VARIATIONAL PRINCIPLE FOR THE ENTROPY

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1. Metric entropy

Let $(X, \mathcal{B}, \mu)$ a measure space and I a countable family of indices.

Definition 1. *We say that $\xi = \{C_i : i \in I\} \subset \mathcal{B}$ is a measurable partition if:*

- $\mu\Big(\bigcup_{i\in I} C_i\Big) = \mu(X)$ *and* $\mu(C_i) > 0$ *for every* $i \in I$*;*
- $\mu(C_i \cap C_j) = 0$ *for every* $i, j \in I$ *with* $i \neq j$*.*

We define also $\xi \vee \eta$ by:

$$\xi \vee \eta = \{C \cap D : C \in \xi, D \in \eta, \mu(C \cap D) > 0\}.$$

Definition 2. *The **entropy of a measurable partition** is given by*

$$H_\mu(\xi) = -\sum_{C\in\xi} \mu(C) \log \mu(C)$$

Let $T : X \to X$ be measurable, μ-T invariant and ξ a measurable partition.

Definition 3. *The **entropy of T with respect to** μ **and the partition** ξ is given by:*

$$h_\mu(T, \xi) = \inf_n \frac{1}{n} H_\mu\Big(\bigvee_{k=0}^{n-1} T^{-k}\xi\Big) = \lim_{n\to\infty} \frac{1}{n} H_\mu\Big(\bigvee_{k=0}^{n-1} T^{-k}\xi\Big) \tag{1}$$

*and the **entropy of T with respect to** μ by:*

$$h_\mu(T) = sup\{h_\mu(T, \xi) : \xi \text{ a measurable partition with } H_\mu(\xi) < \infty\}$$

From (1) we observe that

$$h_\mu(T, \xi) \leq H_\mu(\xi).$$

We can see that in order to calculate the entropy of T with respect to μ we have to consider $h_\mu(T, \xi)$ for every measurable partition ξ with $H_\mu(\xi) < \infty$. Sometimes it is possible to compute the entropy of T with respect to μ using only one special partition described below.

Definition 4. *If T is invertible (mod 0) let $\bigvee_{k=-\infty}^{\infty} T^{-k}\xi$ be the smallest σ-algebra which contains $\bigvee_{k=-n}^{n} T^{-k}\xi$ for every $n \in \mathbb{N}$, then we say that ξ **is a (two-sided) generator** if*

$$\bigvee_{k=-\infty}^{\infty} T^{-k}\xi = \mathcal{B} \quad \text{(mod 0)}.$$

In the same way,

Definition 5. *If T is non-invertible (mod 0) let $\bigvee_{k=0}^{\infty} T^{-k}\xi$ be the smallest σ-algebra which contains $\bigvee_{k=0}^{n-1} T^{-k}\xi$ for every $n \in \mathbb{N}$, then we say that ξ* ***is a (one-sided) generator*** *if*

$$\bigvee_{k=0}^{\infty} T^{-k}\xi = \mathcal{B} \quad \text{(mod 0)}.$$

Now we have

Proposition 6. *If ξ is a generator with $H_\mu(\xi) < \infty$ then*

$$h_\mu(T) = h_\mu(T, \xi)$$

Proposition 7. *The following proprieties hold:*

1. *$h_\mu(id) = 0$;*
2. *$h_\mu(T^k) = k h_\mu(T)$ for every $k \in \mathbb{N}$;*
3. *If an invertible measure preserving transformation possesses a one-sided generator then $h_\mu(T) = 0$.*

2. Examples of calculation of metric entropy

2.1. **Markov measures.** We will study the entropy of the two-sided shift, $\sigma : \{1, ..., k\}^{\mathbb{Z}} \to \{1, ..., k\}^{\mathbb{Z}}$, with respect to a Markov measure i.e. a stochastic pair (P, p) where P is a stochastic matrix, p is a probability vector, $\sum_{i=1}^{k} p_i = 1$, associated to P and:

1. $\sum_{j=1}^{k} p_{ij} = 1$ for each $i = 1, ..., k$;
2. $\sum_{i=1}^{k} p_i p_{ij} = p_j$ for each $j = 1, ..., k$.

