Poly-Bergman spaces on domains Möbius equivalent to a disk

Instituto Superior Técnico - Lisboa

Luís V. Pessoa

International Workshop in Operator Theory and Applications 2010
Technische Universität Berlin, Germany

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Abstract

Let U be a complex domain, which is the image of a disk by a Möbius transformation and let j be a non zero integer. The talk will focusses on explicit representation of the poly-Bergman projection of order j in terms of the canonical two-dimensional singular integral operators $S_{U,j}$. One also discuss how the Lebesgue space $L^2(U, dA)$ decompose on the true poly-Bergman spaces, where dA is the element of Lebesgue area measure. The poly-Bergman kernels of U are explicitly calculated.

Poly Bergman spaces $\mathcal{A}_j^2(U)$, $j \in \mathbb{Z}_\pm$

- $U \subset \mathbb{C}$ open connected set ; $dA(z) = dxdy$ area measure

$$\partial_{\bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \quad \partial_z := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \tag{1.1}$$

Definition (Poly and anti-poly Bergman spaces)

$f \in \mathcal{A}_j^2(U)$ if $f \in L^2(U, dA)$, f is smooth and

$\partial_{\bar{z}}^j f = 0$ and $\partial_z^{-j} f = 0$, *respectively if* $j \in \mathbb{Z}_+$ *and* $j \in \mathbb{Z}_-$

- f is j-analytic function if is smooth and satisfies (1.1) respectively if $j \in \mathbb{Z}_+$ and $j \in \mathbb{Z}_-$

Poly Bergman spaces

$\mathcal{A}_j^2(U)$ are **reproducing kernel Hilbert spaces**. Indeed [Kos-77]

$$|f(z)| \leq \frac{j}{\sqrt{\pi}(1-|z|^2)}\|f\|, \quad \begin{array}{l} f \in \mathcal{A}_j^2(\mathbb{D}) \\ j \in \mathbb{Z}_+ \end{array}, \ (z \in \mathbb{D})$$

If $d_z := \mathrm{dist}(z; \partial U)$ then for every $j \in \mathbb{Z}_\pm$, it follows straightforward that

$$|f(z)| \leq \frac{|j|}{\sqrt{\pi}\, d_z}\|f\|_{L^2(D(z,d_z))} \leq \frac{|j|}{\sqrt{\pi}\, d_z}\|f\|_{L^2(U)}, \quad z \in U, \ f \in \mathcal{A}_j^2(U)$$

Definition (Poly Bergman kernels and projections)

$K_{U,j}(z,w)$, $j \in \mathbb{Z}_\pm$ is the j-Poly Bergman reproducing kernel for U, i.e. the unique function such that $K_{U,j}(z,w) := \overline{k_{U,j,z}(w)}$ and

$$f(z) = \langle f, k_{U,z} \rangle \ ; \ f \in \mathcal{A}_j^2(U), \ j \in \mathbb{Z}_\pm, \ z \in U.$$

$B_{U,j}$ is the **orthogonal projections** from $L^2(U, dA)$ onto $\mathcal{A}_j^2(U)$.

Poly Bergman kernels and projections

- $B_{U,j}$ is an integral operator with kernel given by $K_{U,j}$, $j \in \mathbb{Z}_{\pm}$, i.e.

$$B_{U,j}f(z) = \int_U K_{U,j}(z,w)f(w)dA(w) \; ; \; f \in L^2(U, dA) \, , \, j \in \mathbb{Z}_{\pm}$$

- *Koshelev* formula for poly the Bergman kernel of unit disk [Kos-77]

$$K_{\mathbb{D},j} = \frac{j}{\pi} \frac{\sum_{k=1}^{j}(-1)^{k-1} C_k^j C_j^{j+k-1} \left|1 - \overline{w}z\right|^{2(j-k)} \left|z - w\right|^{2(k-1)}}{(1 - \overline{w}z)^{2j}},$$

With $C_k^j = \frac{j!}{k!(j-k)!}$, $j \in \mathbb{Z}_+$.

