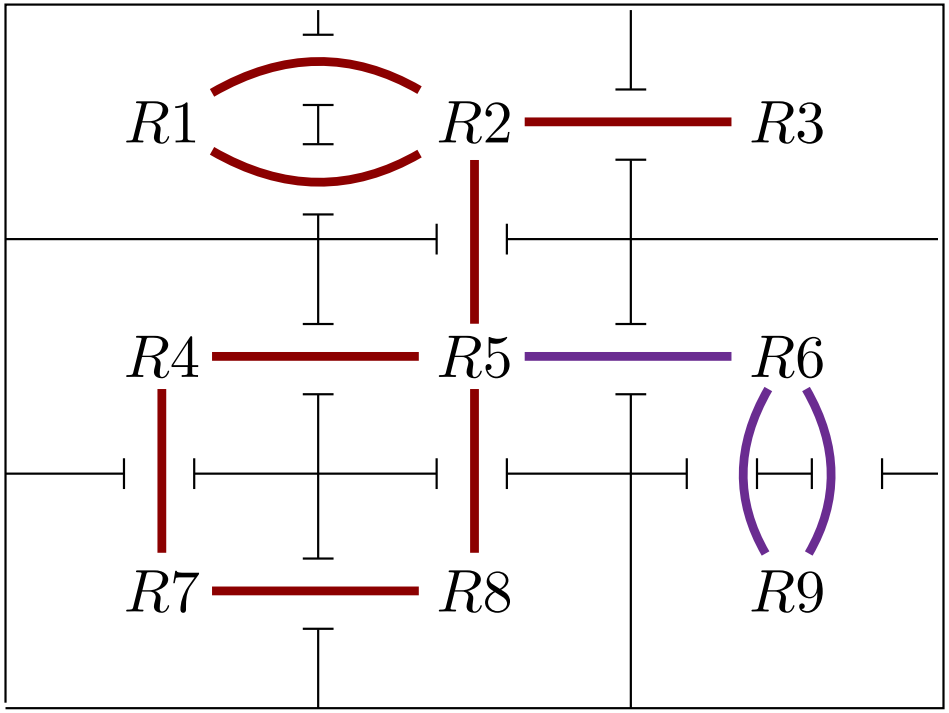
# Plantas



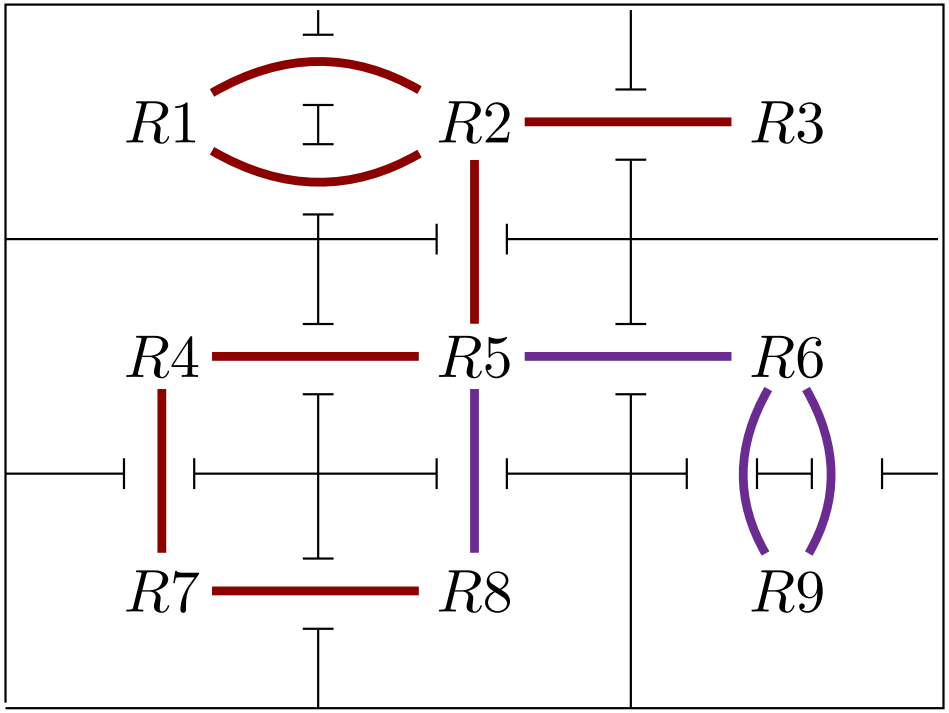
# Plantas



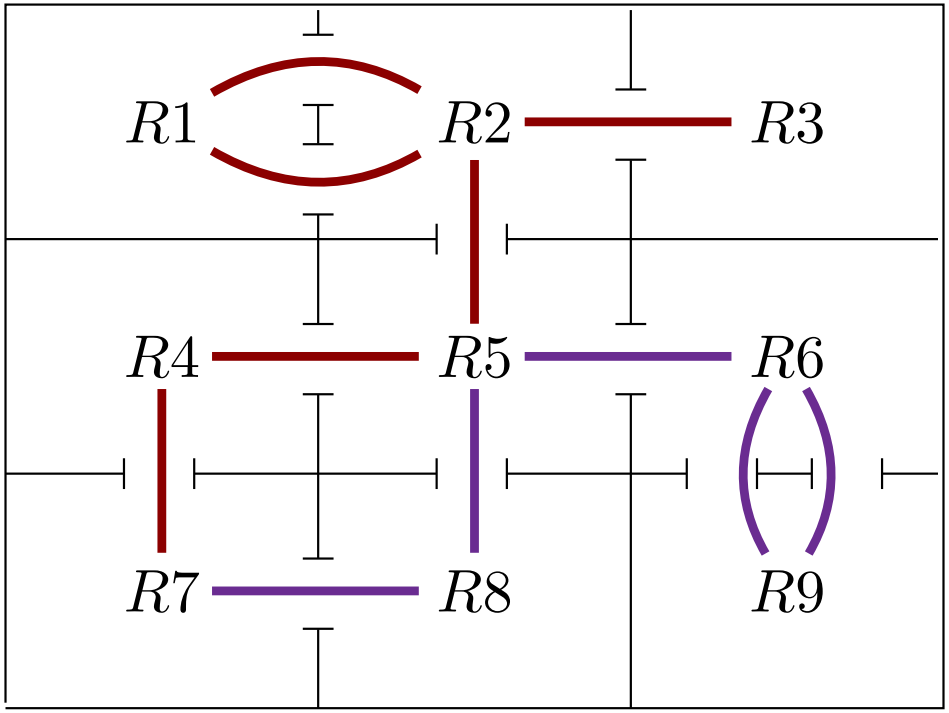
# Plantas



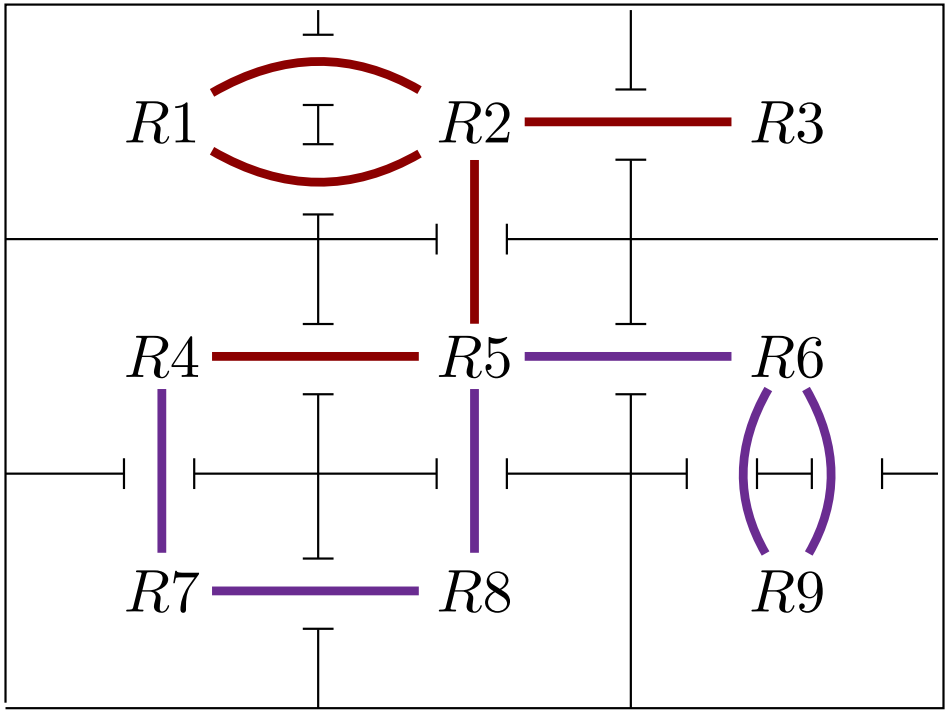
# Plantas



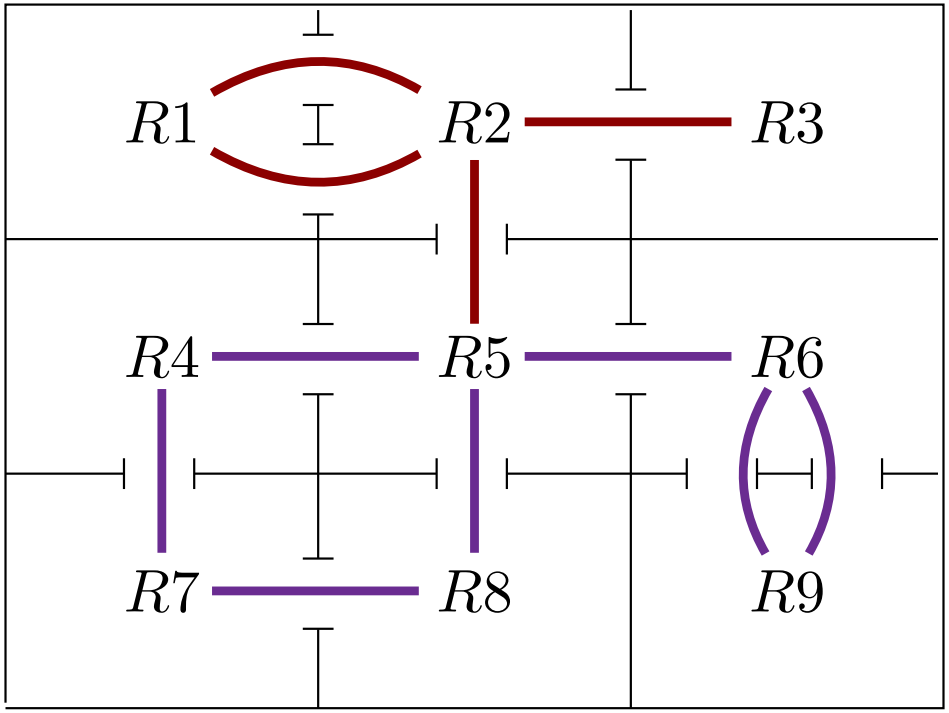
# Plantas



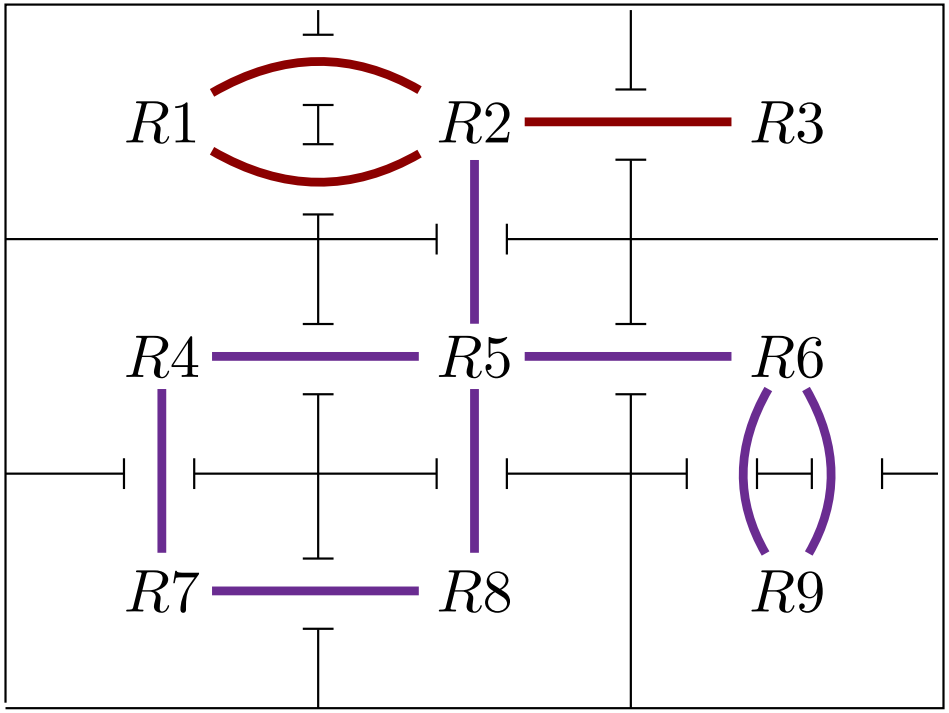
# Plantas



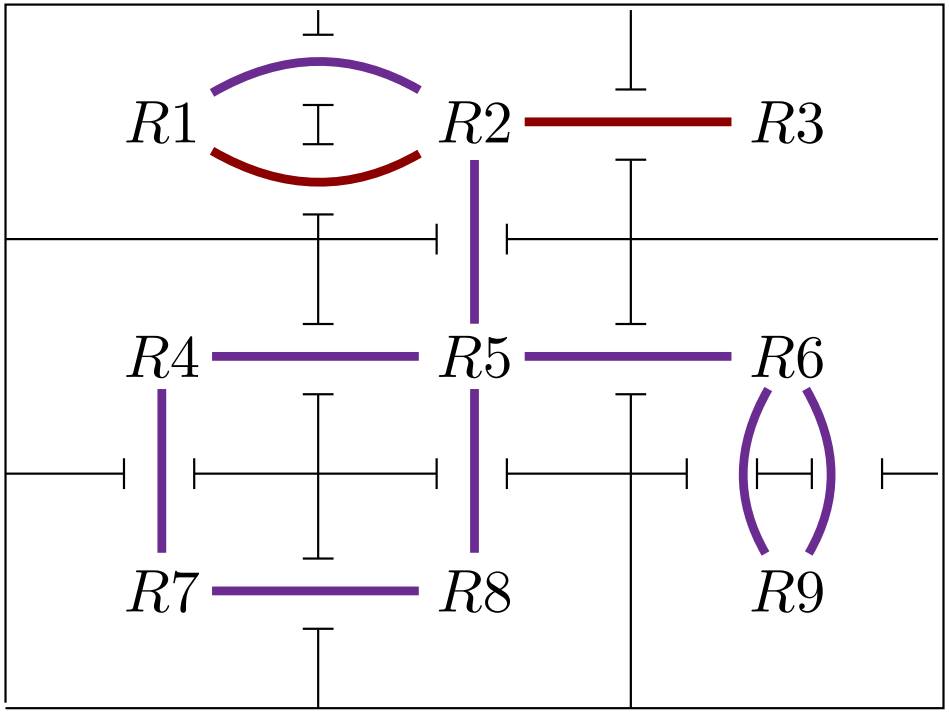
# Plantas



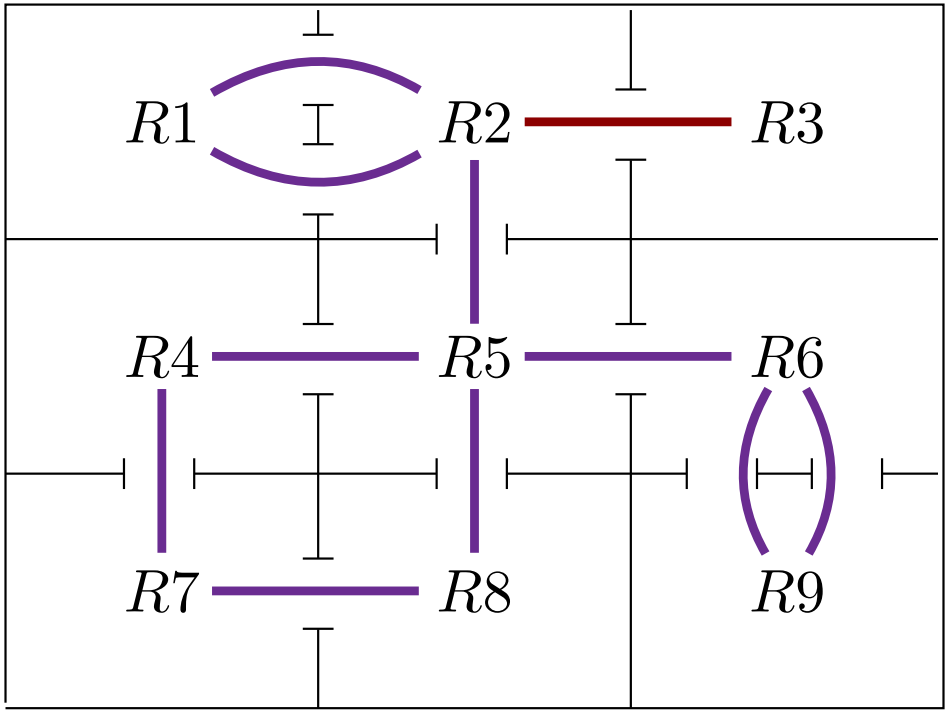
# Plantas



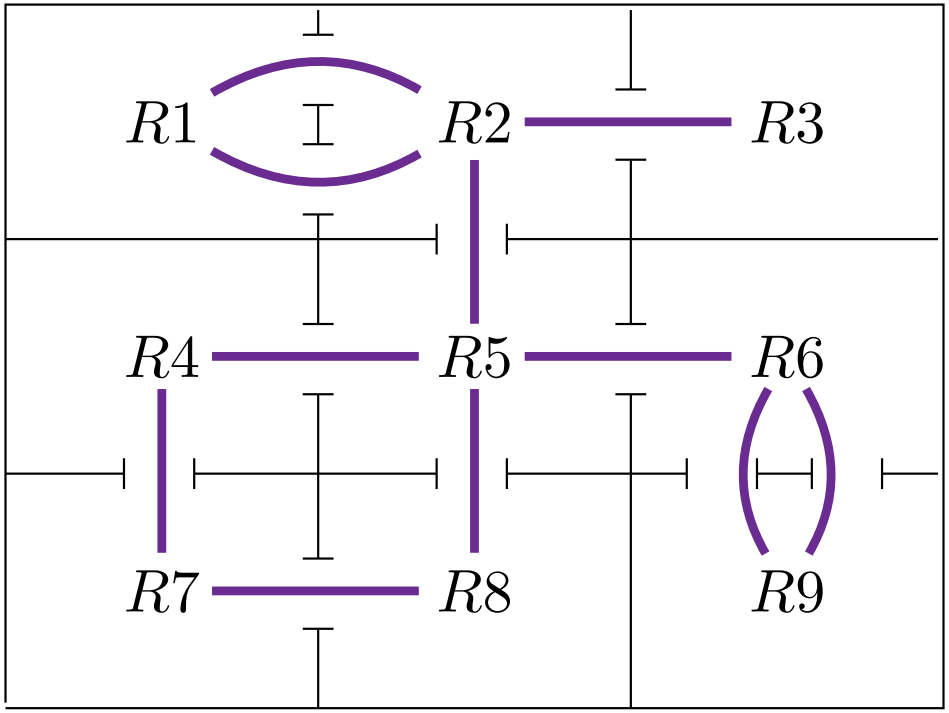
# Plantas



# Plantas



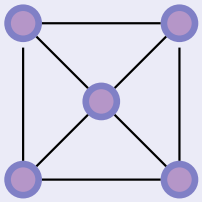
# Plantas

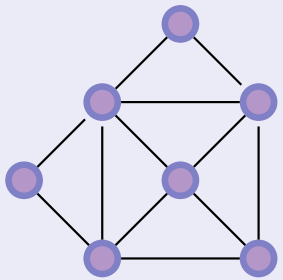


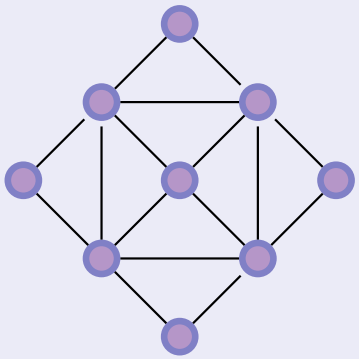
# "Doodleling"

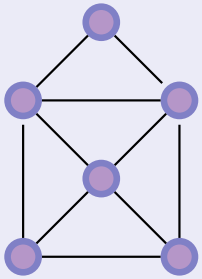
## Casa do Pai Natal

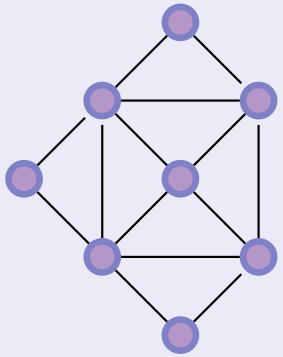
Quais das seguintes figuras podem ser desenhadas sem levantar a caneta do papel, iniciando o desenho num dos vértices e sem repetir arestas?



