MOMENT MAPS, PSEUDO-HOLOMORPHIC CURVES AND SYMPLECTOMORPHISM GROUPS

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Abstract. These are the notes of a 3-lecture mini-course on some basic topics of symplectic geometry and topology, given at the XIV Fall Workshop on Geometry and Physics, September 14–16, 2005, in Bilbao, Spain, to a general mathematical audience.

1. Introduction

The Convexity Theorem for the Moment Map (Atiyah-Guillemin-Sternberg) and the Compactness Theorem for Pseudo-Holomorphic Curves (Gromov) gave rise to a remarkable amount of developments in Symplectic Geometry and Topology in the last 20 years. Sections 2–5 of these notes consist of an introduction to some of the ideas associated with these two fundamental theorems. In sections 6 and 7 it is shown how a combination of these ideas, and of some of the developments they originated, can be used to give a geometric description of the topology of the symplectomorphism group of a symplectic manifold, with particular emphasis on the case of $S^2 \times S^2$.

These notes were written with a general mathematical audience in mind and I hope that any mathematics graduate student is able to understand most of them. For a much broader overview of symplectic geometry see [10]. For a further understanding of the fundamental concepts, ideas and results in symplectic geometry and topology see [11] and [18].

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2. Linear Algebra

In this section we review some linear algebra of euclidean and symplectic vector spaces, pointing out some of their main differences and how they compatibly "intersect" as hermitean vector spaces. The emphasis is on the linear algebra aspects that will be more relevant for the main topics of these lectures.

Euclidean vector spaces. Let V be a real vector space of dimension m. A choice of a basis $\{v_1, \ldots, v_m\}$ for V gives an isomorphism between V and $\mathbb{R}^m$, that sends $X = \sum_{i=1}^m x^i v_i \in V$ to $[x^1 \cdots x^m]^t \in \mathbb{R}^m$.

An *inner product* on V is a bilinear map

$$\begin{array}{rcl} \langle \cdot, \cdot \rangle : V \times V & \longrightarrow & \mathbb{R} \\ (X, Y) & \longmapsto & \langle X, Y \rangle \end{array}$$

that is symmetric (i.e. $\langle X, Y \rangle = \langle Y, X \rangle$, $\forall\, X, Y \in V$) and positive definite (i.e. $\|X\|^2 \equiv \langle X, X \rangle > 0$, $\forall\, 0 \neq X \in V$).

An *euclidean vector space* is a pair $(V, \langle \cdot, \cdot \rangle)$.

Given a basis for V, and hence an isomorphism with $\mathbb{R}^m$, any inner product $\langle \cdot, \cdot \rangle$ can be written as

$$\langle X, Y \rangle = X^t B Y = [x^1 \cdots x^m]\, [b_{ij}]\, [y^1 \cdots y^m]^t = \sum_{i,j=1}^m x^i b_{ij} y^j \, ,$$

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where $b_{ij} = \langle v_i, v_j \rangle$, $\forall\, i,j = 1, \ldots, m$, and $B = [b_{ij}]$ is an $(m \times m)$ symmetric and positive definite matrix representing $\langle \cdot, \cdot \rangle$ in this basis.

The Gram-Schmidt orthogonalization process shows that there always exist *orthonormal basis* for $(V, \langle \cdot, \cdot \rangle)$, i.e. basis $\{e_1, \ldots, e_m\}$ such that $\langle e_i, e_j \rangle = \delta_{ij}$. In such a basis, the matrix B representing $\langle \cdot, \cdot \rangle$ is the $(m \times m)$ identity matrix I. This means that any euclidean vector space $(V, \langle \cdot, \cdot \rangle)$ is isomorphic to $(\mathbb{R}^m, \langle \cdot, \cdot \rangle_0)$ where

$$\langle X, Y \rangle_0 = X^t I Y = [x^1 \cdots x^m][y^1 \cdots y^m]^t = \sum_{i=1}^m x^i y^i \,.$$