Dzhuraev's Formulas

- Two-dimensional Singular Integrals and Dzhuraev's operators

$$S_{U,j}f(z) = \frac{(-1)^j|j|}{\pi}\int_U \frac{(w-z)^{j-1}}{(\overline{w}-\overline{z})^{j+1}} f(w)\,dA(w) \quad , \; j\in\mathbb{Z}_{\pm}$$

$$P_{U,j} = I - S_{U,-j}S_{U,j}$$

- Likewise for all $j \in \mathbb{Z}_+$ we define

$$S_U f(z) = S_{U,-1}f(z) = -\frac{1}{\pi}\int_U \frac{1}{(w-z)^2} f(w)\,dA(w)$$

$$D_{U,j} = I - (S_U)^j(S_U^*)^j \quad \text{and} \quad D_{U,-j} = I - (S_U^*)^j(S_U)^j$$

- If U is bounded finitely multi connected, ∂U is smooth then [D-92]

$$B_{U,j} - D_{U,j} \in \mathcal{K} \quad \text{and} \quad \begin{array}{ccc} S_{U,j} & - & (S_U^*)^j \in \mathcal{K} \\ S_{U,-j} & - & (S_U)^j \in \mathcal{K} \end{array}, \; j\in\mathbb{Z}_{\pm}.$$

Variation of the domain

Definition (Inner exhaustive sequence [P-Submitted])

Let $U \subset \mathbb{C}$ be a domain. $\{U_n\}_{n\in\mathbb{N}}$ is a *Inner exhaustive sequence* for U if

$$U_n \subset U_{n+1} \subset U \quad ; \quad \cup_{n\in\mathbb{N}} U_n = U.$$

Theorem (Inner variation of the domain [P-Submitted])

If $\{U_n\}_{n\in\mathbb{N}}$ is a Inner exhaustive sequence for U then

- $B_{U,j} = \text{s-lim}_n\, \chi_U B_{U_n,j} \chi_U$;
- $\lim_m \|\chi_{U_m} k_{U_m,j,z} - k_{U,j,z}\|_{L^2(U)} = 0$, *uniformly for z within U;*
- $\lim_m \|K_{U_m,j}(z,w) - K_{U,j}(z,w)\|_{L^\infty(F_1\times F_2)} = 0$, $F_1, F_2 \subset U$ *compact;*

Poly Bergman kernel function for Π

- The variation of the domain permits to overcome the non-conformal invariance of poly Bergman spaces
- **In particular, the kernel function $K_{\Pi,j}$ is easily calculated:**
- Indeed, because $\lim_{n\to\infty} K_{D(in,n),j}(z,w) = K_{\Pi,j}(z,w)$, then

Theorem (Kernel Function of upper half plane [P-Submitted], see also [V-99])

$$K_{\Pi,j}(z,w) = \frac{j}{\pi}\sum_{k=1}^{j}(-1)^{k+j-1}C_k^j C_j^{j+k-1}\frac{(w-\overline{z})^{j-k}}{(\overline{w}-z)^{j+k}}|z-w|^{2(k-1)}$$

$$K_{\Pi,-j}(z,w) = \frac{j}{\pi}\sum_{k=1}^{j}(-1)^{k+j-1}C_k^j C_j^{j+k-1}\frac{(z-\overline{w})^{j-k}}{(\overline{z}-w)^{j+k}}|z-w|^{2(k-1)}$$

Explicit Dzhuraev´s formulas on the unit disk

$$\mathrm{cl\,span}\left\{z^m\overline{z}^k : k=0,\cdots,j-1\,;\, m\in\mathbb{N}\right\} = \mathcal{A}_j^2(\mathbb{D})\,,\ j\in\mathbb{Z}_+$$

Proposition ([KP-08])

i) $(S_{\mathbb{D}}^{*}\, w^m\overline{w}^k)(z) = \frac{k}{m+1}\,\overline{z}^{k-1}z^{m+1} + \frac{\min\{0,m+1-k\}}{m+1}\,\overline{z}^{k-m-2}$

ii) $B_{\mathbb{D},j} = I - S_{\mathbb{D},-j}S_{\mathbb{D},-j}\,,\ j\in\mathbb{Z}_\pm$

iii) $S_{\mathbb{D},-j} = S_{\mathbb{D}}^{j}$ and $S_{\mathbb{D},j} = (S_{\mathbb{D}}^{*})^{j}\,,\ j\in\mathbb{Z}_+$

$$N_{j,k} := \mathrm{span}\left\{z^l\overline{z}^s : l=0,\ldots,j-1; s=0,\ldots,k-1\right\}\,,\ \dim N_{j,k} = jk$$

Proposition ([KP-08])

$$\widetilde{B}_{\mathbb{D},j}B_{\mathbb{D},k} = P_{N_{j,k}}$$

Explicit Dzhuraev's formulas on Π. Variation of the domain

$$B_{\Pi,j} = \text{s-lim}\, B_{D(in,n),j} = \text{s-lim}\, D_{D(in,n),j} = D_{\Pi,j}$$

Theorem (Dzhuraev formulas for Π [P-Submitted, RS-03, V-08])

$$\begin{aligned} B_{\Pi,j} &= I - (S_\Pi)^j (S_\Pi^*)^j = I - S_{\Pi,-j} S_{\Pi,j} \\ B_{\Pi,-j} &= I - (S_\Pi^*)^j (S_\Pi)^j = I - S_{\Pi,j} S_{\Pi,-j} \end{aligned} \quad , \; j \in \mathbb{Z}_+$$

Theorem ([V-99, V-08, P-Submitted])

$B_{\Pi,j} B_{\Pi,-k} = 0 \quad$ *for* $\quad j, k \in \mathbb{Z}_+$.

