

## DIMENSION OF MEASURES FOR SUSPENSION FLOWS

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ABSTRACT. We consider hyperbolic flows and more generally suspension flows over two-sided subshifts of finite type, and establish an explicit formula for the pointwise dimension of an arbitrary measure (in particular the measure is not necessarily ergodic and it may not possess a local product structure). The applications of this formula include the description of the Hausdorff dimension of a given measure in terms of an ergodic decomposition, and the proof of the existence of measures of maximal dimension.

### 1. INTRODUCTION

Suspension flows naturally occur as models for flows possessing hyperbolic behavior. In particular, in the case of uniformly hyperbolic flows one can use the Markov systems introduced by Bowen [6] and Ratner [13] to associate suspension flows to the dynamics. This approach conveniently allows us to work with the symbolic dynamics on the base of a suspension flow and then to transfer the results to the original flow (in many cases this transference requires a great care although this problem is of different nature, and essentially depends on the point of view of our approach, namely topological, dimensional, etc).

We consider suspension flows over a subshift of finite type and we introduce an appropriate metric on the base of the suspension that unlike the “standard” symbolic metrics may be “nonuniform”. This corresponds to the existence of possibly nonconstant Lyapunov exponents  $\lambda_s(x)$  and  $\lambda_u(x)$  (or more precisely of an appropriate version of Lyapunov exponents) for the subshift of finite type along the “stable” and “unstable” directions (see Section 2.3 for the definition). This approach also allows us to describe the Hausdorff dimension of a set that is invariant under the suspension flow in terms of a Carathéodory dimensional characteristic that is essentially defined in terms of the Lyapunov exponents.

Our main result is an explicit formula for the pointwise dimension of an invariant measure. Namely, let  $\nu$  be a probability measure invariant under a suspension flow  $\Phi$  on the space  $Y$ . We show that for  $\nu$ -almost every  $x \in Y$  we have (see Theorem 1)

$$\lim_{r \rightarrow 0} \frac{\log \nu(B(x, r))}{\log r} = h_\nu(x) \left( \frac{1}{\lambda_u(x)} - \frac{1}{\lambda_s(x)} \right) + 1. \quad (1)$$

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Here  $B(x, r)$  is the ball of radius  $r$  centered at  $x$  with respect to the so-called Bowen–Walters distance (see Section 2.1 for the definition), which is the natural distance on the phase space of a suspension flow, and  $h_\nu(x)$  denotes the local entropy of  $\nu$  at  $x$  (see Section 3). The limit on the left-hand side of (1), whenever it exists, is called the *pointwise dimension* of  $\nu$  at  $x$ .

In the case of conformal hyperbolic flows (see Section 5.1) Pesin and Sadovskaya [12] established (1) for equilibrium measures of a Hölder continuous potential (note that these measures are ergodic and possess a local product structure), while we consider arbitrary measures. In fact, we follow a different approach developed by Barreira and Wolf in [5] to establish (1) for an arbitrary measure invariant under conformal hyperbolic dynamics. It should be pointed out that (1) is a version of the formula obtained by Young in [16] in the case of ergodic measures invariant under surface diffeomorphisms with nonzero Lyapunov exponents.

We provide two applications of the formula in (1). The first is a description of the Hausdorff dimension of an invariant measure (again not necessarily ergodic) in terms of an ergodic decomposition. We recall that the Hausdorff dimension of a probability measure  $\nu$  on  $Y$  is defined by

$$\dim_H \nu = \inf\{\dim_H Z : \nu(Z) = 1\},$$

where  $\dim_H Z$  denotes the Hausdorff dimension of the set  $Z$ . For any ergodic decomposition of  $\nu$  we have (see Theorem 3)

$$\dim_H \nu = \text{ess sup}_\rho \dim_H \rho,$$

with the essential supremum taken with respect to the ergodic decomposition. The second application is a proof of the existence of ergodic invariant measures of maximal dimension (see Theorem 4). This means that there exists an ergodic invariant measure which attains the supremum of the Hausdorff dimensions over all invariant measures (including the nonergodic measures). The main difficulty of this problem is that the function  $\nu \mapsto \dim_H \nu$  is not upper-semicontinuous.

The following is a description of the contents of the paper. In Section 2 we introduce some basic notions and present the least possible requirements for the suspension flows that allow our approach to work. Section 3 contains a proof of formula (1) and the description of the Hausdorff dimension of invariant measures in terms of ergodic decompositions. Section 4 addresses the existence of measures of maximal dimension. In section 5 we consider the case of hyperbolic flows. Our approach closely follows the approaches developed in [4, 5] for hyperbolic diffeomorphisms and is based on the thermodynamic formalism. All the necessary results from the thermodynamic formalism are recalled or briefly established in the appendix. Instead of repeating the arguments in [4, 5] we carefully highlight the required changes.

## 2. SUSPENSION FLOWS

**2.1. Basic notions.** Let  $\sigma: X \rightarrow X$  be a homeomorphism of the compact metric space  $X$ , and let  $\tau: X \rightarrow (0, \infty)$  be a continuous function. We

consider the space

$$Y = \{(x, t) \in X \times \mathbb{R} : 0 \leq t \leq \tau(x)\},$$

with the points  $(x, \tau(x))$  and  $(\sigma(x), 0)$  identified for each  $x \in X$ . The *suspension flow over  $\sigma$  with height function  $\tau$*  is the flow  $\Phi = \{\varphi_t\}_{t \in \mathbb{R}}$  on  $Y$  with  $\varphi_t: Y \rightarrow Y$  defined by  $\varphi_t(v, s) = (v, s + t)$ . We extend  $\tau$  to a function  $\tau: Y \rightarrow \mathbb{R}$  by

$$\tau(x) = \min\{t > 0 : \varphi_t x \in X \times \{0\}\},$$

and we extend  $\sigma$  to a map  $\sigma: Y \rightarrow X \times \{0\}$  by  $\sigma(x) = \varphi_{\tau(x)}x$ . Since there is no danger of confusion we continue to use the symbols  $\tau$  and  $\sigma$  for the extensions and we identify the base  $X \times \{0\}$  with  $X$ .

One can introduce in a natural way a topology on  $Y$  which makes  $Y$  a compact topological space. This topology is induced by a distance  $d_Y$  introduced by Bowen and Walters in [8] (see for example the appendix of [2] for details), that we call Bowen–Walters distance in this paper.

We briefly recall its definition. Without loss of generality one can assume that the diameter  $\text{diam } X$  of  $X$  is at most 1. If this is not the case then since  $X$  is compact one can simply consider the new distance  $d_X / \text{diam } X$  on  $X$ . We first assume that  $\tau = 1$  on  $X$ , and introduce the Bowen–Walters distance  $d_1$  on the corresponding space  $Y$ . We shall first consider horizontal and vertical segments. Given  $x, y \in X$  and  $t \in [0, 1]$  we define the length of the *horizontal segment*  $[(x, t), (y, t)]$  by

$$\rho_h((x, t), (y, t)) = (1 - t)d_X(x, y) + td_X(\sigma x, \sigma y).$$

Note that

$$\rho_h((x, 0), (y, 0)) = d_X(x, y) \quad \text{and} \quad \rho_h((x, 1), (y, 1)) = d_X(\sigma x, \sigma y).$$

Furthermore, given  $(x, t), (y, s) \in Y$  on the same orbit we define the length of the *vertical segment*  $[(x, t), (y, s)]$  by

$$\rho_v((x, t), (y, s)) = \inf\{|r| : \varphi_r(x, t) = (y, s) \text{ and } r \in \mathbb{R}\}.$$

Finally, given two points  $(x, t), (y, s) \in Y$  the distance  $d_1((x, t), (y, s))$  is defined as the infimum of the lengths of paths between  $(x, t)$  and  $(y, s)$  composed by a finite number of horizontal and vertical segments. We now consider the case of an arbitrary function  $\tau$ , and introduce the Bowen–Walters distance  $d_Y$  on  $Y$ . Given two points  $(x, t), (y, s) \in Y$ , we set

$$d_Y((x, t), (y, s)) = d_1((x, t/\tau(x)), (y, s/\tau(s))).$$

Given  $(x, t), (y, s) \in Y$  we set

$$d_\pi((x, t), (y, s)) = \min \left\{ \begin{array}{l} d_X(x, y) + |t - s|, \\ d_X(\sigma x, y) + \tau(x) - t + s, \\ d_X(x, \sigma y) + \tau(y) - s + t \end{array} \right\}.$$

