On the structure of polyharmonic Bergman spaces.

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Abstract

I will present some new results on the structure of polyharmonic Bergman spaces over some domains in terms of the compression of the Beurling-Ahlfors transform. It will be explained how the results are a consequence of the validity of Dzhuraev's formulas, i.e. how such study can be based on the fact that the compression of the Beurling-Ahlfors transform is a power partial isometry over special domains. Theorems of Paley-Wienner type for polyharmonic Bergman spaces will be given for half-spaces. The talk is partially based on a joint work with A. M. Santos.

Poly-Bergman spaces

$U \subset \mathbb{C}$ non-empty, open and connected ; $dA(z) = dxdy$ area measure

$$\partial_{\bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \quad \partial_z := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

Definition (Poly-Bergman spaces)

$f \in \mathcal{A}_j^2(U)$ if $f \in L^2(U, dA)$, f is smooth and

$$\partial_{\bar{z}}^j f = 0 \text{ and } \partial_z^{-j} f = 0 , \quad \textit{respectively if } j \in \mathbb{Z}_+ \textit{ and } j \in \mathbb{Z}_- \tag{1.1}$$

- if $j \in \mathbb{Z}$ then f is j-polyanalytic if is smooth and satisfies (1.1)
- if $j \in \mathbb{Z}_-$ then it is also usually said that f is $|j|$-anti-polyanalytic
- if $j = 0$ then we have the special case $\mathcal{A}_0^2(U) = \{0\}$

Hilbert spaces of α-Polyanalytic functions, $\alpha = (j, k)$

Now we consider $\alpha := (j, k)$ a pair of non-negative integers

Definition (α-polyanalytic function)

f is smooth on U and $\partial_{\bar{z}}^{j} \partial_{z}^{k} f = 0 \quad (j, k = 0, 1, \ldots)$

Definition (α-polyanalytic Bergman space)

$f \in \mathcal{A}_{\alpha}^{2}(U)$ if $f \in L^2(U, dA)$ and f is α-polyanalytic

- Is $\mathcal{A}_{\alpha}^{2}(U)$ a Hilbert space? The following two results and some analysis will allow to say Yes. First, some definitions.
- Define $N_{j,k} := \mathcal{A}_{j}^{2}(\mathbb{D}) \cap \mathcal{A}_{-k}^{2}(\mathbb{D})$, $j, k \in \mathbb{Z}_{+}$ [see **L.V.P. 14**]
- Then $N_{j,k} = \mathrm{span}\{z^l \bar{z}^n : l = 0, 1, \ldots, k-1;\ n = 0, \ldots, j-1\}$

Hilbert spaces of α-Polyanalytic functions, $\alpha = (j,k)$

Theorem (Yu.I. Karlovich, L.V.P. 08)

The following assertions hold:

i) $B_{\mathbb{D},j}$ *and* $B_{\mathbb{D},k}$ *commute* $\quad (j,k \in \mathbb{Z})$;

ii) $B_{\mathbb{D},j} B_{\mathbb{D},-k}$ *is the projection of* $L^2(\mathbb{D}, dA)$ *onto* $N_{j,k} \quad (j,k \in \mathbb{Z}_+)$.

Lemma

Let $\mathcal{H}$ *be a Hilbert space and let* $M, N \in \mathcal{B}(\mathcal{H})$ *be projections. Then,* $P := M + N - MN$ *is a projection iff* M *and* N *commute. Furthermore, if* P *is a projection, then its range coincides with* $\operatorname{Im} M + \operatorname{Im} N$.

Theorem (L.V.P. 14)

Let $j, k = 0, 1, \ldots$ *and let* $\alpha := (j,k)$. *Then* $\mathcal{A}^2_\alpha(\mathbb{D})$ *is closed in* $L^2(\mathbb{D})$. *If* $B_{\mathbb{D},\alpha}$ *denotes the orthogonal projection of* $L^2(\mathbb{D})$ *onto* $\mathcal{A}^2_\alpha(\mathbb{D})$, *then*

$$B_{\mathbb{D},\alpha} = B_{\mathbb{D},j} + B_{\mathbb{D},-k} - B_{\mathbb{D},j} B_{\mathbb{D},-k}.$$

α-Polyanalytic functions and Singular Integral Operators

- The unitary Beurling-Ahlfors transform and its compression to $L^2(U)$

$$Sf(z) := -\frac{1}{\pi}\int_{\mathbb{C}} \frac{f(w)}{(w-z)^2}\,dA(w) \quad \text{and} \quad S_U := \chi_U S \chi_U$$

- Dzhuraev's Operators (for $j \in \mathbb{Z}_+$)

$$D_{U,j} = I - (S_U)^j (S_U^*)^j \quad \text{and} \quad D_{U,-j} = I - (S_U^*)^j (S_U)^j$$

- If U is bounded finitely connected, ∂U is smooth then

$$B_{U,j} - D_{U,j} \in \mathcal{K} \quad (j \in \mathbb{Z}_\pm).$$

- The existence of Dzhuraev's formulas are strongly dependent on the regularity of the boundary
Yu.I. Karlovich, L.V.P. 08; L.V.P 13