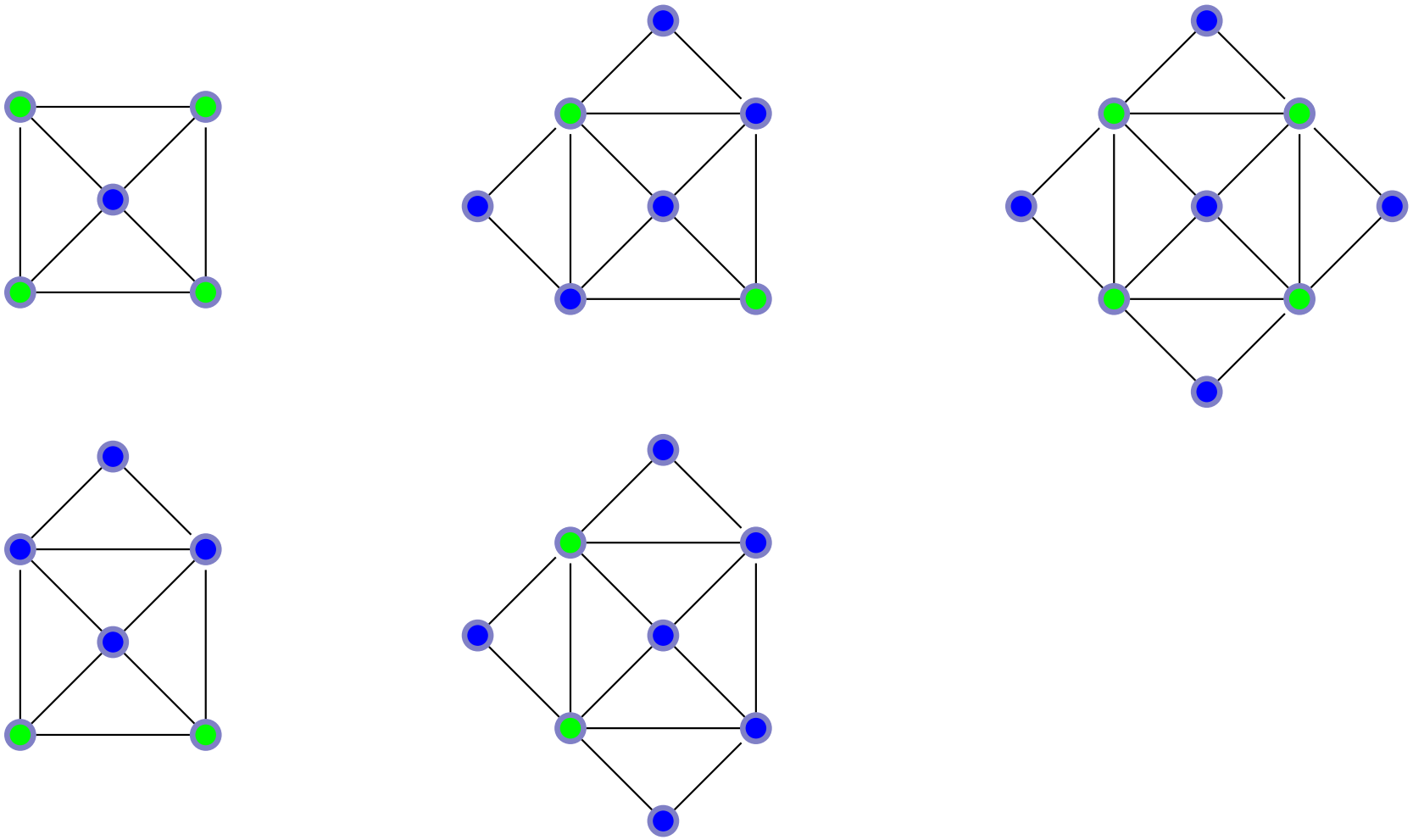




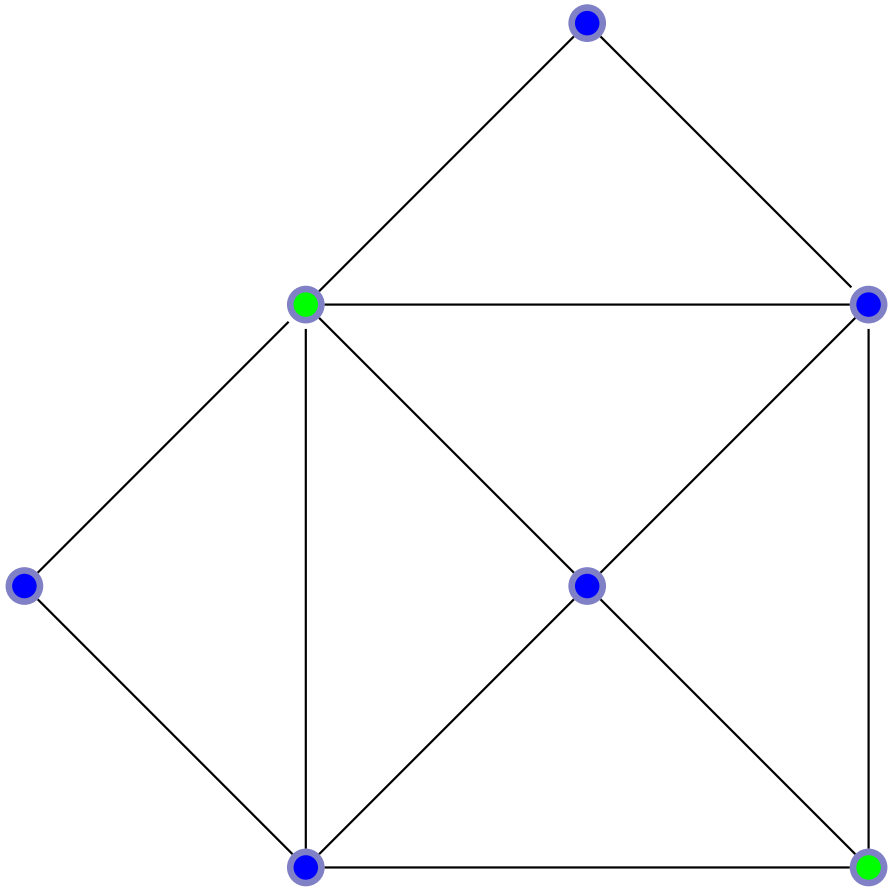




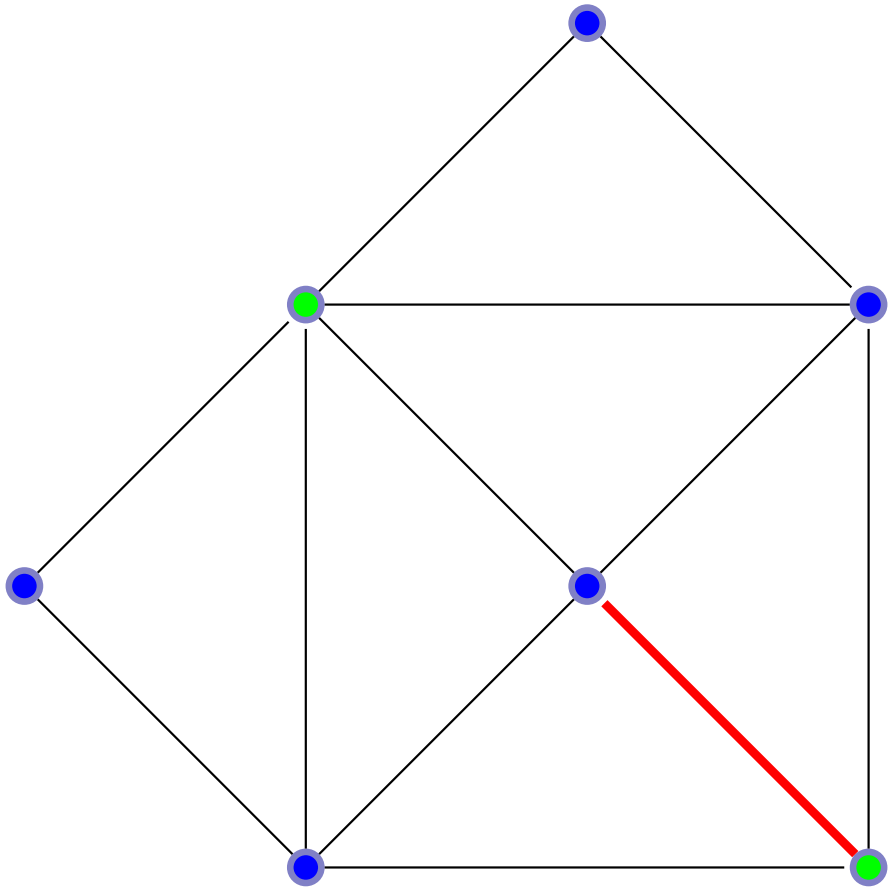
# “Doodleling”