The *orthogonal linear group* $O(m)$ is the group of linear symmetries of an euclidean vector space. By the above, $O(m)$ is isomorphic to the group of linear symmetries of $(\mathbb{R}^m, \langle \cdot, \cdot \rangle_0)$, i.e.

$$\begin{aligned} O(m) \ &\cong \{A : \mathbb{R}^m \to \mathbb{R}^m : \langle AX, AY \rangle_0 = \langle X, Y \rangle_0 \,,\ \forall\, X, Y \in \mathbb{R}^m\} \\ &= \{A \in \mathcal{M}_{m\times m} : X^t A^t A Y = X^t I Y \,,\ \forall\, X, Y \in \mathbb{R}^m\} \\ &= \{A \in \mathcal{M}_{m\times m} : A^t A = I\} \equiv \text{group of orthogonal matrices} \end{aligned}$$

($\mathcal{M}_{m\times m}$ denotes the space of $(m \times m)$ real matrices). The determinant of an orthogonal matrix is ± 1 and these two possibilities correspond to the two connected components of $O(m)$. The subgroup of the orthogonal group formed by the matrices with determinant $+1$ is called the *special orthogonal group* and denoted by $SO(m)$. Both $O(m)$ and $SO(m)$ are Lie groups of dimension

$$\dim(O(m)) = \dim(SO(m)) = \frac{m(m-1)}{2} \,.$$

Example 2.1. The linear symmetries of $(\mathbb{R}^2, \langle \cdot, \cdot \rangle_0)$ with determinant $+1$ are the counter clockwise rotations. Hence, $SO(2)$ can be identified with the group of (2×2) real matrices of the form

$$\begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \,,\ t \in \mathbb{R}/2\pi\mathbb{Z} \cong S^1 \,.$$

This means that $SO(2)$ is isomorphic to the circle S^1, viewed as a connected compact 1-dimensional abelian Lie group.

Symplectic vector spaces. A *symplectic form* on a real vector space V is a bilinear map

$$\begin{aligned} \omega(\cdot, \cdot) : V \times V \ &\longrightarrow\ \mathbb{R} \\ (X, Y) \ &\longmapsto\ \omega(X, Y) \end{aligned}$$

that is skew-symmetric (i.e. $\omega(X, Y) = -\omega(Y, X)$, $\forall\, X, Y \in V$, or equivalently, $\omega(X, X) = 0$, $\forall\, X \in V$) and non-degenerate (i.e. $\forall\, 0 \neq X \in V\ \exists Y \in V :\ \omega(X, Y) \neq 0$).

A *symplectic vector space* is a pair (V, ω).

Given a basis for V, and hence an isomorphism with $\mathbb{R}^m$, any symplectic form ω can be written as

$$\omega(X, Y) = X^t B Y = [x^1 \cdots x^m]\,[b_{ij}]\,[y^1 \cdots y^m]^t = \sum_{i,j=1}^m x^i b_{ij} y^j \,,$$

where $b_{ij} = \omega(v_i, v_j)$, $\forall\, i, j = 1, \ldots, m$, and $B = [b_{ij}]$ is an $(m \times m)$ skew-symmetric and non-degenerate matrix representing ω in this basis.

The skew-symmetric analogue of the Gram-Schmidt process shows that there always exist *symplectic basis* for (V, ω), i.e. basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ $(m = 2n)$ such that $\omega(e_i, f_j) = \delta_{ij}$ and $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$. In such a basis, the matrix B representing ω is

$$\omega_0 \equiv \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \,,$$

where I is the $(n \times n)$ identity matrix. This means that any symplectic vector space (V, ω) is isomorphic to $(\mathbb{R}^{2n}, \omega_0)$ where, for $X = \sum_{i=1}^n (a^i e_i + b^i f_i)$ and $Y = \sum_{i=1}^n (c^i e_i + d^i f_i)$,

$$\omega_0(X, Y) = [a^1 \cdots a^n\ b^1 \cdots b^n] \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} [c^1 \cdots c^n\ d^1 \cdots d^n]^t = \sum_{i=1}^n (a^i d^i - b^i c^i) \,.$$

In particular,

a symplectic vector space is always even dimensional.