The measurable partition $\xi = \{C_1, ..., C_k\}$ is in fact a two-sided generator because:

$$\bigvee_{k=-n}^{n} \sigma^{-k}\xi = \{C_{i_{-n}...i_n} : i_{-n}, ..., i_n \in \{1, ..., k\}\}$$

and the cylinders generate the Borel σ-algebra. Then

$$\begin{aligned}
H_\mu\Big(\bigvee_{k=0}^{n-1} \sigma^{-k}\xi\Big) &= -\sum_{i_0...i_{n-1}} \mu(C_{i_0...i_{n-1}}) \log \mu(C_{i_0...i_{n-1}}) \\
&= -\sum_{i_0...i_{n-1}} p_{i_0} p_{i_0 i_1} ... p_{i_{n-2} i_{n-1}} \log(p_{i_0} p_{i_0 i_1} ... p_{i_{n-2} i_{n-1}}) \\
&= -\sum_{i_0...i_{n-1}} p_{i_0} p_{i_0 i_1} ... p_{i_{n-2} i_{n-1}} \log p_{i_0} \\
&\quad - \sum_{i_0...i_{n-1}} p_{i_0} p_{i_0 i_1} ... p_{i_{n-2} i_{n-1}} \sum_{j=0}^{n-2} \log p_{i_j i_{j+1}} \\
&= -\sum_{i=1}^{k} p_i \log p_i - (n-1) \sum_{i,j=1}^{k} p_i p_{ij} \log p_{ij}
\end{aligned}$$

and so

$$h_\mu(\sigma) = h_\mu(\sigma, \xi) = \lim_{n\to\infty} \frac{1}{n} H_\mu\Big(\bigvee_{k=0}^{n-1} \sigma^{-k}\xi\Big) = -\sum_{i,j=1}^{k} p_i p_{ij} \log p_{ij}$$

2.2. **Rotations.** We will se that any rotation $R_w : \mathbb{T} \to \mathbb{T}$, where $w = e^{2\pi i \tau}$ and $R_w(z) = wz$, has zero entropy with respect to the Lebesgue measure. There are two cases:

- $\tau \in \mathbb{Q}$ or $\{w^n : n \in \mathbb{Z}\}$ is not dense in $\mathbb{T}$ i.e. w is not a root of unity. Them exists $m \in \mathbb{N}$ such that $R_w^m(z) = z$ for every $z \in \mathbb{T}$ so $h_\lambda(R_w) = 0$;
- $\tau \notin \mathbb{Q}$ or $\{w^n : n \in \mathbb{Z}\}$ is dense in $\mathbb{T}$. let $\xi = \{A_1, A_2\}$ the partition of the circle into the upper semi-circle and the lower semi-circle. For every $n > 0$, $R_w^{-n}\xi$ consists of semi-circles beginning at w^{-n} and $-w^{-n}$. Since $\{w^{-n} : n \in \mathbb{N}\}$ is dense, any semi-circle belong to $\bigvee_{n=0}^{\infty} R_w^{-n}\xi$. Hence any arc belong to $\bigvee_{n=0}^{\infty} R_w^{-n}\xi$ and so $h_\lambda(R_w) = 0$.

3. The entropy map

Let (X, d) be a compact metric space $T : X \to X$ continuous. Let $M(X, T)$ be the space of all probability measures on $(X, \mathcal{B}(X))$ that are T-invariant. We know that $M(X, T)$ is a non-empty convex set which is compact in the weak*-topology, by the Krylov-Bogolubov theorem.

Definition 8. *The* ***entropy map*** *is $\mu \mapsto h_\mu(T)$ from $M(X, T)$ to $[0, \infty]$.*

Proposition 9. *The entropy map is affine, i.e. if $\mu, m \in M(X, T)$ and $t \in [0, 1]$ then*

$$h_{t\mu+(1-t)m}(T) = t h_\mu(T) + (1 - t) h_m(T).$$

The entropy map is not in general continuous.

We will give a counterexample in the case of the two-sided shift on $\{0, 1\}^{\mathbb{Z}}$. Let us consider the measures μ_p, for $p \in \mathbb{N}$, concentrated on the p periodic points, giving to each such a point measure $1/2^p$. We have that $\mu_p \in M(X, \sigma)$ and $h_{\mu_p}(\sigma) = 0$, for every p, because the measure is concentrated on a finite set of points. And let μ be the $(1/2, 1/2)$-Bernoulli measure, which we know, has $h_\mu(\sigma) = \log 2$. Now the collection of functions that depends only on a finite number of coordinates form a dense subset $F(X)$ of $C(X)$ by the Stone-Weierstrass theorem. If $f \in F(X)$ then exists N such that $\int_X f d\mu_p = \int_X f d\mu$ if $p \geq N$. Therefor $\mu_p \to \mu$ and so the entropy map is not continuous.