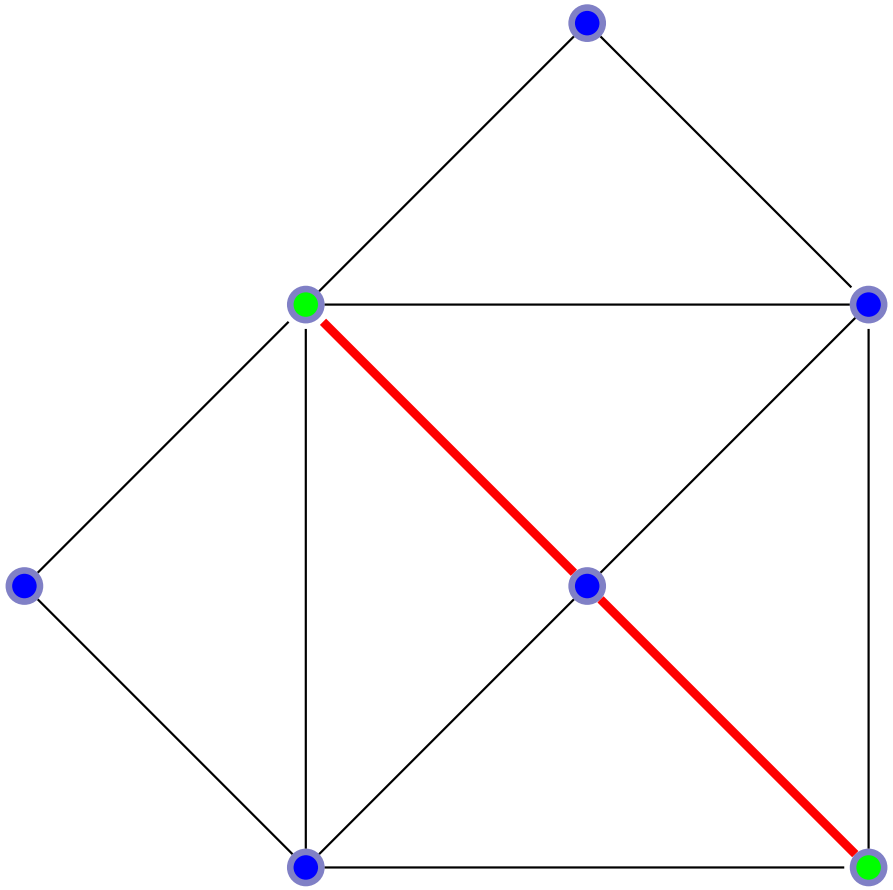
# Casa do Pai Natal



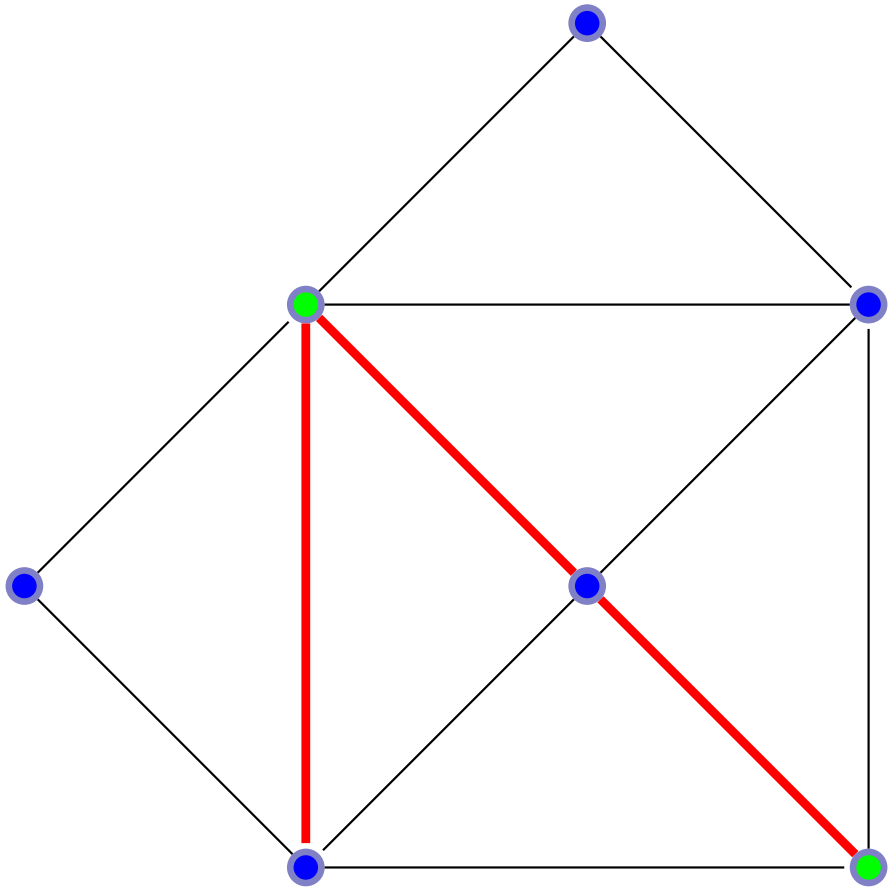
# Casa do Pai Natal



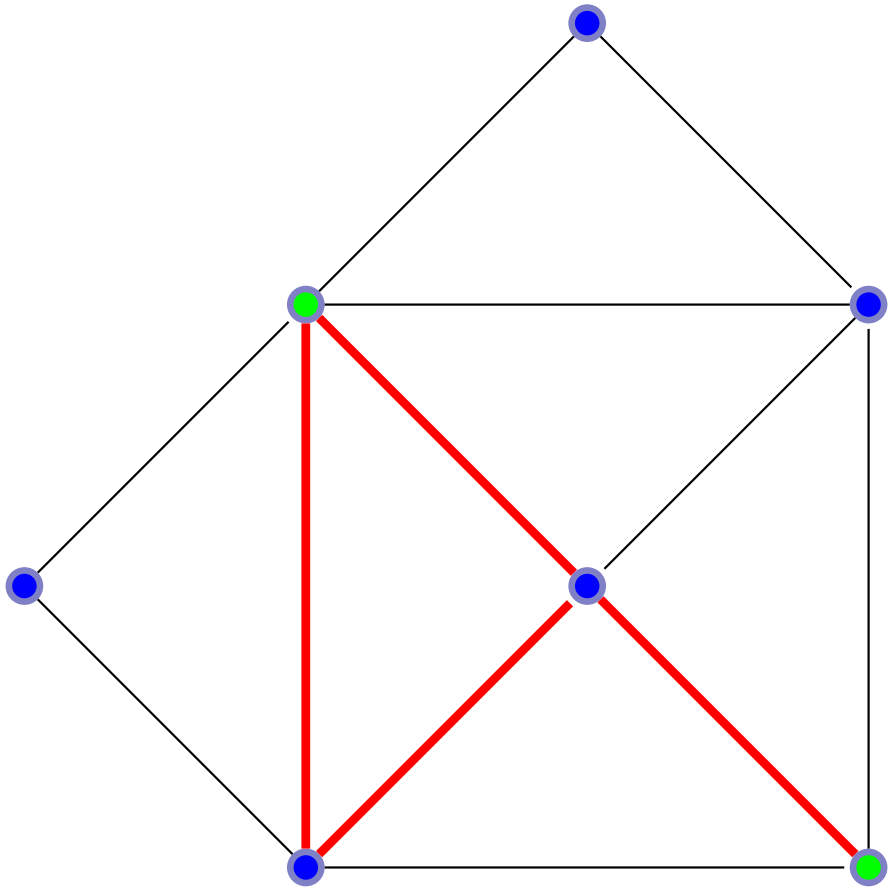
# Casa do Pai Natal



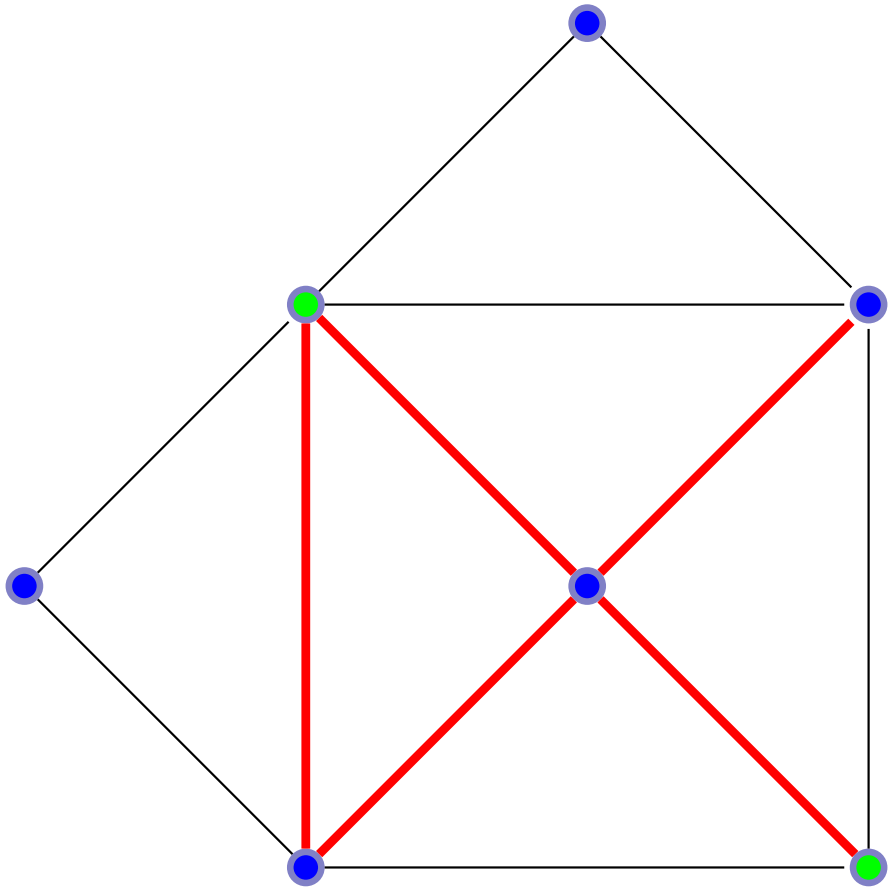
# Casa do Pai Natal



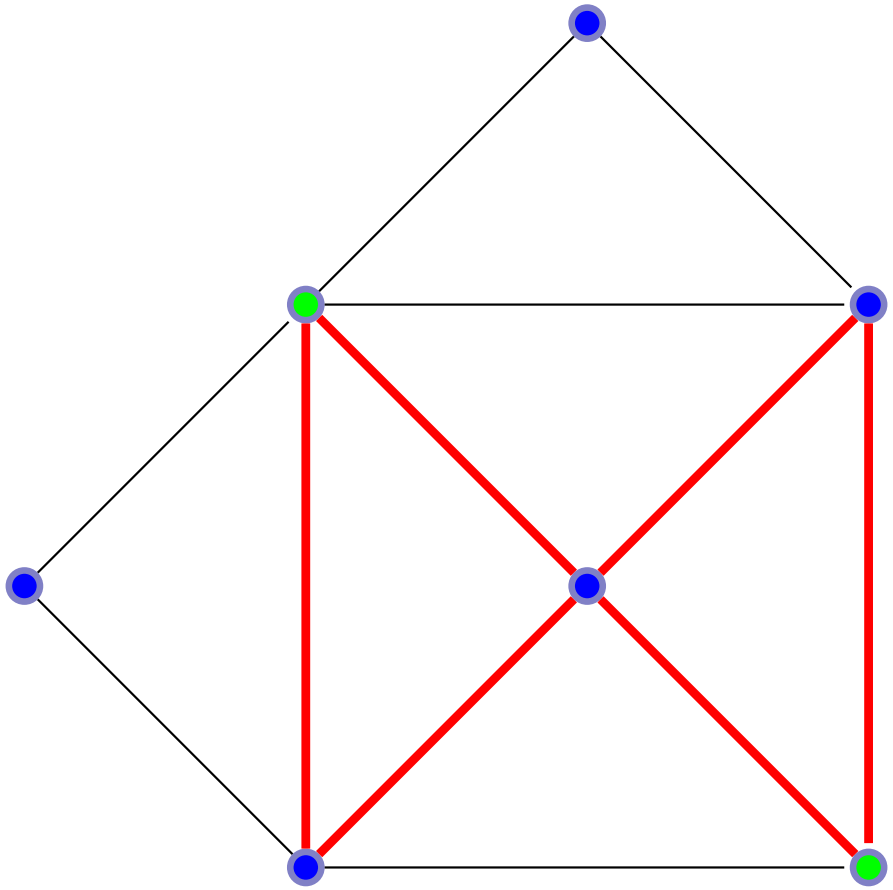
# Casa do Pai Natal



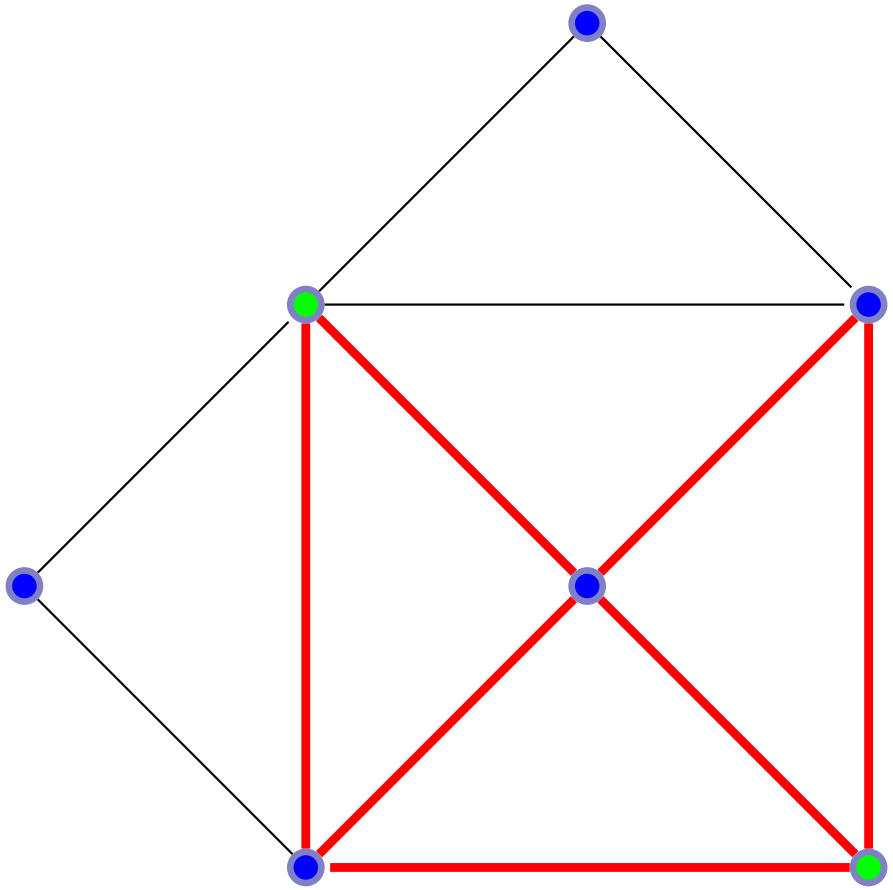
# Casa do Pai Natal



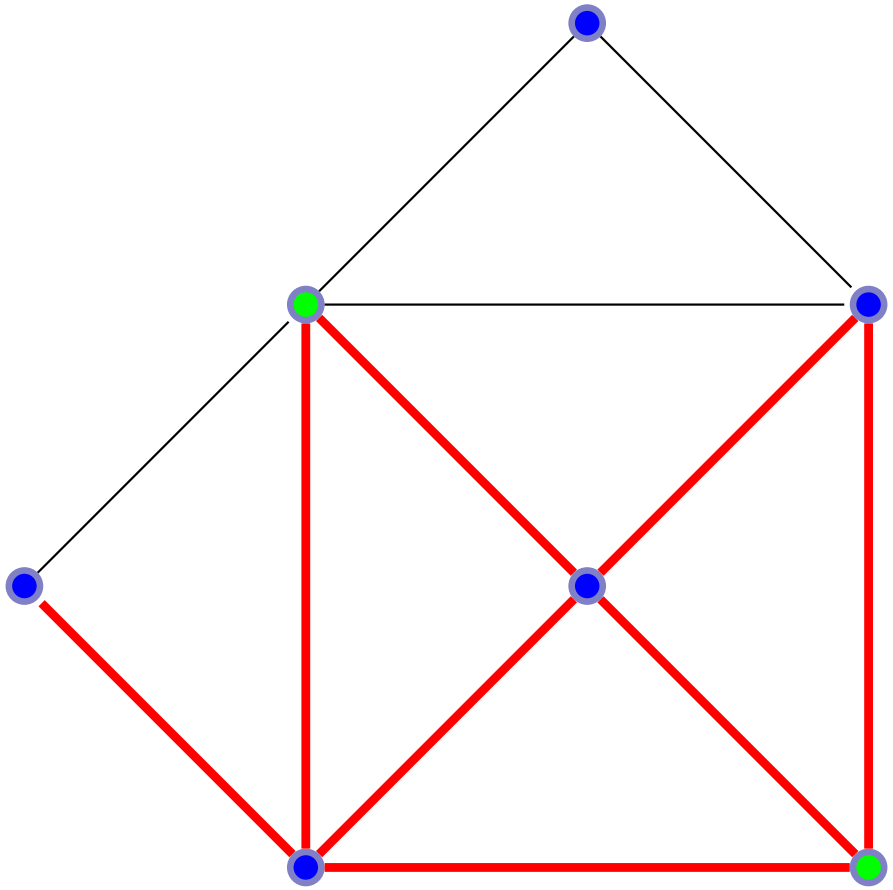
# Casa do Pai Natal



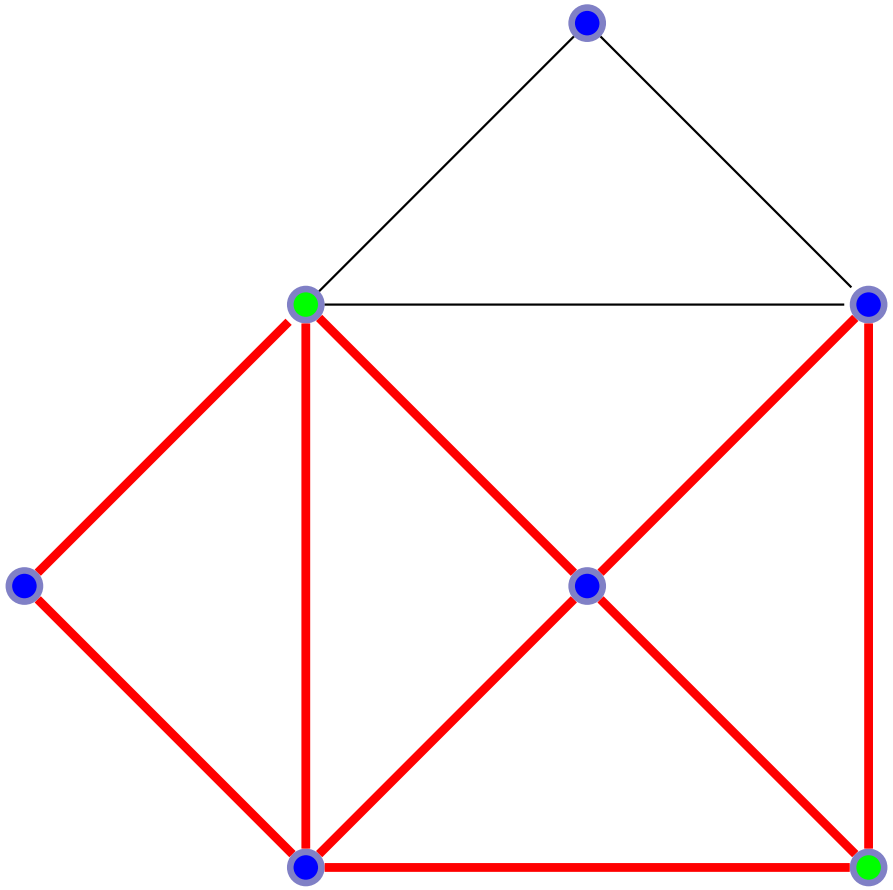
# Casa do Pai Natal



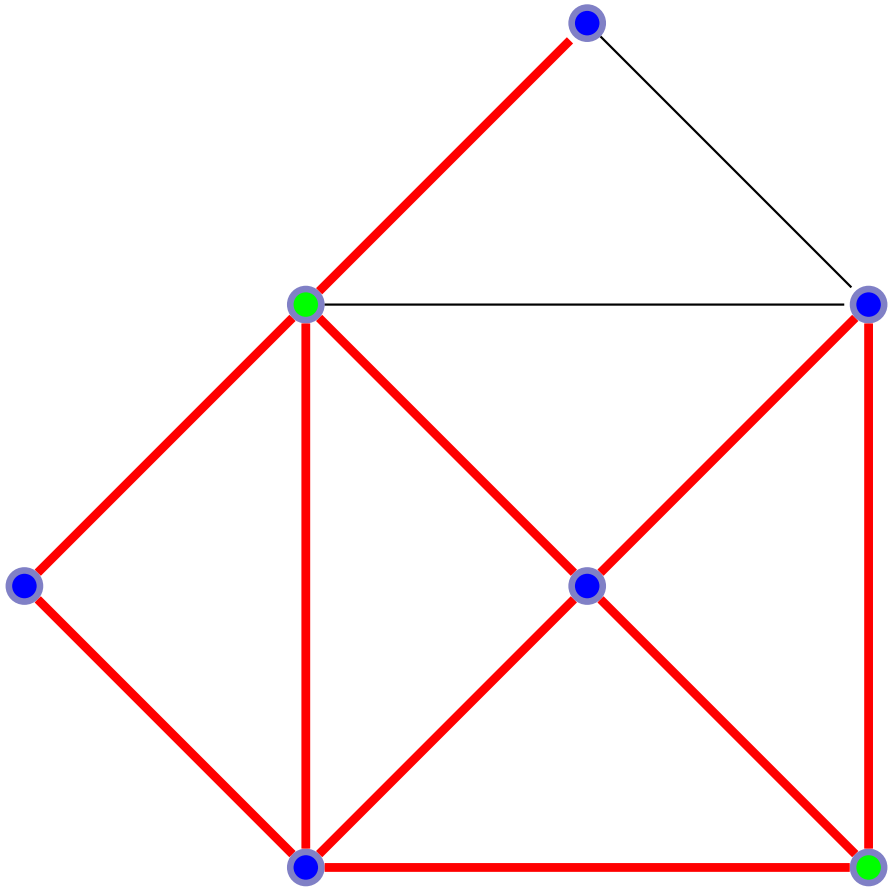
# Casa do Pai Natal



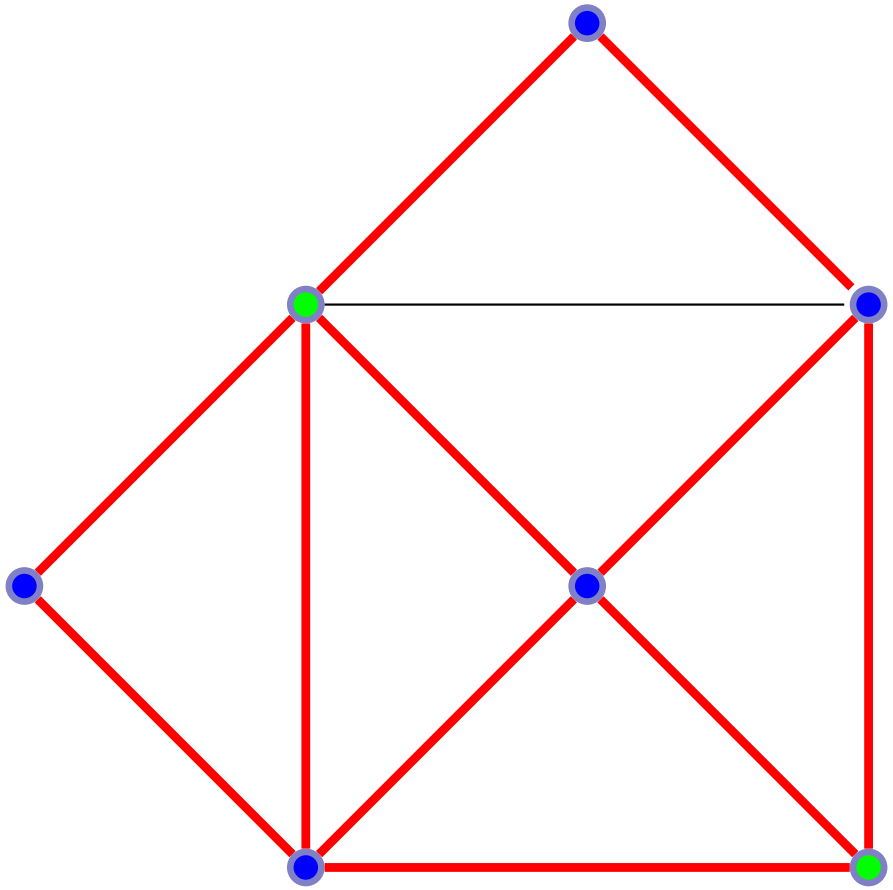
# Casa do Pai Natal



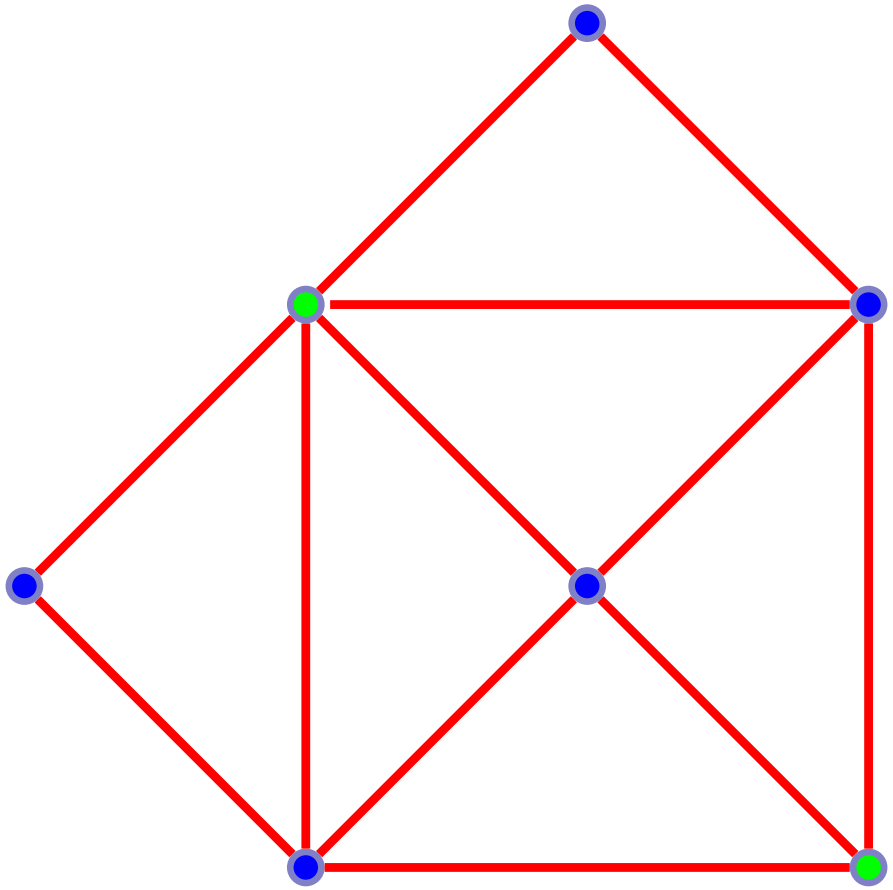
# Casa do Pai Natal



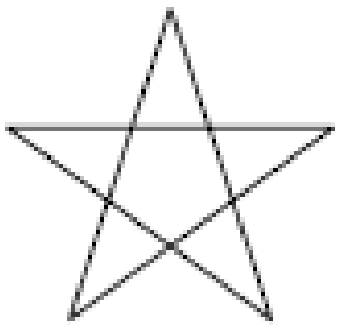
# Casa do Pai Natal

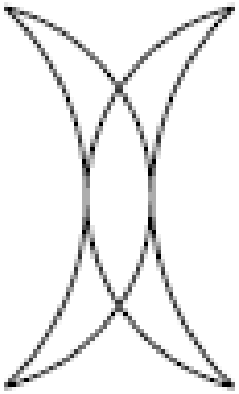


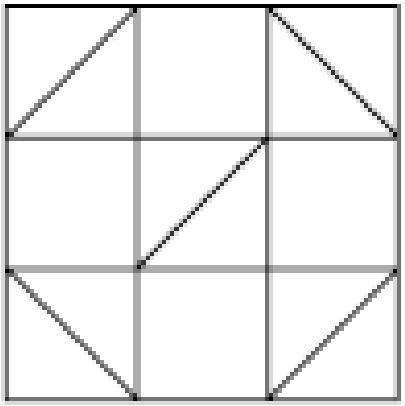
# Casa do Pai Natal



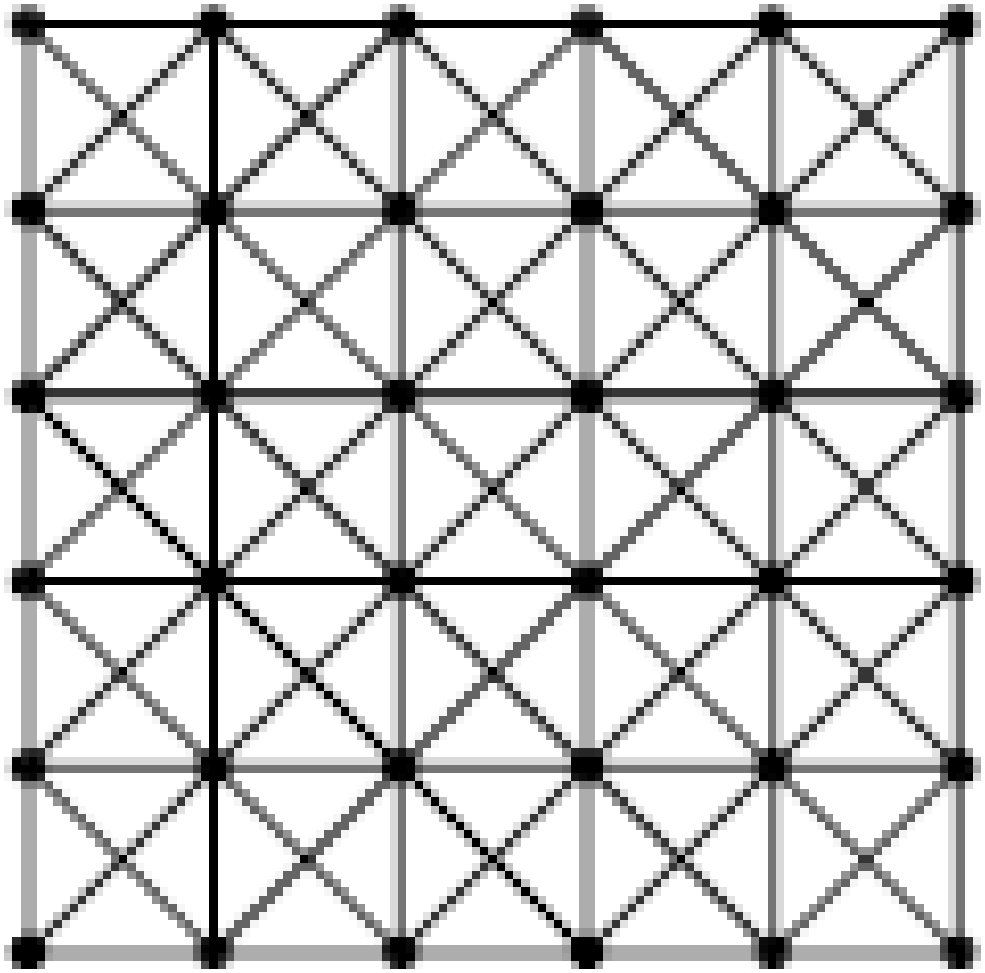
# “Doodling a lot”: Pitágoras, Maomé e Tutte (*vide* [BC87])