It was established by Barreira and Saussol in [2] that there exists a constant  $c > 1$  such that if  $p, q \in Y$  then

$$c^{-1}d_\pi(p, q) \leq d_Y(p, q) \leq cd_\pi(p, q). \quad (2)$$

We denote by  $\mathcal{M}_\Phi(Y)$  the space of  $\Phi$ -invariant probability measures on  $Y$  and by  $\mathcal{M}_\sigma(X)$  the space of  $\sigma$ -invariant probability measures on  $X$ . There is a canonical identification between these two spaces. Namely, for each

measure  $\mu \in \mathcal{M}_\sigma(X)$  and  $m$  the Lebesgue measure on  $\mathbb{R}$  the product measure  $\mu \times m$  attributes zero measure to the base  $X \times \{0\}$  and we can define a map  $T: \mathcal{M}_\sigma(X) \rightarrow \mathcal{M}_\Phi(Y)$  by

$$T(\mu) = T_\mu = (\mu \times m)|_Y / (\mu \times m)(Y). \quad (3)$$

The relation between the measures  $\mu$  and  $T_\mu$  is given by

$$\int_Y a dT_\mu = \frac{\int_X \Delta_a d\mu}{\int_X \tau d\mu} \quad (4)$$

for each continuous function  $a: Y \rightarrow \mathbb{R}$ , where  $\Delta_a: Y \rightarrow \mathbb{R}$  is defined by

$$\Delta_a(x) = \int_0^{\tau(x)} a(\varphi_t x) dt.$$

It is straightforward to verify that  $T$  is onto and one-to-one. In particular any measure in  $\mathcal{M}_\Phi(Y)$  is of the form  $T_\mu$  for some measure  $\mu \in \mathcal{M}_\sigma(X)$ .

See Section 5.1 for the principal examples of suspension flows over a topological Markov chain, obtained from hyperbolic flows and the associated so-called Markov systems.

**2.2. Metric on the base and Hausdorff dimension.** We now consider the case when  $\sigma: X \rightarrow X$  is a two-sided topological Markov chain on  $X \subset \{1, \dots, p\}^{\mathbb{Z}}$ . We equip  $X$  with the topology induced by the cylinders

$$C_{i_{-n} \dots i_m} = \{(\dots j_0 \dots) : j_k = i_k \text{ for } -n \leq k \leq m\},$$

where  $i_{-n}, \dots, i_m \in \{1, \dots, p\}$ . Let  $\beta_s: X \rightarrow \mathbb{R}$  and  $\beta_u: X \rightarrow \mathbb{R}$  be continuous negative functions and write

$$d_s(C_{i_{-n} \dots i_m}) = \sup_{x \in C_{i_{-n} \dots i_m}} \exp \sum_{k=0}^m \beta_s(\sigma^k(x)),$$

$$d_u(C_{i_{-n} \dots i_m}) = \sup_{x \in C_{i_{-n} \dots i_m}} \exp \sum_{k=0}^n \beta_u(\sigma^{-k}(x)).$$

We define a distance  $d$  on  $X$  by

$$d((\dots i_0 \dots), (\dots j_0 \dots)) = |i_0 - j_0| + d_s(C_{i_{-n_u} \dots i_{n_s}}) + d_u(C_{i_{-n_u} \dots i_{n_s}}),$$

where

$$n_s = \max\{n \in \mathbb{N} : i_k = j_k \text{ for } k \leq n\},$$

$$n_u = \max\{n \in \mathbb{N} : i_k = j_k \text{ for } k \geq -n\}.$$

It is straightforward to verify that the diameter of a cylinder  $C$  computed with respect to the distance  $d$  is given by

$$\text{diam}_d C = d_s(C) + d_u(C). \quad (5)$$

The distance  $d$  was introduced by Barreira and Saussol in [2]. The topology induced by  $d$  coincides with the topology generated by the cylinders. It makes  $X$  a compact metric space and it is straightforward to verify that  $\beta_s$  and  $\beta_u$  are Hölder continuous with respect to the distance  $d$ .

The distance  $d$  also induces a Bowen–Walters distance  $d_Y$  on  $Y$ . In this paper we always compute the Hausdorff dimension of subsets of  $Y$  with respect to this distance.

We now give an alternative definition of the Hausdorff dimension as a Carathéodory dimensional characteristic (for a detailed introduction to the theory of Carathéodory dimensional characteristics we refer to the book by Pesin [11]). Given  $\alpha \in \mathbb{R}$  and  $Z \subset X$  we consider the function

$$M(Z, \alpha) = \liminf_{\ell \rightarrow 0} \sum_{\Gamma} \exp(-\alpha d_s(C) - \alpha d_u(C)),$$

where the infimum is taken over all covers  $\Gamma$  of  $Z$  by cylinders  $C_{i_{-n} \dots i_m}$  with  $m > \ell$  and  $n > \ell$ . The  $(\beta_s, \beta_u)$ -dimension of  $Z$  is defined by

$$\dim_{\beta_s, \beta_u} Z = \inf\{\alpha : M(Z, \alpha) = 0\}.$$

It follows from (5) that the  $(\beta_s, \beta_u)$ -dimension coincides with the Hausdorff dimension with respect to  $d$ .

**2.3. Lyapunov exponents.** We now introduce numbers that play the role of Lyapunov exponents in the case of suspension flows. We note that given a function  $\beta: X \rightarrow \mathbb{R}$  there always exists a continuous function  $a: Y \rightarrow \mathbb{R}$  such that  $\Delta_a|X = \beta$ . For example, we can define  $a: Y \rightarrow \mathbb{R}$  by

$$a(\varphi_t x) = \frac{\beta(x)}{\tau(x)} \psi' \left( \frac{t}{\tau(x)} \right)$$

for each  $x \in X$  and  $t \in [0, \tau(x)]$ , where  $\psi: [0, 1] \rightarrow [0, 1]$  is any fixed nondecreasing  $C^1$  function such that  $\psi(0) = 0$ ,  $\psi(1) = 1$ , and  $\psi'(0) = \psi'(1) = 0$ . We now consider continuous functions  $\zeta_s: Y \rightarrow \mathbb{R}^-$  and  $\zeta_u: Y \rightarrow \mathbb{R}^+$  such that

$$\Delta_{\zeta_s}|X = \beta_s \quad \text{and} \quad \Delta_{\zeta_u}|X = -\beta_u. \quad (6)$$

With the help of Proposition 17 in [2] one can easily verify that if  $\tau$  is Hölder continuous then  $\zeta_s$  and  $\zeta_u$  are also Hölder continuous.

For each  $x \in Y$  we define the numbers

$$\lambda_s(x) = \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \zeta_s(\varphi_r x) dr \quad \text{and} \quad \lambda_u(x) = \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \zeta_u(\varphi_r x) dr, \quad (7)$$

whenever the corresponding limit exists. By Birkhoff's ergodic theorem  $\lambda_s(x)$  and  $\lambda_u(x)$  are well-defined  $\nu$ -almost everywhere with respect to any  $\Phi$ -invariant probability measure  $\nu$  on  $Y$ , and

$$\int_Y \lambda_s d\nu = \int_Y \zeta_s d\nu \quad \text{and} \quad \int_Y \lambda_u d\nu = \int_Y \zeta_u d\nu. \quad (8)$$

It is straightforward to verify that the domains of these functions are  $\Phi$ -invariant and that each of the functions is  $\Phi$ -invariant on its domain. We observe that

$$\sum_{k=0}^{n-1} \Delta_a(\sigma^k(x)) = \int_0^{\tau_n(x)} a(\varphi_t x) dt,$$

where  $\tau_n(x) = \sum_{k=0}^{n-1} \tau(\sigma^k(x))$ . This implies that

$$\begin{aligned} \lambda_s(\varphi_t x) &= \lim_{n \rightarrow +\infty} \frac{1}{\tau_n(x)} \sum_{k=0}^{n-1} \beta_s(\sigma^k(x)), \\ \lambda_u(\varphi_t x) &= - \lim_{n \rightarrow +\infty} \frac{1}{\tau_n(x)} \sum_{k=0}^{n-1} \beta_u(\sigma^k(x)), \end{aligned} \quad (9)$$

whenever the corresponding limit exists. These identities show that the definition of the functions  $\lambda_s$  and  $\lambda_u$  in (7) is independent of the particular extensions  $\Delta_{\zeta_s}$  of  $\beta_s$  and  $\Delta_{\zeta_u}$  of  $-\beta_u$  to the space  $Y$ .