α-Polyanalytic functions and Singular Integral Operators

Theorem (Yu.I. Karlovich, L.V.P. 08; L.V.P. 14)

$$B_{\mathbb{D},j} = D_{\mathbb{D},j} \quad , \quad B_{\Pi,j} = D_{\Pi,j} \quad , \quad B_{\mathbb{E},j} = D_{\mathbb{E},j}$$

If $U \in \{\mathbb{D}, \Pi, \mathbb{E}\}$ then S_U is a $*$-power partial isometry

Theorem (L.V.P. 14)

Let j and k be nonnegative integers and let $\alpha := (j, k)$. Then,

$$B_{\mathbb{D},\alpha} = I - (S_{\mathbb{D}})^j (S_{\mathbb{D}}^*)^{j+k} (S_{\mathbb{D}})^k = I - (S_{\mathbb{D}}^*)^k (S_{\mathbb{D}})^{j+k} (S_{\mathbb{D}}^*)^j .$$

Some results are then easily generalised to L^p, $1 < p < +\infty$

Theorem (L.V.P. 14)

Let $j, k = 0, 1, \ldots$ and let $\alpha := (j, k)$. Then, $B_{\mathbb{D},\alpha}$ defines a bounded idempotent acting on $L^p(\mathbb{D})$, for $1 < p < +\infty$.

α-Polyanalytic functions and Singular Integral Operators

The compression of the Riesz transforms of **even** order $(S_j = R_{-2j})$

$$S_{\mathbb{D},j} f(z) := \frac{(-1)^j |j|}{\pi} \int_{\mathbb{D}} \frac{(w-z)^{j-1}}{(\overline{w}-\overline{z})^{j+1}} f(w) dA(w) \ , \ j \in \mathbb{Z}_{\pm}$$

From results in **Yu.I. Karlovich; L.V.P. 08** we known that

$$S_{\mathbb{D},j} = (S_{\mathbb{D}}^*)^j \quad \text{and} \quad S_{\mathbb{D},-j} = (S_{\mathbb{D}})^j \, .$$

Theorem (L.V.P. 14)

Let $j, k = 0, 1, \ldots$ *and let* $\alpha := (j, k)$. *Then,*

$$B_{\mathbb{D},\alpha} = I - S_{\mathbb{D},-j} S_{\mathbb{D},j+k} S_{\mathbb{D},-k} = I - S_{\mathbb{D},k} S_{\mathbb{D},-j-k} S_{\mathbb{D},j} .$$

α-Polyanalytic Bergman spaces are RKHS

α-Polyanalytic Bergman spaces are **reproducing kernel Hilbert spaces**

Theorem (L.V.P. 14)

Let $U \subset \mathbb{C}$ be a domain, let $j, k = 0, 1, \dots$ let $\alpha := (j, k)$. Then $\mathcal{A}^2_\alpha(U)$ is a RKHS. For every $n, m = 0, 1, \dots$ and every $z \in U$, one has

$$|\partial_z^n \partial_{\bar{z}}^m f(z)| \leq \frac{M}{d_z^{n+m+1}} \|f\| \ , \ f \in \mathcal{A}^2_\alpha(U)$$

where M is a positive constant only depending on n, m, j and k.

$B_{\mathbb{D},\alpha}$ is integral operator with kernel given by the α-polyanalytic Bergman kernel $K_{\mathbb{D},\alpha}(z, w)$, which has a non-friendly representation.

True Poly-Bergman Spaces and More $N_{j,k}$ Type Spaces

- For $j \in \mathbb{Z}_{\pm}$, the true poly Bergman spaces, which were introduced over half-spaces in **N. Vasilevski 99**

$$\mathcal{A}^2_{(\pm 1)}(\mathbb{D}) := \mathcal{A}^2_{\pm 1}(\mathbb{D}) \quad \text{and} \quad \mathcal{A}^2_{(j)}(\mathbb{D}) := \mathcal{A}^2_{j}(\mathbb{D}) \ominus \mathcal{A}^2_{j-\operatorname{sgn} j}(\mathbb{D})$$

- Then it is clear that

$$B_{\mathbb{D},(j)} = B_{\mathbb{D},j} - B_{\mathbb{D},j-1} \;, j > 1 \quad \text{and} \quad B_{\mathbb{D},(j)} = B_{\mathbb{D},j} - B_{\mathbb{D},j+1} \;, j < -1.$$

- We introduce the following spaces like in the definition of $N_{j,k}$

$$\begin{aligned} N_{(j),k} &:= \mathcal{A}^2_{(j)}(\mathbb{D}) \cap \mathcal{A}^2_{-k}(\mathbb{D}) = \operatorname{Im} B_{\mathbb{D},(j)} B_{\mathbb{D},-k} \\ N_{j,(k)} &:= \mathcal{A}^2_{j}(\mathbb{D}) \cap \mathcal{A}^2_{(-k)}(\mathbb{D}) = \operatorname{Im} B_{\mathbb{D},j} B_{\mathbb{D},(-k)} \\ N_{(j),(k)} &:= \mathcal{A}^2_{(j)}(\mathbb{D}) \cap \mathcal{A}^2_{(-k)}(\mathbb{D}) = \operatorname{Im} B_{\mathbb{D},(j)} B_{\mathbb{D},(-k)} \end{aligned}$$

Unitary Operators on True Poly Bergman Type Spaces

Theorem (L.V.P. 14)