## Quantas vezes é necessário levantar a caneta do papel? (*Vide* [BC87])



# 2. Labirintos

# Catedral de Notre Dame d'Amiens

# Catedral de Notre Dame d'Amiens

# Catedral de Notre Dame de Chartres

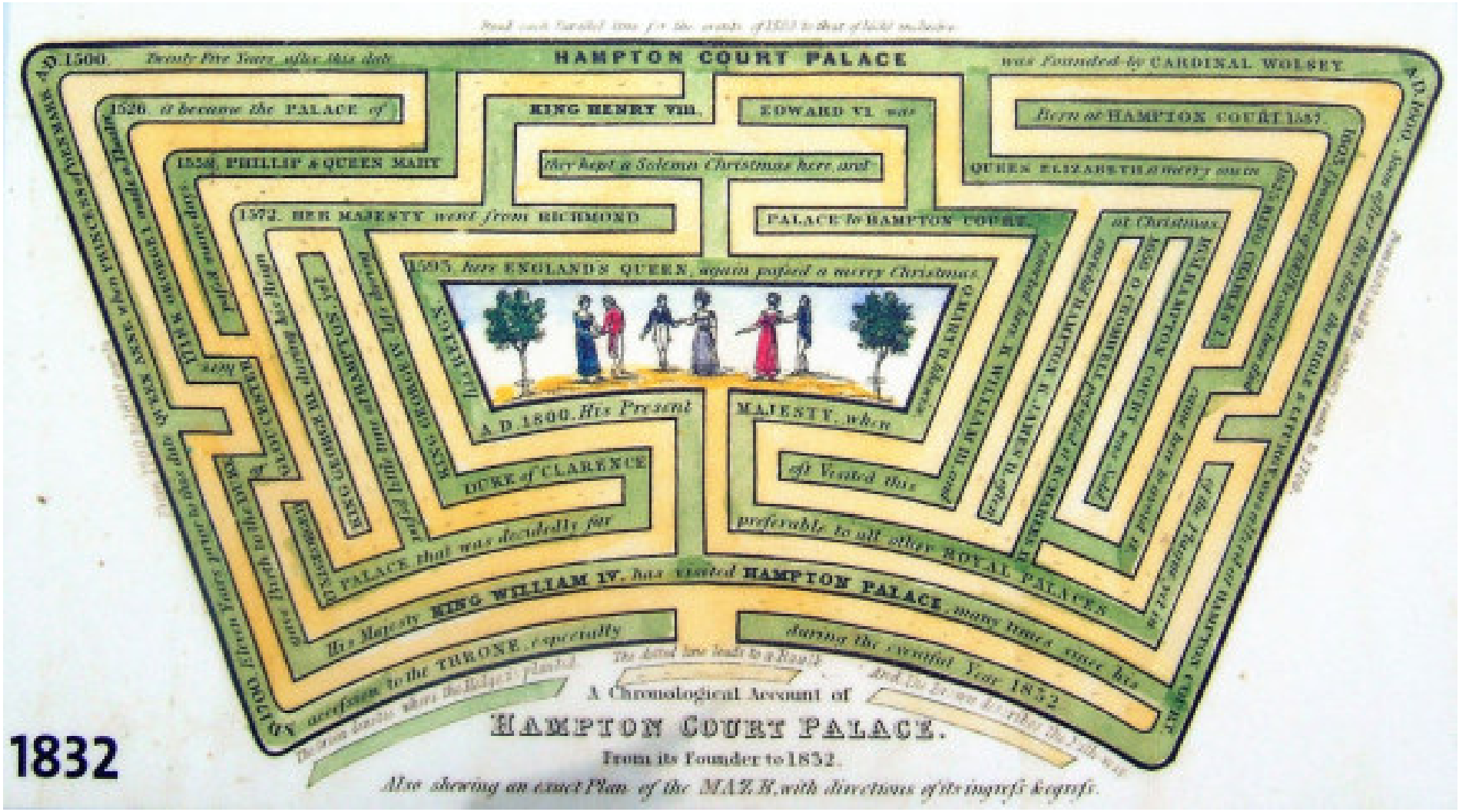
# Labirinto de Hever Castle

# Labirinto de New Hampton

# Labirinto de Wiltshire (Somerset)

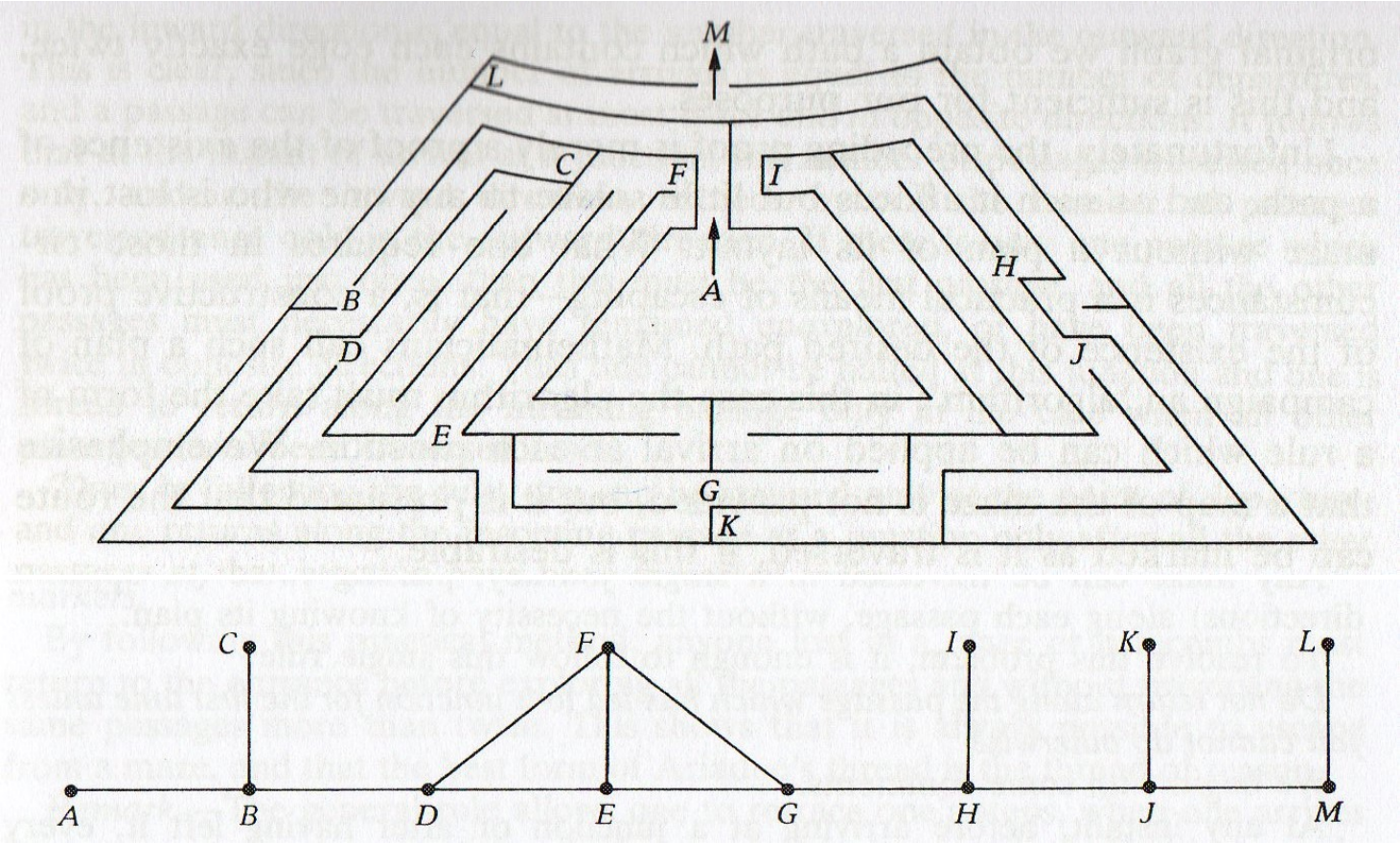
# Labirinto de Reignac sur Indre (França)

# Labirinto de New Hampton




1832

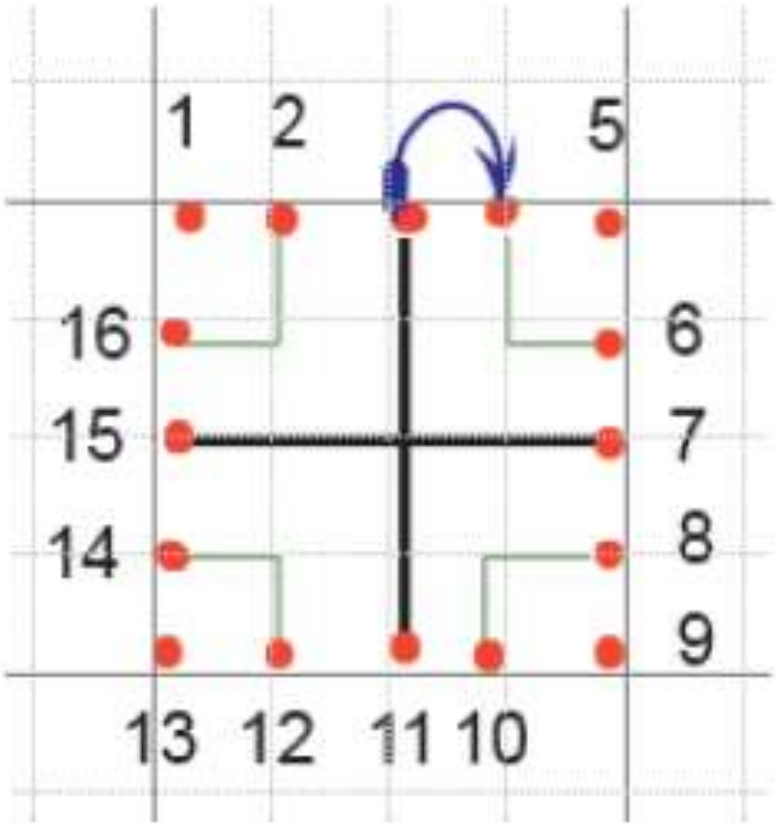
# Labirinto de New Hampton (*vide* [BLW06])