### 3. POINTWISE DIMENSION AND ERGODIC DECOMPOSITIONS

Let  $\Phi$  be a suspension flow on  $Y$  over the map  $\sigma: X \rightarrow X$  and let  $\nu \in \mathcal{M}_\Phi(Y)$ . For  $\nu$ -almost every  $x \in Y$  there exists the limit

$$h_\nu(x) = \lim_{\varepsilon \rightarrow 0} \lim_{t \rightarrow \infty} -\frac{1}{t} \log \nu(B(x, t, \varepsilon)), \quad (10)$$

where

$$B(x, t, \varepsilon) = \{y \in Y : d(\varphi_r y, \varphi_r x) < \varepsilon \text{ whenever } 0 \leq r \leq t\}. \quad (11)$$

The number  $h_\nu(x)$  is called the *local entropy* at  $x$ . The function  $x \mapsto h_\nu(x)$  is  $\nu$ -integrable,  $\Phi$ -invariant  $\nu$ -almost everywhere, and the Kolmogorov–Sinai entropy  $h_\nu(\Phi)$  of  $\Phi$  with respect to  $\nu$  is given by

$$h_\nu(\Phi) = \int_Y h_\nu(x) d\nu(x).$$

The *lower* and *upper pointwise dimensions* of  $\nu$  at the point  $x \in Y$  are defined by

$$\underline{d}_\nu(x) = \liminf_{r \rightarrow 0} \frac{\log \nu(B(x, r))}{\log r} \quad \text{and} \quad \bar{d}_\nu(x) = \limsup_{r \rightarrow 0} \frac{\log \nu(B(x, r))}{\log r},$$

where  $B(x, r)$  is the ball of radius  $r$  centered at  $x$  with respect to the Bowen–Walters distance (see Section 2.1). It follows from (2) that if  $x = (y, t)$  with  $y \in X$  and  $t \in [0, \tau(y)]$  then

$$B_X(y, r/c) \times (t - r/c, t + r/c) \subset B(x, r) \subset B_X(y, cr) \times (t - cr, t + cr)$$

for all sufficiently small  $r > 0$ . Using (3), this implies that if  $\mu \in \mathcal{M}_\sigma(Y)$  is the unique  $\sigma$ -invariant probability measure on  $X$  such that  $T(\mu) = \nu$  (see Section 2.1) then

$$\frac{\mu(B_X(y, r/c) \times (t - r/c, t + r/c))}{(\mu \times m)(Y)} \leq \nu(B(x, r)) \leq \frac{\mu(B_X(y, cr) \times (t - cr, t + cr))}{(\mu \times m)(Y)},$$

and hence

$$\underline{d}_\nu(x) = \underline{d}_\mu(y) + 1 \quad \text{and} \quad \bar{d}_\nu(x) = \bar{d}_\mu(y) + 1.$$

Our first result is an explicit formula for the pointwise dimension of  $\nu$  in terms of the local entropy and of the Lyapunov exponents.

**Theorem 1.** *Let  $\Phi$  be a suspension flow on  $Y$  over a two-sided topological Markov chain with Hölder continuous height function, and  $\nu$  a  $\Phi$ -invariant probability measure on  $Y$ . Then, for  $\nu$ -almost every  $x \in Y$  we have*

$$\underline{d}_\nu(x) = \bar{d}_\nu(x) = h_\nu(x) \left( \frac{1}{\lambda_u(x)} - \frac{1}{\lambda_s(x)} \right) + 1.$$

*Proof.* By Birkhoff's ergodic theorem, for  $\nu$ -almost every  $x \in Y$  there exists the limit

$$\chi(x) = \lim_{n \rightarrow \infty} \frac{\tau_n(x)}{n}. \quad (12)$$

Using the Shannon–McMillan–Breiman theorem, we can show that for  $\nu$ -almost every  $x = (v, s) \in Y$  we have

$$h_\nu(x) = \frac{1}{\chi(x)} \lim_{n,m \rightarrow \infty} -\frac{1}{n+m} \log \mu(R_{n,m}(v)), \quad (13)$$

where  $\mu$  denotes the measure induced by  $\nu$  on  $X$ , and  $R_{n,m}(v) = C_{i_{-m} \cdots i_n}$  is any cylinder such that  $v \in R_{n,m}(v)$ . Let now  $Z$  be a full  $\nu$ -measure  $\Phi$ -invariant set of points  $x \in Y$  such that:

1. the numbers  $\lambda_s(x)$  and  $\lambda_u(x)$  in (7) and  $\chi(x)$  in (12) are well-defined;
2. the number  $h_\nu(x)$  in (10) is well-defined and satisfies (13).

Fix  $\varepsilon > 0$  sufficiently small. For each  $x = (v, s) \in Z$  there exists  $p(x) \in \mathbb{N}$  such that if  $m, n \geq p(x)$  then

$$\tau_m(x)(\lambda_s(x) - \varepsilon) < \sum_{k=0}^{m-1} \beta_s(\sigma^k(v)) < \tau_m(x)(\lambda_s(x) + \varepsilon),$$

$$\tau_n(x)(\lambda_u(x) - \varepsilon) < -\sum_{k=0}^{n-1} \beta_u(\sigma^k(v)) < \tau_n(x)(\lambda_u(x) + \varepsilon),$$

$$n(\chi(x) - \varepsilon) < \tau_n(x) < n(\chi(x) + \varepsilon),$$

$$-h_\nu(x)\chi(x) - \varepsilon < \frac{1}{n+m} \log \mu(R_{n,m}(v)) < -h_\nu(x)\chi(x) + \varepsilon.$$

Fix now  $\ell \in \mathbb{N}$  and define the set  $Q_\ell = \{x \in Z : p(x) \leq \ell\}$ . This plays the role of a Pesin set. For each  $x = (v, s) \in Z$  there exists  $r(x) > 0$  such that for every  $r \in (0, r(x))$  we can choose  $m = m(x, r)$  and  $n = n(x, r)$  with  $\tau_m(x), \tau_n(x) \geq p(x)$  such that

$$\begin{aligned} \exp \sum_{k=0}^{m-1} \beta_s(\sigma^k(v)) &\geq r \quad \text{and} \quad \exp \sum_{k=0}^m \beta_s(\sigma^k(v)) < r, \\ \exp \sum_{k=0}^{n-1} \beta_u(\sigma^k(v)) &\geq r \quad \text{and} \quad \exp \sum_{k=0}^n \beta_u(\sigma^k(v)) < r. \end{aligned}$$

Set  $R(x, r) = R_{n(x,r), m(x,r)}(v)$ . By approximating the ball  $B(x, r)$  by the set  $R(x, r) \times I_r(x)$ , where  $I_r(x)$  is some interval of length  $2r$ , we can proceed in a similar way to that in [5, Theorem 15] to establish the desired result. For completeness we include the detailed argument here.

It follows from (2) and the construction of  $R(x, r)$  that there exists  $\gamma > 0$  (independent of  $x$  and  $r$ ) such that  $B(x, \gamma r) \supset R(x, r) \times I_r(x)$ . Thus, for each  $x \in Z \setminus X$  and each sufficiently small  $r$  we obtain

$$\begin{aligned} \nu(B(x, \gamma r)) &\geq \mu(R(x, r))2r \geq \exp[(-h_\nu(x)\chi(x) - \varepsilon)(n+m)]2r \\ &\geq \exp[-h_\nu(x)(\tau_n(x) + \tau_m(x)) - (h_\nu(x) + 1)\varepsilon(n+m)]2r \\ &\geq \exp[(-h_\nu(x) - a(x)\varepsilon)(\tau_n(x) + \tau_m(x))]2r, \end{aligned}$$

where  $a(x) = (h_\nu(x) + 1)\kappa$  with  $\kappa = 1/\min\{\min_X \tau, \tau(x)\}$ . Therefore

$$\nu(B(x, \gamma r)) \geq \exp \left[ (h_\nu(x) + a(x)\varepsilon) \left( \frac{\log r}{\lambda_u(x) - \varepsilon} - \frac{\log r}{\lambda_s(x) + \varepsilon} \right) \right] 2r.$$

Taking logarithms, dividing by  $\log r$ , and letting  $r \rightarrow 0$  it follows readily from the arbitrariness of  $\varepsilon$  that

$$\bar{d}_\nu(x) \leq h_\nu(x) \left( \frac{1}{\lambda_u(x)} - \frac{1}{\lambda_s(x)} \right) + 1$$

for every  $x \in Z \setminus X$  and hence for  $\nu$ -almost every  $x \in Y$ .