Sometimes is not even upper semi-continuous, but for a special class of maps we will prove that the entropy map is upper semi-continuous.

Definition 10. *$T : X \to X$ is called an* ***expansive homeomorphism*** *if $\exists \delta$ with the property that if $x \neq y$ then $\exists n \in \mathbb{Z}$ with $d(T^n x, T^n y) > \delta$.*

Teorem 11. *If T is an expansive homeomorphism then the entropy map is upper semi-continuous, i.e., if $\mu \in M(X, T)$ and $\varepsilon > 0$ then exists a neighborhood U of μ in $M(X, T)$ such that $m \in U$ implies that*

$$h_m(T) < h_\mu(T) + \varepsilon.$$

Proof. Let δ be an expansive constant for T,$\mu \in M(X, T)$ and $\varepsilon > 0$.

Let now $\xi = \{C_1, ..., C_k\}$ be a finite partition with $\operatorname{diam}(C_j) < \delta$. Then $h_\mu(T) = h_\mu(T, \xi)$. Let N so that

$$\frac{1}{N} H_\mu\Big(\bigvee_{k=0}^{N-1} T^{-k}\xi\Big) < h_\mu(T) + \varepsilon$$

Fix $\varepsilon_1 > 0$ to be chosen later. As μ is regular we choose compact sets:

$$K_{i_0 \dots i_{N-1}} \subset \bigcap_{k=0}^{N-1} T^{-k} C_{i_k}$$

with $\mu\Big(\bigcap_{k=0}^{N-1} T^{-k} C_{i_k} \setminus K_{i_0 \dots i_{N-1}}\Big) < \varepsilon_1$.

Then

$$\bigcup_{k=0}^{N-1} \bigcup_{i_k = j} T^k K_{i_0, \dots i_{N-1}} \subset C_j$$

The sets $L_j := \bigcup_{k=0}^{N-1} \bigcup_{i_k=j} T^k K_{i_0,\dots i_{N-1}}$ are compact and disjoint so there is a partition $\eta = \{D_1, ..., D_k\}$ with $\operatorname{diam}(D_j) < \delta$ and $L_j \subset \operatorname{int}(D_j)$. We have

$$K_{i_0 \dots i_{N-1}} \subset \operatorname{int}\Big(\bigcap_{k=0}^{N-1} T^{-k} D_{i_k}\Big)$$

By Uryson's lemma we can choose $f_{i_0 \dots i_{N-1}} \in C(X)$ such that:

- $0 \le f_{i_0 \dots i_{N-1}} \le 1$
- equals 1 on $K_{i_0 \dots i_{N-1}}$
- vanishes on $X \setminus \operatorname{int}\Big(\bigcap_{k=0}^{N-1} T^{-k} D_{i_k}\Big)$

Let now

$$U_{i_0 \dots i_{N-1}} := \Big\{ m \in M(X,T) : \Big| \int f_{i_0 \dots i_{N-1}} dm - \int f_{i_0 \dots i_{N-1}} d\mu \Big| < \varepsilon_1 \Big\}$$

The set $U_{i_0 \dots i_{N-1}}$ is open is $M(X,T)$ and if $m \in U_{i_0 \dots i_{N-1}}$ then

$$m\Big(\bigcap_{k=0}^{N-1} T^{-k} D_{i_k}\Big) \ge \int f_{i_0 \dots i_{N-1}} dm > \int f_{i_0 \dots i_{N-1}} d\mu - \varepsilon_1 \ge \mu(K_{i_0 \dots i_{N-1}}) - \varepsilon_1$$

and

$$\mu\Big(\bigcap_{k=0}^{N-1} T^{-k} C_{i_k}\Big) - m\Big(\bigcap_{k=0}^{N-1} T^{-k} D_{i_k}\Big) < 2\varepsilon_1$$

Now if $U := \bigcap_{i_0 \dots i_{N-1}} U_{i_0 \dots i_{N-1}}$ and $m \in U$ then:

$$\Big|\mu\Big(\bigcap_{k=0}^{N-1} T^{-k} C_{i_k}\Big) - m\Big(\bigcap_{k=0}^{N-1} T^{-k} D_{i_k}\Big)\Big| < 2\varepsilon_1 k^N$$

because if $\sum_{i=1}^m a_i = 1 = \sum_{i=1}^m b_i$ and also exists $c > 0$ with $a_i - b_i < c$ foe every i then $|a_i - b_i| < cm \quad \forall i$, as $b_i - a_i = \sum_{j \neq i}(a_j - b_j) < cm$.