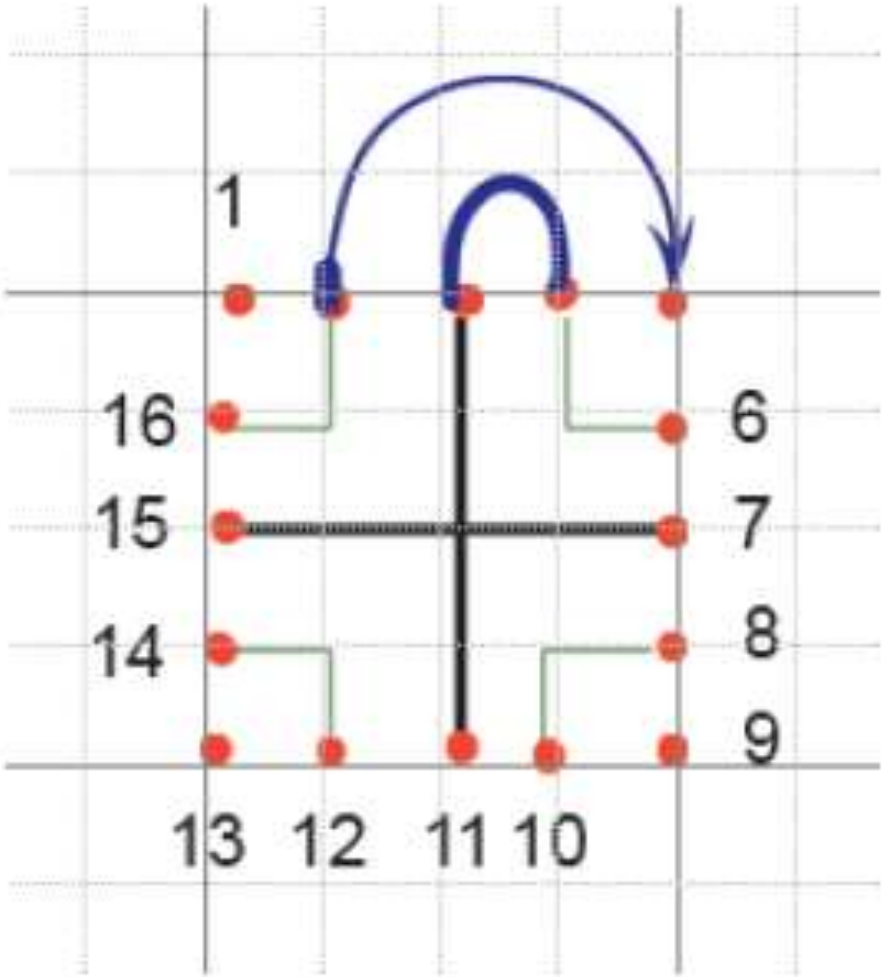
# 4. O Labirinto de Dédalo

# O labirinto de Dédalo

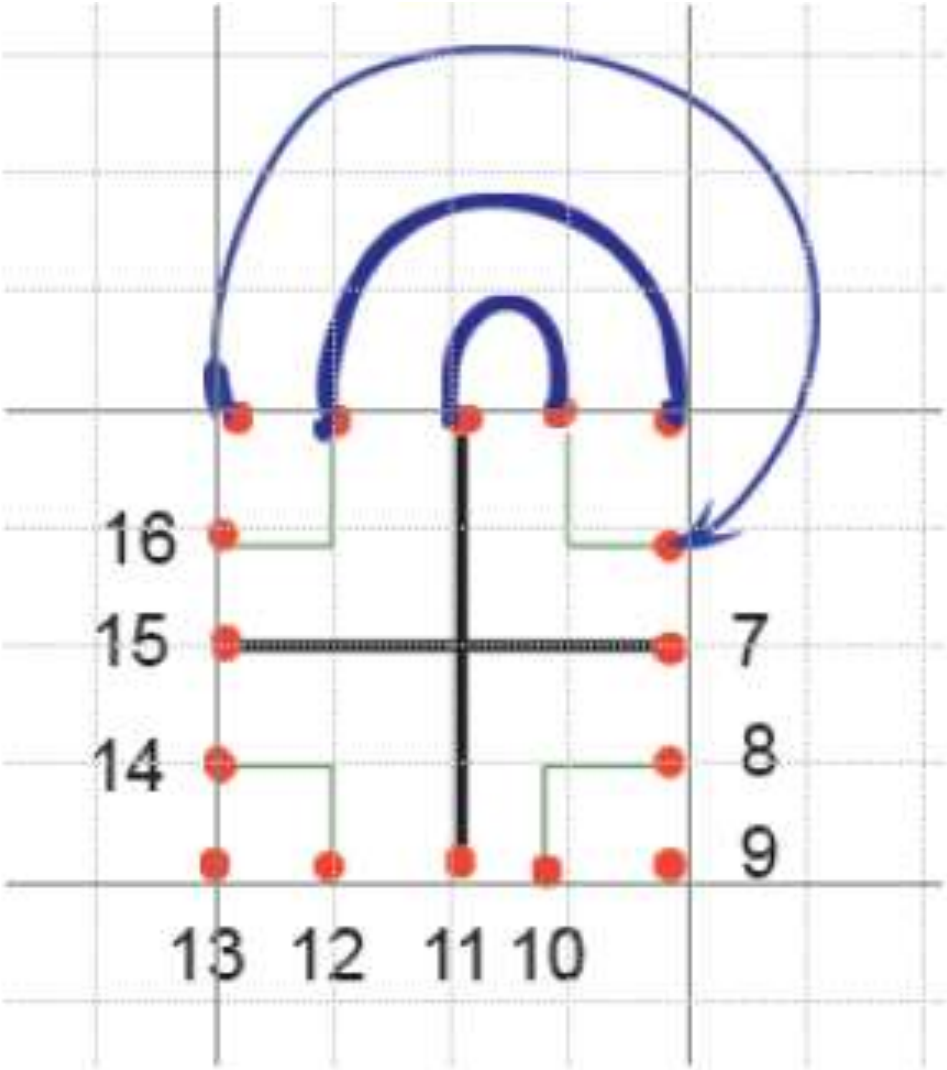
# O Labirinto de Dédalo



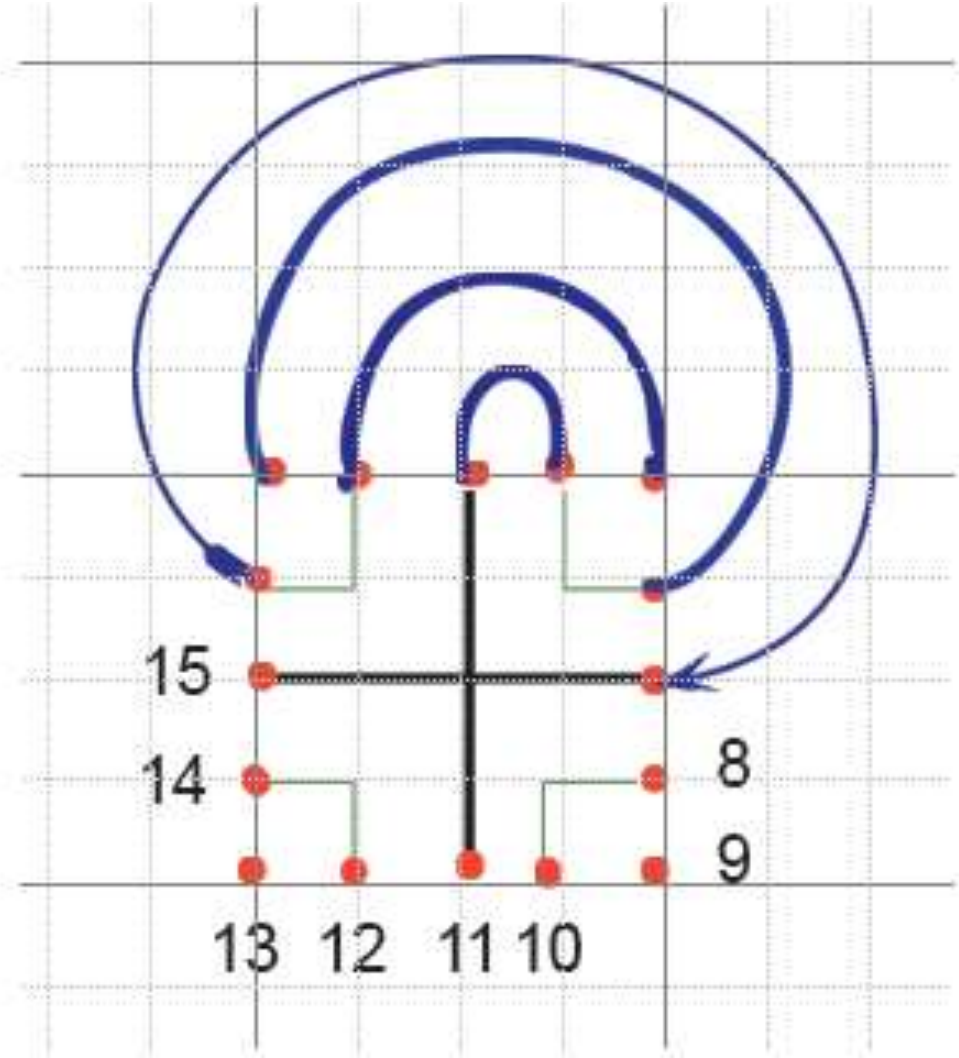
# O labirinto de Dédalo



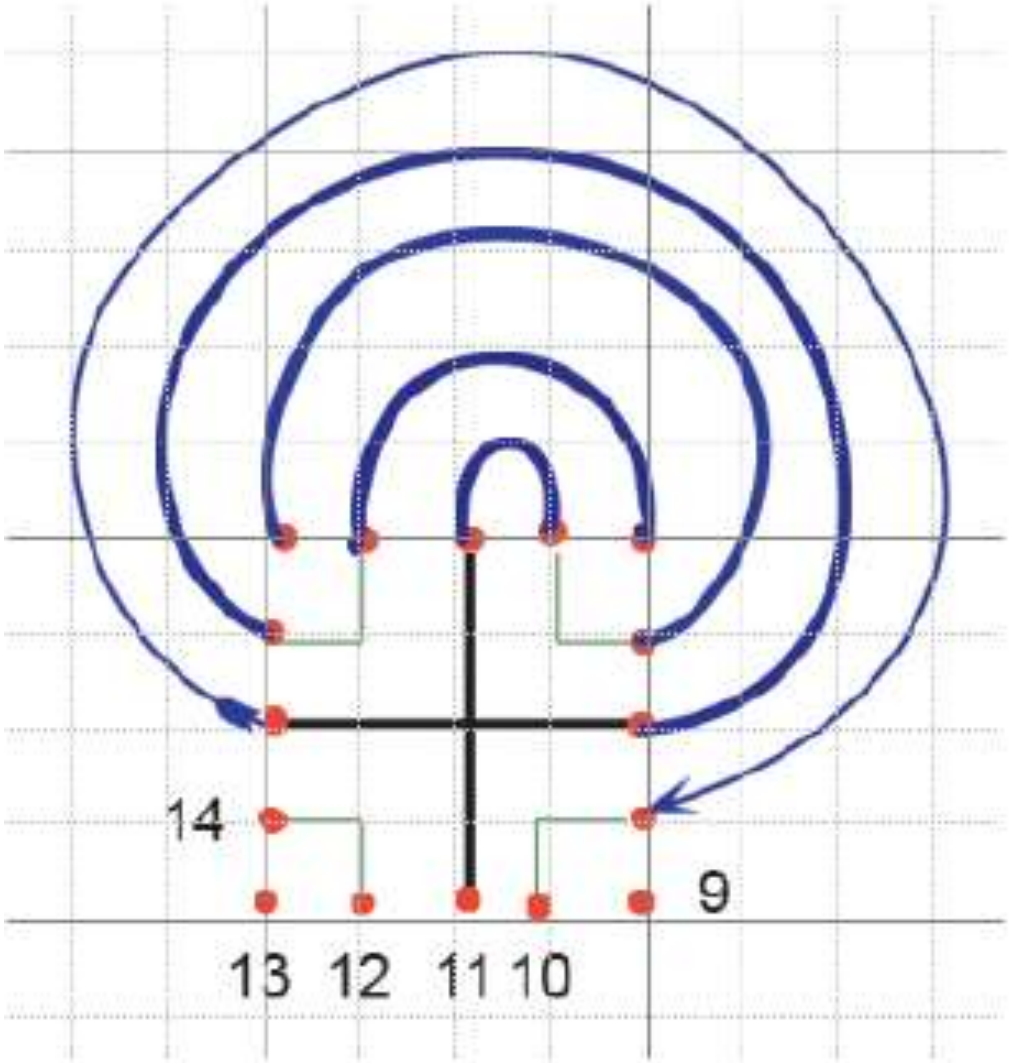
# Labirinto de Creta



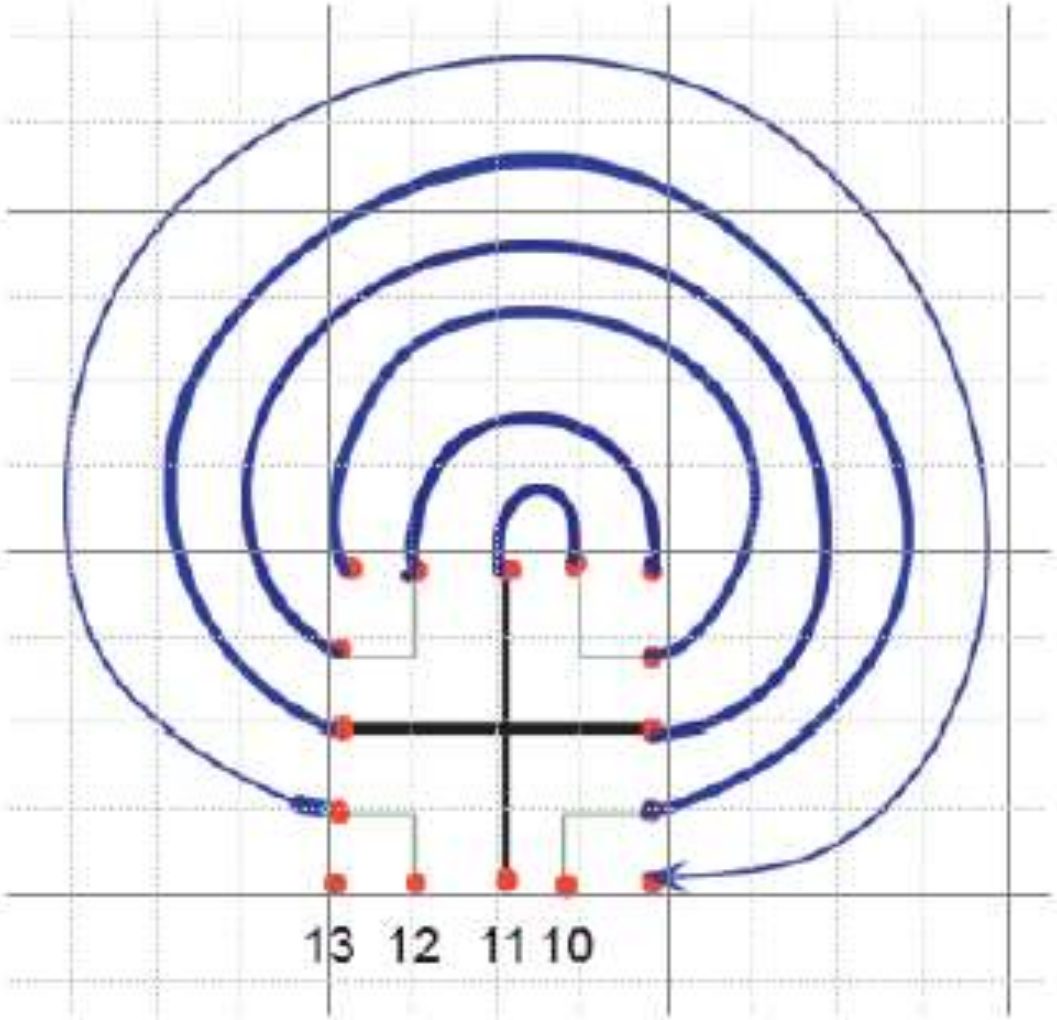
# O labirinto de Dédalo



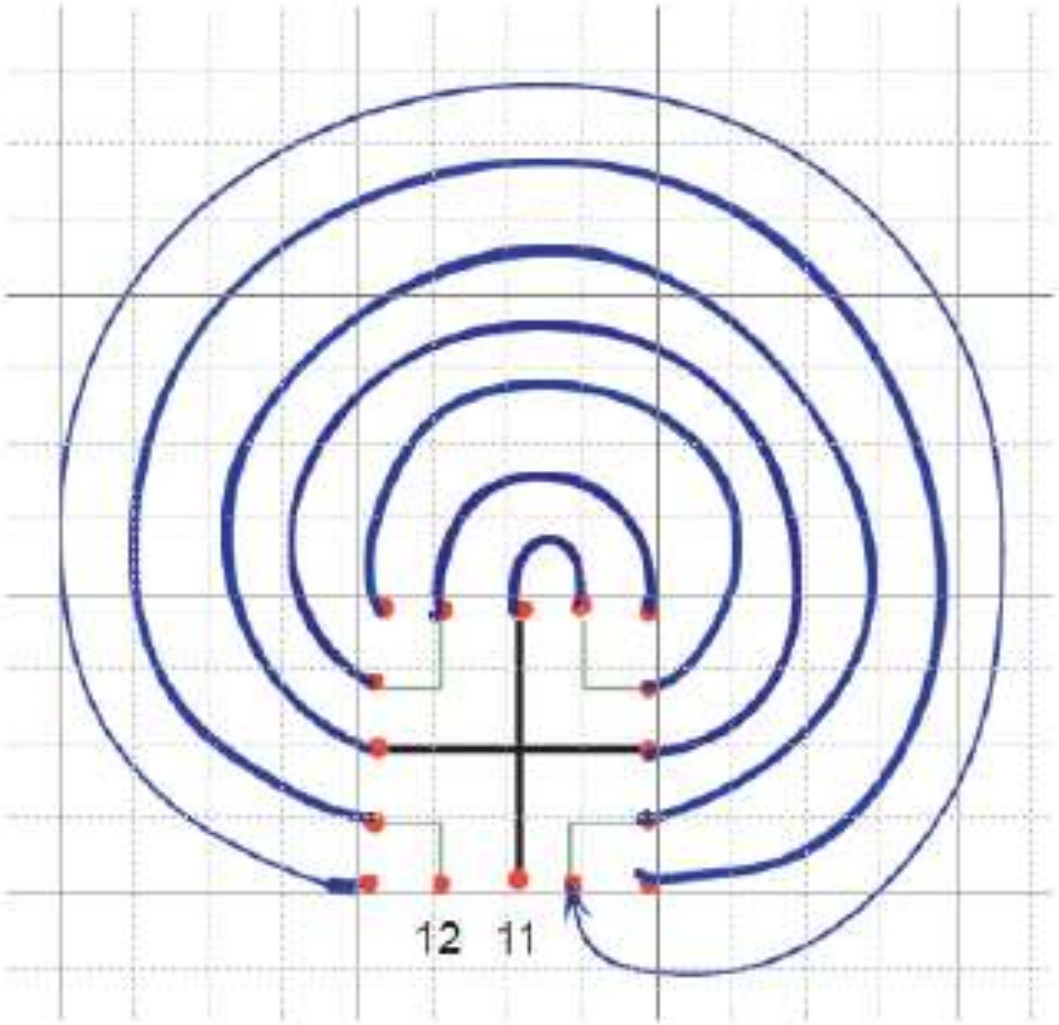
# O labirinto de Dédalo



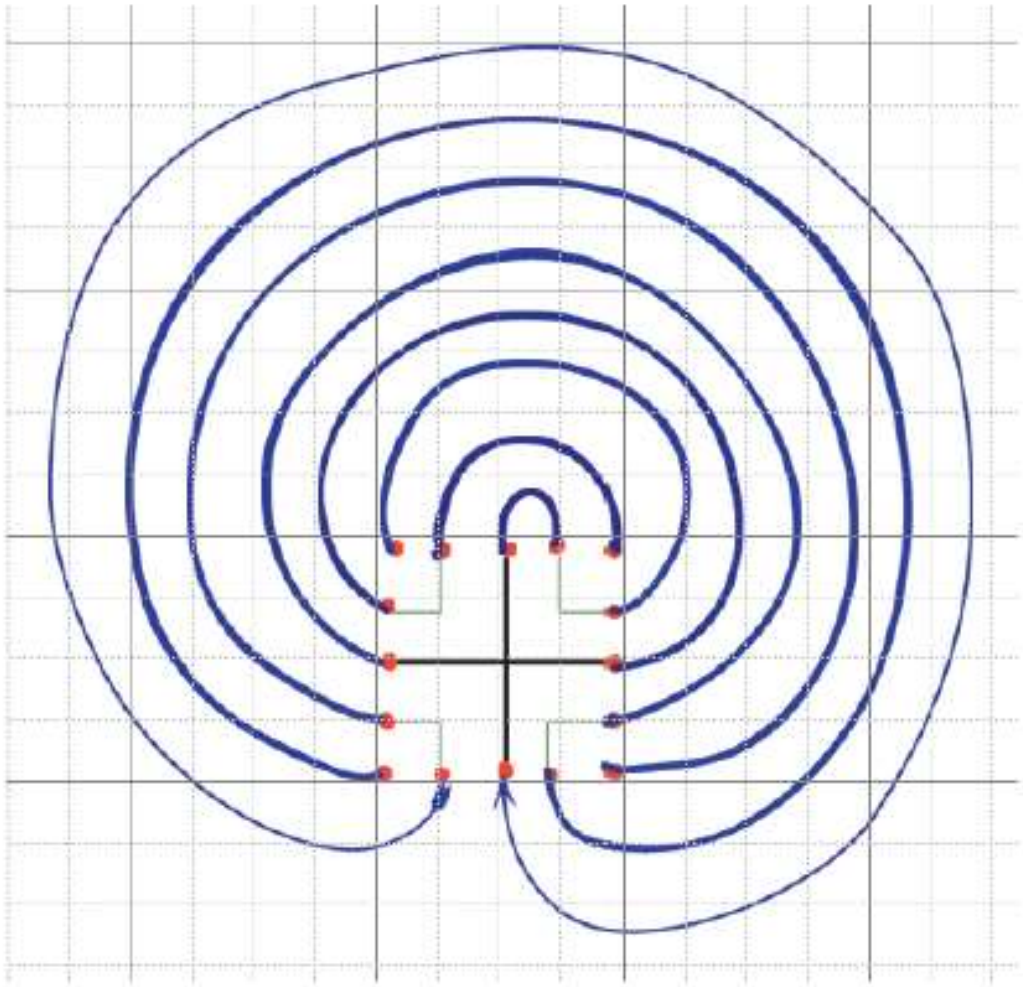
# O labirinto de Dédalo



# O labirinto de Dédalo



# O labirinto de Dédalo



# O labirinto de Dédalo na *Eneida* de Virgílio

## *Eneida*, Livro V, trad. João Franco Barreto

*Como já na alta Creta o monstruoso*
*Labirinto se diz teve tecido*
*Um caminho tão cego, e portentoso,*
*Com outros mil caminhos dividido;*
*Que era assaz intrincado e duvidoso*
*Error, e tão confuso e incompreendido,*
*Que aos que entravam nele era impossível*
*Sair daquela confusão terrível:*

# Labirinto da Quinta da Regaleira em Sintra

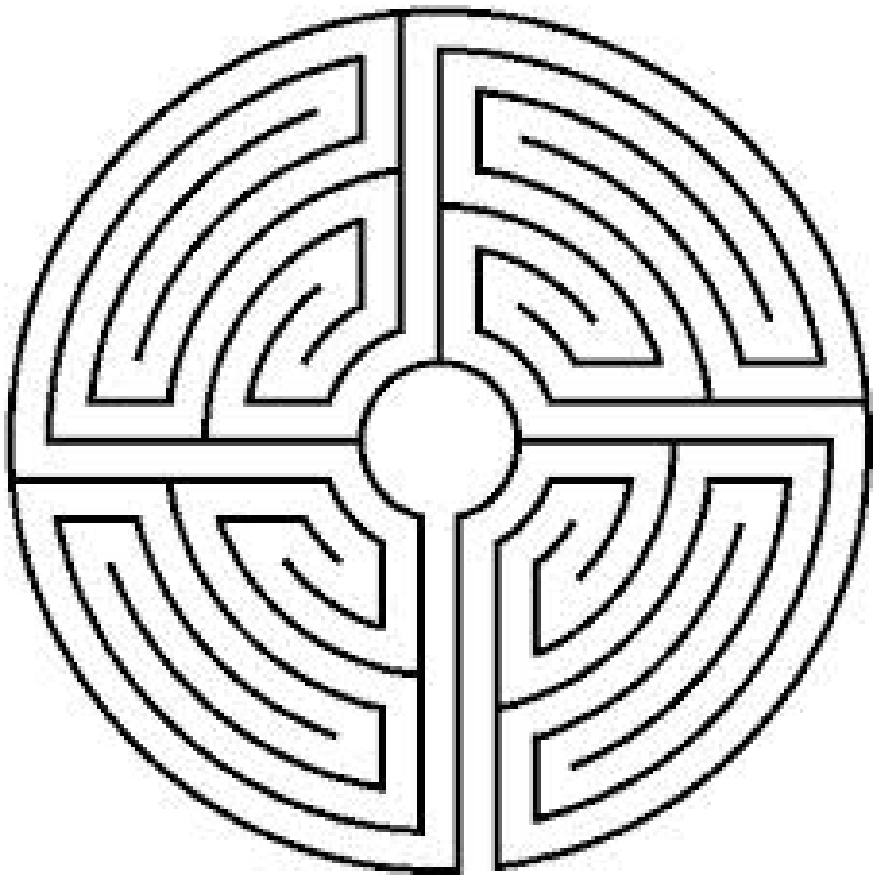
# O labirinto de Dédalo nas *Metamorfoses* de Ovídio

## *Metamorfoses*, Livro VIII, trad. Paulo Farmhouse Alberto

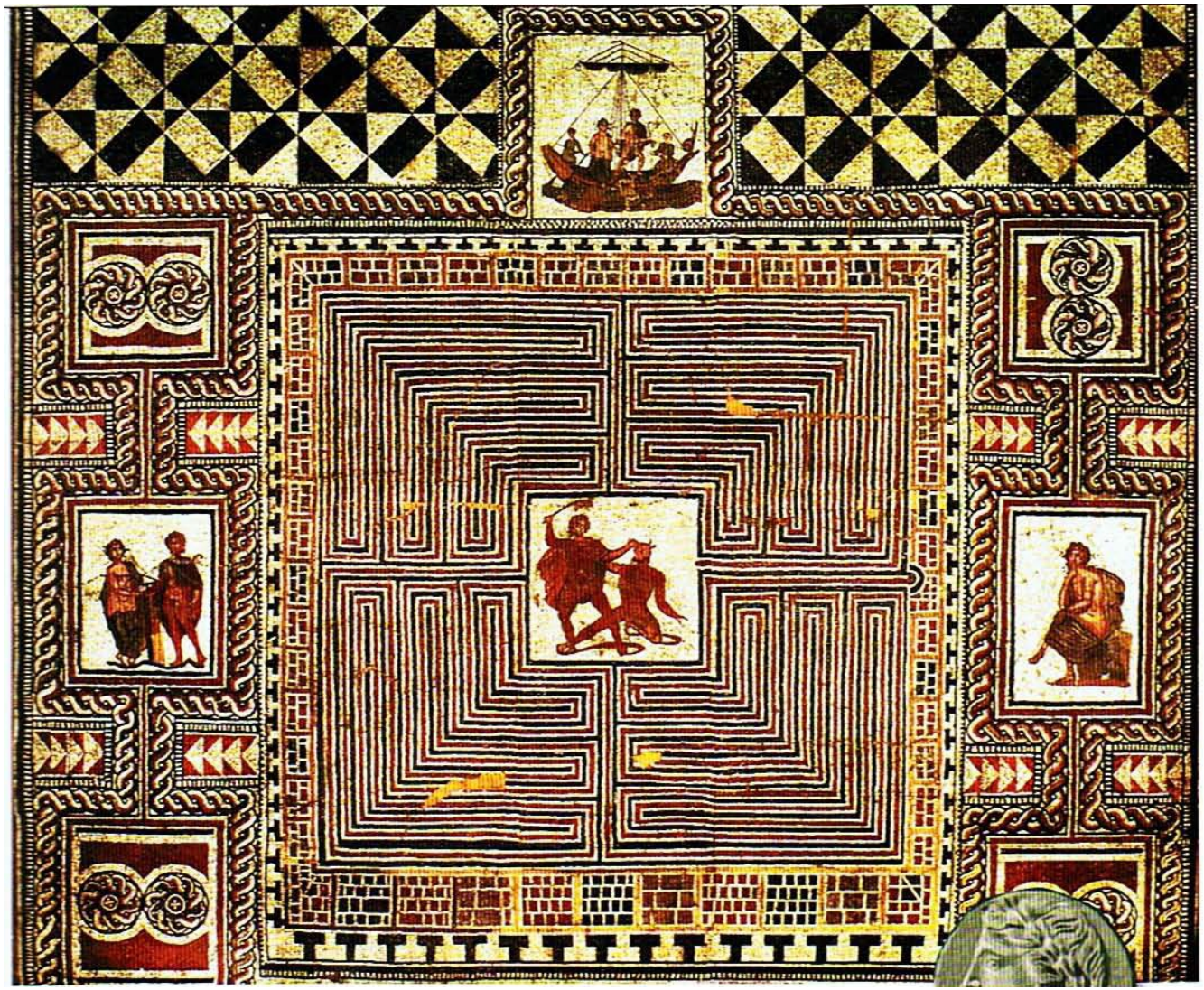
*Celebérrimo pelo seu talento na arte da arquitetura, Dédalo*
*encarrega-se da obra, baralha os sinais e faz o olhar enganar-se*
*em retorcidas curvas e contracurvas de corredores sem conta.*
*Tal como na Frígia o Meandro nas límpidas águas se diverte*
*fluindo e refluindo num deslizar que confunde, e, correndo*
*ao encontro de si próprio, contempla a água que há-de vir,*
*e, voltando-se ora para nascente, ora para o mar aberto, empurra a sua corrente*
*sem rumo certo, assim enche Dédalo*
*os inumeráveis corredores de equívocos. A custo ele próprio*
*logrou voltar à entrada: de tal modo enganador era o edifício.*

