Projecções poly Bergman e operadores de Calderon-Zygmund

ENSPM 08 - ISEC

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27 de Junho de 2008

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Resumo

- Ir-se-á discorrer acerca igualdades explícitas entre projecções do tipo *Bergman* e operadores na álgebra-$*$ gerada por o operador de *Calderon-Zygmund* S. As mencionadas igualdades permitirão decompor o espaço das funções de quadrado *Lebesgue* integrável em espaços ortogonais de funções de tipo analítico (ou anti-analítico). Os operadores de *Calderon-Zygmund* S e S^* desempenham o papel de operadores unitários entre os espaços de funções de tipo analítico envolvidos.
- Igualdades entre projecções de tipo *Bergman* e operadores na álgebra de operadores integrais singulares, encontram-se dependentes da regularidade da fronteira do domínio considerado. A referida dependência encontrasse afastada do entendimento actual. Apresentaremos exemplos de domínios que não admitem fórmulas de *Dzhuraev*, exemplificamos como a variação interior do domínio permite deduzir novos resultados, estabelecemos explícitas igualdades entre operadores integrais singulares e projecções de*Bergman* em sectores mensuráveis com o semi-plano e mostramos que o semi-plano é o único sector que admite formulas de *Dzhuraev*. Simples observações permitirão estabelecer resultados clássicos tanto quanto demonstrações elementares das fórmulas de *Dzhuraev* em semi-espaços.

UTL

Espaços poly Bergman $\mathcal{A}_j^2(U)$, $j \in \mathbb{Z}_\pm$

- $U \subset \mathbb{C}$ open connected set ; $dA(z) = dxdy$ area measure

$$\partial_{\bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \quad \partial_z := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad (1.1)$$

Definition (Poly and anti-poly Bergman spaces)

$f \in \mathcal{A}_j^2(U)$ if $f \in L^2(U, dA)$, f is smooth and

$\partial_{\bar{z}}^j f = 0$ and $\partial_z^{|j|} f = 0$, *respectively if* $j \in \mathbb{Z}_+$ *and* $j \in \mathbb{Z}_-$

- f is j-analytic and $|j|$-anti-analytic function if is smooth and satisfies (1.1) respectively if $j \in \mathbb{Z}_+$ and $j \in \mathbb{Z}_-$

Poly Bergman spaces $\mathcal{A}_j^2(U), j \in \mathbb{Z}_\pm$

- $\mathcal{A}_j^2(U)$ are **reproducing kernel Hilbert spaces**. Indeed

$$|f(z)| \leq \frac{j}{\sqrt{\pi}\, d_z} \|f\|_{L^2(D(z,d_z))} \leq \frac{j}{\sqrt{\pi}\, d_z} \|f\|_{L^2(U)}, \quad \begin{array}{l} f \in \mathcal{A}_j^2(U), \\ z \in U. \end{array}$$

- $\widetilde{\mathcal{A}}_j(U) := \mathcal{A}_{-j}(U)$, $j \in \mathbb{Z}_+$ is also used (N. Vasilevski notation)

Definition (Poly Bergman kernels and projections)

$K_j(z,w)$, $j \in \mathbb{Z}_\pm$ is the j-Poly Bergman reproducing kernel for U, i.e. the unique function such that $\overline{k_{U,j,z}(w)} := K_j(z,w)$ and

$$f(z) = \langle f, k_{U,z} \rangle \ , \ f \in \mathcal{A}_j^2(U) \, , \, j \in \mathbb{Z}_\pm \, , \, z \in U \, .$$

$B_{U,j}$ is the **orthogonal projections** from $L^2\,(U, d\,A)$ onto $\mathcal{A}_j^2\,(U)$

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Poly Bergman kernels and projections

- $B_{U,j}$ is an integral operator with kernel given by $K_{U,j}$, $j \in \mathbb{Z}_{\pm}$, i.e.

$$B_{U,j}f(z) = \int_U K_{U,j}(z,w)f(w)dA(z)\ ,\ f \in L^2(U,dA)\,,\, j \in \mathbb{Z}_{\pm}$$