So if $m \in U$ and ε_1 small enough the continuity of $x \log x$ gives:

$$\frac{1}{N} H_m\Big(\bigvee_{k=0}^{N-1} T^{-k}\eta\Big) < \frac{1}{N} H_\mu\Big(\bigvee_{k=0}^{N-1} T^{-k}\xi\Big) + \frac{\varepsilon}{2}$$

Hence, $m \in U$ and ε_1 small enough then:

$$h_m(T) = h_m(T,\eta) \leq \frac{1}{N} H_m\Big(\bigvee_{k=0}^{N-1} T^{-k}\eta\Big)$$
$$< \frac{1}{N} H_\mu\Big(\bigvee_{k=0}^{N-1} T^{-k}\xi\Big) + \frac{\varepsilon}{2} < h_\mu(T) + \varepsilon.$$

□

4. Topological entropy

We can define topological entropy in the same way we defined the metric entropy, with open covers, α, taking the place of the measurable partitions and $H(\alpha) = \log N(\alpha)$, where $N(\alpha)$ is the minimal cardinality of a subcover of α. But in order to prove the variational principle we will use also Bowen's definition. Let (X,d) be a compact metric space, $T : X \to X$ a continuous map:

$$h_{top}(T) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N_d(T,\varepsilon,n)$$

Where $N_d(T,\varepsilon,n)$ is the maximum number of point in X with pairwise d_n^T-distances at least ε, with

$$d_n^T := max_{0 \leq i \leq n-1} d(T^i x, T^i y).$$

We will call such a set of points (n,ε)**-separated**.

5. Variational principle

In this section we prove the relationship between topological entropy and the metric entropy for a continuous map T of a compact metric space i.e. :

$$h_{top}(T) = sup\{h_\mu(T) : \mu \in M(X,T)\}.$$

- In 1968 L.W.Goodwyn proved the inequality:
$$sup\{h_\mu(T) : \mu \in M(X,T)\} \leq h_{top}(T).$$
- In 1970 E.I.Dinaburg proved equality when X has finite covering dimension.
- In 1970 T.N.T.Goodman proved equality in the general case.

The elegant proof we will give is due to M.Misiurewicz .

Lemma 1. *Let X be a compact metric space, μ a Borel probability measure on X.*

1. *For $x \in X$, $\delta > 0$ there exists $\delta' \in (0,\delta)$ such that $\mu(\partial B(x,\delta')) = 0$.*
2. *For $\delta > 0$ there exists a finite measurable partition $\xi = \{C_1, ..., C_k\}$ with $diam(C_i) < \delta$ for all i and $\mu(\partial\xi) = 0$.*

Proof. 1. $\bigcup_{\delta' \in (0,\delta)} \partial B(x,\delta')$ is an uncountable disjoint union with finite measure.

2. By 1) (and X compact) there is a finite open cover $\{B_1, ..., B_k\}$ by balls of radius less then $\delta/2$ with $\mu(\partial B_j) = 0$ for all j. Let $C_1 = \overline{B_1}$, $C_i = \overline{B_i} \setminus \bigcup_{j=1}^{i-1} \overline{B_j}$ for $i > 1$ and $\xi = \{C_1, ..., C_k\}$. Then ξ is as desired as $\partial\xi \subset \bigcup_{i=1}^{k} \partial B_i$.

□

Note that if $\mu(\partial\xi) = 0$ then $\mu\Big(\partial \bigvee_{k=0}^{n-1} T^{-k}\xi\Big) = 0$.

Now we will describe a method of constructing measures of large entropy.