Simple proof based on the variation of the domain

Consider W_{φ_n} with $\varphi_n : \mathbb{D} \to D(in, n) \quad , \quad \varphi_n(z) = nz + in$

Invariance for dilations and translations gives that

$$B_{\Pi,j} B_{\Pi,-k} = \text{s-lim}\, W_{\varphi_n}^* B_{\mathbb{D},j} B_{\mathbb{D},-k} W_{\varphi_n} = \text{s-lim}\, W_{\varphi_n}^* P_{N_{j,k}} W_{\varphi_n}$$

$\text{w-lim}_{n\to\infty} W_{\varphi_n} = 0$ and $P_{N_{j,k}} \in \mathcal{K}$ imply that $B_{\Pi,j} B_{\Pi,-k} = 0$

The complement of a disk

- Let $\varphi(z) := (az + b)/(cz + d)$ be a non constant Möbius map
- $\varphi(\mathbb{D})$ is either a disk, a half-space or the complement of a disk
- We have considered the disk $\mathbb{D} := \{z : |z| < 1\}$ and the half-space $\Pi := \{z : \mathrm{Im}\, z > 0$
- Define $\Omega = \{z : |z| > 1\}$

The guidelines for what follows will be

- Exact Dzhuraev's formulas for Ω;
- The poly Bergman kernel for Ω.

Recall some examples of domains not admitting *Dzhuraev's* formulas

Violation of Dzhuraev's formulas

- Let U be bounded domain admitting Dzhuraev's formulas
- l_z a half-Straight line outgoing from $z \in U$

Proposition ([KP-08])

$U \backslash l_z$ does not admit Dzhuraev's formulas.

- Π_ϕ is the sector $\{z : 0 < \arg z < \phi\}$ for $0 < \phi \leq 2\pi$

Proposition ([P-Submitted])

- *Π_ϕ admits Dzhuraev's formulas iff $\phi = \pi$.*
- *Let $U \subset \mathbb{C}$ be in the following conditions*

$$\lambda U \subset \nu U \text{ if } 1 \leq \lambda \leq \nu \quad \text{and} \quad \cup_{\lambda \geq 1} \lambda U = \Pi_\phi .$$

If U admits Dzhuraev's formulas then $\phi = \pi$.

SIO and poly Bergman projections on some sectors

If $\Pi_m := \{z : \mathrm{Im} z^m > 0\}$, $m = 2, \cdots$ then [KP-08]

$$\begin{aligned} B_{\Pi_m} &= I - (S_{\Pi_m} + R_{\Pi_m})(S_{\Pi_m} + R_{\Pi_m})^* \\ \widetilde{B}_{\Pi_m} &= I - (S_{\Pi_m} + R_{\Pi_m})^* (S_{\Pi_m} + R_{\Pi_m}) \end{aligned}$$

where $W_{\varepsilon_m} f(z) := \varepsilon_m f(\varepsilon_m z)$ for $\varepsilon_m := e^{i\pi/m}$ and

$$R_{\Pi_m} = \sum_{k=1}^{m-1} R_{m,k} \quad \text{and} \qquad R_{m,k} = \chi_{\Pi_m} W_{\varepsilon_m}^{2k} S \chi_{\Pi_m}$$

Isomorphism with the punctured disk

- Define $\nu_j(z) := (\overline{z}/z)^j$ and $\varphi(z) = 1/z$
- $W : L^2(\mathbb{D}) \to L^2(\Omega) \quad , \quad Wf(z) = f(\varphi(z))\varphi'(z)$
- The punctured domain $U_\xi = U \backslash \{\xi\}$

Proposition ([P-Submitted-2])

The following operator is an onto unitary operator

$$V_j : \mathcal{A}_j^2(\mathbb{D}_0) \to \mathcal{A}_j^2(\Omega) \quad , \quad V_j = \nu_{j-1} W.$$