- *Koshelev* formula for poly Bergman kernel of unit disk [B, 91]

$$K_{\mathbb{D},j} = \frac{j}{\pi}\frac{\sum_{k=1}^{j}(-1)^{k-1}C_k^j C_j^{j+k-1}\left|1-\overline{w}z\right|^{2(j-k)}\left|z-w\right|^{2(k-1)}}{(1-\overline{w}z)^{2j}},$$

$$\text{With } C_k^j = \tfrac{j!}{k!(j-k)!}\ ,\ j \in \mathbb{Z}_+ .$$

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Dzhuraev formulas

- The *Calderon-Zygmund* operators

$$\begin{aligned} S_{U,j}f(z) &= \frac{(-1)^j|j|}{\pi}\int_U \frac{(w-z)^j}{(\overline{w}-\overline{z})^j}\frac{f(w)}{|w-z|^2}dA(w) \\ D_{U,j} &= I - S_{U,-j}S_{U,j} \end{aligned} \quad , \ j \in \mathbb{Z}_\pm$$

- In [D, 92] we find that if U is finite multi connected, ∂U with C^2 boundary then we have the Dzhuraev formulas

$$\begin{aligned} B_{U,j} &= D_{U,j} + T_{U,j} \\ (T_{U,j} &\in \mathcal{K}\,,\ j \in \mathbb{Z}_\pm) \end{aligned} \quad \text{and} \quad \begin{aligned} S_{U,j} &= (S_U^*)^j \\ S_{U,-j} &= S_U^j \end{aligned} \quad , \ j \in \mathbb{Z}_+ \,.$$

- Boundary with Holder condition on derivative it is sufficient [KP, 08]

Easy violation of Dzhuraev formulas

- Suppose U is bounded domain admitting *Dzhuraev* formulas
- Consider $z \in U$, l_z a half-Straight line outgoing from z and the *slitted* domains $U_{l_z} := U \backslash l_z$

Proposition ([KP, 08])

U_{l_z} does not admit Dzhuraev formulas

(Idea for a possible proof:)

- $$D_{U_{l_z},j} = D_{U,j} = B_{U,j} + T_{U,j} \, , \; T_{U,j} \in \mathcal{K} \, , \; j \in \mathbb{Z}_+$$
- It is sufficient to prove that $L_j := B_{U,j} - B_{U_{l_z},j} \notin \mathcal{K}$
- $g_k = L_j f_k$ are linearly independent functions in $\mathrm{Im}\, L_j$, being

$$f_k(w) = (w - \xi)^{1/k} \, , \; k \in \mathbb{Z}_+ \, , \; \xi \in lz$$

Explicit Dzhuraev formulas

$$\mathrm{cl\,span}\left\{z^m\overline{z}^k : k = 0, \cdots, j-1\,;\ m \in \mathbb{N}\right\} = \mathcal{A}_j^2(\mathbb{D})\,,\ j \in \mathbb{Z}_+$$

Proposition ([KP, 08])

i) $(S^*_{\mathbb{D}}\, w^m\overline{w}^k)(z) = \frac{k}{m+1}\ \overline{z}^{k-1}z^{m+1} + \frac{\min\{0,m+1-k\}}{m+1}\ \overline{z}^{k-m-2}$

ii) $B_{\mathbb{D},j} = I - S_{\mathbb{D},-j}S_{\mathbb{D},-j}\ ,\ j \in \mathbb{Z}_\pm$

iii) $S_{\mathbb{D},-j} = S^j_{\mathbb{D}}\ ,\ j \in \mathbb{Z}_+$ and $S_{\mathbb{D},j} = (S^*_{\mathbb{D}})^j$

$$N_{j,k} := \mathrm{span}\left\{z^l\overline{z}^s : l = 0, \ldots, j-1; s = 0, \ldots, k-1\right\}\ ,\ \dim N_{j,k} = jk$$

Proposition ([KP, 08])

$$\widetilde{B}_{\mathbb{D},j}B_{\mathbb{D},k} = P_{N_{j,k}}$$

Unitary operators between analytical type spaces

$L_k := \left\{\lambda \overline{z}^{k-1} : \lambda \in \mathbb{C}\right\}$ and $\widetilde{L}_k := \left\{\lambda z^{k-1} : \lambda \in \mathbb{C}\right\}$

The density of polynomials $p(z, \overline{z})$ of degree $j-1$ in $\overline{z}$, are important for

Proposition ([KP, 08])