Lemma 2. *Let (X, d) be a compact metric space, $T : X \to X$ a homeomorphism, $E_n \subset X$ an (n, ε)-separated set, $\nu_n := (1/card(E_n)) \sum_{x\in E_n} \delta_x$ the uniform δ-measure on E_n, and $\mu_n := (1/n) \sum_{i=0}^{n-1} T_*^i \nu_n$. Then there is an accumulation point μ of $\{\mu_n\}_{n\in\mathbb{N}}$ (in the weak* topology) that is T-invariant and satisfies*

$$\limsup_{n\to\infty} \frac{1}{n} \log card(E_n) \leq h_\mu(T).$$

Proof. Since $M(X)$ is compact we can choose a subsequence $\{n_k\}$ such that:

- $\lim_{k\to\infty} \frac{1}{n_k} \log card(E_{n_k}) = \limsup_{n\to\infty} \frac{1}{n} \log card(E_n)$
- μ_{n_k} converges in $M(X)$ to some $\mu \in M(X)$

In fact $\mu \in M(X, T)$ because $T_*\mu_n - \mu_n = (T_*^n \nu_n - \nu_n)/n$ (T_* cont. as in the proof of K-B theorem) and ν_n are probability measures.

We choose now a partition ξ with diameter less then ε and $\mu(\partial\xi) = 0$ as in Lemma 1. Then

$$H_{\nu_n}\Big(\bigvee_{k=0}^{n-1} T^{-k}\xi\Big) = \log card(E_n)$$

Since each $C \in \bigvee_{k=0}^{n-1} T^{-k}\xi$ contains at most one $x \in E_n$ so there are $card(E_n)$ elements of ν_n-measure $1/card(E_n)$ in $\bigvee_{k=0}^{n-1} T^{-k}\xi$.

Now suppose $0 < q < n$ and for every $0 \leq k < q$ define $a(k) := [(n-k)/q]$.

- If we fix $0 \leq k < q$ then

$$\{0, 1, ..., n-1\} = \{k + rq + i : 0 \leq r \leq a(k), 0 < i \leq q\} \cup S$$

where $S = \{0, 1, ..., k, k + a(k)q + 1, ..., n-1\}$ and

$$card(S) \leq 2q$$

since $k + a(k)q \geq k + [\frac{n-k}{q} - 1]q \geq n - q$.

Then we have

$$\bigvee_{k=0}^{n-1} T^{-k}\xi = \Big(\bigvee_{r=0}^{a(k)-1} T^{-(rq+k)}\Big(\bigvee_{i=0}^{q-1} T^{-i}\xi\Big)\Big) \vee \Big(\bigvee_{j\in S} T^{-j}\xi\Big)$$

and

$$\begin{aligned}\log card(E_n) =\ & H_{\nu_n}\Big(\bigvee_{h=0}^{n-1} T^{-h}\xi\Big)\\ \le & \sum_{r=0}^{a(k)-1} H_{\nu_n}\Big(T^{-(rq+k)}\bigvee_{i=0}^{q-1} T^{-i}\xi\Big) + \sum_{j\in S} H_{\nu_n}\Big(T^{-j}\xi\Big)\\ \le & \sum_{r=0}^{a(k)-1} H_{T_*^{rq+k}\nu_n}\Big(\bigvee_{i=0}^{q-1} T^{-i}\xi\Big) + 2q\log card(\xi).\end{aligned}$$

Then

$$\begin{aligned}q\log card(E_n) &= \sum_{k=0}^{q-1} H_{\nu_n}\Big(\bigvee_{h=0}^{n-1} T^{-h}\xi\Big)\\ &\le \sum_{k=0}^{q-1}\left(\sum_{r=0}^{a(k)-1} H_{T_*^{rq+k}\nu_n}\left(\bigvee_{i=0}^{q-1} T^{-i}\xi\right) + 2q\log card(\xi)\right)\\ &\le nH_{\mu_n}\left(\bigvee_{i=0}^{q-1} T^{-i}\xi\right) + 2q^2\log card(\xi)\end{aligned}$$

Thus

$$\limsup_{n\to\infty}\frac{1}{n}\log card(E_n) \le \lim_{k\to\infty}\frac{H_{\mu_{n_k}}\Big(\bigvee_{i=0}^{q-1} T^{-i}\xi\Big)}{q} =^{!} \frac{H_{\mu}\Big(\bigvee_{i=0}^{q-1} T^{-i}\xi\Big)}{q}$$

and that implies

$$\limsup_{n\to\infty}\frac{1}{n}\log card(E_n) \le h_\mu(T,\xi) \le h_\mu(T).$$

□

Teorem 12. *Let (X,d) be a compact metric space, $T : X \to X$ a homeomorphism, then*

$$h_{top}(T) = sup\{h_\mu(T) : \mu \in M(X,T)\}.$$

Proof. (1) Let $\mu \in M(X,T)$ we will show that $h_\mu(T) \le h_{top}(T)$.