Let $j \in \mathbb{Z}_+$ *and let* $k \in \mathbb{Z}_\pm$. *The operators*

$$(S_{\mathbb{D}})^j : \mathcal{A}^2_{(k)}(\mathbb{D}) \ominus N_{(k),j} \to \mathcal{A}^2_{(k+j)}(\mathbb{D}) \,, \qquad k > 0$$

$$(S_{\mathbb{D}})^j : \mathcal{A}^2_{(k)}(\mathbb{D}) \to \mathcal{A}^2_{(k+j)}(\mathbb{D}) \ominus N_{j,(-k-j)} \,, \qquad 0 < j < -k$$

as well as the following ones

$$(S^*_{\mathbb{D}})^j : \mathcal{A}^2_{(k)}(\mathbb{D}) \ominus N_{j,(-k)} \to \mathcal{A}^2_{(k-j)}(\mathbb{D}) \,, \qquad k < 0$$

$$(S^*_{\mathbb{D}})^j : \mathcal{A}^2_{(k)}(\mathbb{D}) \to \mathcal{A}^2_{(k-j)}(\mathbb{D}) \ominus N_{(k-j),j} \,, \qquad 0 < j < k$$

are isometric isomorphisms. Furthermore ▽

$$\operatorname{Ker}(S^*_{\mathbb{D}})^j = \mathcal{A}^2_j(\mathbb{D}) \quad \textit{and} \quad \operatorname{Ker}(S_{\mathbb{D}})^j = \mathcal{A}^2_{-j}(\mathbb{D}) \,.$$

Unitary Operators on Poly-Bergman Type Spaces

Theorem (L.V.P. 14)

Let $j \in \mathbb{Z}_+$ *and* $k \in \mathbb{Z}_\pm$. *The operators*

$$(S_{\mathbb{D}})^j : \mathcal{A}_k^2(\mathbb{D}) \ominus N_{k,j} \to \mathcal{A}_{k+j}^2(\mathbb{D}) \ominus \mathcal{A}_j^2(\mathbb{D}) \,, \qquad k > 0$$

$$(S_{\mathbb{D}})^j : \mathcal{A}_k^2(\mathbb{D}) \ominus \mathcal{A}_{-j}^2(\mathbb{D}) \to \mathcal{A}_{k+j}^2(\mathbb{D}) \ominus N_{j,-k-j} \,, \qquad 0 < j < -k$$

as well as the following ones

$$(S_{\mathbb{D}}^*)^j : \mathcal{A}_k^2(\mathbb{D}) \ominus N_{j,-k} \to \mathcal{A}_{k-j}^2(\mathbb{D}) \ominus \mathcal{A}_{-j}^2(\mathbb{D}) \,, \qquad k < 0$$

$$(S_{\mathbb{D}}^*)^j : \mathcal{A}_k^2(\mathbb{D}) \ominus \mathcal{A}_j^2(\mathbb{D}) \to \mathcal{A}_{k-j}^2(\mathbb{D}) \ominus N_{k-j,j} \,, \qquad 0 < j < k$$

are isometric isomorphisms.

Isomorphisms From the Bergman to the True Poly-Bergman Spaces and Differential Operators

For $k = 0, 1, \dots,$ let us consider the following operators in $\mathcal{B}(L^2(\mathbb{D}))$

$$\mathcal{S}_k := (S_{\mathbb{D}})^k z^k I \quad \text{and} \quad \mathcal{S}_{*k} := (S_{\mathbb{D}}^*)^k \overline{z}^k I$$

Theorem (L.V.P. 14)

Let $k = 0, 1, \dots$. Then, the following bounded operators

$$\mathcal{S}_k : \mathcal{A}^2(\mathbb{D}) \to \mathcal{A}^2_{(k+1)}(\mathbb{D}) \quad \textit{and} \quad \mathcal{S}_{*k} : \mathcal{A}^2_{-1}(\mathbb{D}) \to \mathcal{A}^2_{(-k-1)}(\mathbb{D})$$

are one-to-one and onto. If f lies in $\mathcal{A}^2(\mathbb{D})$ and in $\mathcal{A}^2_{-1}(\mathbb{D})$, then it respectively holds that

$$(\mathcal{S}_k f)(z) = \frac{\partial_z^k[(\overline{z}z - 1)^k f(z)]}{k!}$$

$$(\mathcal{S}_{*k} f)(z) = \frac{\partial_{\overline{z}}^k[(\overline{z}z - 1)^k f(z)]}{k!}$$

A Remark on some A. K. Ramazanov Results

- **A. K. Ramazanov 99** defines the following spaces
$$\mathcal{A}_k L_2^0(\mathbb{D}) := \left\{\partial_z^{k-1}[(1-z\overline{z})^{k-1}F(z)] : F \in \mathcal{A}^2(\mathbb{D})\right\} \ , \ k = 1, \ldots .$$
- The main result in that paper is the following assertion
$$\mathcal{A}_j^2(\mathbb{D}) = \bigoplus_{k=1}^{j} \mathcal{A}_k L_2^0(\mathbb{D}) \ , \ j = 1, \ldots .$$
- Considering the following Theorem, the A. K. Ramazanov result is nothing more that a **geometric evidence**