# Labirinto de Avaris

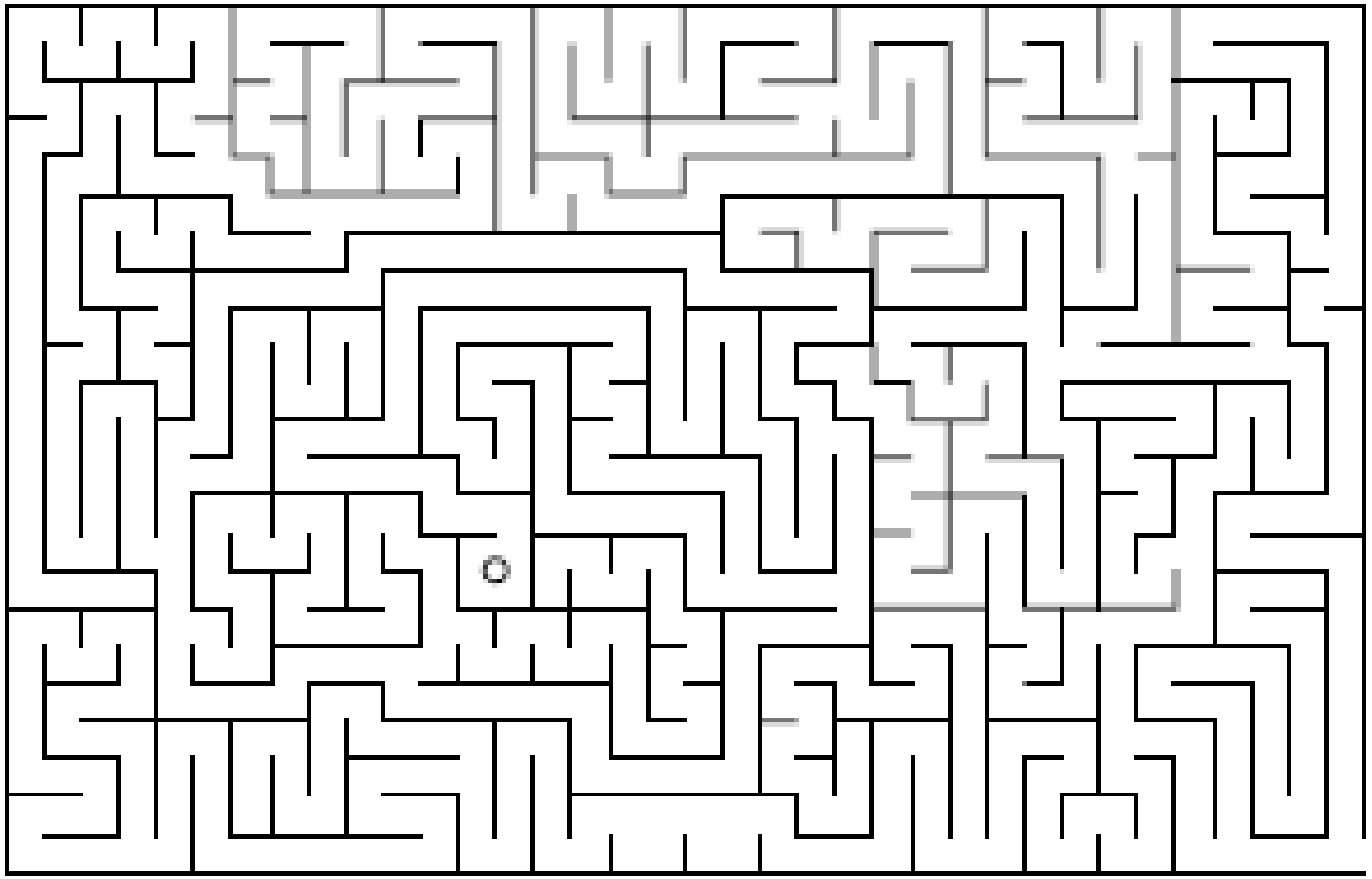
# Labirinto romano



# Labirinto romano



# Labirinto moderno (*vide* [BC87])



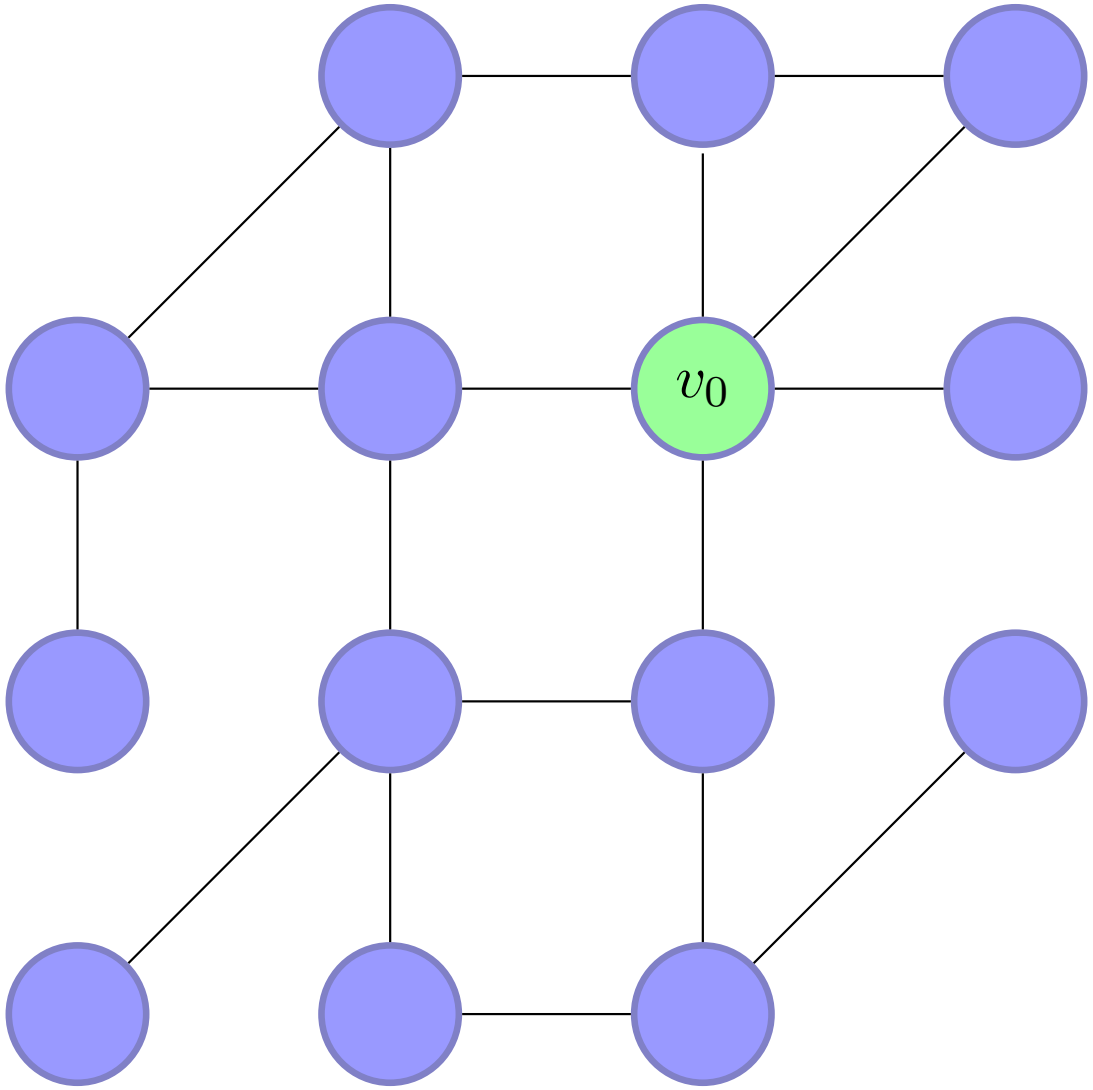
# Pesquisa em profundidade

## Pesquisa em profundidade

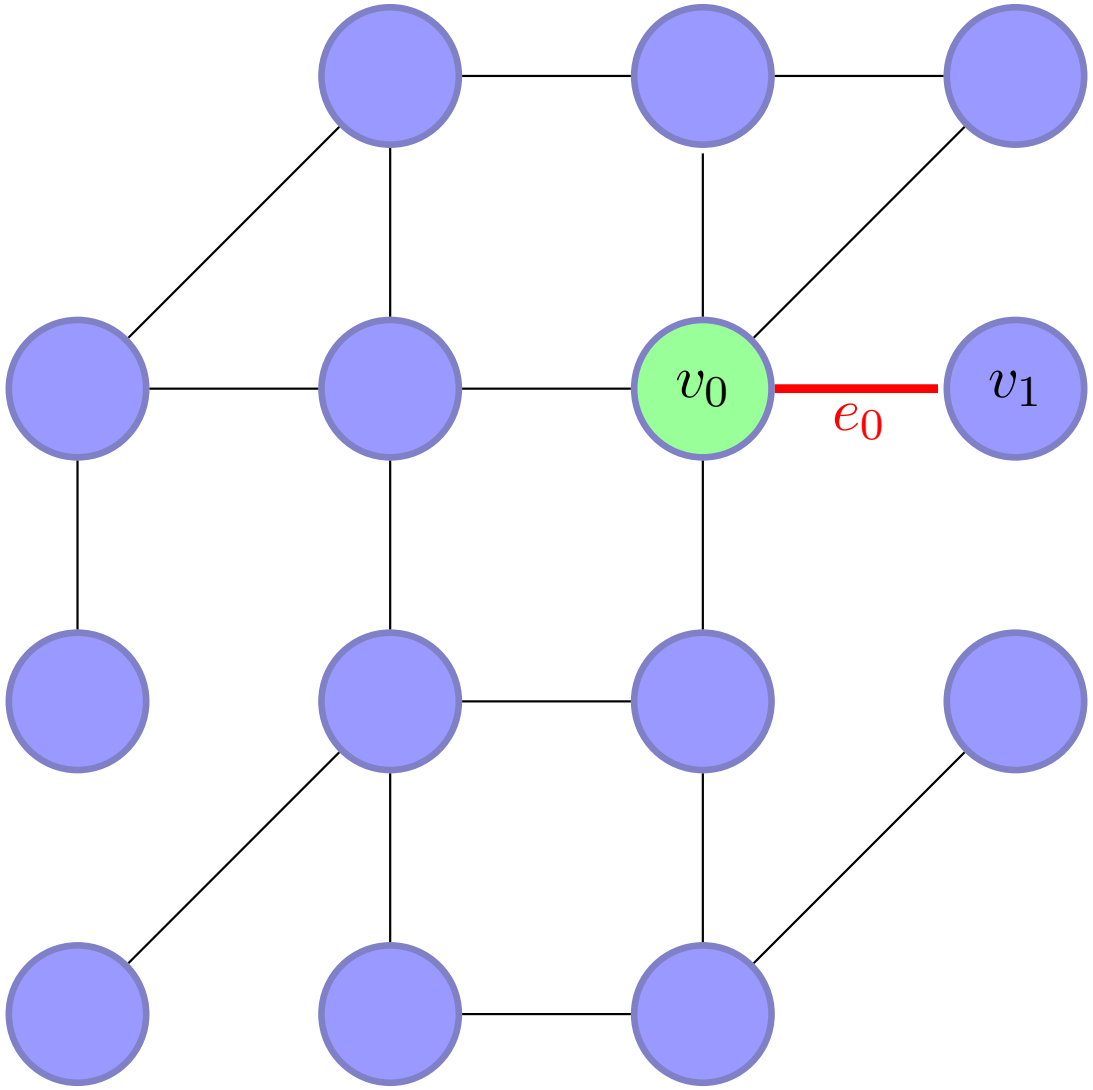
Seja $\mathcal{G}$ um grafo conexo e $v_0$ um vértice de $\mathcal{G}$. A árvore de cobertura de $\mathcal{G}$ obtida por pesquisa em profundidade constroi-se da seguinte maneira:

1. Toma-se $u = v_0$ como vértice corrente da pesquisa e $n = 0$. $\mathcal{T}_0$ é a árvore que contém somente o vértice $v_0$.
2. Escolhe-se, caso exista, um vértice $v_{n+1}$ ainda não incluído em $\mathcal{T}_n$ adjacente a $u$; toma-se para $\mathcal{T}_{n+1}$ a árvore que espande $\mathcal{T}_n$ com a nova aresta $v_n v_{n+1}$; toma-se $u := v_{n+1}$ e $n := n + 1$.
3. Repete-se o passo $2$ até que o vértice corrente $u$ não seja adjacente a nenhum vértice de $\mathcal{G}$ ainda não visitado. Se a árvore obtida $\mathcal{T}_n$ cobre todo o grafo, então a pesquisa termina; se não, retrocede-se para o vértice corrente anterior, i.e. toma-se $u = v_{n-1}$ e $n := n + 1$; repete-se o passo $2$.