Let $\xi = \{C_1, ..., C_k\}$ be a finite measurable partition and $0 < \varepsilon < \frac{1}{k\log k}$. Since μ is regular there exist compact sets $B_i \subset C_i$ with $\mu(C_i \setminus B_i) < \varepsilon$.

Let $\eta = \{B_0, B_1, ..., B_k\}$ the partition with $B_0 = X \setminus \cup_{i=1}^{k} B_i$.

We have $\mu(B_0) < k\varepsilon$ and

$$\begin{aligned}H_\mu(\xi|\eta) &< -\sum_{i=0}^{k}\sum_{j=1}^{k}\mu(B_i)\phi\left(\frac{\mu(B_i\cap A_j)}{\mu(B_i)}\right)\\ &= -\mu(B_0)\sum_{j=1}^{k}\phi\left(\frac{\mu(B_0\cap A_j)}{\mu(B_0)}\right)\\ &\le \mu(B_0)\log k < k\varepsilon\log k < 1\end{aligned}$$

So

$$h_\mu(T,\xi) \le h_\mu(T,\eta) + H_\mu(\xi|\eta) \le h_\mu(T,\eta) + 1$$

The reason we can bring in the topological entropy is that $\alpha := \{B_0 \cup B_1, ..., B_0 \cup B_k\}$ is an open cover of X. We have

$$H_\mu\left(\bigvee_{k=0}^{n-1} T^{-k}\eta\right) \le \log card\left(\bigvee_{k=0}^{n-1} T^{-k}\eta\right) \le \log\left(2^n card\left(\bigvee_{k=0}^{n-1} T^{-k}\alpha\right)\right)$$

If δ_0 is the Lebesgue number of α i.e., the supremum of $\delta > 0$ such that every δ-ball is contained in an element of α, then δ_0 is the Lebesgue number of $\bigvee_{k=0}^{n-1} T^{-k}\alpha$ with respect to the d_n^T.

Since $\bigvee_{k=0}^{n-1} T^{-k}\alpha$ is a minimal cover every $C \in \bigvee_{k=0}^{n-1} T^{-k}\alpha$ contain a point x_C not in any other element of $\bigvee_{k=0}^{n-1} T^{-k}\alpha$.

The x_C set form a (δ_0, n)-separated set.

Then

$$h_\mu(T,\eta) \le h_{top}(T) + \log 2$$

and

$$h_\mu(T,\xi) \le h_\mu(T,\eta) + 1 \le h_{top}(T) + 1 + \log 2,$$

as this is also true for any iterate of T we have

$$h_\mu(T) \le \frac{h_\mu(T^n)}{n} \le \frac{h_{top}(T^n) + 1 + \log 2}{n} = h_{top}(T) + \frac{1 + \log 2}{n}$$

for all $n \in \mathbb{N}$ and hence

$$h_\mu(T) \le h_{top}(T)$$

(2) On the other hand applying Lemma 2 to maximal (n,ε)-separated sets in X we have

$$\limsup_{n\to\infty} \frac{1}{n} \log N_d(T,\varepsilon,n) \le h_\mu(T) \tag{2}$$

for a corresponding accumulation point $\mu \in M(X,T)$. Thus

$$\limsup_{n\to\infty} \frac{1}{n} \log N_d(T,\varepsilon,n) \le \sup_\mu h_\mu(T).$$

and letting $\varepsilon \to 0$ we get

$$h_{top}(T) \le \sup\{h_\mu(T) : \mu \in M(X,T)\}.$$

□

A measure $\mu \in M(X,T)$ is called a **measure of maximal entropy** for T if $h_\mu(T) = h_{top}(T)$.

We observe that for expansive homeomorphisms the existence of such a measure is assured by the upper semi-continuity of the entropy map and the compactness of $M(X,T)$. But the same is true because in (2) the left hand side is equal to the topological entropy if ε is less then half its expansive constant, and so by Lemma 2 we can construct a measure of maximal entropy for such an ε.

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