Guidelines

- To study the poly Bergman space of a punctured domain
- To estimate the norm of every point evaluation of a derivative

Point evaluations of derivatives

One has that [P-Submitted-2]

$$\partial_z^n f(0) = \langle f, p_{n,j} \rangle \ , \ f \in \mathcal{A}_j^2(\mathbb{D}) \quad (n \in \mathbb{N})$$

where the polynomial $p_{n,j}(z,\overline{z})$ is given by the following recursive formula

$$p_{n,1} := \frac{(n+1)!}{\pi} z^n$$

$$p_{n,j}(z,\overline{z}) := p_{n,1} - (n+1)! \sum_{k=1}^{j-1} \frac{(S_{\mathbb{D}})^k p_{n+2k,j-k}}{(n+k+1)!k!} \ ; \ j = 2, \cdots$$

Observe that [KP-08]

$$\left(S_{\mathbb{D}} \, \overline{w}^n w^m\right)(z) = \frac{m}{n+1} \, \overline{z}^{n+1} z^{m-1} + \frac{\min\{0, n+1-m\}}{m+1} \, z^{m-n-2}.$$

For derivatives in order to $\overline{z}$ we consider Vekua's derivation formulas

$$\partial_{\overline{z}} S_U f = \partial_z f \quad , \quad \partial_z S_U^* f = \partial_{\overline{z}} f \quad f \in C^\infty(U) \cap L^2(U)$$

Poly Bergman spaces on punctured domains

Proposition ([P-Submitted-2])

Let $U \subset \mathbb{C}$ *be a domain and* $z \in U$. *For every* $k, n = 0, 1, \cdots$

$$|\partial_z^n \partial_{\overline{z}}^k f(z)| \leq \frac{M}{d_z^{k+n+1}} \|f\| \ , \ f \in \mathcal{A}_j^2(U) \quad (j \in \mathbb{Z}_\pm),$$

where M *is a positive constant only depending on* n, k *and* j.

Proposition ([P-Submitted-2])

Let $U \subset \mathbb{C}$ *be a bounded domain and* $\xi \in U$. *If* $j = 2, \cdots$ *then*

$$\mathcal{A}_j^2(U_\xi) = \mathrm{span}\left\{\psi, \frac{(\overline{z}-\overline{\xi})^k}{(z-\xi)^l} : \psi \in \mathcal{A}_j^2(U);\ k = 1, \cdots, j-1;\ l = 1, \cdots k\right\}.$$

The Hilbert space $\mathcal{A}_j^2(U_\xi) \ominus \mathcal{A}_j^2(U)$ *has finite dimension* $|j|(|j|-1)/2$.

The Dzhuraev's operator $P_{\Omega,j}$

Let $\varphi : U \to V$ be an analytic bijection, $\varphi(z) = (az+b)/(cz+d)$

$$W_\varphi : L^2(V) \to L^2(U) \quad , \quad W_\varphi f(z) := f(\varphi(z))\varphi'(z)$$

Proposition (Möbius Change of Variable in SIO [P-Submitted-2])

$$W_\varphi S_{V,j} W_\varphi^* = c_{j+1} S_{U,j} c_{j-1}, \quad j \in \mathbb{Z}_\pm$$

where for every $j \in \mathbb{Z}$ the unitary functions c_j are defined by the following

$$c_j(z) := \frac{\Delta^j}{|\Delta|^j} \left(\frac{\overline{cz} + \overline{d}}{cz+d} \right)^j, \quad \Delta := ad - bc \neq 0.$$

Proposition (Dzhuraev's formulas [P-Submitted-2])

$$V_j B_{\mathbb{D},j} V_j^* = P_{\Omega,j} \quad \text{and} \quad B_{\Omega,j} = P_{\Omega,j} + Q_j$$

Q_j is the orthogonal projection of $L^2(\Omega)$ onto the $|j|(|j|-1)/2$ dimensional subspace $\mathcal{A}_j^2(\Omega) \ominus V_j(\mathcal{A}_j^2(\mathbb{D}))$

Orthogonality between poly and anti-poly Bergman spaces

[KP-08]
$$\begin{aligned} \mathrm{cl\ span}\left\{z^k\overline{z}^l : l=0,\cdots,j-1;\ k=0,1,\cdots\right\} &= \mathcal{A}_j^2(\mathbb{D}) \\ \mathrm{cl\ span}\left\{\overline{z}^k z^l : l=0,\cdots,j-1;\ k=0,1,\cdots\right\} &= \mathcal{A}_{-j}^2(\mathbb{D}) \end{aligned}, \quad j\in\mathbb{Z}_+$$