Theorem (L.V.P. 14)

Let $k \in \mathbb{Z}_+$. Then $\mathcal{A}_{(k)}^2(\mathbb{D})$ coincides with $\mathcal{A}_k L_2^0(\mathbb{D})$ and
$$\mathcal{A}_{(-k)}^2(\mathbb{D}) = \left\{\partial_{\overline{z}}^{k-1}[(1-z\overline{z})^{k-1}F(z)] : F \in \mathcal{A}_{-1}^2(\mathbb{D})\right\} .$$

The polyharmonic Bergman space

Definition (Polyharmonic Bergman Space)

For $k = 1, 2, \ldots$ let $\mathcal{H}_k^2(\mathbb{D}) := \mathcal{A}_\alpha^2(\mathbb{D})$, where $\alpha := (k, k)$.
That is, $f \in \mathcal{H}_k^2(\mathbb{D})$ iff f is smooth, $f \in L^2(U, dA)$ and $\Delta^k f = 0$.

- $\mathcal{H}_k^2(\mathbb{D})$ is a RKHS of functions on $\mathbb{D}$.
- $Q_{\mathbb{D},k}$ is defined to be the projection from $L^2(\mathbb{D})$ onto $\mathcal{H}_k^2(\mathbb{D})$;
- Let $j \in \mathbb{Z}_+$. Define $P_{j,k}$, $P_{(j),k}$, $P_{j,(k)}$ and $P_{(j),(k)}$ as the projections of $L^2(\mathbb{D})$ onto $N_{j,k}$, $N_{(j),k}$, $N_{j,(k)}$ and $N_{(j),(k)}$, respectively.

Theorem (L.V.P. 14)

Let k be a positive integer. Then

$$Q_{\mathbb{D},k} = B_{\mathbb{D},k} + B_{\mathbb{D},-k} - P_{k,k}.$$

Furthermore, $Q_{\mathbb{D},k} = I - (S_{\mathbb{D}})^k (S_{\mathbb{D}}^*)^{2k} (S_{\mathbb{D}})^k = I - (S_{\mathbb{D}}^*)^k (S_{\mathbb{D}})^{2k} (S_{\mathbb{D}}^*)^k.$

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The True Polyharmonic Bergman Spaces

- $\mathcal{H}^2_{(1)}(\mathbb{D}) := \mathcal{H}^2_1(\mathbb{D}) =: \mathcal{H}^2(\mathbb{D})$
- $\mathcal{H}^2_{(k)}(\mathbb{D}) := \mathcal{H}^2_k(\mathbb{D}) \ominus \mathcal{H}^2_{k-1}(\mathbb{D})\ ,\ k > 1.$
- $Q_{\mathbb{D},(1)} = Q_{\mathbb{D},1} =: Q_{\mathbb{D}}$ and $Q_{\mathbb{D},(k)} = Q_{\mathbb{D},k} - Q_{\mathbb{D},k-1}\ ,\ k > 1$

Theorem (L.V.P. 14)

Let $k = 2, \ldots$. *Then*

$$Q_{\mathbb{D},(k)} = B_{\mathbb{D},(k)} + B_{\mathbb{D},(-k)} - P_{(k),k} - P_{k-1,(k)}.$$

Furthermore, for $k = 1, \ldots,$ *one has*

$$\mathcal{H}^2_{(k)}(\mathbb{D}) = \left(\mathcal{A}^2_{(k)}(\mathbb{D}) \ominus N_{(k),k}\right) \oplus \left(\mathcal{A}^2_{(-k)}(\mathbb{D}) \ominus N_{k,(k)}\right) \oplus N_{(k),(k)}.$$

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A Hilbert Basis and (Generalized) Zernike Polynomials

- $\phi_m(z) := \sqrt{\frac{m}{\pi}}\, z^{m-1}$ and $\phi_{n,m} := (S_{\mathbb{D}})^{n-1}\phi_{n+m-1} \quad (n, m = 1, \ldots).$ **L.V.P. 14**
- $\phi_{n,m}(z) = \frac{\sqrt{n+m-1}}{\sqrt{\pi}(n+m-2)!}\partial_z^{n-1}\partial_{\bar{z}}^{m-1}(\bar{z}z-1)^{n+m-2}$ **Koshelev 77**
- The two previous definitions coincide **L.V.P. 14**
- $\{\phi_{n,m} : n \leq j\}$ and $\{\phi_{n,m} : n = j\}$ are Hilbert basis for $\mathcal{A}_j^2(\mathbb{D})$ **Koshelev 77** and $\mathcal{A}_j L_2^0(\mathbb{D})$ **Ramazanov 99**. This is evident from **L.V.P. 14** definition and from previous $\triangle$ theorem
- **Torre 08** and **Wunche 05** have defined the *disc polynomials* $p_{m,n}^{\alpha}$ without mention to its relations with the (weighted) poly-Bergman spaces. It is easily seen that $p_{n,m}^{0}$ coincide with $\phi_{n,m}$. Their properties were used to solve problem of Quantum Optics.
- in **L.V.P. 14** we find additional properties concerning the poly-Bergman spaces of negative order and also the spaces $N_{j,k}$.