The *symplectic linear group* $Sp(V,\omega)$ is the group of linear symmetries of a symplectic vector space (V,ω). By the above, $Sp(V,\omega)$ is isomorphic to the group $Sp(2n)$ of linear symmetries of $(\mathbb{R}^{2n},\omega_0)$, i.e.

$$\begin{aligned} Sp(2n) &\cong \left\{A:\mathbb{R}^{2n}\to\mathbb{R}^{2n} : \omega_0(AX,AY)=\omega_0(X,Y)\,,\ \forall X,Y\in\mathbb{R}^{2n}\right\} \\ &= \left\{A\in\mathcal{M}_{2n\times 2n} : X^tA^t\omega_0AY = X^t\omega_0Y\,,\ \forall X,Y\in\mathbb{R}^{2n}\right\} \\ &= \{A\in\mathcal{M}_{2n\times 2n} : A^t\omega_0A=\omega_0\} \equiv \text{group of real symplectic matrices..} \end{aligned}$$

The determinant of a real symplectic matrix is always $+1$ and so $Sp(2n)$ is always a subgroup of $SL(2n)$. Its dimension is

$$\dim Sp(2n) = \frac{2n(2n+1)}{2} > \frac{2n(2n-1)}{2} = \dim O(2n)\,.$$

Example 2.2. In dimension 2 a symplectic form is the same as an area form. Hence, $Sp(2)$ can be identified with $SL(2)\equiv$ area preserving linear transfornations of $\mathbb{R}^2$, a group of dimension 3.

Euclidean gradient vector fields. Let us come back to the standard euclidean vector space $(\mathbb{R}^m,\langle\cdot,\cdot\rangle_0)$. The inner product $\langle\cdot,\cdot\rangle_0$ provides an isomorphism between R^m and its dual

$$(\mathbb{R}^m)^* \equiv \{\text{linear functionals } \alpha:\mathbb{R}^m\to\mathbb{R}\}$$

given by

$$\begin{aligned} \mathbb{R}^m &\longrightarrow (\mathbb{R}^m)^* \\ X &\longmapsto \alpha_X(\cdot)\equiv\langle X,\cdot\rangle_0\,. \end{aligned}$$

Considering $(\mathbb{R}^m,\langle\cdot,\cdot\rangle_0)$ as a Riemannian manifold, one can use the previous isomorphism at each point $p\in\mathbb{R}^m$ to identify the tangent space $T_p\mathbb{R}^m\cong\mathbb{R}^m$ with the cotangent space $T_p^*\mathbb{R}^m\cong(\mathbb{R}^m)^*$. This pointwise linear algebra isomorphism provides a global isomorphism between vector fields and one forms on $\mathbb{R}^m$:

$$\begin{aligned} \mathcal{X}(\mathbb{R}^m) &\longrightarrow \Omega^1(\mathbb{R}^m) \\ X &\longmapsto \alpha_X(\cdot)\equiv\langle X,\cdot\rangle_0\,. \end{aligned}$$

Let $h\in C^\infty(\mathbb{R}^m)$ be a smooth function on $\mathbb{R}^m$. Under the previous isomorphism, one can associate to the one form $dh\in\Omega^1(\mathbb{R}^m)$ a unique vector field $\nabla h\in\mathcal{X}(\mathbb{R}^m)$, characterized by the property

$$\langle\nabla h,Y\rangle_0 = dh(Y)\,,\ \forall Y\in\mathcal{X}(\mathbb{R}^m)\,.$$

This vector field ∇h is called the *euclidean gradient* of the function h. In standard linear coordinates $(x^1,\ldots,x^m)$ of $\mathbb{R}^m$ and with respect to the standard orthonormal basis of each $T_p\mathbb{R}^m\cong\mathbb{R}^m$, it is given by

$$\nabla h = \left(\frac{\partial h}{\partial x^1},\ldots,\frac{\partial h}{\partial x^m}\right)\,.$$

One of the most important properties of the euclidean gradient ∇h is the fact that it is always orthogonal to the level sets of the function h.