Because $V_j(\mathcal{A}_j^2(\mathbb{D}_0)) = \mathcal{A}_j(\Omega)$ and $\mathcal{A}_j^2(\mathbb{D}_0) = \mathcal{A}_j(\mathbb{D}) \oplus \widetilde{F}_j$ then

Proposition ([P-Submitted-2])

$$\begin{aligned} \mathrm{cl\ span}\left\{\frac{1}{z^2}\frac{\overline{z}^l}{z^k} : l=0,\cdots,j-1;\ k=l,\cdots\right\} &= \mathcal{A}_j^2(\Omega) \\ \mathrm{cl\ span}\left\{\frac{1}{\overline{z}^2}\frac{z^l}{\overline{z}^k} : l=0,\cdots,j-1;\ k=l,\cdots\right\} &= \mathcal{A}_{-j}^2(\Omega) \end{aligned}, \quad j\in\mathbb{Z}_+$$

Proposition ([P-Submitted-2])

Let $j,k\in\mathbb{Z}_\pm$. If $jk<0$ then $B_{\Omega,j}B_{\Omega,k}=0$.

Singular integral operators and partial isometries

$$S_j = F^{-1}(\xi/\bar{\xi})^j F \ , \ j \in \mathbb{Z}_\pm \quad (\mathrm{Mikhlin\ \ Symbol})$$

Proposition

The application $\mathbb{Z} \ni j \to S_j$ *is a group homomorphism* $\quad (S_0 := I)$

$$P_{\Omega,j} = I - \chi_\Omega S_{-j}(1-\chi_{\mathbb{D}})S_j\chi_\Omega = (\chi_{\mathbb{D}} S_j \chi_\Omega)^*(\chi_{\mathbb{D}} S_j \chi_\Omega) \ , \ j \in \mathbb{Z}_\pm .$$

Recall: $P : \mathcal{H}_1 \to \mathcal{H}_2$ is a **partial isometry** with **initial space** $\mathcal{A} \subset \mathcal{H}$ if $\mathrm{Ker}\, P = \mathcal{A}^\perp$ and P acts unitarily on $\mathcal{A}$; $\mathrm{Im}\, P$ is its **final space**.

Proposition (Well known)

Let $P : \mathcal{H}_1 \to \mathcal{H}_2$ *be a bounded operator. The assertions are equivalent:*

- P *is a partial isometry with initial space* N *and final space* M*;*
- P^* *is a partial isometry with initial space* M *and final space* N*;*
- P^*P *and* PP^* *are projections of* $\mathcal{H}_1$ *onto* N *and of* $\mathcal{H}_2$ *onto* M*.*

Different Dzhuraev's formulas

Definition

$$\mathcal{P}_j(\Omega) := \operatorname{Im} P_{\Omega,j} = V_j\left[\mathcal{A}_j^2(\mathbb{D})\right] \subset \mathcal{A}_j^2(\Omega)\ ,\ j \in \mathbb{Z}_\pm .$$

Proposition ([P-Submitted-2])

- $\mathcal{P}_j(\Omega)$ *and* $\mathcal{A}_{-j}^2(\mathbb{D})$ *are respectively the initial and final spaces of the partial isometry* $\chi_{\mathbb{D}} S_j \chi_\Omega$*;*
- $\mathcal{A}_j^2(\mathbb{D})$ *and* $\mathcal{P}_{-j}(\Omega)$ *are respectively the initial and final spaces of the partial isometry* $\chi_\Omega S_j \chi_{\mathbb{D}}$*.*

Proposition ([P-Submitted-2])

$D_{\Omega,j} := I - (S_\Omega)^j (S_\Omega^*)^j$ *is an orthogonal projection and* $\operatorname{Im} D_{\Omega,j} \subset \mathcal{A}_j^2(\Omega)$.

$$D_{\Omega,j} D_{\Omega,k} = D_{\Omega,m}\ ,\ jk > 0 \quad \text{and} \quad D_{\Omega,j} D_{\Omega,k} = 0\ ,\ jk < 0$$

where $m := \operatorname{sgn}(j) \min\{|j|, |k|\}$.