Hilbert Basis for the Polyharmonic Bergman Type Spaces

Theorem (Koshelev 77; Ramazanov 99; L.V.P. 14)

Let j and k be positive integers. Then

$$\{\phi_{n,m}\} \quad , \quad \{\phi_{n,m} : n = j\} \quad , \quad \{\phi_{n,m} : m = k\} \quad \text{and} \quad \{\phi_{n,m} : n = j,\, m = k\}$$

are Hilbert bases for $L^2(\mathbb{D})$, $\mathcal{A}^2_{(j)}(\mathbb{D})$, $\mathcal{A}^2_{(-k)}(\mathbb{D})$ and $N_{(j),(k)}$, respectively.

Theorem (L.V.P. 14)

Let $j, k \in \mathbb{Z}_+$ and let $\alpha := (j, k)$. The following sets

$$\{\phi_{n,m} : (n \leq j) \vee (m \leq k)\}$$

and

$$\{\phi_{n,m} : (n = k\,;\, m \geq k) \vee (m = k\,;\, n \geq k)\}$$

are Hilbert bases for the spaces $\mathcal{A}^2_\alpha(\mathbb{D})$ and $\mathcal{H}^2_{(k)}(\mathbb{D})$, respectively.

Isomorphisms from the Harmonic Bergman to the True Harmonic Poly-Bergman Spaces and Differential Operators

Definition (Some Differential Operator - L.V.P. 14)

Let $k \in \mathbb{Z}_+$ and let $\mathcal{R}_k$ be the operator defined on $C^\infty(\mathbb{D})$ by

$$(\mathcal{R}_k u)(z) := \frac{\Delta^{k-1}[(1-\bar{z}z)^{2k-2}u(z)]}{4^{k-1}(2k-2)!}.$$

Theorem (L.V.P. 14)

Let k be a positive integer. Then,

$$\mathcal{R}_k : \mathcal{H}^2(\mathbb{D}) \to \mathcal{H}^2_{(k)}(\mathbb{D})$$

is a one-to-one and onto bounded operator. Furthermore,

$$\mathcal{H}^2_{(k)}(\mathbb{D}) = \left\{\Delta^{k-1}[(1-\bar{z}z)^{2k-2}h(z)] : h \in \mathcal{H}^2(\mathbb{D})\right\}.$$

Decomposition of Polyharmonic Functions

- Geometrically evident $\mathcal{H}_k^2(\mathbb{D}) = \bigoplus_{n=1}^{k} \mathcal{H}_{(n)}^2(\mathbb{D})$.
- It follows a decomposition of polyharmonic functions, different from the classical ones named by **Pavlović**, **Fischer** and **Almansi**
- Also note that from **Yu.I. Karlovich; L.V.P. 08** and the evident inclusion $\mathcal{A}_k^2(\mathbb{D}) \subset \mathcal{H}_k^2(\mathbb{D})$, for $k = 1, \ldots,$ it easily follows that

$$L^2(\mathbb{D}) = \bigoplus_{n=1}^{+\infty} \mathcal{H}_{(n)}^2(\mathbb{D}).$$

Theorem (L.V.P. 14)

Let $k \in \mathbb{Z}_+$ and let $f \in \mathcal{H}_k^2(\mathbb{D})$. For $n = 1, \ldots, k$ there exists unique functions h_n in the harmonic Bergman space such that

$$f(z) = \sum_{n=1}^{k} \Delta^{n-1}[(1 - \bar{z}z)^{2n-2} h_n(z)].$$

In particular, the following decomposition holds

$$\mathcal{H}_k^2(\mathbb{D}) = \bigoplus_{n=1}^{k} \Delta^{n-1}[(1 - \bar{z}z)^{2n-2}\mathcal{H}^2(\mathbb{D})].$$

Unitary Operators on True Polyharmonic Bergman Type Spaces

Definition (L.V.P. 14)

$$M^m_{(k),n} := \left(N_{(k),n} \ominus N_{(k),k}\right) \oplus \left(N_{m,(k)} \ominus N_{k-1,(k)}\right) \qquad n, m \geq k$$

$M^m_{(k),n}$ is $[(n-k)+(m-k+1)]$-dimensional space

Theorem (L.V.P. 14)

Let $k, j \in \mathbb{Z}_+$ be such that $k \leq j$. The following bounded operator

$$(S_{\mathbb{D}})^j + (S^*_{\mathbb{D}})^j : \mathcal{H}^2_{(k)}(\mathbb{D}) \ominus M^{k+2j-1}_{(k),k+2j} \to \mathcal{H}^2_{(k+j)}(\mathbb{D})$$

is a isometric isomorphism and $M^{k+2j-1}_{(k),k+2j}$ is a $4j$-dimensional space.