Symplectic gradient vector fields. On the standard symplectic vector space $(\mathbb{R}^{2n},\omega_0)$, the symplectic form ω_0 also provides an isomorphism between $\mathbb{R}^{2n}$ and its dual $\left(\mathbb{R}^{2n}\right)^*$, given by

$$\begin{aligned} \mathbb{R}^{2n} &\longrightarrow \left(\mathbb{R}^{2n}\right)^* \\ X &\longmapsto \alpha_X(\cdot)\equiv\omega_0(X,\cdot)\equiv(X\lrcorner\,\omega_0)\,. \end{aligned}$$

As before, one can use the previous isomorphism at each point $p\in\mathbb{R}^{2n}$ to identify the tangent space $T_p\mathbb{R}^{2n}\cong\mathbb{R}^{2n}$ with the cotangent space $T_p^*\mathbb{R}^{2n}\cong\left(\mathbb{R}^{2n}\right)^*$. This pointwise linear algebra isomorphism provides a global isomorphism between vector fields and one forms on $\mathbb{R}^{2n}$:

$$\begin{aligned} \mathcal{X}(\mathbb{R}^m) &\longrightarrow \Omega^1(\mathbb{R}^m) \\ X &\longmapsto \alpha_X(\cdot)\equiv\omega_0(X,\cdot)\equiv(X\lrcorner\,\omega_0)(\cdot)\,. \end{aligned}$$

Let $h \in C^\infty(\mathbb{R}^{2n})$ be a smooth function on $\mathbb{R}^{2n}$. Under the previous isomorphism, one can associate to the one form $dh \in \Omega^1\left(\mathbb{R}^{2n}\right)$ a unique vector field $X_h \in \mathcal{X}\left(\mathbb{R}^{2n}\right)$, characterized by the property

$$(X_h \lrcorner\, \omega_0)(\cdot) = dh(\cdot)\,.$$

This vector field X_h is called the *symplectic gradient* or *Hamiltonian vector field* of the function h. In standard linear coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n) \equiv (x, y)$ of $\mathbb{R}^{2n}$ and with respect to the standard symplectic basis of each $T_x\mathbb{R}^{2n} \cong \mathbb{R}^{2n}$, it is given by

$$X_h = \left(\frac{\partial h}{\partial y^1}, \ldots, \frac{\partial h}{\partial y^n}, -\frac{\partial h}{\partial x^1}, \ldots, -\frac{\partial h}{\partial x^n}\right) \equiv \left(\frac{\partial h}{\partial y}, -\frac{\partial h}{\partial x}\right)\,.$$

Any Hamiltonian vector field X_h has the following fundamental properties:

(i) it is always tangent to the level sets of the function h;

(ii) its flow, i.e. one parameter group of diffeomorphisms $(X_h)_t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, preserves the symplectic form ω_0.

We will come back to this property (ii) when we discuss general symplectic manifolds (see in particular Example 3.3).

Example. Consider the plane $\mathbb{R}^2$ equipped with the standard inner product $\langle \cdot, \cdot \rangle_0$ and symplectic form ω_0. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be the smooth function

$$h(x, y) = -\frac{1}{2}(x^2 + y^2)\,.$$

The level sets of this function are the circles centered at the origin. Its euclidean gradient vector field ∇h is given by

$$(\nabla h)_{p=(x,y)} = -(x, y)\,,$$

while its symplectic gradient or Hamiltonian vector field X_h is given by

$$(X_h)_{p=(x,y)} = (-y, x)\,.$$

The flow $(X_h)_t : \mathbb{R}^2 \to \mathbb{R}^2$ of X_h is given by

$$(X_h)_t(x_0, y_0) = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \equiv \text{ rotations of } \mathbb{R}^2.$$

One says that the function $h(x, y) = -(x^2 + y^2)/2$ generates in $\mathbb{R}^2$, through its Hamiltonian vector field X_h, an action of the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ that preserves the symplectic form, i.e. a Hamiltonian S^1-action. We will come back to this type of actions later on. Note that in this very particular case the S^1-action preserves not only the symplectic form ω_0, but also the inner product $\langle \cdot, \cdot \rangle_0$.