The lower bound for the pointwise dimension requires additional arguments. Given  $x$  in the full  $\nu$ -measure set  $Z$  we consider the set  $\Gamma(x)$  of points  $y \in Z$  such that

$$|\lambda_s(y) - \lambda_s(x)| < \varepsilon, |\lambda_u(y) - \lambda_u(x)| < \varepsilon, |\chi(y) - \chi(x)| < \varepsilon, |h_\nu(y) - h_\nu(x)| < \varepsilon.$$

The  $\Phi$ -invariant sets  $\Gamma(x)$  cover  $Z$  and we can choose points  $y_i \in Z$  for  $i = 1, 2, \dots$  such that  $\Gamma_i = \Gamma(y_i)$  satisfy  $\nu(\Gamma_i) > 0$  for each  $i$ , and  $\bigcup_{i \in \mathbb{N}} \Gamma_i$  has full  $\nu$ -measure. Proceeding in a similar way to that in [11, Section 22], for each  $x \in \Gamma_i \cap Q_\ell$  and  $r > 0$ , we denote by  $R'(x, r)$  the largest rectangle containing  $x$  with the property that  $R'(x, r) = R(y, r)$  for some  $y \in R'(x, r) \cap \Gamma_i \cap Q_\ell$  and that  $R(z, r) \subset R'(x, r)$  for any  $z \in R'(x, r) \cap \Gamma_i \cap Q_\ell$ . Two sets  $R'(x, r)$  and  $R'(y, r)$  either coincide or intersect at most along their boundaries.

The Borel density lemma (see for example [9, Theorem 2.9.11]) tells us that for  $\nu$ -almost every  $x \in \Gamma_i \cap Q_\ell$  we have

$$\lim_{r \rightarrow 0} \frac{\nu(B(x, r) \cap \Gamma_i \cap Q_\ell)}{\nu(B(x, r))} = 1.$$

Therefore, for  $\nu$ -almost every  $x \in \Gamma_i \cap Q_\ell$  there exists  $r(x) > 0$  such that

$$\nu(B(x, r)) \leq 2\nu(B(x, r) \cap \Gamma_i \cap Q_\ell)$$

for each  $r \in (0, r(x))$ . Furthermore, there exist a constant  $K > 0$  (independent of  $x$  and  $r$ ) and points  $x_1, \dots, x_k \in \Gamma_i \cap Q_\ell$  with  $k \leq K$  such that the sets  $R'(x_j, r) \times I_r(x_j)$  for  $j = 1, \dots, k$  cover  $B(x, r) \cap \Gamma_i \cap Q_\ell$ . We obtain

$$\begin{aligned} \nu(B(x, r)) &\leq 2\nu(B(x, r) \cap \Gamma_i \cap Q_\ell) \leq 4r \sum_{j=1}^k \nu(R'(x_j, r)) \\ &\leq 4r \sum_{j=1}^k \exp[(-h_\nu(x_j)\chi(x_j) + \varepsilon)(n(x_j, r) + m(x_j, r))] \\ &\leq 4r \sum_{j=1}^k \exp[(-h_\nu(x_j) + a(x_j)\varepsilon)(\tau_n(x_j) + \tau_m(x_j))] \\ &\leq 4r \sum_{j=1}^k \exp \left[ b(y_i) \left( \frac{\log r - \min \beta_u}{\lambda_u(x_j) + \varepsilon} - \frac{\log r - \min \beta_s}{\lambda_s(x_j) - \varepsilon} \right) \right] \\ &\leq 4rK \exp \left[ c(x) \left( \frac{\log r - \min \beta_u}{\lambda_u(x) + 2\varepsilon} - \frac{\log r - \min \beta_s}{\lambda_s(x) - 2\varepsilon} \right) \right], \end{aligned}$$

where

$$\begin{aligned} b(y_i) &= h_\nu(y_i) - \varepsilon - (h_\nu(y_i) + \varepsilon + 1)\varepsilon\kappa, \\ c(x) &= h_\nu(x) - 2\varepsilon - (h_\nu(x) + 2\varepsilon + 1)\varepsilon\kappa. \end{aligned}$$



Taking logarithms, dividing by  $\log r$ , and letting  $r \rightarrow 0$  we conclude from the arbitrariness of  $\varepsilon$  that

$$\underline{d}_\nu(x) \geq h_\nu(x) \left( \frac{1}{\lambda_u(x)} - \frac{1}{\lambda_s(x)} \right) + 1 \quad (14)$$

for  $\nu$ -almost every  $x \in \Gamma_i \cap Q_\ell$ . Letting  $\ell \rightarrow \infty$  we conclude that (14) holds for  $\nu$ -almost every  $x \in \Gamma_i$ . Since  $\bigcup_{\ell \in \mathbb{N}} \Gamma_i$  has full  $\nu$ -measure (14) holds for  $\nu$ -almost every  $x \in Y$ .  $\square$

We emphasize that we do not require the measure  $\nu$  to be ergodic and that it may have only an ‘‘almost’’ local product structure (in the sense of [1]) instead of a local product structure as in the case of Gibbs measures.

The following is a simple consequence of Theorem 1.

**Corollary 2.** *If  $\Phi$  is a suspension flow on  $Y$  over a two-sided topological Markov chain with Hölder continuous height function, and  $\nu$  is a  $\Phi$ -invariant probability measure on  $Y$ , then*

$$\dim_H \nu = \text{ess sup} \left\{ h_\nu(x) \left( \frac{1}{\lambda_u(x)} - \frac{1}{\lambda_s(x)} \right) + 1 : x \in Y \right\}. \quad (15)$$

If, in addition,  $\nu$  is ergodic then

$$\begin{aligned} \dim_H \nu &= h_\nu(\Phi) \left( \frac{1}{\int_Y \lambda_u d\nu} - \frac{1}{\int_Y \lambda_s d\nu} \right) + 1 \\ &= h_\mu(\sigma) \left( -\frac{1}{\int_X \beta_u d\mu} - \frac{1}{\int_X \beta_s d\mu} \right) + 1, \end{aligned} \quad (16)$$

where  $\mu$  denotes the measure on  $X$  induced by  $\nu$ .

*Proof.* We have (see for example [5, Proposition 3])

$$\dim_H \nu = \text{ess sup} \{ \underline{d}_\nu(x) : x \in Y \}, \quad (17)$$

and hence the identity in (15) follows immediately from Theorem 1. When  $\nu$  is ergodic, the first identity in (16) follows immediately from (15) and from the  $\Phi$ -invariance of the functions  $h_\nu$ ,  $\lambda_s$ , and  $\lambda_u$ . For the second identity we observe that by (4), (6), and (8),

$$\int_Y \lambda_s d\nu = \int_X \beta_s \mu / \int_X \tau d\mu \quad \text{and} \quad \int_Y \lambda_u d\nu = \int_X \beta_u \mu / \int_X \tau d\mu.$$

Furthermore, by Abramov’s formula,  $h_\nu(\Phi) = h_\mu(\sigma) / \int_X \tau d\mu$ , and we obtain the second identity in (16). This completes the proof.  $\square$

In the case of conformal hyperbolic flows and equilibrium measures with a Hölder continuous potential, the identity in (16) was established by Pesin and Sadovskaya in [12].

We now describe the behavior of the Hausdorff dimension of an invariant measure under an ergodic decomposition.