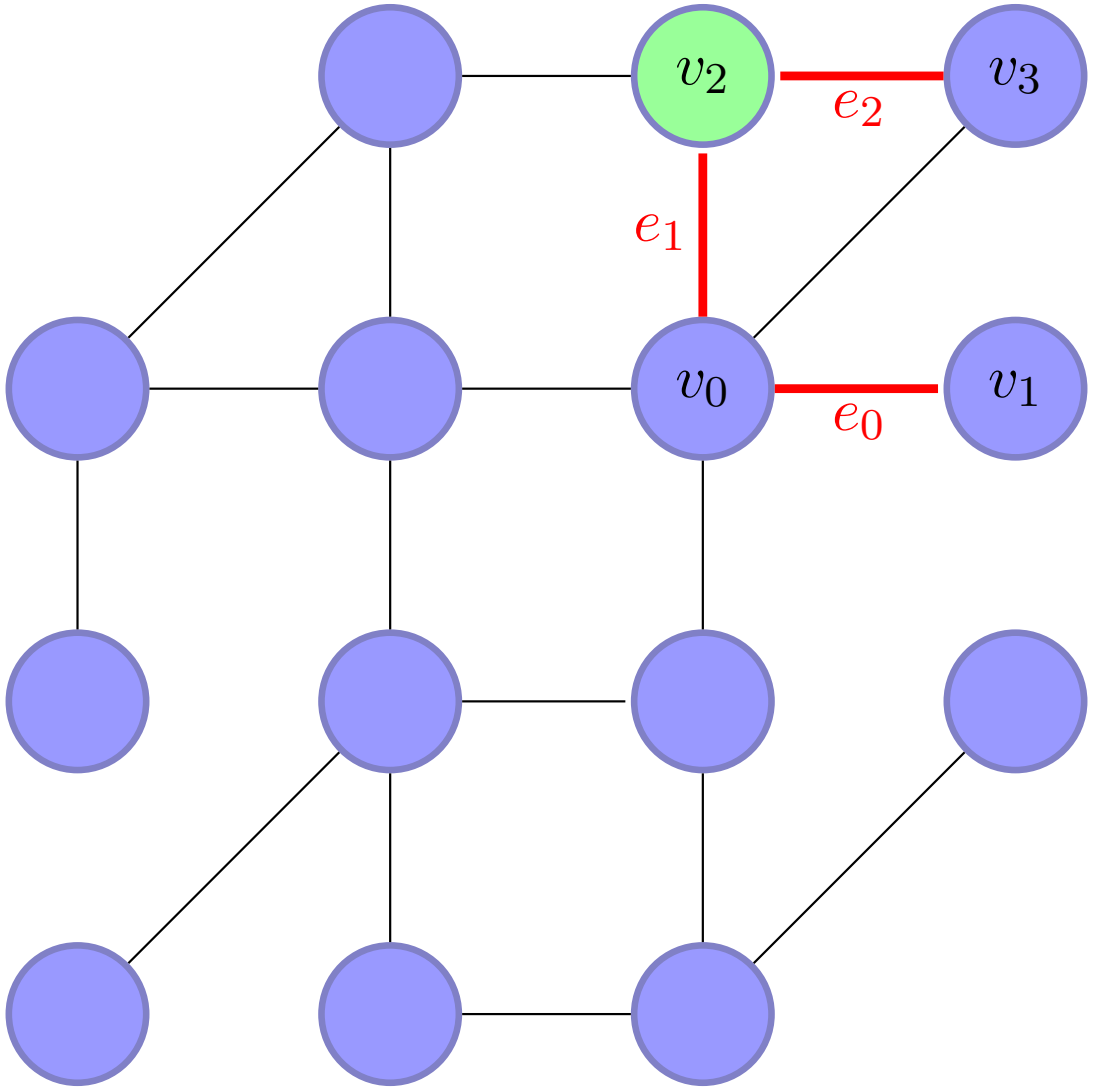
# Pesquisa em profundidade



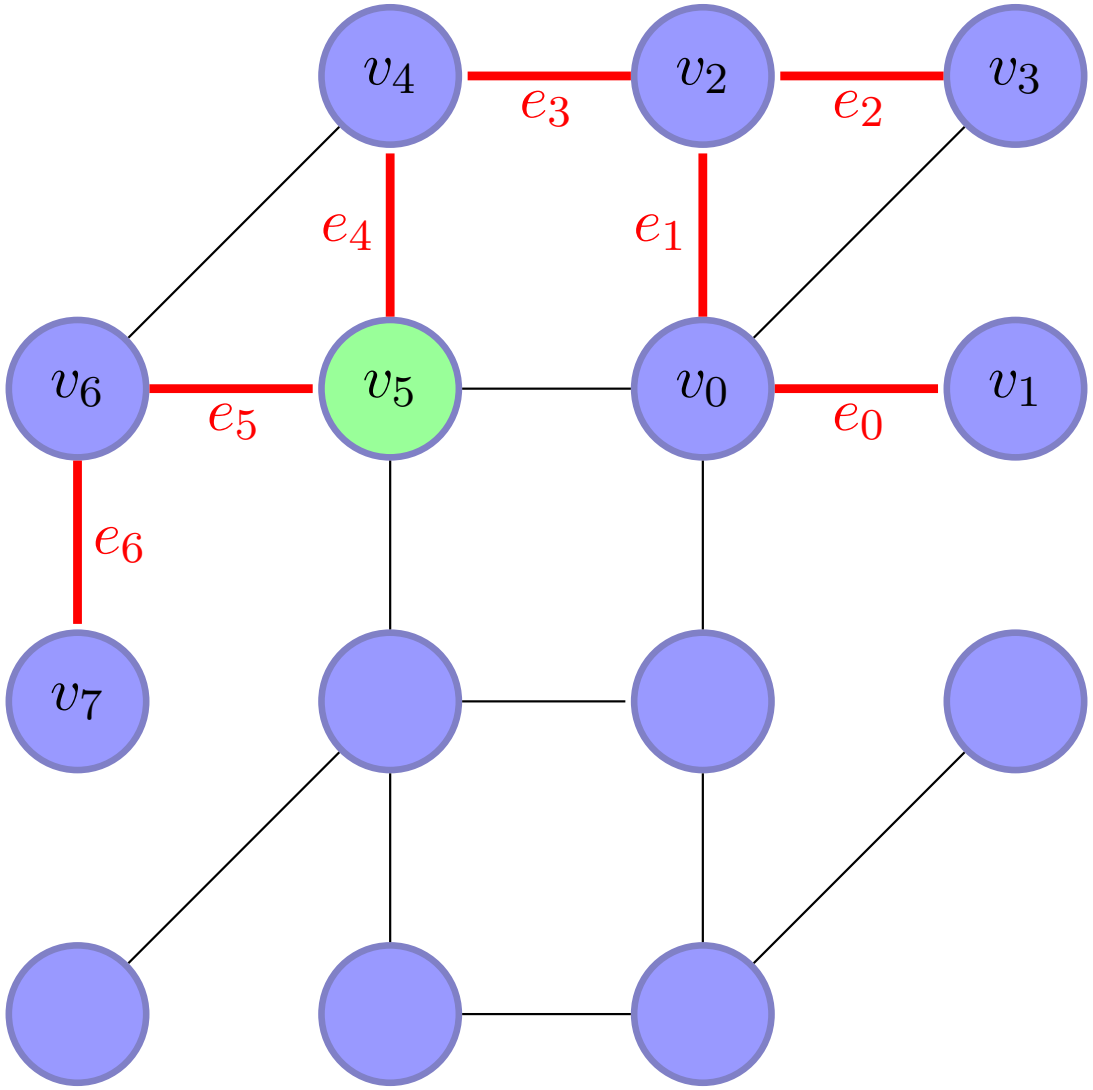
# Pesquisa em profundidade



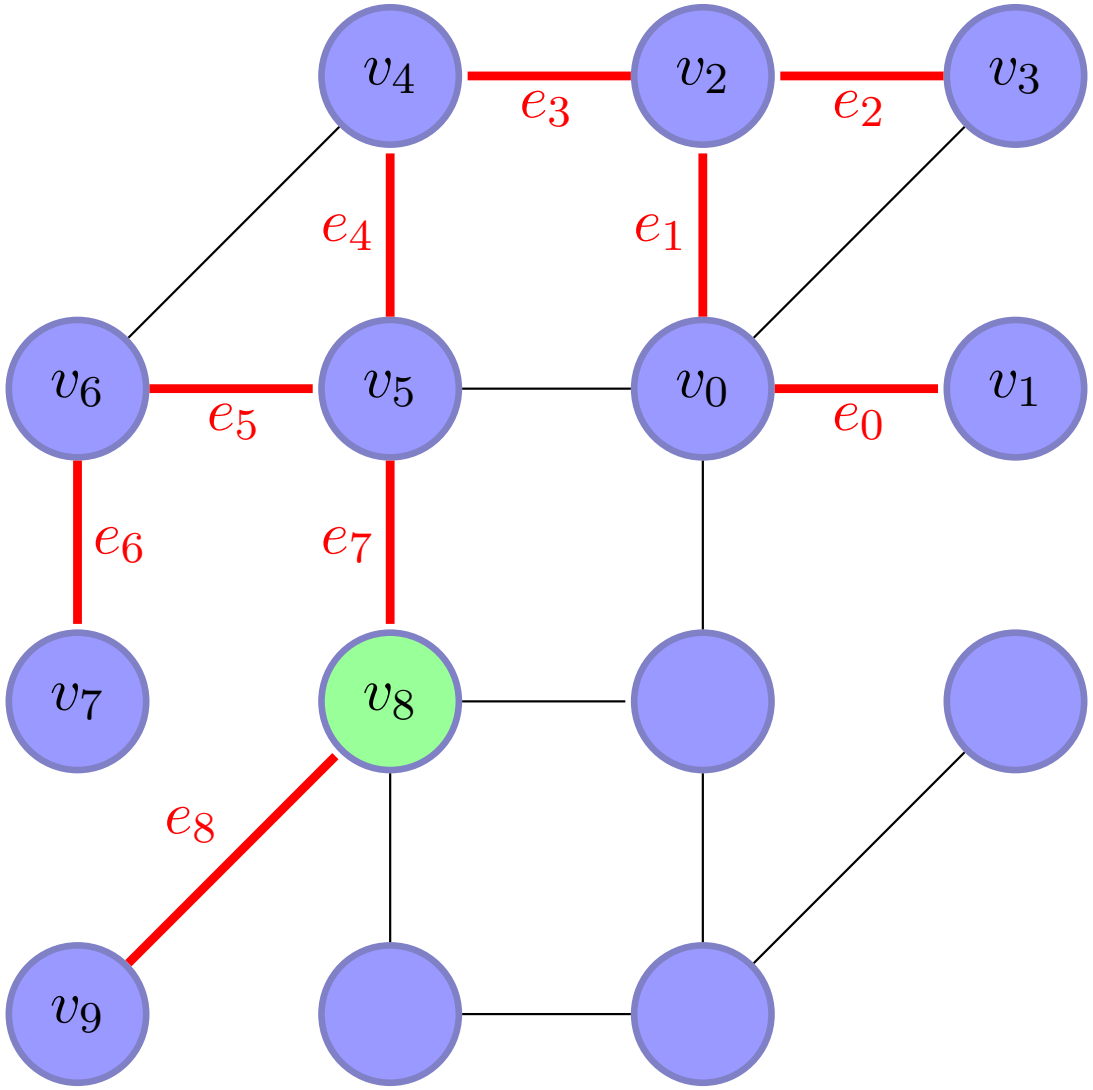
# Pesquisa em profundidade



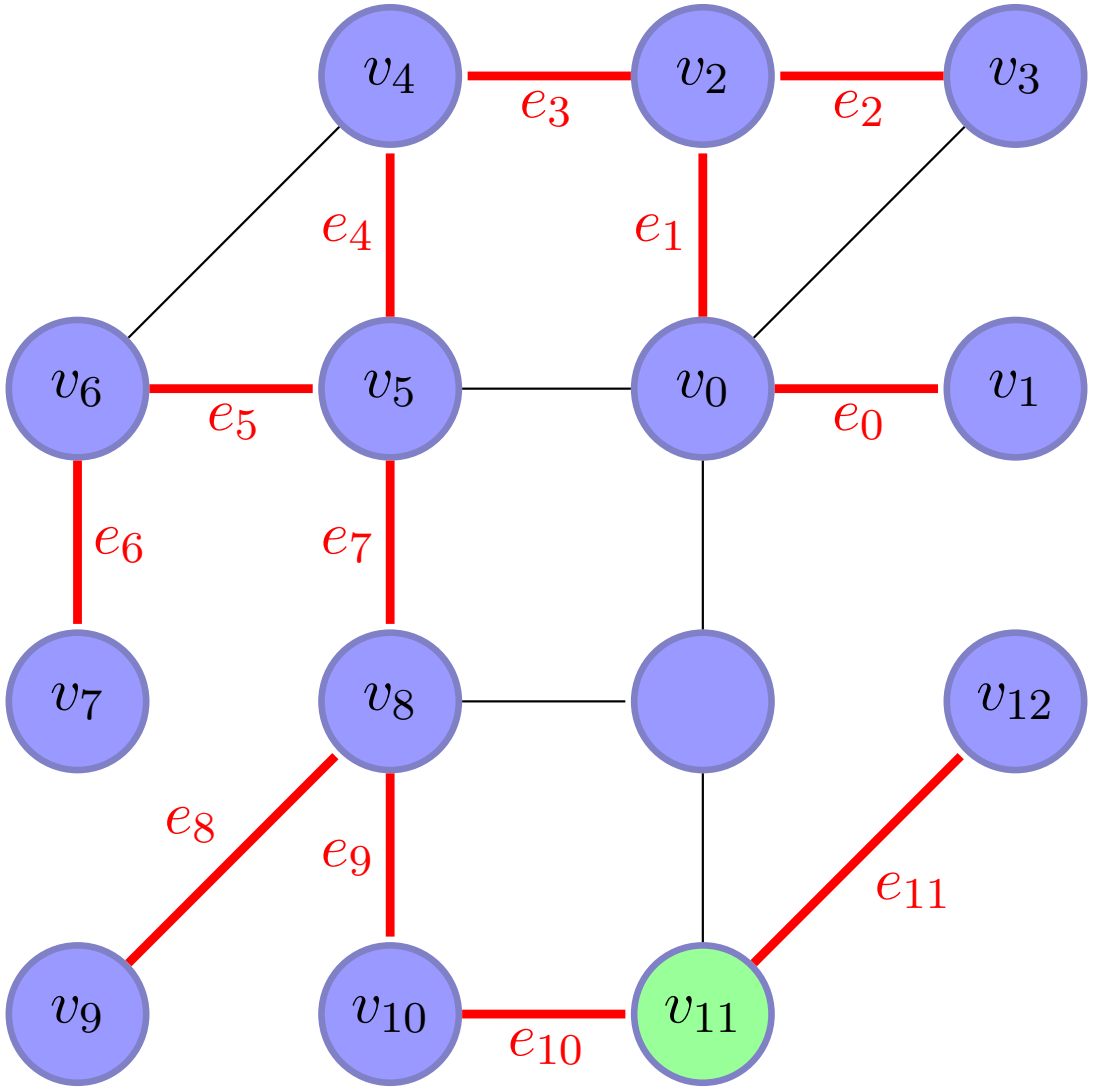
# Pesquisa em profundidade



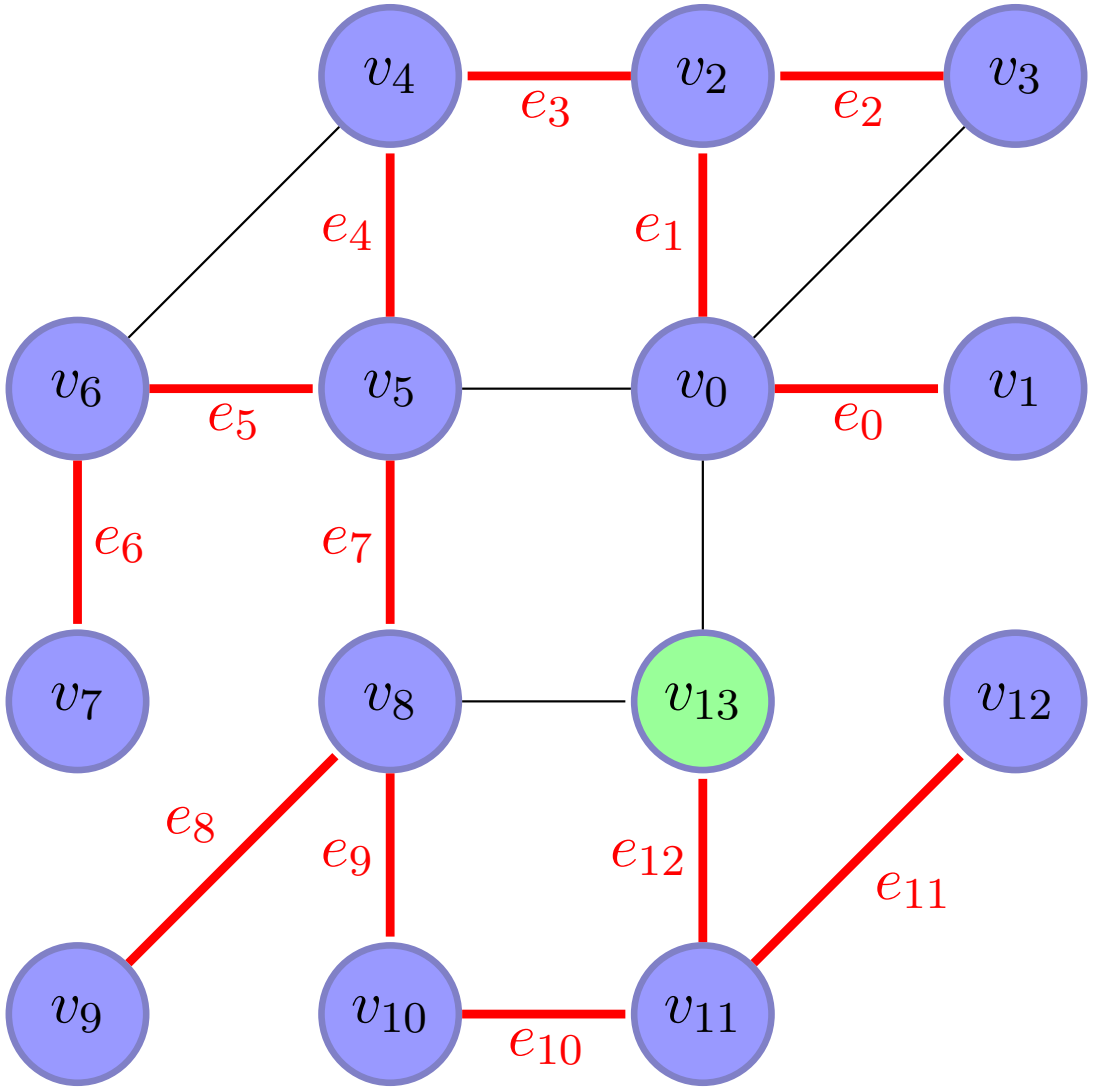
# Pesquisa em profundidade



# Pesquisa em profundidade



# Pesquisa em profundidade



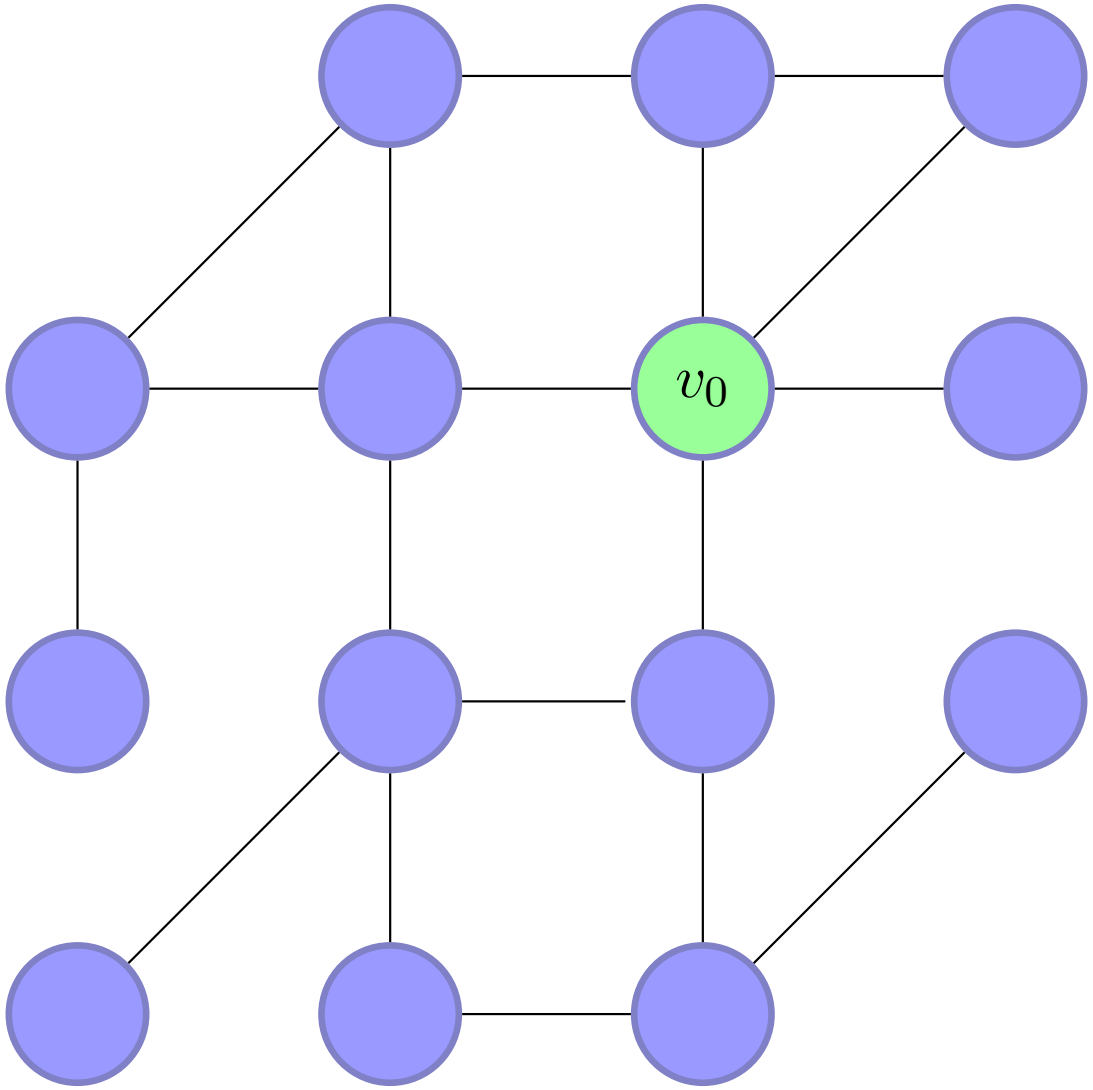
# Pesquisa em largura

## Pesquisa em largura

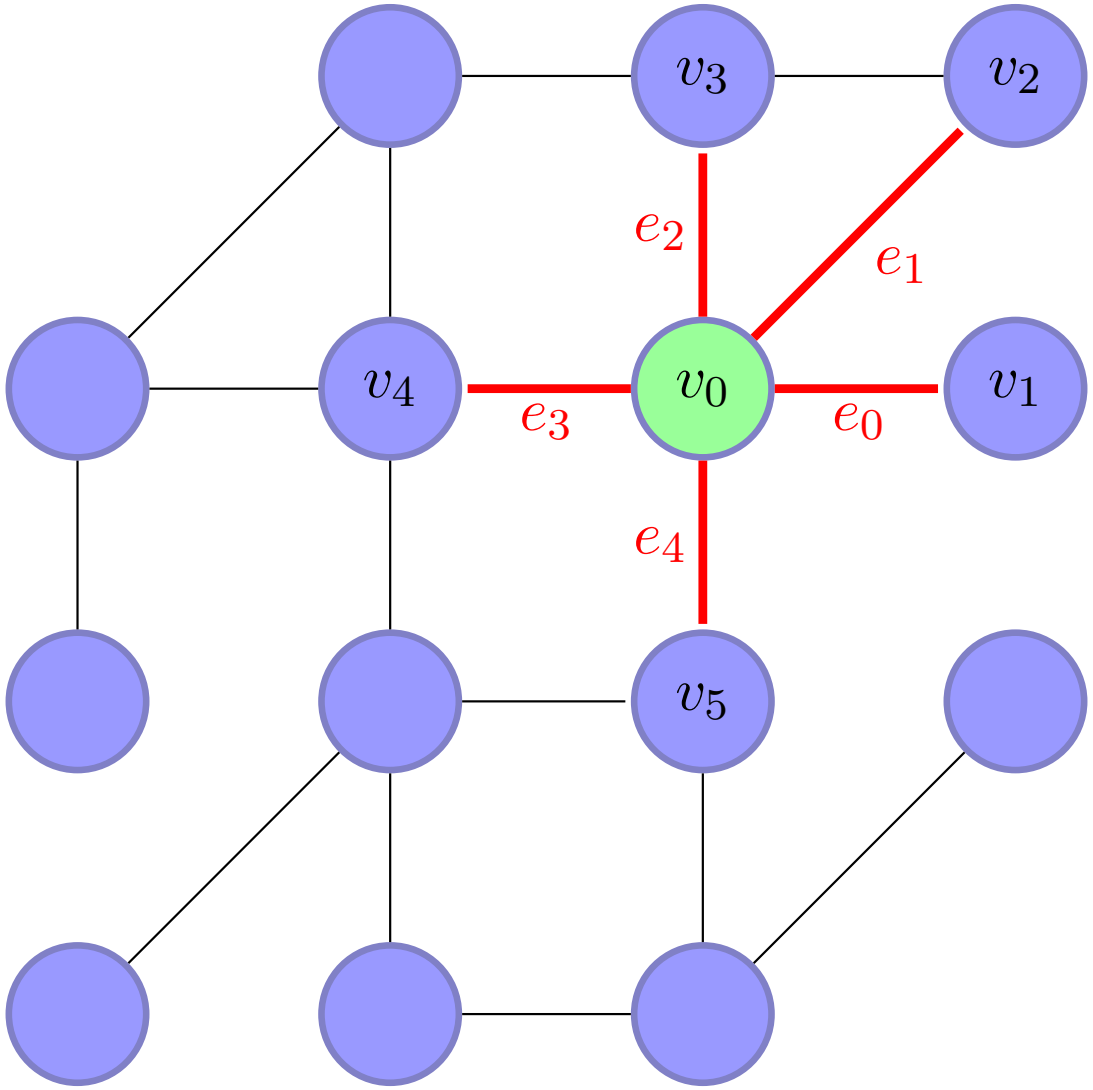
Seja $\mathcal{G}$ um grafo conexo e $v_0$ um vértice de $\mathcal{G}$. A árvore de cobertura de $\mathcal{G}$ obtida por pesquisa em largura constroi-se da seguinte maneira:

1. Toma-se $u = v_0$ como vértice corrente da pesquisa, $n = 0$ e $m = 1$. $\mathcal{T}_0$ é a árvore que contém somente o vértice $v_0$.
2. Escolhe-se, caso exista, um vértice $v_{n+1}$ ainda não incluído em $\mathcal{T}_n$ adjacente a $u$; toma-se para $\mathcal{T}_{n+1}$ a árvore que espande $\mathcal{T}_n$ com a nova aresta $v_n v_{n+1}$; toma-se $n := n + 1$.
3. Repete-se o passo $2$ até que não existam mais vértices não visitados adjacentes a $u$. Se a árvore obtida $\mathcal{T}_n$ cobre já todo o grafo, então a pesquisa termina; se não, toma-se o primeiro dos vértices $v_m$ que ainda não foi vértice corrente e toma-se $m := m + 1$; repete-se o passo $2$.