Hermitean vector spaces. Consider $\mathbb{R}^{2n}$ equipped with the standard inner product $\langle \cdot, \cdot \rangle_0$ and symplectic form $\omega_0(\cdot, \cdot)$, and recall that the connected groups of linear transformations that preserve each of these structures are

$$SO(2n) = \left\{A \in \mathcal{M}_{2n\times 2n} : A^t = A^{-1} \text{ and } \det(A) = +1\right\}$$

and

$$Sp(2n) = \left\{A \in \mathcal{M}_{2n\times 2n} : A^t\omega_0 A = \omega_0\right\}\,, \text{ with } \omega_0 = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}\,.$$

We then have that

$$A \in SO(2n) \cap Sp(2n) \Rightarrow A^{-1}\omega_0 A = \omega_0 \Rightarrow A \text{ preserves } \omega_0 \text{ as a linear map of } \mathbb{R}^{2n}.$$

We will consider instead the linear map $J_0 = -\omega_0$, i.e.

$$J_0 = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}\,,$$

which has the following important properties:

(i) $J_0^2 = -$ identity, i.e. J_0 gives $\mathbb{R}^{2n}$ the structure of a complex vector space;

(ii) J_0 relates the inner product $\langle \cdot, \cdot \rangle_0$ and symplectic form $\omega_0(\cdot, \cdot)$ through

$$\langle X, Y \rangle_0 = \omega_0(X, J_0 Y)\,,\ \forall\, X, Y \in \mathbb{R}^{2n}\,;$$

(iii) $A \in SO(2n) \cap Sp(2n)$ iff A is a complex linear map of $(\mathbb{R}^{2n}, J_0)$ that preserves the *Hermitean form*

$$h_0(\cdot,\cdot) = \langle\cdot,\cdot\rangle_0 + i\omega_0(\cdot,\cdot)\,.$$

This means that the *Hermitean vector space* $(\mathbb{R}^{2n}, J_0, h_0(\cdot,\cdot))$ can be seen as the compatible "intersection" of the euclidean vector space $(\mathbb{R}^{2n}, \langle\cdot,\cdot\rangle_0)$ and the symplectic vector space $(\mathbb{R}^{2n}, \omega_0(\cdot,\cdot))$. Moreover, there is a natural isomorphism

$$SO(2n) \cap Sp(2n) \cong U(n) \equiv \text{group of unitary linear maps.}$$

The inclusion $U(n) \subset Sp(2n)$. We will now derive one geometrical and one topological consequence of the inclusion $U(n) \subset Sp(2n)$, that will be relevant later on.

$U(n)$ is a maximal compact subgroup of $Sp(2n)$. This means in particular that a maximal abelian compact subgroup of $Sp(2n)$ is isomorphic to a maximal abelian compact subgroup of $U(n)$, and these are well known to be isomorphic to

$$\mathbb{T}^n \equiv S^1 \times \cdots \times S^1 \equiv n\text{-dimensional torus.}$$

The natural action of $\mathbb{T}^n$ on $\mathbb{R}^{2n}$ can be easily described by considering the factorization

$$\begin{array}{ccc} \mathbb{R}^{2n} & \cong & \mathbb{R}^2 \times \cdots \times \mathbb{R}^2 \\ (x,y) & \cong & (x^1,y^1) \times \cdots \times (x^n,y^n) \end{array}$$