**Theorem 3.** *Let  $\Phi$  be a suspension flow on  $Y$  over a two-sided topological Markov chain with a Hölder continuous height function, and  $\nu$  a  $\Phi$ -invariant probability measure on  $Y$ . Then, for any ergodic decomposition  $\tau$  of  $\nu$  we have*

$$\dim_H \nu = \text{ess sup} \{ \dim_H \rho : \rho \in \mathcal{M}_\Phi(Y) \},$$

with the essential supremum taken with respect to  $\tau$ .

*Proof.* If  $\nu(Y \setminus B) = 0$  then  $\rho(Y \setminus B) = 0$  for  $\tau$ -almost every  $\rho \in \mathcal{M}_\Phi(Y)$ . Hence,  $\dim_H B \geq \dim_H \rho$  for  $\tau$ -almost every  $\rho \in \mathcal{M}_\Phi(Y)$ , and thus

$$\dim_H B \geq \operatorname{ess\,sup}\{\dim_H \rho : \rho \in \mathcal{M}_\Phi(Y)\}.$$

Taking the infimum over all the sets  $B$  with  $\nu(Y \setminus B) = 0$  we obtain

$$\dim_H \nu \geq \operatorname{ess\,sup}\{\dim_H \rho : \rho \in \mathcal{M}_\Phi(Y)\}.$$

The opposite inequality can be obtained from a straightforward modification of arguments in [5, Theorem 17] using Corollary 2.  $\square$

#### 4. MEASURES OF MAXIMAL DIMENSION

Let now  $\Phi$  be a suspension flow on  $Y$  over a two-sided topological Markov chain  $\sigma : X \rightarrow X$  with Hölder continuous height function. Let  $\beta_s$  and  $\beta_u$  be (Hölder) continuous negative functions on  $X$ , and let  $\zeta_s$  and  $\zeta_u$  be defined as in Section 2.3. For each  $\nu \in \mathcal{M}_\Phi(Y)$  we define

$$\lambda_s(\nu) \stackrel{\text{def}}{=} \int_Y \lambda_s(x) d\nu(x) \quad \text{and} \quad \lambda_u(\nu) \stackrel{\text{def}}{=} \int_Y \lambda_u(x) d\nu(x). \quad (18)$$

Our approach to establish the existence of measures of maximal dimension is based on the study of the topological pressure of the two-parameter family  $(p, q) \mapsto -p\zeta_u + q\zeta_s$  (see the appendix for details about topological pressure). Consider the function  $Q : \mathbb{R}^2 \rightarrow \mathbb{R}$  defined by

$$Q(p, q) = P_\Phi(-p\zeta_u + q\zeta_s).$$

Since  $\zeta_s$  and  $\zeta_u$  are Hölder continuous, the results in the appendix imply that  $Q$  is real-analytic. Furthermore, for each  $(p, q) \in \mathbb{R}^2$  the function  $-p\zeta_u + q\zeta_s$  possesses a unique equilibrium measure  $\nu_{p,q}$  which is ergodic.

Since the maps  $\nu \mapsto \lambda_s(\nu)$  and  $\nu \mapsto \lambda_u(\nu)$  defined by (18) are continuous on  $\mathcal{M}_\Phi(Y)$ , and  $\mathcal{M}_\Phi(Y)$  is compact, we can define

$$\lambda_s^{\min} = \min \lambda_s(\mathcal{M}_\Phi(Y)), \quad \lambda_s^{\max} = \max \lambda_s(\mathcal{M}_\Phi(Y)),$$

and

$$\lambda_u^{\min} = \min \lambda_u(\mathcal{M}_\Phi(Y)), \quad \lambda_u^{\max} = \max \lambda_u(\mathcal{M}_\Phi(Y)).$$

Consider the intervals

$$I_s = (\lambda_s^{\min}, \lambda_s^{\max}) \quad \text{and} \quad I_u = (\lambda_u^{\min}, \lambda_u^{\max}).$$

Note that  $I_s \neq \emptyset$  (respectively  $I_u \neq \emptyset$ ) if and only if  $\zeta_s$  (respectively  $\zeta_u$ ) is not cohomologous to a constant (see the appendix for the definition).

We are now ready to establish the existence of ergodic measures of maximal dimension.

**Theorem 4.** *Let  $\Phi$  be a suspension flow on  $Y$  over a topologically mixing two-sided topological Markov chain with Hölder continuous height function. Then there exists an ergodic measure  $\nu \in \mathcal{M}_\Phi(Y)$  such that*

$$\dim_H \nu = \sup\{\dim_H \rho : \rho \in \mathcal{M}_\Phi(Y)\}. \quad (19)$$

*Proof.* The proof of this theorem goes along the lines of the proof of the corresponding result in the case of hyperbolic diffeomorphisms (see [4]). Therefore, we will only provide a sketch.

We denote by  $\mathcal{M}_E \subset \mathcal{M}_\Phi(Y)$  the set of ergodic  $\Phi$ -invariant probability measures on  $Y$ . Since any ergodic decomposition attributes full measure to  $\mathcal{M}_E$ , we conclude from Theorem 3 that

$$\sup\{\dim_H \rho : \rho \in \mathcal{M}_\Phi(Y)\} = \sup\{\dim_H \rho : \rho \in \mathcal{M}_E\}.$$

Therefore, it is sufficient to establish the existence of  $\nu \in \mathcal{M}_E$  with

$$\dim_H \nu = \sup\{\dim_H \rho : \rho \in \mathcal{M}_E\}.$$

We consider a sequence  $(\nu_n)_{n \in \mathbb{N}}$  of measures in  $\mathcal{M}_E$  such that

$$\lim_{n \rightarrow \infty} \dim_H \nu_n = \sup\{\dim_H \nu : \nu \in \mathcal{M}_E\}. \quad (20)$$

Since  $\mathcal{M}_\Phi(Y)$  is compact in the weak\* topology, we can also assume that  $(\nu_n)_{n \in \mathbb{N}}$  converges to some measure  $m \in \mathcal{M}_\Phi(Y)$ . Since the map  $\mathcal{M}_\Phi(Y) \ni \nu \mapsto h_\nu(\Phi)$  is upper semi-continuous, it follows from (16) and the continuity of  $\nu \mapsto \lambda_u(\nu)$  and  $\nu \mapsto \lambda_s(\nu)$  that

$$\lim_{n \rightarrow \infty} \dim_H \nu_n \leq h_m(\Phi) \left( \frac{1}{\lambda_u(m)} - \frac{1}{\lambda_s(m)} \right) + 1 \stackrel{\text{def}}{=} d(m). \quad (21)$$

Using (20) and (21) we obtain

$$\sup\{\dim_H \nu : \nu \in \mathcal{M}_E\} \leq d(m).$$

Therefore, in order to establish the existence of a measure  $\nu \in \mathcal{M}_E$  satisfying (19), it is sufficient to show that there exists  $\rho \in \mathcal{M}_E$  with

$$\dim_H \rho = d(m). \quad (22)$$

We note that when  $m$  is ergodic, it follows from (16) that  $\dim_H m = d(m)$ , and hence (19) holds for the measure  $m$ . However, a priori it is not clear whether  $m$  must be ergodic.

Analogously as in [4] we can show that it is enough to consider the following four cases:

1.  $\lambda_s(m) \in I_s$  and  $\lambda_u(m) \in I_u$ ;
2.  $\lambda_s(m) \in I_s$  and  $\zeta_u$  is cohomologous to a constant;
3.  $\lambda_u(m) \in I_u$  and  $\zeta_s$  is cohomologous to a constant;
4.  $\lambda_s(m) \notin I_s$  and  $\lambda_u(m) \notin I_u$ .

For the proof of the existence of  $\rho \in \mathcal{M}_E$  satisfying (22) we can now proceed in an analogous manner to that in the proof of Theorem 6 in [4]. For this we use in particular results from the thermodynamic formalism which are established in Theorem 11 in the appendix.