The following applications are isometric isomorphisms

$$S_{\mathbb{D}}^* : \mathcal{A}^2_{(k+1)}(\mathbb{D}) \to \mathcal{A}^2_{(k)}(\mathbb{D}) \ominus L_k \quad , \quad S_{\mathbb{D}} : \mathcal{A}^2_{(k)}(\mathbb{D}) \ominus L_k \to \mathcal{A}^2_{(k+1)}(\mathbb{D})$$
$$S_{\mathbb{D}}^* : \widetilde{\mathcal{A}}^2_{(k)}(\mathbb{D}) \ominus \widetilde{L}_k \to \widetilde{\mathcal{A}}^2_{(k+1)}(\mathbb{D}) \quad , \quad S_{\mathbb{D}} : \widetilde{\mathcal{A}}^2_{(k+1)}(\mathbb{D}) \to \widetilde{\mathcal{A}}^2_{(k)}(\mathbb{D}) \ominus \widetilde{L}_k$$

In addition,

$$S_{\mathbb{D}}^* : \mathcal{A}^2_{(1)}(\mathbb{D}) \to \{0\} \quad , \quad S_{\mathbb{D}} : \widetilde{\mathcal{A}}^2_{(1)}(\mathbb{D}) \to \{0\}$$

The following ≪nice≫ equalities are important for the proof

$$\widetilde{B}_{\mathbb{D}} B_{\mathbb{D},(k)} g = \tfrac{k}{\pi} \langle g \, , \, \overline{z}^{k-1} \rangle \overline{z}^{k-1} \quad , \quad B_{\mathbb{D}} \widetilde{B}_{\mathbb{D},(k)} g = \tfrac{k}{\pi} \langle g \, , \, z^{k-1} \rangle z^{k-1}$$

$$\langle g \, , \, \overline{z}^{k-1} \rangle = \tfrac{\pi}{k!} \tfrac{\partial^{k-1} g}{\partial \overline{z}^{k-1}}(0) \quad , \quad \left(\widetilde{B}_{\mathbb{D}} B_{\mathbb{D}} f\right)(z) = \tfrac{1}{\pi} \int_{\mathbb{D}} f(w) \, dA(w)$$

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Decomposition of $L^2(\mathbb{D})$ in analytical type spaces

Proposition ([KP, 08])

If U is non empty bounded domain, then

$$L^2(U) = \bigoplus_{k=1}^{\infty} \mathcal{A}^2_{(k)}(U) = \bigoplus_{k=1}^{\infty} \widetilde{\mathcal{A}}^2_{(k)}(U)$$
$$\text{s-lim}_{n\to\infty} D_{U,n} = I, \quad \text{s-lim}_{n\to\infty} \widetilde{D}_{U,n} = I.$$

From the unitary actions of $S_{\mathbb{D}}$ and $S^*_{\mathbb{D}}$ on poly Bergman spaces

$$L^2(\mathbb{D}) = \bigoplus_{k=0}^{\infty} S^k_{\mathbb{D}}\left(\mathcal{A}^2(\mathbb{D})\right) = \bigoplus_{k=0}^{\infty} (S^*_{\mathbb{D}})^k\left(\widetilde{\mathcal{A}}^2(\mathbb{D})\right).$$

Later we will remark some differences with half spaces

Inside approximation of analytical type projections

Definition (Inner exhaustive sequence)

Let $U \subset \mathbb{C}$ be a domain. $\{U_n\}_{n\in\mathbb{N}}$ is a *Inner exhaustive sequence* for U if

$$U_n \subset U_{n+1} \subset U \quad ; \quad \cup_{n\in\mathbb{N}} U_n = U;$$

Theorem (Inside aproximation [P, 08])

$B_{n,j}$, $K_{n,j}(z,w)$, B_j, $K_j(z,w)$ the j-Poly Bergman projection and reproducing kernels respectively for U_n and U. Then

- $B_j = \text{s-lim}_n \, \chi_U B_{n,j} \chi_U$;
- $\lim_m \|\chi_{U_m} k_{m,j,z} - k_{j,z}\|_{L^2(U)} = 0$, *uniformly for z within U.*
- $\lim_m \|K_{m,j}(z,w) - K_j(z,w)\|_{L^\infty(F_1\times F_2)} = 0$, $F_1, F_2 \subset U$ *compact;*

Inside approximation of analytical type projections

- *The proof is valid in more general reproducing kernel Hilbert spaces with analytical type structure, for instance in reproducing Hilbert spaces of Modules of the poly analytic type*
- **Some History:**
  - *Bergman, Schiffer* In case of domains bounded by smooth boundaries, they had investigated variation of Kernel functions with variation of domains
  - *Burbea* made use of such kind result - 1980. A result in the Bergman case, and with the domains evolved having analytical boundaries - Riemann Surface methods - *Vitranen; Sario*