The Singular integral operators $S_{\Omega,j}$ and $(S_\Omega^*)^j$

The proof of the previous Proposition is technical and only depends on:

- $\partial_{\bar z} S_U f = \partial_z f \quad , \quad \partial_z S_U^* f = \partial_{\bar z} f \quad f \in C^\infty(U) \cap L^2(U)$
- orthogonality between poly and anti-poly Bergman spaces of Ω
- the evident equality $D_{\Omega,j} = D_{\Omega,j-1} + (S_\Omega)^{j-1} B_\Omega (S_\Omega^*)^{j-1}$

In what follows we relate the operators $S_{\Omega,j}$ and $(S_\Omega^*)^j$, for $j \in \mathbb{Z}_+$

$$S_{\Omega,j} = \chi_\Omega S^* S_{j-1} \chi_\Omega = \chi_\Omega S^* (\chi_\Omega + \chi_\mathbb{D}) S_{j-1} \chi_\Omega = S_\Omega^* S_{\Omega,j-1} + \chi_\Omega S^* \chi_\mathbb{D} S_{j-1} \chi_\Omega .$$

For every $j \in \mathbb{Z}_+$ define the operator

$$T_j : L^2(\Omega) \to L^2(\Omega) \quad , \quad T_j = \chi_\Omega S^* \chi_\mathbb{D} S_j \chi_\Omega .$$

Thus

$$S_{\Omega,j} = S_\Omega^* S_{\Omega,j-1} + T_{j-1} = \cdots = (S_\Omega^*)^j + \sum_{k=0}^{j-2} (S_\Omega^*)^k T_{j-1-k} .$$

The Singular integral operators $S_{\Omega,j}$ and $(S_\Omega^*)^j$

Definition

$$N_j := \mathcal{A}^2(\mathbb{D})\cap\mathcal{A}^2_{-j}(\mathbb{D}),\ X_j := \chi_\Omega S_{-j}\chi_{\mathbb{D}}\,(N_j)\,,\ Y_j := \chi_\Omega S^*\chi_{\mathbb{D}}\,(N_j)\,;\ j\in\mathbb{Z}_+$$

Proposition ([P-Submitted-2])

- $N_j = \mathrm{span}\,\{z^l : l = 0,1,\ldots,j-1\}$;
- X_j *and* Y_j *are* j*-dimensional and* $X_j \subset \mathcal{A}^2_j(\Omega)$ and $Y_j \subset \widetilde{\mathcal{A}}^2(\Omega)$;
- $X_j \perp X_k$ *for every* $j,k\in\mathbb{Z}_+$ *such that* $j\neq k$;
- T_j *is a partial isometry with initial space* X_j *and final space* Y_j.

The Singular integral operators $S_{\Omega,j}$ and $(S_{\Omega}^{*})^{j}$

Proposition ([P-Submitted-2])

$(S_{\Omega}^{*})^{k}$ and $(S_{\Omega})^{k}$ are partial isometries with initial spaces $[\mathcal{D}_{k}(\Omega)]^{\perp}$ and $[\mathcal{D}_{-k}(\Omega)]^{\perp}$ and final spaces $[\mathcal{D}_{-k}(\Omega)]^{\perp}$ and $[\mathcal{D}_{k}(\Omega)]^{\perp}$, for $k \in \mathbb{Z}_{+}$.

Proposition ([P-Submitted-2])

$(S_{\Omega}^{*})^{k} T_{n}$ is a partial isometry with initial and final spaces respectively given by X_{n} and $(S_{\Omega}^{*})^{k}(Y_{n})$, for $n, k = 0, \cdots$.

What about $L_{j} := \sum_{k=0}^{j-2} (S_{\Omega}^{*})^{k} T_{j-1-k} := \sum_{k=0}^{j-2} L_{j,k}$?

Definition

- Define the true projections $D_{\Omega,(\pm1)} := D_{\Omega,\pm1}$ jointly with

$$D_{\Omega,(j)} := D_{\Omega,j} - D_{\Omega,j-1} \; ; \; j = 2, \cdots$$
$$D_{\Omega,(j)} := D_{\Omega,j} - D_{\Omega,j+1} \; ; \; -j = 2, \cdots$$

- The true images $\mathcal{D}_{(j)}(\Omega) := \mathrm{Im}\, D_{\Omega,(j)}$, $j \in \mathbb{Z}_{\pm}$

The Singular integral operators $S_{\Omega,j}$ and $(S_\Omega^*)^j$

Proposition ([P-Submitted-2])