In the cases 1, 2, and 3 we obtain that  $m = \nu_{p,q}$  for some  $p, q \in \mathbb{R}$ , and in particular (22) holds for  $m$ . In the case 4 we can establish the existence of a measure  $\nu \in \mathcal{M}_E$  with  $\dim_H \nu = d(\nu)$  possibly differing from  $m$  in (22). We refer to [4, 5] for full details.  $\square$

## 5. HYPERBOLIC FLOWS

**5.1. Hyperbolic flows and Markov systems.** We now describe an example of a suspension flow over a topological Markov chain. Let  $\Phi = \{\varphi_t\}_{t \in \mathbb{R}}$  be a  $C^1$  flow on a Riemannian manifold. A compact  $\Phi$ -invariant set  $\Lambda \subset M$  is *hyperbolic* for  $\Phi$  if there exist a continuous splitting of the tangent bundle

$T_\Lambda M = E^s \oplus E^u \oplus E^0$  and constants  $c > 0$  and  $\lambda \in (0, 1)$  such that for each  $x \in \Lambda$  the following properties hold:

1. the vector  $\frac{d}{dt}(\varphi_t x)|_{t=0}$  generates  $E_x^0$ ;
2.  $d_x \varphi_t(E_x^s) = E_{\varphi_t x}^s$  and  $d_x \varphi_t(E_x^u) = E_{\varphi_t x}^u$  for each  $t \in \mathbb{R}$ ;
3.  $\|d_x \varphi_t v\| \leq c\lambda^t \|v\|$  for every  $v \in E_x^s$  and every  $t > 0$ ;
4.  $\|d_x \varphi_{-t} v\| \leq c\lambda^t \|v\|$  for every  $v \in E_x^u$  and every  $t > 0$ .

For example, geodesic flows on compact Riemannian manifolds with strictly negative sectional curvature have the whole unit tangent bundle as a hyperbolic set. Furthermore, time changes and small  $C^1$  perturbations of flows with a hyperbolic set also possess a hyperbolic set.

Let  $\Lambda$  be a compact invariant locally maximal hyperbolic set for  $\Phi$ . Consider a Markov system  $R_1, \dots, R_p$  for  $\Phi$  on  $\Lambda$  (see for example [2] for the definition) and its associated transfer function  $\tau: \Lambda \rightarrow [0, \infty)$  defined by  $\tau(x) = \min\{t > 0 : \varphi_t x \in \bigcup_{i=1}^p R_i\}$ . We also define a map  $T: \Lambda \rightarrow \bigcup_{i=1}^p R_i$  by  $T(x) = \varphi_{\tau(x)} x$ . By work of Bowen [6] and Ratner [13], there exist Markov systems of arbitrarily small diameter.

We define a  $p \times p$  matrix  $A$  with entries  $a_{ij} = 1$  if  $\text{int } TR_i \cap \text{int } R_j \neq \emptyset$ , and  $a_{ij} = 0$  otherwise. Consider the set  $X \subset \{1, \dots, p\}^{\mathbb{Z}}$  defined by

$$X = \{(\dots i_{-1} i_0 i_1 \dots) : a_{i_n i_{n+1}} = 1 \text{ for every } n \in \mathbb{Z}\},$$

and the topological Markov chain  $\sigma: X \rightarrow X$ . The *coding map*  $\pi: X \rightarrow \bigcup_{i=1}^p R_i$  defined by  $\pi(\dots i_0 \dots) = \bigcap_{j \in \mathbb{Z}} \overline{T^{-j} \text{int } R_{i_j}}$  satisfies  $\pi \circ \sigma = T \circ \pi$ . This construction shows that with the help of a Markov system we can naturally associate a suspension flow (the suspension flow over  $\sigma$  with height function  $\tau \circ \pi$ ) to each given compact invariant locally maximal hyperbolic set. In many situations it is convenient to establish a result for the hyperbolic flow  $\Phi$  by first establishing an appropriate version on a suspension flow associated to  $\Phi$  and then to transfer this result to  $\Phi$ .

Assume now that the flow  $\Phi = \{\varphi_t\}_{t \in \mathbb{R}}$  is *conformal* on  $\Lambda$ . This means that the maps

$$d_x \varphi_t|_{E_x^u}: E_x^u \rightarrow E_{\varphi_t x}^u \quad \text{and} \quad d_x \varphi_t|_{E_x^s}: E_x^s \rightarrow E_{\varphi_t x}^s$$

are multiples of isometries for each  $x \in \Lambda$  and  $t \in \mathbb{R}$ . In this case the Hausdorff dimension of subsets of  $\Lambda$  coincides with a certain  $(\beta_s, \beta_u)$ -dimension. To see this we consider the functions  $\beta_s: X \rightarrow \mathbb{R}$  and  $\beta_u: X \rightarrow \mathbb{R}$  defined by

$$\beta_s(x) = \log \|d_{\pi x} \varphi_{\tau(\pi x)}|_{E^s(\pi x)}\|, \quad \beta_u(x) = -\log \|d_{\pi x} \varphi_{\tau(\pi x)}|_{E^u(\pi x)}\|. \quad (23)$$

Note that

$$\begin{aligned} \sum_{k=0}^{n-1} \beta_s(\sigma^k(x)) &= \log \|d_{\pi x} \varphi_{\tau_n(\pi x)}|_{E^s(\pi x)}\|, \\ \sum_{k=0}^{n-1} \beta_u(\sigma^{-k}(x)) &= -\log \|d_{\pi x} \varphi_{-\tau_n(\pi x)}|_{E^u(\pi x)}\|. \end{aligned} \quad (24)$$

By work of Schmeling in [15], for every  $\Phi$ -invariant set  $B \subset \Lambda$ ,

$$\dim_H B = 1 + \dim_{\beta_s, \beta_u} \pi^{-1} B.$$

Let  $\nu$  be a  $\Phi$ -invariant probability measure on  $\Lambda$ . By Birkhoff's ergodic theorem, for  $\nu$ -almost every  $x \in \Lambda$  there exist the limits

$$\kappa_u(x) = \lim_{t \rightarrow +\infty} \frac{1}{t} \log \|d_x \varphi_t | E_x^u\| \quad \text{and} \quad \kappa_s(x) = \lim_{t \rightarrow +\infty} \frac{1}{t} \log \|d_x \varphi_t | E_x^s\|.$$

As observed by Pesin and Sadovskaya in [12] we have

$$\kappa_u(x) = \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \zeta_u(\varphi_r(x)) dr \quad \text{and} \quad \kappa_s(x) = \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \zeta_s(\varphi_r(x)) dr,$$

where

$$\begin{aligned} \zeta_u(x) &= \left. \frac{\partial}{\partial t} \log \|d_x \varphi_t | E_x^u\| \right|_{t=0} = \lim_{t \rightarrow 0} \frac{1}{t} \log \|d_x \varphi_t | E_x^u\|, \\ \zeta_s(x) &= \left. \frac{\partial}{\partial t} \log \|d_x \varphi_t | E_x^s\| \right|_{t=0} = \lim_{t \rightarrow 0} \frac{1}{t} \log \|d_x \varphi_t | E_x^s\|. \end{aligned}$$

For the functions  $\beta_s$  and  $\beta_u$  in (23), it follows from (9) and (24) that  $\kappa_u(x) = \lambda_u(x)$  and  $\kappa_s(x) = \lambda_s(x)$ , whenever any of the corresponding limits exist.

**5.2. Dimension of measures.** We now consider hyperbolic flows and formulate corresponding statements to those in the former sections in the case of suspension flows. The first two statements are respectively versions of Theorems 1 and 3, and were established by Barreira and Wolf in [5]. The third statement is new and gives a version of Theorem 4 for hyperbolic flows. Namely, it establishes the existence of measures of maximal dimension for hyperbolic flows.

**Theorem 5** ([5, Theorem 15]). *Let  $\Phi$  be a  $C^{1+\varepsilon}$  flow with a hyperbolic set  $\Lambda \subset M$  on which  $\Phi$  is conformal, and  $\nu$  a  $\Phi$ -invariant probability measure on  $\Lambda$ . Then, for  $\nu$ -almost every  $x \in \Lambda$  we have*

$$\underline{d}_\nu(x) = \bar{d}_\nu(x) = h_\nu(x) \left( \frac{1}{\lambda_u(x)} - \frac{1}{\lambda_s(x)} \right) + 1.$$

It follows from Theorem 5 and (17) that

$$\dim_H \nu = \operatorname{ess\,sup} \left\{ h_\nu(x) \left( \frac{1}{\lambda_u(x)} - \frac{1}{\lambda_s(x)} \right) + 1 : x \in \Lambda \right\}.$$

This identity can then be used to establish the following.