Each S^1-factor of $\mathbb{T}^n$ acts on the corresponding $\mathbb{R}^2$-factor of $\mathbb{R}^{2n}$ by counter clockwise rotations, leaving the other $\mathbb{R}^2$-factors fixed. In fact, the action of the i-th S^1-factor on $\mathbb{R}^{2n}$ is generated by the function

$$\begin{array}{rcl} h_i : \mathbb{R}^{2n} & \to & \mathbb{R} \\ (x,y) & \mapsto & h_i(x,y) = -\frac{1}{2}\left((x^i)^2 + (y^i)^2\right) \end{array}$$

through the corresponding Hamiltonian vector field. One can put these n functions h_i together and get a map

$$\begin{array}{rcl} \mu : \mathbb{R}^{2n} & \to & \mathbb{R}^n \\ (x,y) & \mapsto & \mu(x,y) = (h_1(x,y), \ldots, h_n(x,y)) \end{array}$$

This map μ is a basic example of a moment map. It generates in this case the above Hamiltonian action of $\mathbb{T}^n$ on the symplectic vector space $(\mathbb{R}^{2n}, \omega_0)$.

The quotient space $Sp(2n)/U(n)$ is a contractible symmetric space of dimension $n(n+1)$, known as the Siegel upper half space [21]. It generalizes the usual upper half plane ($n = 1$), and can be identified with the following space:

$$\begin{array}{rcl} \mathcal{J}(\mathbb{R}^{2n}, \omega_0) & = & \{J : \mathbb{R}^{2n} \to \mathbb{R}^{2n} : J \text{ linear}, J^2 = -\text{ id. and } \langle\cdot,\cdot\rangle \equiv \omega_0(\cdot, J\cdot) \text{ is an inner product}\} \\ & \equiv & \text{linear complex structures on } \mathbb{R}^{2n} \text{ compatible with } \omega_0. \end{array}$$

In fact, this space $\mathcal{J}(\mathbb{R}^{2n}, \omega_0)$ has the following properties:
(i) $J_0 \in \mathcal{J}(\mathbb{R}^{2n}, \omega_0)$;
(ii) if $J \in \mathcal{J}(\mathbb{R}^{2n}, \omega_0)$ then $(A^{-1}JA) \in \mathcal{J}(\mathbb{R}^{2n}, \omega_0)$, for any $A \in Sp(2n)$;
(iii) it follows from (ii) that $Sp(2n)$ acts on $\mathcal{J}(\mathbb{R}^{2n}, \omega_0)$, with isotropy at J_0 given by $U(n)$;
(iv) this action of $Sp(2n)$ on $\mathcal{J}(\mathbb{R}^{2n}, \omega_0)$ is transitive (hermitean analogue of the Gram-Schmidt process).

It follows from (iii) and (iv) that

$$Sp(2n)/U(n) \cong \mathcal{J}(\mathbb{R}^{2n}, \omega_0) \text{ and so, in particular, } \mathcal{J}(\mathbb{R}^{2n}, \omega_0) \text{ is always contractible.}$$

3. Symplectic Manifolds

In this section we discuss some basic features of symplectic manifolds and their symplectomorphism groups.

Definition and basic examples.

Definition 3.1. *A* symplectic manifold *is a pair* (M,ω) *where* M *is a smooth manifold and* ω *is a closed and non-degenerate* 2*-form, i.e.*

(i) $\omega \in \Omega^2(M)$ *is such that* $d\omega = 0$ *and*
(ii) *for any* $p \in M$ *and* $0 \neq X \in T_pM$*, there exists* $Y \in T_pM$ *such that* $\omega_p(X,Y) \neq 0$*.*

As in the linear case, the non-degeneracy condition (ii) implies that a symplectic manifold is always even dimensional. If M has dimension $2n$, the non-degeneracy condition (ii) is equivalent to requiring that

$$\omega^n \equiv \omega \wedge \cdots \wedge \omega \in \Omega^{2n}(M) \text{ is a volume form.}$$

Hence, a symplectic manifold (M,ω) is always oriented.

Any 2-dimensional surface equipped with an area form is a symplectic manifold. For example, the sphere S^2 or any other compact orientable surface Σ_g of genus g.