Inside approximation of analytical type projections

Some easy consequences:

- Because $\lim_{n\to\infty} K_{D(in,n),j}(z,w) = K_{\Pi,j}(z,w)$, then

Theorem (Kernel Function of upper half plane [P, 08])

$$K_{\Pi,j}(z,w) = \frac{j}{\pi}\sum_{k=1}^{j}(-1)^{k+j-1} C_k^j C_j^{j+k-1} \frac{(w-\overline{z})^{j-k}}{(\overline{w}-z)^{j+k}} |z-w|^{2(k-1)}$$

$$K_{\Pi,-j}(z,w) = \frac{j}{\pi}\sum_{k=1}^{j}(-1)^{k+j-1} C_k^j C_j^{j+k-1} \frac{(z-\overline{w})^{j-k}}{(\overline{z}-w)^{j+k}} |z-w|^{2(k-1)}$$

- Because $B_{\Pi,j} = \text{s-lim}\, B_{D(in,n),j} = \text{s-lim}\, D_{D(in,n),j} = D_{\Pi,j}$ then

Theorem (Dzhuraev formulas for upper half plane [P, 08, RS, 03])

$$\begin{array}{rcl} B_{\Pi,j} & = & I - S_\Pi^j (S_\Pi^*)^j = I - S_{\Pi,-j} S_{\Pi,j} \\ B_{\Pi,-j} & = & I - (S_\Pi^*)^j S_\Pi^j = I - S_{\Pi,j} S_{\Pi,-j} \end{array} \quad , j \in \mathbb{Z}_+$$

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Inside approximation of analytical type projections

- Properties of Calderon-Zygmund operators S_j , $j \in \mathbb{Z}$

$$\begin{aligned} S_j &= \operatorname*{s-lim}_n S_{n\mathbb{D},j} = \operatorname*{s-lim}_n (S^*_{n\mathbb{D}})^j = (S^*)^j \\ I - SS^* &= \operatorname*{s-lim}_n (I_n - S_{n\mathbb{D}} S^*_{n\mathbb{D}}) = \operatorname*{s-lim}_n B_{n\mathbb{D}} = B_{\mathbb{C},j} = 0 \end{aligned}$$

If $S_0 = I$, then the application

$$\mathbb{Z} \ni j \to S_j \quad \text{is additive group homomorphism}$$

- **Such properties of** *Calderon-Zygmund* **operators are classical**

UTL

Inside approximation of analytical type projections

- Orthogonality of poly and anti-poly Bergman spaces on Π

Theorem ([V, 99, P, 08])

$B_{\Pi,j} B_{\Pi,-k} = 0 \quad \textit{for} \quad j,k \in \mathbb{Z}_\pm \,.$

Idea for simple proof based on inside approximation

Consider W_{φ_n} with $\varphi_n : \mathbb{D} \to D(in,n) \quad , \quad \varphi_n(z) = nz + in$

Invariance for dilations and translations gives that
$B_{\Pi,j} B_{\Pi,-k} = \text{s-lim}\, W^*_{\varphi_n} B_{\mathbb{D},j} B_{\mathbb{D},-k} W_{\varphi_n} = \text{s-lim}\, W^*_{\varphi_n} P_{N_{j,k}} W_{\varphi_n}$

Because $\text{w-lim}_{n\to\infty} W_{\varphi_n} = 0$ and $P_{N_{j,k}} \in \mathcal{K}$ we get $B_{\Pi,j} B_{\Pi,-k} = 0$

$\left\| B_{\Pi,j} B_{\Pi,-k} f \right\| = \text{s-lim} \left\| W^*_{\varphi_n} P_{N_{j,k}} W_{\varphi_n} f \right\| = \text{s-lim} \left\| P_{N_{j,k}} W_{\varphi_n} f \right\| = 0\,.$

- *Dzhuraev* formulas and orthogonality is the essential to achieve the unitary action of Calderon-Zygmund on Bergman kind spaces.