**Theorem 6** ([5, Theorem 17]). *Let  $\Phi$  be a  $C^{1+\varepsilon}$  flow with a hyperbolic set  $\Lambda \subset M$  on which  $\Phi$  is conformal, and  $\nu$  a  $\Phi$ -invariant probability measure on  $\Lambda$ . Then, for any ergodic decomposition  $\tau$  of  $\nu$  we have*

$$\dim_H \nu = \operatorname{ess\,sup} \{ \dim_H \rho : \rho \in \mathcal{M}_\Phi(\Lambda) \},$$

with the essential supremum taken with respect to  $\tau$ .

Theorems 5 and 6 could alternatively be obtained in a straightforward manner as consequences respectively of Theorems 1 and 3.

We finally discuss the existence of measures of maximal dimension.

**Theorem 7.** *Let  $\Phi$  be a  $C^{1+\varepsilon}$  flow with a hyperbolic set  $\Lambda \subset M$  on which  $\Phi$  is conformal and topologically mixing. Then there exists an ergodic measure  $\nu \in \mathcal{M}_\Phi(\Lambda)$  such that*

$$\dim_H \nu = \sup \{ \dim_H \rho : \rho \in \mathcal{M}_\Phi(\Lambda) \}.$$

*Proof.* The proof can be obtained with slight changes from the proof of Theorem 4, using the information given by Theorems 1 and 3.  $\square$

#### APPENDIX A. THERMODYNAMIC FORMALISM

We first recall some basic notions from the thermodynamic formalism for suspension flows. Let  $\sigma$  be a homeomorphism on a compact metric space  $X$  and  $\tau : X \rightarrow (0, \infty)$  a continuous function. Let  $\Phi = \{\varphi_t\}_{t \in \mathbb{R}}$  be a continuous suspension flow over  $\sigma$  with height function  $\tau$  on  $Y$ . Let  $u : Y \rightarrow \mathbb{R}$  be a continuous function and given  $x \in Y$ ,  $t > 0$ , and  $\varepsilon > 0$  write

$$u(x, t, \varepsilon) = \sup \left\{ \int_0^t u(\varphi_s y) ds : y \in B(x, t, \varepsilon) \right\},$$

with  $B(x, t, \varepsilon)$  as in (11). We define the *topological pressure of  $u$*  (with respect to  $\Phi$ ) by

$$P_\Phi(u) = \lim_{\varepsilon \rightarrow 0} \limsup_{t \rightarrow \infty} \frac{1}{t} \log \inf_{\Gamma} \sum_{x \in \Gamma} \exp(u(x, t, \varepsilon)),$$

where the infimum is taken over all finite or countable sets  $\Gamma = \{x_i\} \subset Y$  such that  $\bigcup_i B(x_i, t, \varepsilon) = Y$ . The topological pressure of  $u$  satisfies the following variational principle.

**Proposition 8.** *For every continuous suspension flow  $\Phi$  on the compact metric space  $Y$  and every continuous function  $u : Y \rightarrow \mathbb{R}$  we have*

$$P_\Phi(u) = \sup \left\{ h_\nu(\Phi) + \int_Y u d\nu : \nu \in \mathcal{M}_\Phi(Y) \right\}. \quad (25)$$

We say that  $\nu \in \mathcal{M}_\Phi(Y)$  is an *equilibrium measure* for  $u$  (with respect to the flow  $\Phi$ ) if the supremum in (25) is attained by this measure, that is,

$$P_\Phi(u) = h_\nu(\Phi) + \int_Y u d\nu.$$

When the map  $\nu \mapsto h_\nu(\Phi)$  is upper semi-continuous each continuous function has an equilibrium measure (which need not be unique). In particular, if  $\Phi$  is expansive then the Kolmogorov–Sinai entropy depends upper semi-continuously on the measure. Recall that  $\Phi$  is *expansive* if there exists  $\varepsilon > 0$  such that for any continuous function  $s : \mathbb{R} \rightarrow \mathbb{R}$  with  $s(0) = 0$  and points  $x, y \in M$  with

$$d(\varphi_t x, \varphi_{s(t)} x) < \varepsilon \text{ and } d(\varphi_t x, \varphi_{s(t)} y) < \varepsilon \text{ for every } t \in \mathbb{R}$$

we have  $x = y$ . We refer to [10, 14] for more details.

Let  $C^\alpha(Y)$  denote the space of Hölder continuous functions on  $Y$  with Hölder exponent  $\alpha$ . The following proposition gathers results from [2, 3, 7].

**Proposition 9.** *Let  $\Phi$  be a suspension flow on  $Y$  over a topologically mixing two-sided topological Markov chain with Hölder continuous height function. If  $a \in C^\alpha(Y)$  then:*

1.  $\Delta_a \in C^\alpha(X)$  and  $P_\sigma(\Delta_a - P_\Phi(a)\tau) = 0$ ;
2. there is a unique equilibrium measure  $\nu_a$  of  $a$  with respect to  $\Phi$ ;
3. the measure  $\nu_a$  is ergodic and  $\nu_a = T(\mu_{\Delta_a - P_\Phi(a)\tau})$ , where  $\mu_{\Delta_a - P_\Phi(a)\tau}$  is the unique equilibrium measure of  $\Delta_a - P_\Phi(a)\tau$  with respect to  $\sigma$ .

We say that a function  $a: Y \rightarrow \mathbb{R}$  is  $\Phi$ -cohomologous to a function  $b: Y \rightarrow \mathbb{R}$  if there exists a bounded measurable function  $q: Y \rightarrow \mathbb{R}$  such that

$$a(x) - b(x) = \lim_{t \rightarrow 0} \frac{q(\varphi_t x) - q(x)}{t}$$

for every  $x \in Y$ . Recall that two functions  $A: X \rightarrow \mathbb{R}$  and  $B: X \rightarrow \mathbb{R}$  are said to be  $\sigma$ -cohomologous if there exists a bounded measurable function  $q: X \rightarrow \mathbb{R}$  such that  $A(x) - B(x) = q(\sigma(x)) - q(x)$  for every  $x \in X$ .

**Proposition 10** (see [2]). *Let  $\Phi$  be a suspension flow on  $Y$  over a topologically mixing two-sided topological Markov chain with Hölder continuous height function. Then for all  $a, b \in C^\alpha(Y)$  we have  $\nu_a = \nu_b$  if and only if  $a - b$  is  $\Phi$ -cohomologous to a constant. Moreover, if  $a, b \in C^\alpha(Y)$  and  $q: Y \rightarrow \mathbb{R}$  is bounded and measurable, then the following conditions are equivalent:*

1.  $a$  is  $\Phi$ -cohomologous to  $b$  with

$$a(y) - b(y) = \lim_{t \rightarrow 0} \frac{q(\varphi_t y) - q(y)}{t} \text{ for every } y \in Y;$$

2.  $\Delta_a$  is  $\sigma$ -cohomologous to  $\Delta_b$  on  $X$  with

$$\Delta_a(x) - \Delta_b(x) = q(\sigma(x)) - q(x) \text{ for every } x \in X.$$

This proposition allows us to characterize the cohomological properties of the flow  $\Phi$  in terms of the cohomological properties of the map  $\sigma$  on the base.

Combining properties of the map  $u \mapsto P_\Phi(u)$  with the Implicit Function Theorem we obtain the following.