If (M_1,ω_1) and (M_2,ω_2) are symplectic manifolds, then $(M = M_1 \times M_2, \omega = \omega_1 \times \omega_2)$ is also a symplectic manifold ($\omega_1 \times \omega_2$ means the sum of the pullbacks of the symplectic forms ω_1 and ω_2 from the factors M_1 and M_2).

The imaginary part of the hermitean metric on any Kähler manifold is a symplectic form. Hence, any Kähler manifold is a symplectic manifold. In particular, the complex projective space $\mathbb{CP}^n$ equipped with its Fubini-Study form ω_{FS} is a symplectic manifold.

When (M,ω) is a compact symplectic manifold we have that

$$\omega^n = \text{volume form} \Rightarrow 0 \neq [\omega^n] \in H^{2n}(M,\mathbb{R}) \Rightarrow 0 \neq [\omega] \in H^2(M,\mathbb{R})\,.$$

In particular, the spheres S^{2n} have no symplectic form when $n > 1$, since $H^2(S^{2n},\mathbb{R}) = 0$ when $n > 1$.

Symplectomorphisms and Darboux's Theorem.

Definition 3.2. *Let* (M,ω) *be a symplectic manifold. A* symplectomorphism *of* M *is a diffeomorphism* $\varphi : M \to M$ *such that* $\varphi^*(\omega) = \omega$*. These form the* symplectomorphism group*, a subgroup of* $\mathrm{Diff}(M)$ *that will be denoted by* $\mathrm{Diff}(M,\omega)$*.*

Example 3.3. Consider a symplectic manifold (M,ω) and let $h : M \to \mathbb{R}$ be a smooth function on M. The non-degeneracy of ω implies that there exists a unique vector field $X_h \in \mathcal{X}(M)$ such that $X_h \lrcorner\, \omega = dh$. As in the particular case of $(M = \mathbb{R}^{2n}, \omega = \omega_0)$, this vector field X_h is called the *symplectic gradient* or *Hamiltonian vector field* of the function h and has the following fundamental property:

$$\text{the flow } \varphi_t \equiv (X_h)_t : M \to M \text{ consists of symplectomorphisms of } M.$$

This can be proved using Cartan's formula to compute

$$\mathcal{L}_{X_h}\omega = X_h \lrcorner\, d\omega + d(X_h \lrcorner\, \omega) = X_h \lrcorner\, 0 + d(dh) = 0\,.$$

Hence, on a symplectic manifold (M,ω) any smooth function $h \in C^\infty(M)$ gives rise, through the flow of the corresponding Hamiltonian vector field $X_h \in \mathcal{X}(M)$, to a 1-parameter group of symplectomorphisms.

One can use the symplectomorphisms constructed in the previous example to prove that:

(i) the symplectomorphism group $\mathrm{Diff}(M,\omega)$ is always infinite-dimensional;
(ii) the action of $\mathrm{Diff}(M,\omega)$ on the manifold M is always k-transitive, for any $k \in \mathbb{N}$;
(iii) in particular, any point of a symplectic manifold (M,ω) looks locally like any other point of (M,ω).

This last statement is made more precise in the following

Theorem 3.4. [Darboux] *Let* (M,ω) *be a symplectic manifold of dimension* $2n$*. Then, any point* $p \in M$ *has a neighborhood* $U \subset M$ *symplectomorphic to a neighborhood* V *of the origin in* $(\mathbb{R}^{2n}, \omega_0)$*, i.e.*

$$\textit{there exists a diffeomorphism } \phi : U \subset M \to V \subset \mathbb{R}^{2n} \textit{ such that } \phi(p) = 0 \textit{ and } \phi^*(\omega_0) = \omega.$$

In other words,

there are no local invariants in symplectic geometry,

which is in sharp contrast with what happens, for example, in Riemannian geometry.