Decomposition of $L^2(\Pi)$ in analytical type spaces and unitary character of Calderon-Zygmund operators

Theorem ([KP, 07, V, 99, V, 06])

The following applications

$$S_\Pi^* : \mathcal{A}^2_{(k+1)}(\Pi) \to \mathcal{A}^2_{(k)}(\Pi) \quad , \quad S_\Pi : \mathcal{A}^2_{(k)}(\Pi) \to \mathcal{A}^2_{(k+1)}(\Pi)$$
$$S_\Pi^* : \widetilde{\mathcal{A}}^2_{(k)}(\Pi) \to \widetilde{\mathcal{A}}^2_{(k+1)}(\Pi) \quad , \quad S_\Pi : \widetilde{\mathcal{A}}^2_{(k+1)}(\Pi) \to \widetilde{\mathcal{A}}^2_{(k)}(\Pi)$$

are isometric isomorphisms. In addition,

$$S_\Pi^* : \ \mathcal{A}^2_{(1)}(\Pi) \to \{0\}, \quad S_\Pi : \ \widetilde{\mathcal{A}}^2_{(1)}(\Pi) \to \{0\}$$

We also have the following decomposition

$$L^2(\Pi) = \bigoplus_{k=0}^{\infty} S_\Pi^k\left(\mathcal{A}^2(\Pi)\right) \bigoplus_{k=0}^{\infty} (S_\Pi^*)^k\left(\widetilde{\mathcal{A}}^2(\Pi)\right).$$

Compare with the unit disk case.Vasilevski had published a second proof [V, 06] for the first part of theorem and a first proof for the second part[V, 99].

Dzhuraev type formulas on sectors

For sectors $\Pi_m := \{z : \operatorname{Im} z^m > 0\}$, $m = 2, \cdots$ mensurable with π we have

Theorem ([KP, 08])

$$\begin{aligned} B_{\Pi_m} &= I - (S_{\Pi_m} + R_{\Pi_m})(S_{\Pi_m} + R_{\Pi_m})^* \\ \widetilde{B}_{\Pi_m} &= I - (S_{\Pi_m} + R_{\Pi_m})^*(S_{\Pi_m} + R_{\Pi_m}) \end{aligned}$$

being $\varepsilon := e^{i\pi/m}$, $W_{\varepsilon_m} f(z) = \varepsilon_m f(\varepsilon_m z)$ *and*

$$R_{m,k} = \chi_{\Pi_m} W_{\varepsilon_m}^{2k} S \chi_{\Pi_m} I\,,\ k = 1, \cdots, m-1 \quad \text{and} \quad R_{\Pi_m} = \sum_{k=1}^{m-1} R_{m,k}\,.$$

Dzhuraev type formulas on sectors

Consider $\arg z \in [\,0, 2\pi\,[$ and

$$\Pi_\varphi^\phi = \{z : \varphi < \arg z < \phi\}\,,\ \Pi_\phi := \Pi_0^\phi\,,\ 0 \le \varphi < \phi < 2\pi$$

- $B_{\Pi_\varphi^\phi, j} = D_{\Pi_\varphi^\phi, j} + K_{\Pi_\varphi^\phi, j} \Rightarrow K_{\Pi_\varphi^\phi, j} = 0\ ,\ j \in \mathbb{Z}_\pm\,.$

(Simple proof) Let V_τ be the dilatation operators.

Then $\text{w-lim}_{\tau\to 0} V_\tau = 0$ and $B_{\Pi_\varphi^\phi, j}, D_{\Pi_\varphi^\phi, j}$ commute with V_τ . Thus

$$\left\| K_{\Pi_\varphi^\phi, j} f \right\| = \left\| K_{\Pi_\varphi^\phi, j} V_\tau f \right\| \to 0$$

- In the general setting of Hilbert spaces we have

Proposition

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{A} \subset \mathcal{H}$ a non trivial closed subspace and B the orthogonal projection of $\mathcal{H}$ onto $\mathcal{A}$. If $S \in \mathcal{B}(\mathcal{H})$ and $D := I - SS^*$ is such that $\operatorname{Im} D \subset \mathcal{A}$ then $B = D$ iff $S^* f = 0\,,\ f \in \mathcal{A}$.