**Theorem 11.** *Let  $\Phi$  be a suspension flow on  $Y$  over a topologically mixing two-sided topological Markov chain with Hölder continuous height function. Then the following properties hold:*

1. the map  $a \mapsto P_\Phi(a)$  is real-analytic on  $C^\alpha(Y)$ ;
2. for each  $a, b \in C^\alpha(Y)$ ,

$$\left. \frac{d}{dt} P_\Phi(a + tb) \right|_{t=0} = \int_Y b d\nu_a;$$

3. for each  $a, e_1, e_2 \in C^\alpha(Y)$  we have

$$D_a^2 P_\Phi(e_1, e_2) = \frac{Q_{\Delta_a - P_\Phi(a)\tau}(\Delta_{e_1} - \tau \int_Y e_1 d\nu_a, \Delta_{e_2} - \tau \int_Y e_2 d\nu_a)}{\int_X \tau d\mu_{\Delta_a - P_\Phi(a)\tau}},$$

where

$$Q_h(f_1, f_2) = \sum_{k=-\infty}^{\infty} \left( \int_X f_1(f_2 \circ T^k) d\mu_h - \int_X f_1 d\mu_h \int_X f_2 d\mu_h \right) \quad (26)$$

for all  $h, f_1, f_2 \in C^\alpha(X)$ ;

4. for each  $a, b \in C^\alpha(Y)$  and  $t \in \mathbb{R}$  we have

$$\frac{d^2}{dt^2} P_\Phi(a + tb) \geq 0,$$

with equality if and only if  $b$  is  $\Phi$ -cohomologous to a constant.

*Proof.* Define  $g: C^\alpha(Y) \times \mathbb{R} \rightarrow C^\alpha(X)$  by  $g(a, t) = \Delta_a - t\tau$ . Since  $\tau$  is Hölder continuous, Proposition 9 implies that  $g$  is well-defined. Furthermore, we define  $G: C^\alpha(X) \times \mathbb{R} \rightarrow \mathbb{R}$  and  $f: C^\alpha(Y) \rightarrow C^\alpha(Y) \times \mathbb{R}$  by

$$G = P_\sigma \circ g \quad \text{and} \quad f(a) = (a, P_\Phi(a)).$$

It is well-known that  $G$  is analytic (see [14]). It follows from Ruelle's formulas for the derivatives of the pressure (see again [14]) that

$$D_\varphi P_\sigma \psi = \int_X \psi d\mu_\varphi \quad \text{and} \quad D_h^2 P_\sigma(f_1, f_2) = Q_h(f_1, f_2),$$

with  $Q_h$  as in (26). We have

$$\partial_t G = D_{g(a,t)} P_\sigma \partial_t g = D_{g(a,t)} P_\sigma(-\tau) = - \int_X \tau d\mu_{g(a,t)} < 0.$$

By Proposition 9 we have  $G(a, P_\Phi(a)) = 0$  for all  $a \in C^\alpha(Y)$  and hence the Implicit Function Theorem ensures that  $a \mapsto P_\Phi(a)$  is real-analytic.

To prove the second statement we note that

$$\partial_a G b = D_{g(a,t)} P_\sigma \circ \partial_a g b = D_{g(a,t)} P_\sigma(\Delta_b) = \int_X \Delta_b d\mu_{g(a,t)}.$$

Here we use the fact that  $a \mapsto \Delta_a$  is linear and continuous. Since

$$D_a P_\Phi = -[D_{2,(a,P_\Phi(a))} G]^{-1} D_{1,(a,P_\Phi(a))} G,$$

we may conclude that

$$D_a P_\Phi(b) = - \frac{\int_X \Delta_b d\mu_{(g \circ f)(a)}}{- \int_X \tau d\mu_{(g \circ f)(a)}} = \int_Y b d\nu_a.$$

In order to prove the third statement we notice that  $G(f(a)) = 0$  for all  $a \in C^\alpha(Y)$ . Therefore

$$0 = D_a^2(G \circ f)(e_1, e_2) = D_{f(a)} G(D_a^2 f(e_1, e_2)) + D_{f(a)}^2 G(D_a f e_1, D_a f e_2). \quad (27)$$

Using the identities  $D_a f e = (e, D_a P_\Phi e)$  and  $D_a^2 f = (0, D_a^2 P_\Phi)$ , we conclude that

$$\begin{aligned} D_{f(a)} G(D_a^2 f(e_1, e_2)) &= D_{2,f(a)} G(D_a^2 P_\Phi(e_1, e_2)) \\ &= -D_a^2 P_\Phi(e_1, e_2) \int_X \tau d\mu_{(g \circ f)(a)} \end{aligned} \quad (28)$$

Set now  $u = f(a)$  and  $v_i = D_a f e_i$  for  $i = 1, 2$ . To calculate  $D_u^2 G(v_1, v_2)$  we use the identities  $G(a, t) = P_\sigma(\Delta_a - t\tau) = P_\sigma(g(a, t))$ . We obtain

$$D_u^2 G(v_1, v_2) = D_{g(u)} P_\sigma(D_u^2 g(v_1, v_2)) + D_{g(u)}^2 P_\sigma(D_u g v_1, D_u g v_2).$$

Since  $g$  is linear and continuous we obtain  $D_u g = g$  and  $D_u^2 g = 0$ , which implies

$$D_u^2 G(v_1, v_2) = D_{g(u)}^2 P_\sigma(g(v_1), g(v_2)) = Q_{g(u)}(g(v_1), g(v_2)).$$

Therefore

$$D_{f(a)}^2 G(D_a f e_1, D_a f e_2) = Q_{(g \circ f)(a)}(g(e_1, D_a P_\Phi e_1), g(e_2, D_a P_\Phi e_2)).$$



It follows from (27) and (28) that

$$\begin{aligned} D_a^2 P_\Phi(e_1, e_2) &= -\frac{D_{f(a)}G(D_a^2 f(e_1, e_2))}{\int_X \tau d\mu_{(g \circ f)(a)}} \\ &= \frac{D_{f(a)}^2 G(D_a f e_1, D_a f e_2)}{\int_X \tau d\mu_{(g \circ f)(a)}} \\ &= \frac{Q_{(g \circ f)(a)}(\Delta_{e_1} - \tau \int_Y e_1 d\nu_a, \Delta_{e_2} - \tau \int_Y e_2 d\nu_a)}{\int_X \tau d\mu_{(g \circ f)(a)}}, \end{aligned}$$

where  $(g \circ f)(a) = \Delta_a - P_\Phi(a)\tau$ .

The fourth statement follows from the fact that  $Q_h(f, f) \geq 0$  for all  $h$ ,  $f \in C^\alpha(X)$ , where equality holds if and only if  $f$  is  $\sigma$ -cohomologous to a constant. Observe first that

$$\begin{aligned} \frac{d^2}{dt^2} P_\Phi(a + tb) &= D_{a+tb}^2 P_\Phi(b, b) \\ &= \frac{Q_{(g \circ f)(a+tb)}(\Delta_b - \tau \int_Y b d\nu_{a+tb}, \Delta_b - \tau \int_Y b d\nu_{a+tb})}{\int_X \tau d\mu_{(g \circ f)(a+tb)}} \geq 0 \end{aligned}$$

for each  $a, b \in C^\alpha(Y)$  and  $t \in \mathbb{R}$ . The equality holds if and only if  $\Delta_b - \tau \int_Y b d\nu_{a+tb}$  is  $\sigma$ -cohomologous to a constant. We will show that this constant is zero. Therefore  $\Delta_b$  is  $\sigma$ -cohomologous to  $\tau \int_Y b d\nu_{a+tb}$ , which is equivalent (by Proposition 10) to  $b$  being  $\Phi$ -cohomologous to the constant  $\int_Y b d\nu_{a+tb}$ .

If  $\Delta_b - \tau \int_Y b d\nu_{a+tb}$  is  $\sigma$ -cohomologous to a constant  $c$ , then for every  $\mu \in \mathcal{M}_\sigma(X)$  we have  $\int_X (\Delta_b - \tau \int_Y b d\nu_{a+tb}) d\mu = c$ . By Proposition 9 and Abramov's formula we obtain

$$\int_Y b d\nu_{a+tb} = \frac{\int_X \Delta_b d\mu_{(g \circ f)(a+tb)}}{\int_X \tau d\mu_{(g \circ f)(a+tb)}}.$$

Since  $\mu_{(g \circ f)(a+tb)} \in \mathcal{M}_\sigma(X)$  we conclude that

$$\begin{aligned} c &= \int_X \left( \Delta_b - \tau \int_Y b d\nu_{a+tb} \right) d\mu_{(g \circ f)(a+tb)} \\ &= \int_X \left( \Delta_b - \tau \frac{\int_X \Delta_b d\mu_{(g \circ f)(a+tb)}}{\int_X \tau d\mu_{(g \circ f)(a+tb)}} \right) d\mu_{(g \circ f)(a+tb)} = 0. \end{aligned}$$

This completes the proof.  $\square$

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