Let $j \in \mathbb{Z}_+$. The operators

$$\begin{array}{lllllll} (S_\Omega)^j & : & \mathcal{D}_{(k)}(\Omega) & \to & \mathcal{D}_{(k+j)}(\Omega) & ; & k \in \mathbb{Z}_+ \\ (S_\Omega)^j & : & \mathcal{D}_{(k)}(\Omega) & \to & \mathcal{D}_{(k+j)}(\Omega) & ; & k \in \mathbb{Z}_- \quad , \quad j < -k \end{array}$$

jointly with the following ones

$$\begin{array}{lllllll} (S_\Omega^*)^j & : & \mathcal{D}_{(k)}(\Omega) & \to & \mathcal{D}_{(k-j)}(\Omega) & ; & k \in \mathbb{Z}_- \\ (S_\Omega^*)^j & : & \mathcal{D}_{(k)}(\Omega) & \to & \mathcal{D}_{(k-j)}(\Omega) & ; & k \in \mathbb{Z}_+ \quad , \quad j < k \end{array}$$

are isometric isomorphisms. Furthermore

$$\operatorname{Ker}(S_\Omega^*)^j = \mathcal{D}_j(\Omega) \supset \mathcal{A}^2(\Omega) \quad \text{and} \quad \operatorname{Ker}(S_\Omega)^j = \mathcal{D}_{-j}(\Omega) \supset \widetilde{\mathcal{A}}^2(\Omega)\,.$$

- $L_{j,k} := (S_\Omega^*)^k T_{j-1-k}$ has initial and final spaces given by
$$X_{j-1-k} \quad \text{and} \quad (S_\Omega^*)^k(Y_{j-1-k}) \subset (S_\Omega^*)^k(\mathcal{D}_{-1})$$
- initial and final spaces of $L_{j,k}$ and of $L_{j,l}$ are orthogonal $(k \neq l)$

Different Dzhuraev's operators

Proposition ([P-Submitted-2])

$S_{\Omega,j} = (S_\Omega^*)^j + L_j$, where $L_j := \sum_{k=0}^{j-2} L_{j,k}$ is a partial isometry with initial and final spaces having dimension $j(j-1)/2$ and given by

$$\bigoplus_{k=0}^{j-2} X_{j-1-k} \quad \text{and} \quad \bigoplus_{k=0}^{j-2} (S_\Omega^*)^k (Y_{j-1-k}).$$

$$S_{\Omega,-j} S_{\Omega,j} = (S_\Omega)^j (S_\Omega^*)^j + (S_\Omega)^j L_j + L_j^* (S_\Omega^*)^j + L_j^* L_j.$$

From the action of $(S_\Omega^*)^k$ on $\mathcal{D}_{-1}(\Omega)$ we deduce

$$\begin{aligned} \mathrm{Im}\, L_j &= \bigoplus_{k=0}^{j-2} (S_\Omega^*)^k (Y_{j-1-k}) \subset \bigoplus_{k=0}^{j-2} \mathcal{D}_{(-k-1)}(\Omega) \\ &= \mathcal{D}_{-j+1}(\Omega) \subset \mathcal{D}_{-j}(\Omega)\ ;\ j = 2, \cdots . \end{aligned}$$

Different Dzhuraev's formulas

Proposition ([P-Submitted-2])

$$D_{\Omega,j} = P_{\Omega,j} + F_j \ , \ j \in \mathbb{Z}_\pm$$

F_j is an orthogonal projection onto a $|j|(|j|-1)/2$ dimensional space and

$$\mathrm{Im}\, F_j = \bigoplus_{k=1}^{j-1} X_k \ , \ j > 1 \quad \text{and} \quad \mathrm{Im}\, F_j = \bigoplus_{k=1}^{|j|-1} \overline{X_k} \ , \ j < -1$$

Proposition ([P-Submitted-2])

If $j \in \mathbb{Z}_\pm$ then the Dzhuraev's formula $B_{\Omega,j} = D_{\Omega,j}$ is valid.

The Singular integral operators $S_{\Omega,j}$ and $(S_\Omega^*)^j$

Proposition ([P-Submitted-2])

Let $j \in \mathbb{Z}_+$. The operators

$$\begin{array}{lllllll} (S_\Omega)^j & : & \mathcal{A}^2_{(k)}(\Omega) & \to & \mathcal{A}^2_{(k+j)}(\Omega) & ; & k \in \mathbb{Z}_+ \\ (S_\Omega)^j & : & \mathcal{A}^2_{(k)}(\Omega) & \to & \mathcal{A}^2_{(k+j)}(\Omega) & ; & k \in \mathbb{Z}_- \quad , \quad j < -k \end{array}$$

jointly with the following ones

$$\begin{array}{lllllll} (S_\Omega^*)^j & : & \mathcal{A}^2_{(k)}(\Omega) & \to & \mathcal{A}^2_{(k-j)}(\Omega) & ; & k \in \mathbb{Z}_- \\ (S_\Omega^*)^j & : & \mathcal{A}^2_{(k)}(\Omega) & \to & \mathcal{A}^2_{(k-j)}(\Omega) & ; & k \in \mathbb{Z}_+ \quad , \quad j < k \end{array}$$

are isometric isomorphisms. Furthermore

$$\mathrm{Ker}\,(S_\Omega^*)^j = \mathcal{A}^2_j(\Omega) \supset \mathcal{A}^2(\Omega) \quad \text{and} \quad \mathrm{Ker}\,(S_\Omega)^j = \mathcal{A}^2_{-j}(\Omega) \supset \widetilde{\mathcal{A}}^2(\Omega)\,.$$