Dzhuraev type formulas on sectors

Having in attention

- *Vékua* Derivation Formulas

$$\partial_{\overline{z}} S_U f(z) = \partial_z f(z) \quad , \quad \partial_z S_U^* f(z) = \partial_{\overline{z}} f(z) \, , \; z \in U \, .$$

- and that

$$S_{D,j} = S_D^{*j} \quad \text{and} \quad S_{D,-j} = S_D^j \, , \; j \in \mathbb{Z}_+$$

- we achieve for general domain U that

$$\mathrm{Im}\, D_{U,j} \subset \mathcal{A}_j^2(U) \, , \; j \in \mathbb{Z}_\pm \, .$$

- Thus Π_ϕ admit *j-Dzhuraev* formulas iff

$$S_{\Pi_\phi, j}(\mathcal{A}_j^2(\Pi_\phi)) = \{0\} \, .$$

Dzhuraev type formulas on sectors

It will be important to consider special dense set of Bergman spaces

Lemma

Suppose that $0 < \phi < 2\pi$. *There exists* $\delta > 0$ *and a set* Λ_ϕ *with dense span in the space* $\mathcal{A}^2(\Pi_\phi)$ *such that*

$$\Lambda_\phi \subset C^\infty(\overline{\Pi}_\phi)\,;\; f(z) = \mathcal{O}\left(z^{-(1+\delta)}\right)\;,\; |z| \to +\infty\;,\; z \in \overline{\Pi}_\phi\;, f \in \Lambda_\phi\,.$$

- Suppose $j \in \mathbb{Z}$. Having in attention Green formulas
$$\frac{1}{2i}\int_{\partial U} f(w)dw = \int_U \partial_{\overline{w}} f(w)dA(w)\;,\; f \in C^1(\overline{U})\,,$$
- if $l_\theta = \{re^{i\theta} : r \geq 0\}$, $\theta \in \mathbb{R}$ we obtain for function in Λ_ϕ that

$$S_{\Pi^\phi_\varphi,j} f(z) = \frac{1}{2j\pi i}\int_{l_\phi} - \int_{l_\varphi} \frac{(w-z)^{j-1}}{(\overline{w}-\overline{z})^j} f(w)dw\;,\; f \in \Lambda_\phi\,,\, j \in \mathbb{Z}_+$$

Dzhuraev type formulas on sectors

Having in attention that if z is a non isolated zero of the non zero j-analytical or $|j|$-anti-analytical function f, then in a sufficiently small neighborhood of $z \in U$, the zero set of f is the union of not more than $2|j|-2$ OSCAR's outgoing from point z [B, 91] and

Gathering all results with some analysis one achieve that

Theorem ([P, 08])

Π_ϕ, $0 < \phi \leq 2\pi$ *admit j-Dzhuraev formulas,* $j \in \mathbb{Z}_\pm$ *iff* $\phi = \pi$.

For Further Reading

M. B. Balk,
Polyanalytic Functions.
Akademie Verlag, Berlin, 1991.

A. Dzhuraev,
Methods of Singular Integral Equations.
Longman Scientific Technical, 1992.

N. L. Vasilevski, Poly-Bergman spaces and two-dimensional singular integral operators. In: The Extended Field of Operator Theory (ed. M. A. Dritschel), Operator Theory: Advances and Applications **171** *(2006), 349–359.*

N. L. Vasilevski, On the structure of Bergman and poly Bergman spaces. Integral Equations Operator Theory **33** *(1999), 471–488.*

For Further Reading

- Yu. I. Karlovich and L. Pessoa, *Algebras generated by Bergman and anti-Bergman projections and by multiplications by piecewise continuous coefficients.* Integral Equations and Operator Theory **52** (2005), 219–270.
- Yu. I. Karlovich and L. V. Pessoa, *C^*-algebras of Bergman type operators with piecewise continuous coefficients.* Integral Equations and Operator Theory **57** (2007), 521–565.
- Yu. I. Karlovich and Luís V. Pessoa, *C^*-algebras of Bergman type operators with piecewise continuous coefficients on bounded domains*, Proceedings of the Fifth ISAAC Congress (Catania, Italy, July 25-30, 2005). Eds. H.G.W. Begehr, F. Nicolosi. World Scientific, Singapure, 2008, pp. 25–30.
- Yu. I. Karlovich and Luís V. Pessoa, *Poly-Bergman projections and orthogonal decompositions of L^2-spaces over bounded domains*, Operator Theory: Advances and Applications, **181** (2008), 263-282.

For Further Reading

- Luís V. Pessoa, *Inside variation of domain in analytical type spaces, to be submitted probably on the next week*
- J. Ramírez and I. M. Spitkovsky, *On the algebra generated by a poly-Bergman projection and a composition operator*. Factorization, Singular Operators and Related Problems, Proc. of the Conf. in Honour of Professor Georgii Litvinchuk (eds. S. Samko, A. Lebre, and A. F. dos Santos), Kluwer Academic Publishers, Dordrecht, 2003, 273–289.