For $j \geq k$, we give an isometry between a subspace of $\mathcal{H}^2_{(k)}(\mathbb{D})$ with codimension $4j$, and the true polyharmonic Bergman space of order $k+j$

Unitary Operators on Polyharmonic Bergman Type Spaces

Theorem (L.V.P. 14)

Let $k, j \in \mathbb{Z}_+$ *be such that* $k \leq j$. *The following operator*

$$(S_{\mathbb{D}})^j + (S_{\mathbb{D}}^*)^j : \mathcal{H}_k^2(\mathbb{D}) \ominus M_{j,k} \to \mathcal{H}_{k+j}^2(\mathbb{D}) \ominus \mathcal{H}_j^2(\mathbb{D})$$

is an isometric isomorphism, where $M_{j,k}$ *is the* $4jk$*-dimensional space given by*

$$M_{j,k} = \bigoplus_{n=1}^{k} M_{(n),n+2j}^{n+2j-1}.$$

Unitary Operators on the True Poly-Bergman Spaces

Theorem (Yu.I. Karlovich, L.V.P. 07; N. Vasilevski)

(L.V.P. 14 also for $\mathbb{E}$) *Let j be a positive integer. The operators*

$$(S_\Pi)^j : \mathcal{A}^2_{(k)}(\Pi) \to \mathcal{A}^2_{(k+j)}(\Pi)\ ,\ k \in \mathbb{Z}_+$$

$$(S^*_\Pi)^j : \mathcal{A}^2_{(k)}(\Pi) \to \mathcal{A}^2_{(k-j)}(\Pi)\ ,\ k \in \mathbb{Z}_-$$

and the operators

$$(S^*_\Pi)^j : \mathcal{A}^2_{(k)}(\Pi) \to \mathcal{A}^2_{(k-j)}(\Pi)\ ;\ k \in \mathbb{Z}_+, j < k$$

$$(S_\Pi)^j : \mathcal{A}^2_{(k)}(\Pi) \to \mathcal{A}^2_{(k+j)}(\Pi)\ ;\ k \in \mathbb{Z}_-, j < -k$$

are isometric isomorphisms. Furthermore

$$\mathrm{Ker}\,(S^*_\Pi)^j = \mathcal{A}^2_j(\Pi) \quad \text{and} \quad \mathrm{Ker}\,(S_\Pi)^j = \bar{\mathcal{A}}^2_j(\Pi).$$

Unitary Operators on Poly-Bergman Spaces

Corollary (L.V.P., A.M. Santos 14)

Let $j \in \mathbb{Z}_+$ *and* $k \in \mathbb{Z}_\pm$. *The operators*

$$(S_\Pi)^j : \mathcal{A}_k^2(\Pi) \to \mathcal{A}_{k+j}^2(\Pi) \ominus \mathcal{A}_j^2(\Pi), \qquad k > 0$$

$$(S_\Pi^*)^j : \mathcal{A}_k^2(\Pi) \to \mathcal{A}_{k-j}^2(\Pi) \ominus \mathcal{A}_{-j}^2(\Pi), \qquad k < 0$$

as well as the following ones

$$(S_\Pi)^j : \mathcal{A}_k^2(\Pi) \ominus \mathcal{A}_{-j}^2(\Pi) \to \mathcal{A}_{k+j}^2(\Pi), \qquad 0 < j < -k$$

$$(S_\Pi^*)^j : \mathcal{A}_k^2(\Pi) \ominus \mathcal{A}_j^2(\Pi) \to \mathcal{A}_{k-j}^2(\Pi), \qquad 0 < j < k$$

are isometric isomorphisms. Furthermore

$$\mathrm{Ker}\,(S_\Pi^*)^j = \mathcal{A}_j^2(\Pi) \quad \text{and} \quad \mathrm{Ker}\,(S_\Pi)^j = \bar{\mathcal{A}}_j^2(\Pi)\,.$$

Polyharmonic Spaces and Calderon-Zygmund Operators

Theorem (L.V.P., A.M. Santos 14)

Let $j = 1, 2, \ldots$. *The following direct sum decomposition holds*

$$\mathcal{H}_j^2(\Pi) = \mathcal{A}_j^2(\Pi) \oplus \bar{\mathcal{A}}_j^2(\Pi).$$

Corollary (L.V.P., A.M. Santos 14)

Let $j = 1, 2, \ldots$. *The polyharmonic Bergman projections* $Q_{\Pi,j}$ *is given by*

$$Q_{\Pi,j} = B_{\Pi,j} + \widetilde{B}_{\Pi,j} = 2I - (S_\Pi)^j (S_\Pi^*)^j - (S_\Pi^*)^j (S_\Pi)^j.$$

Corollary (L.V.P., A.M. Santos 14)

Let $j = 1, 2, \ldots$. *Then,* $Q_{\Pi,j}$ *defines a bounded idempotent acting from* $L^p(\Pi)$, *for* $1 < p < +\infty$, *onto* $\mathcal{H}_j^p(\Pi)$.

Unitary Operators on Polyharmonic Bergman spaces

Theorem (L.V.P., A.M. Santos 14)

Let j be a positive integer. Then

$$\mathcal{H}^2_{(j)}(\Pi) = \mathcal{A}^2_{(j)}(\Pi) \oplus \bar{\mathcal{A}}^2_{(j)}(\Pi) \quad \text{and} \quad Q_{\Pi,(j)} = B_{\Pi,(j)} + \widetilde{B}_{\Pi,(j)}$$

Theorem (L.V.P., A.M. Santos 14)

Let $j, k = 1, 2, \ldots$. If $0 < k \leq j$, then the following operator is an isometric isomorphism

$$(S_\Pi)^j + (S_\Pi^*)^j : \mathcal{H}^2_{(k)}(\Pi) \to \mathcal{H}^2_{(j+k)}(\Pi).$$