Poly Bergman kernel function

$$V_j B_{\mathbb{D}_0,j} V_j^* f(z) = \int_\Omega \frac{\overline{z}^{j-1}}{z^{j+1}} K_{\mathbb{D}_0,j}\left(\frac{1}{w},\frac{1}{z}\right)\frac{w^{j-1}}{\overline{w}^{j+1}} f(w)\,dA(w) = B_{\Omega,j} f(z),$$

Thus

$$K_{\Omega,j}(z,w) = \frac{(\overline{z}w)^{j-1}}{(z\overline{w})^{j+1}} K_{\mathbb{D}_0,j}\left(\frac{1}{z},\frac{1}{w}\right)\ ;\ z,w\in\Omega.$$

If $\{\varphi_{j,k}\}$ is an orthonormal base for the space $\mathcal{A}_j^2(\mathbb{D}_0)\ominus\mathcal{A}_j^2(\mathbb{D})$ then

$$\begin{aligned} K_{\Omega,j}(z,w) &= \frac{(\overline{z}w)^{j-1}}{(z\overline{w})^{j+1}} K_{\mathbb{D},j}\left(\frac{1}{z},\frac{1}{w}\right) + \frac{(\overline{z}w)^{j-1}}{(z\overline{w})^{j+1}} \sum_k \varphi_{j,k}\left(\frac{1}{z}\right)\overline{\varphi}_{j,k}\left(\frac{1}{w}\right) \\ &= \frac{(\overline{z}w)^{j-1}}{(z\overline{w})^{j+1}} K_{\mathbb{D},j}\left(\frac{1}{z},\frac{1}{w}\right) + \sum_k V_j\varphi_{j,k}(z)\overline{V_j\varphi_{j,k}}(w) \end{aligned}$$

Poly Bergman kernel function

Thus $\{V_j\varphi_{j,k}\}_k$ is an orthonormal base for the space

$$\mathcal{A}_j^2(\Omega) \ominus V_j(\mathcal{A}_j^2(\mathbb{D})) = \bigoplus_{k=1}^{j-1} X_k \ ; \ j = 2, \cdots$$

Due to $X_k := \chi_\Omega S_{-k}\chi_{\mathbb{D}}\,(N_k)$ we obtain an orthonormal base for X_k

$$\psi_{k,l}(z) = -\frac{\bar{z}^k}{z^l} F(-k, l; 1; 1 - |z|^{-2}), \ \ l = 1, \cdots, k$$

where $F(-k, b; c; z)$ is the $(2,1)$-hypergeometric function given for $b, z \in \mathbb{C}$, $c \in \mathbb{C} \setminus \{0, -1, -2, \ldots, -k+1\}$ and $k = 0, 1, 2, \ldots$ by

$$F(-k, b; c; z) = \sum_{n=0}^{k} \frac{(-k)_n (b)_n}{(c)_n n!} z^n.$$

Poly Bergman kernel function

Proposition ([P-Submitted-2])

Let $j \in \mathbb{Z}_+$. The j-poly-Bergman kernel of Ω is given by

$$K_{\Omega,j}(z,w) = \frac{j}{\pi}\frac{\sum_{n=1}^{j}(-1)^{n-1}\binom{j}{n}\binom{j+n-1}{j}|1-\overline{w}z|^{2(j-n)}|z-w|^{2(n-1)}}{(1-\overline{w}z)^{2j}}$$
$$+\sum_{k=1}^{j-1}\sum_{l=1}^{k}\frac{k-l+1}{\pi}\frac{(\overline{z}w)^k}{(z\overline{w})^l}F(-k,l;1;1-|z|^{-2})F(-k,l;1;1-|w|^{-2})$$

Additionally, for every $j \in \mathbb{Z}_\pm$ one has that

$$K_{\Omega,j}(z,w) = K_{\Omega,-j}(w,z)\,; \;\; z,w \in \Omega.$$

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