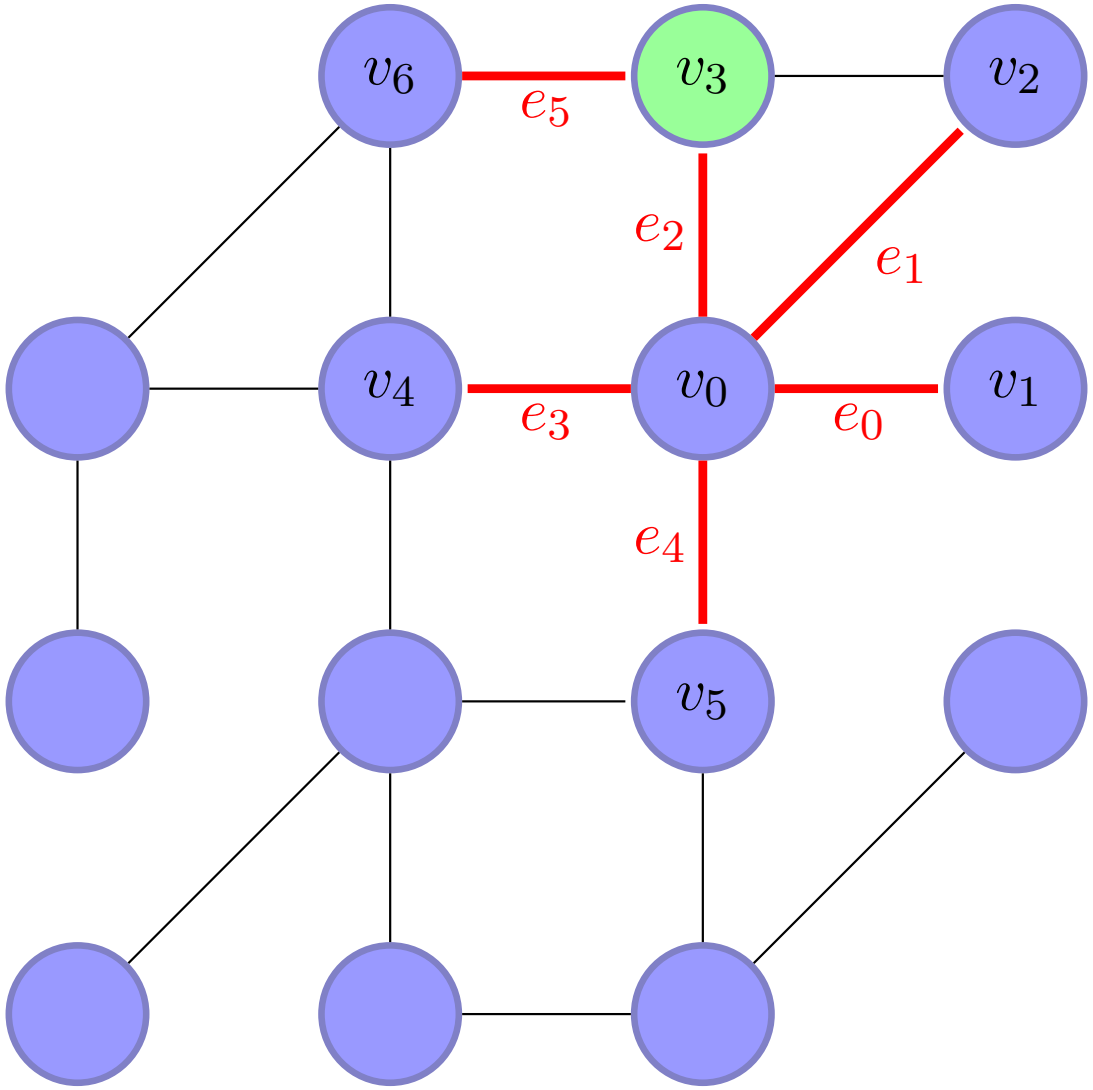
# Pesquisa em largura



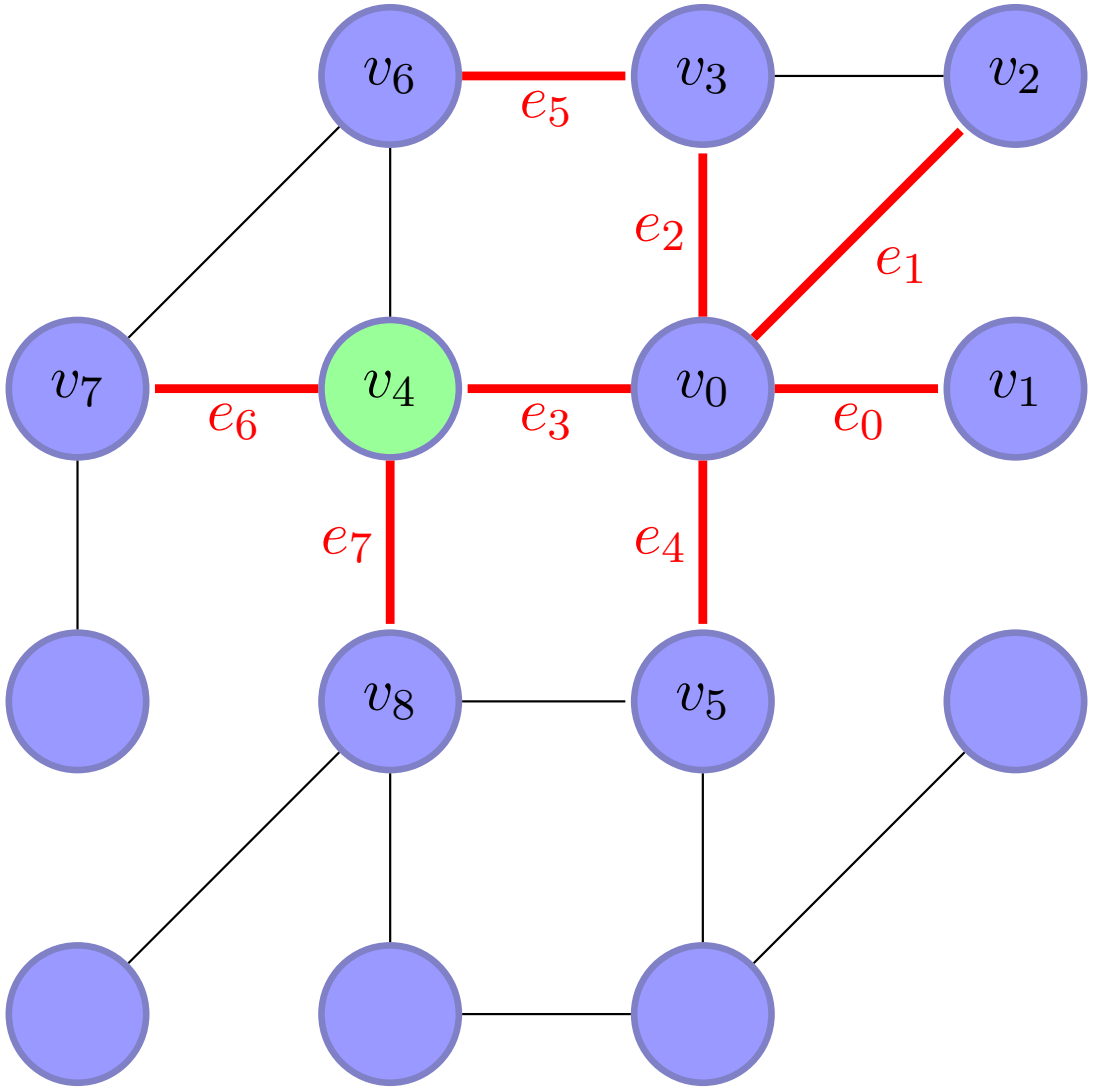
# Pesquisa em largura



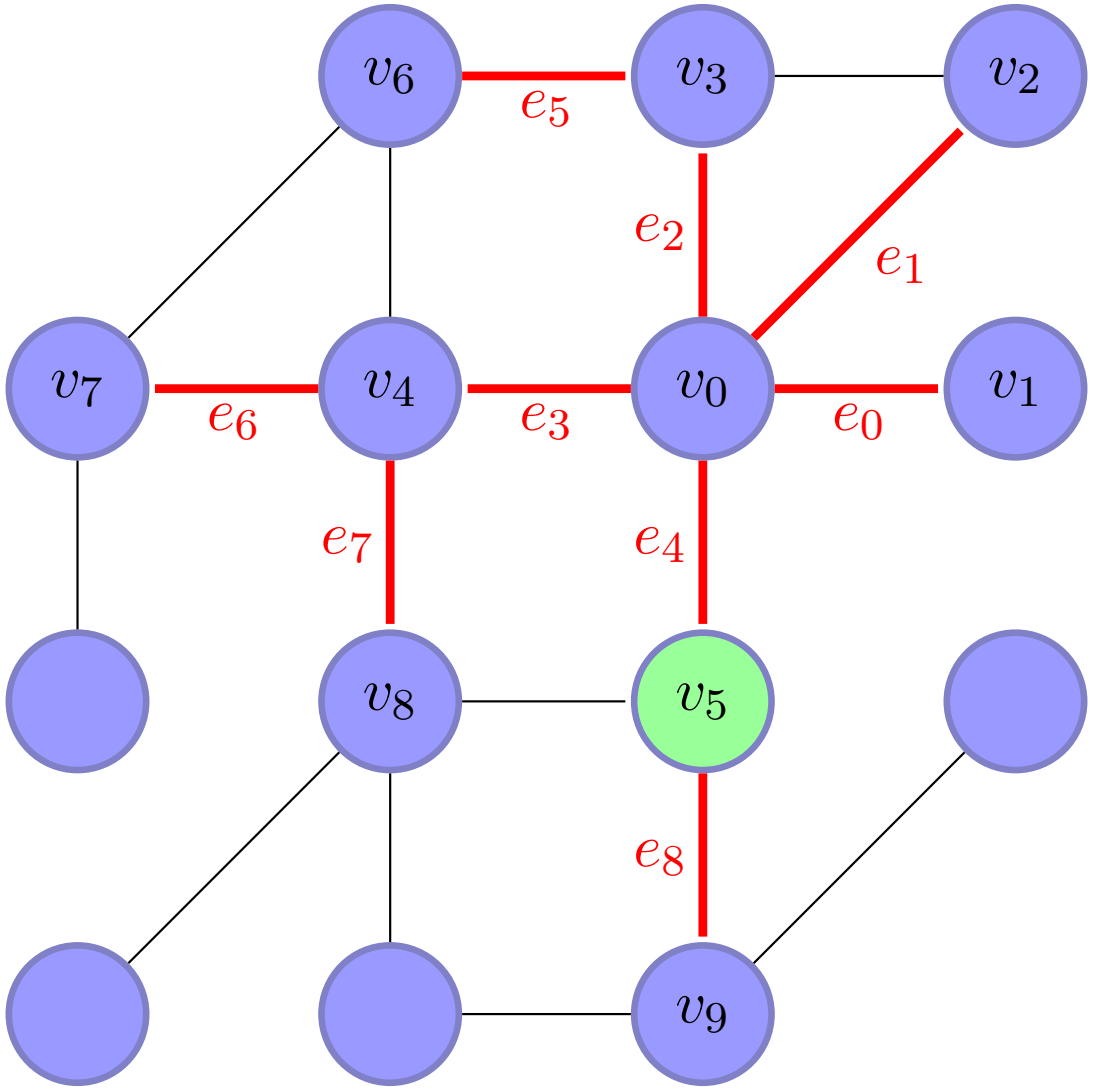
# Pesquisa em largura



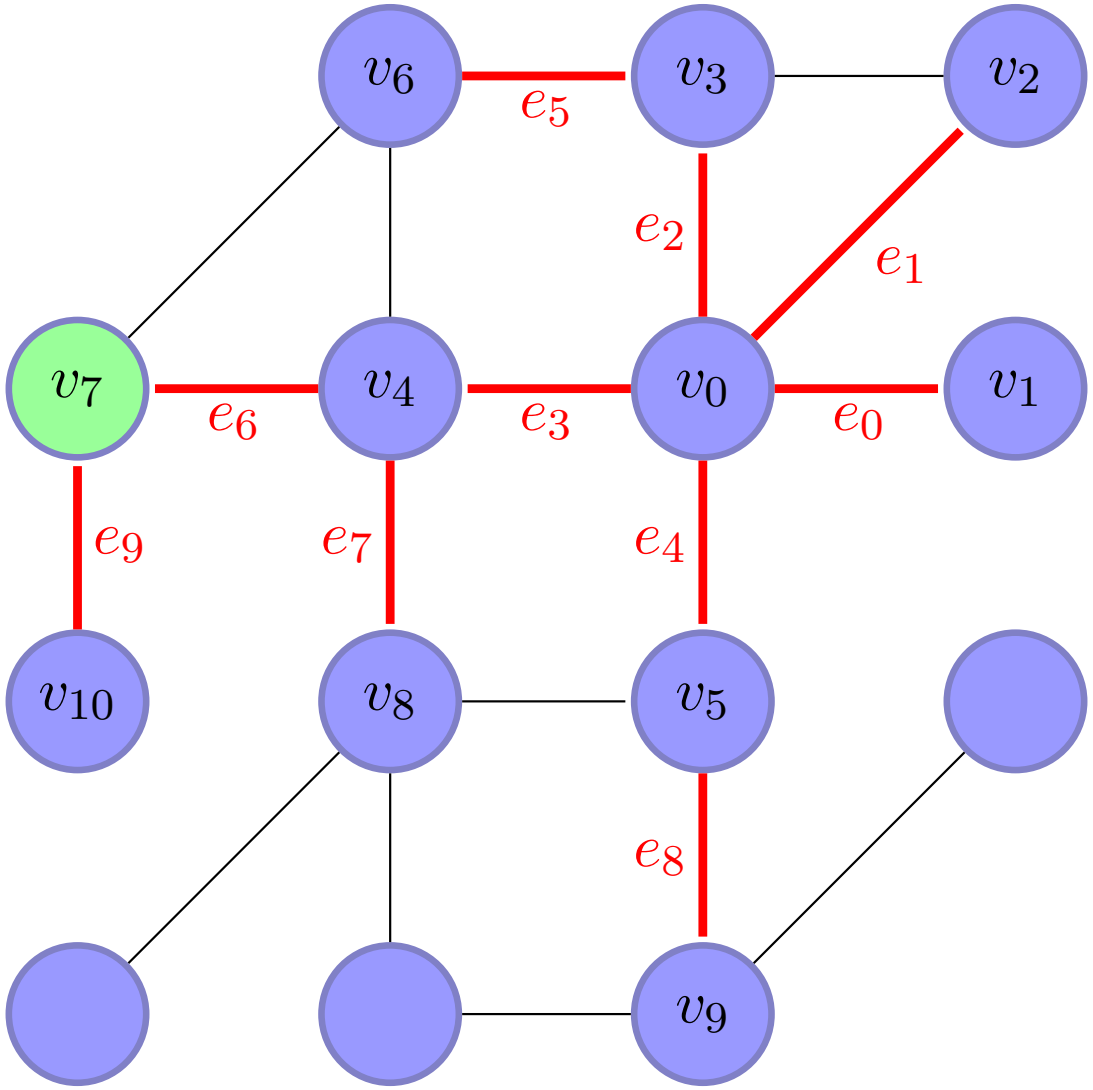
# Pesquisa em largura



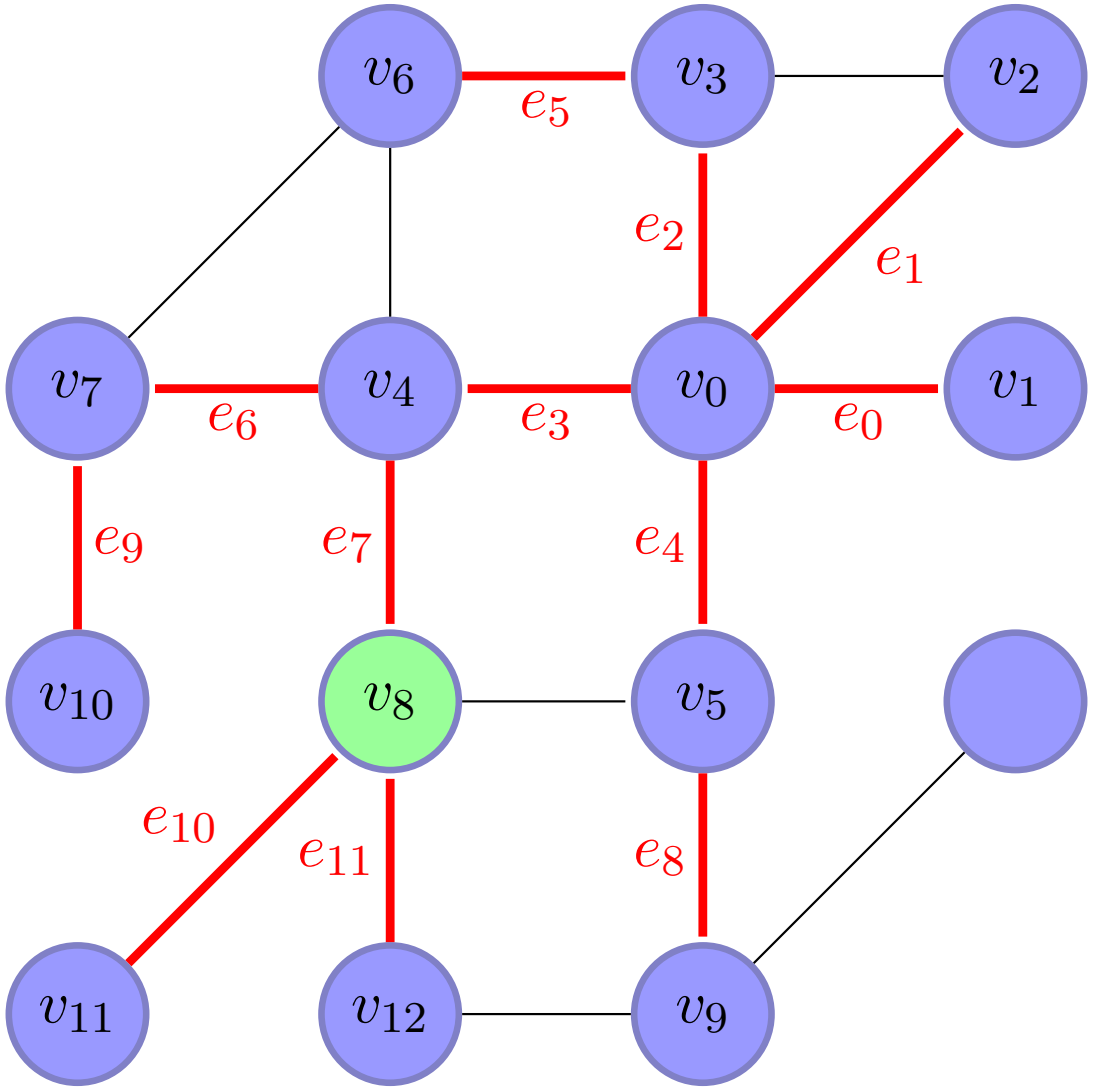
# Pesquisa em largura



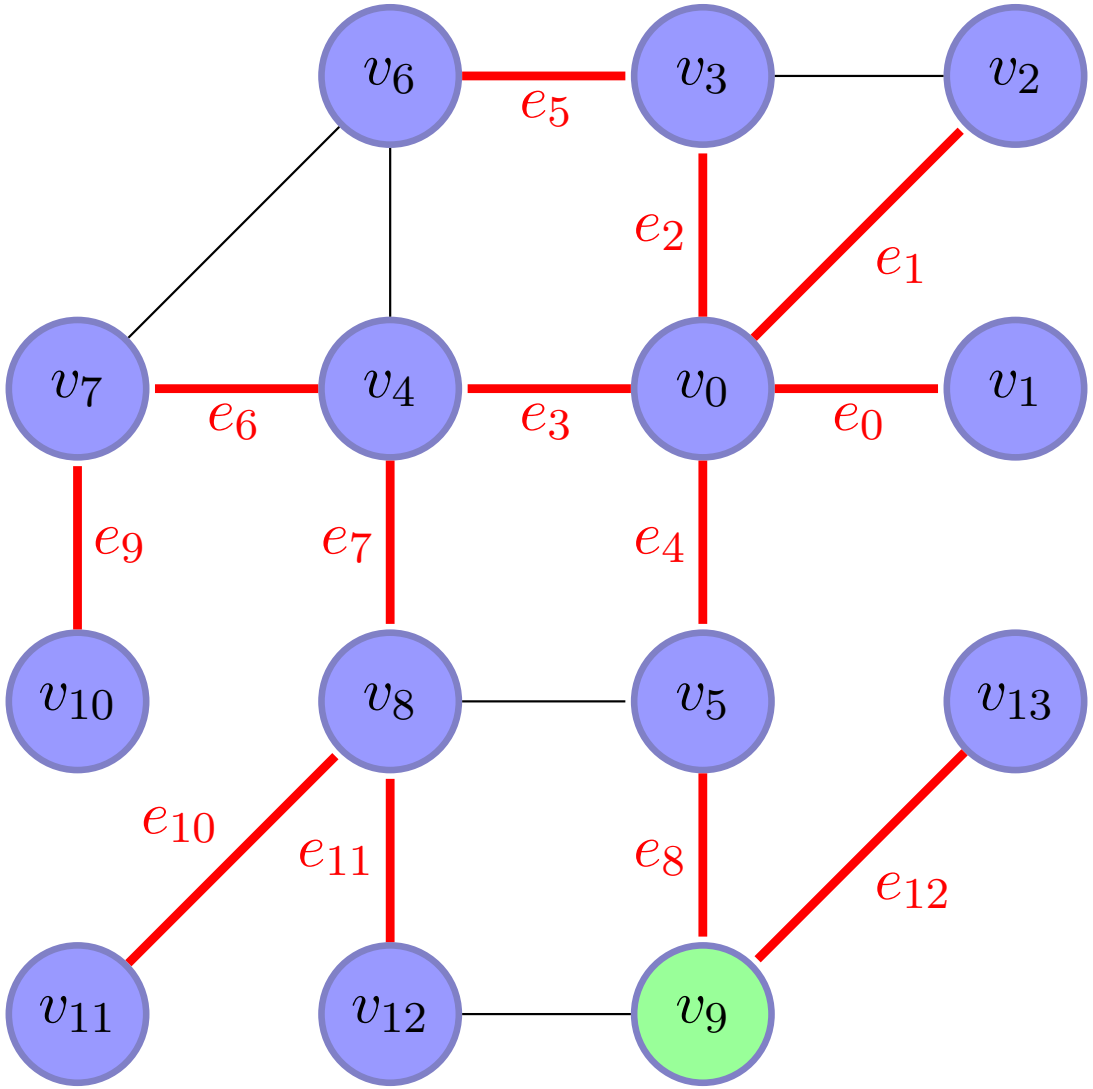
# Pesquisa em largura



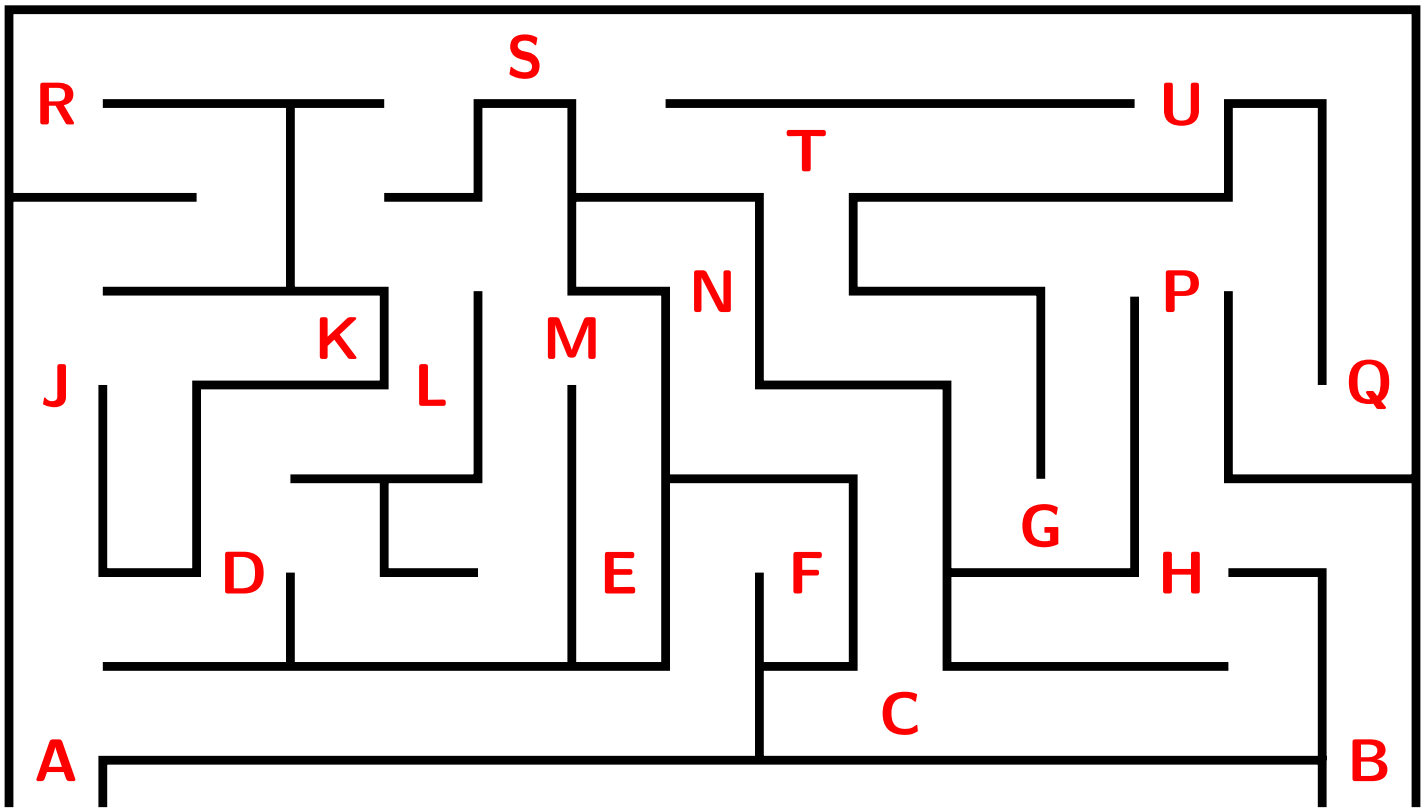
# Pesquisa em largura



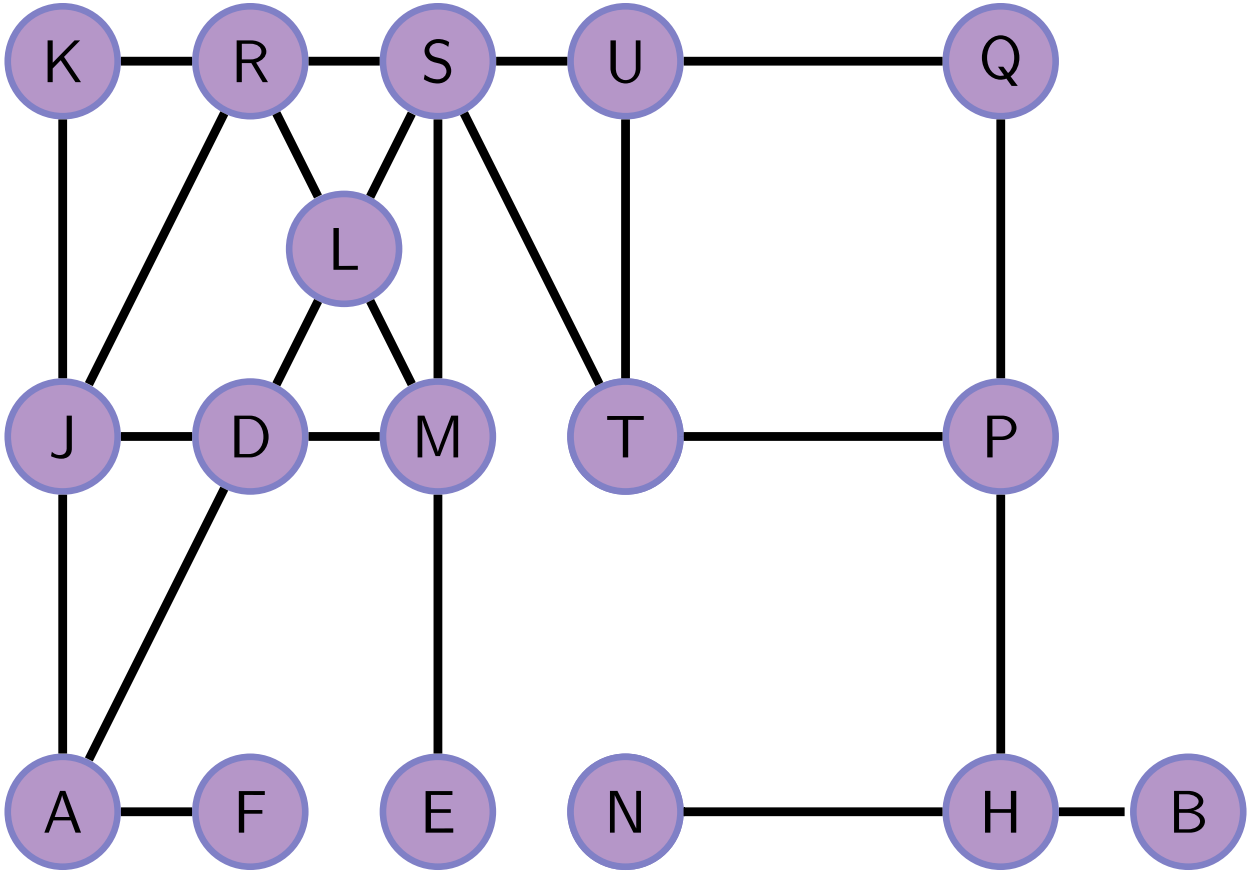
# Pesquisa em largura



# Labirinto de Gopal



# Labirinto de Gopal



# Algoritmo dos labirintos

## Algoritmo dos labirintos

1. Sempre que chegar a um vértice não visitado, siga pela aresta ainda não percorrida nele incidente que lhe aprouver;
2. Sempre que, por aresta ainda não percorrida, chegar a vértice já visitado ou a vértice de grau $1$ (beco sem saída), recue pela mesma aresta até ao vértice precedente;
3. Sempre que por aresta já percorrida chegar a vértice já visitado, siga por nova aresta, se esta existir;
4. Uma aresta já percorrida duas vezes não pode mais ser mais usada.

# O tema do labirinto na poesia (*vide* compilação em [Fer08])

## Miguel Torga, Diário VII

*Perdi-me nos teus braços, alamedas*
*Onde o tempo caminha e descaminha.*
*Pus a força que tinha*
*Na instintiva defesa*
*De encontrar a saída, a liberdade.*
*Mas agora Teseu era um poeta,*
*E Ariane a poesia, o labirinto.*
*Desajudado,*
*Só me resta cantar, deixar marcado*
*O pânico que sinto.*

# Bibliography I


[BC87] W. W. Rouse Ball and H. S. M. Coxeter. *Mathematical Recreations and Essays*. Dover Publications, Inc., New York, thirteenth edition, 1892, 1974, 1987.

[BLW06] N. L. Biggs, E. K. Lloyd, and R. J. Wilson. *Graph Theory 1736–1936*. Clarendon Press, Oxford, 1976, 1986, 2006.

[Cha85] Garry Chartrand. *Introduction to Graph Theory*. Dover Publications, Inc., New York, 1977, 1985.

[Fer08] José Ribeiro Ferreira. *Labirinto e Minotauro, Mito de Ontem e de Hoje*. UI&D Centro de Estudos Clássicos e Humanísticos da Universidade de Coimbra, 2008.