Unitary Operators and Differential Operators

Definition

For $j = 0, 1, \dots,$ the operator $\mathcal{R}_j$ is defined to be the following operator

$$(S_\Pi)^j + (S_\Pi^*)^j : \mathcal{H}^2(\Pi) \to \mathcal{H}^2_{(j+1)}(\Pi)$$

Theorem (L.V.P., A.M. Santos 14)

Let $j = 0, 1, \dots,$ *then the following operator*

$$\mathcal{R}_j : \mathcal{H}^2(\Pi) \to \mathcal{H}^2_{(j+1)}(\Pi) \quad , \quad \mathcal{R}_j u(z) = \frac{\Delta^j \left[y^{2j} u(z)\right]}{(2j)!}.$$

is unitary, where $z = x + iy$ *are cartesian coordinates.*

Unitary Operators and Differential Operators

Theorem (L.V.P., A.M. Santos 14)

Let $j = 1, 2, \ldots$ *and let* $u \in \mathcal{H}_j^2(\Pi)$. *For* $k = 0, \ldots, j-1$ *there exists unique functions* ν_k *in the harmonic Bergman space such that*

$$u(z) = \sum_{k=0}^{j-1} \Delta^k[(z-\overline{z})^{2k}\nu_k(z)].$$

In particular, the following decomposition holds

$$\mathcal{H}_j^2(\Pi) = \bigoplus_{k=0}^{j-1} \Delta^k[(z-\overline{z})^{2k}\mathcal{H}^2(\Pi)].$$

Hilbert basis

Definition (L.V.P. 14)

Let $k \in \mathbb{Z}_+$ and $j \in \mathbb{Z}_\pm$. Then we define the following functions

$$\psi_{j,k}(z) := \frac{2i\sqrt{k}}{\sqrt{\pi}(j-1)!}\partial_z^{j-1}\left[\frac{(\overline{z}-z)^{j-1}(z-i)^{k-1}}{(z+i)^{k+1}}\right], \ j \in \mathbb{Z}_+$$

$$\psi_{j,k}(z) := \overline{\psi}_{-j,k}(z), \ j \in \mathbb{Z}_-.$$

Theorem (L.V.P., A.M. Santos 14)

$$L^2(\Pi) = \bigoplus_{j=1}^{+\infty} \mathcal{H}^2_{(j)}(\Pi).$$

Furthermore, for a positive integer j, the following sets

$$\{\psi_{n,m}\} \quad , \quad \{\psi_{n,m} : n = \pm j\} \quad \text{and} \quad \{\psi_{n,m} : n = \pm 1, \ldots, \pm j\}$$

are Hilbert bases for $L^2(\Pi)$, $\mathcal{H}^2_{(j)}(\Pi)$ and $\mathcal{H}^2_j(\Pi)$, respectively.

Kernel Functions

- $K_{\Pi,j}(z,w)$ Kernel function for poly-Bergman space **L.V.P. 13**

$$K_{\Pi,j}(z,w) = -\frac{j}{\pi}\frac{\sum_{k=1}^{j}(-1)^{j-k}\binom{j}{k}\binom{j+k-1}{j}|z-\overline{w}|^{2(j-k)}|z-w|^{2(k-1)}}{(z-\overline{w})^{2j}}.$$

  - follows from Koshelev formula in the disk by means of the variation of the domain technique.
  - **N. Vasilevski 99** also has a closed formula with the summation of j^3 terms
  - also see **L.D. Abreu 12**

- $K_{\Pi,(j)}(z,w)$ Kernel function for true poly-Bergman space **L.V.P. 14**

$$K_{\Pi,(j+1)}(z,w) = -\frac{\partial_z^j\partial_{\overline{w}}^j}{\pi(j!)^2}\left[\frac{(\overline{z}-z)^j(w-\overline{w})^j}{(z-\overline{w})^2}\right], \; j=0,1,\ldots.$$

Different Closed formulas for Kernel Functions

- **L.V.P., A.M. Santos 14**

$$K_{\Pi,(j+1)}(z,w) = \Delta_z^j \Delta_w^j \left[\frac{y^{2j}}{(2j)!} \frac{s^{2j}}{(2j)!} K_\Pi(z,w) \right]$$

$$K_{\Pi,(-j-1)}(z,w) = \Delta_z^j \Delta_w^j \left[\frac{y^{2j}}{(2j)!} \frac{s^{2j}}{(2j)!} \overline{K_\Pi}(z,w) \right],$$

where $z := x + iy$ and $w := t + is$ are cartesian coordinates.

- The classical reproducing kernel for the harmonic Bergman space

$$K_\Pi^h(z,w) = \frac{2}{\pi} \frac{|z-\overline{w}|^2 - 2(x-t)^2}{|z-\overline{w}|^4}$$

- **L.V.P., A.M. Santos 14**

$$K_{\Pi,(j+1)}^h(z,w) = \Delta_z^j \Delta_w^j \left[\frac{y^{2j}}{(2j)!} \frac{s^{2j}}{(2j)!} K_\Pi^h(z,w) \right].$$

Paley-Wienner and Bargmann Type Transforms

Theorem (N. Vasilevski 99; L.V.P., A.M. Santos 14)

The following operators are unitary operators

$$R : L^2(\mathbb{R}^+, dt) \to \mathcal{A}^2(\Pi) \ , \ Ra(z) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \sqrt{t}a(t)e^{izt}dt,$$

$$\widetilde{R} : L^2(\mathbb{R}^+, dt) \to \bar{\mathcal{A}}^2(\Pi) \ , \ \widetilde{R}a(z) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \sqrt{t}a(t)e^{-i\bar{z}t}dt.$$

Theorem (L.V.P., A.M. Santos 14)

The following operator is a unitary operator

$$R^h : L^2(\mathbb{R}, dt) \to \mathcal{H}^2(\Pi) \ , \ R^h a(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \sqrt{|t|}a(t)e^{ixt}e^{-y|t|}dt,$$

where $z := x + iy$ *are cartesian coordinates.*

Paley-Wienner and Bargmann Type Transforms

Theorem (P. Duren, E.A. Gallardo-Guitíerrez, A. Montes-Rodrígues 07)

For $\lambda > -1$ *and* $dA_\lambda := y^\lambda dA$, *where* $z := x + iy$ *are cartesian coordinates, the complex Fourier transform*

$$F_\lambda^c a(z) = \frac{2^{\lambda/2}}{\sqrt{\pi\Gamma(\lambda+1)}} \int_0^{+\infty} a(t) e^{izt} dt \ , \ z \in \Pi,$$

is an isometric isomorphism from $L^2(\mathbb{R}^+, dt/t^{\lambda+1})$ *onto* $\mathcal{A}^2(\Pi, dA_\lambda)$.

Proposition (L.V.P., A.M. Santos 14)

Let $\lambda > -1$ *and let* $z := x + iy$ *be cartesian coordinates. The map*

$$F_\lambda^h : L^2(\mathbb{R}, dt/|t|^{\lambda+1}) \to \mathcal{H}^2(\Pi, dA_\lambda)$$

$$F_\lambda^h a(z) = \frac{2^{\lambda/2}}{\sqrt{\pi\Gamma(\lambda+1)}} \int_{-\infty}^{+\infty} a(t) e^{ixt} e^{-y|t|} dt$$

defines an isometric isomorphism. (Harmonic Fourier Transform?)

Bargmann Type Transforms for polyharmonic spaces

Theorem (L.V.P., A.M. Santos 14)

Let $j = 0, 1, \ldots$ and let $z := x + iy$ be cartesian coordinates. Then

$$R^h_{(j)} : L^2(\mathbb{R}) \to \mathcal{H}^2_{(j+1)}(\Pi) \quad ; \quad R^h_{(j)} a(z) = \frac{\Delta^j \left[y^{2j} R^h a(z) \right]}{(2j)!},$$

is an isometric isomorphisms. Furthermore,

$$R^h_{(j)} a(z) = \sum_{k=0}^{j-1} y^k \nu_k(z) = \sum_{k=0}^{j-1} L_k(y) \mu_k(z),$$

where the harmonic components ν_k and μ_k satisfy the following

$$\nu_k = F^h_{2k} a_k \in \mathcal{A}^2(\Pi, dA_{2k}) \quad \text{and} \quad \mu_k = F^h_{2j-2} b_k \in \mathcal{A}^2(\Pi, dA_{2j-2}),$$

and the functions a_k and b_k are respectively given by

$$a_k(t) := (-1)^k \binom{j}{k} \frac{\sqrt{(2k)!}}{k!} |t|^k \sqrt{|t|} a(t) \ , \ t \in \mathbb{R}$$

$$b_k(t) := \binom{j}{k} \frac{\sqrt{(2j)!}}{2^{j-k}} |t|^k (1 - 2|t|)^{j-k} \sqrt{|t|} b(t) \ , \ t \in \mathbb{R}.$$

Bargmann Type Transforms for polyharmonic spaces

We have manage with the help of the Laguerre polynomials

$$L_n(z) := \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k!} z^k \ , \ n = 0, 1, \ldots .$$

Theorem (L.V.P., A.M. Santos 14)

For $j = 1, 2, \ldots,$ the following operators are unitary operators

$$R_j^h : \left[L^2(\mathbb{R})\right]_j \to \mathcal{H}_j^2(\Pi) \quad , \quad R_j^h(f_k)_k(z) = \sum_{k=0}^{j-1} R_{(k)}^h f_k(z).$$

TÉCNICO LISBOA

Isomorphism between copies of the Hardy space

$$\mathcal{F} : \mathcal{A}^2_{\bar\partial}(\Pi) \to L^2(\mathbb{R}^+) \quad , \quad \mathcal{F}f(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ixt} dx, \quad t \in \mathbb{R}^+$$

Theorem (L.V.P., A.M. Santos 14)

For $j = 1, 2, \dots,$ *the operator*

$$W_j^h : \left[\mathcal{A}^2_{\bar\partial}(\Pi)\right]_{2j} \to \mathcal{H}^2_j(\Pi) \quad , \quad W_j^h (f_k)_{k=1}^{2j} = R_j^h (g_k)_{k=1}^{j}$$

where

$$g_k(t) = \chi_+ \mathcal{F} f_{2k-1}(t) + \chi_- \mathcal{F} f_{2k}(-t) \quad ; \quad k = 1, \dots, j.$$

Thanks all!

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