Symplectic and Hamiltonian vector fields. The Lie algebra of the symplectomorphism group $\mathrm{Diff}(M,\omega)$, viewed as an infinite-dimensional Lie group, is naturally identified with the vector space $\mathcal{X}(M,\omega)$ of *symplectic vector fields*, i.e. vector fields $X \in \mathcal{X}(M)$ such that $\mathcal{L}_X\omega = 0$, with Lie bracket $[\cdot,\cdot]$ given by the usual Lie bracket of vector fields. As before, we can use Cartan's formula to obtain

$$\mathcal{L}_X\omega = X\lrcorner\, d\omega + d(X\lrcorner\,\omega) = X\lrcorner\, 0 + d(X\lrcorner\,\omega) = d(X\lrcorner\,\omega)\,.$$

Hence, the vector space of symplectic vector fields is given by

$$\mathcal{X}(M,\omega) = \{X \in \mathcal{X}(M) \,:\, \text{the 1-form } X\lrcorner\,\omega \text{ is closed}\}\ ,$$

while its subspace of *Hamiltonian vector fields* is given by

$$\mathcal{X}_H(M,\omega) = \{X \in \mathcal{X}(M) \,:\, \text{the 1-form } X\lrcorner\,\omega \text{ is exact}\}\ .$$

In fact, as the following theorem shows, $\mathcal{X}_H(M,\omega)$ is a Lie subalgebra of $\mathcal{X}(M,\omega)$.

Theorem 3.5. *If $X, Y \in \mathcal{X}(M,\omega)$ are symplectic vector fields, then $[X,Y]$ is the Hamiltonian vector field of the function $\omega(Y,X) : M \to \mathbb{R}$, i.e.*

$$[X,Y] = X_{\omega(Y,X)} \in \mathcal{X}_H(M,\omega)\,.$$

Proof. It suffices to compute $[X,Y]\lrcorner\,\omega$, using standard formulas from differential geometry and the defining properties of X, Y and ω:

$$\begin{aligned}[X,Y]\lrcorner\,\omega &= \mathcal{L}_X(Y\lrcorner\,\omega) - Y\lrcorner\,(\mathcal{L}_X\omega)\\ &= d(X\lrcorner\,(Y\lrcorner\,\omega)) + X\lrcorner\,(d(Y\lrcorner\,\omega)) - Y\lrcorner\,(d(X\lrcorner\,\omega)) - Y\lrcorner\,(X\lrcorner\, d\omega)\\ &= d(\omega(Y,X))\,.\end{aligned}$$

□

Remark 3.6. *$\mathcal{X}_H(M,\omega)$ is the Lie algebra of a fundamental subgroup of the symplectomorphism group: the subgroup* $\mathrm{Ham}(M,\omega) \subset \mathrm{Diff}(M,\omega)$ *of* Hamiltonian symplectomorphisms *of* (M,ω)*. It follows from Theorem 3.5 that this Lie algebra can be naturally identified with the vector space $C^\infty(M)/\mathbb{R}$, i.e. smooth functions on M modulo constants, equipped with a bracket $\{\cdot,\cdot\}$ known as the Poisson bracket:*

$$\{f,g\} \equiv \omega(X_f, X_g)\,.$$

Note that when $H^1(M,\mathbb{R}) = 0$ we have that $\mathcal{X}_H(M,\omega) = \mathcal{X}(M,\omega)$.

Moser's Theorem. The following theorem shows that, locally in the space of closed non-degenerate 2-forms, the only invariant of a symplectic structure is its cohomology class.

Theorem 3.7. [Moser] *Let M be a compact manifold and ω_t, $0 \le t \le 1$, a smooth family of symplectic forms with fixed cohomology class, i.e. $\frac{d}{dt}[\omega_t] = 0$ in $H^2(M,\mathbb{R})$. Then, there exists a smooth family of diffeomorphisms of $\varphi_t : M \to M$, such that $\varphi_0 =$ identity and $\varphi_t^*(\omega_t) = \omega_0$, $\forall\, t \in [0,1]$.*

Proof. The argument we will use to prove this theorem, one of the two presented by Moser in his original paper [20], is very useful. It is known as Moser's method or trick.

We first observe that

$$\frac{d}{dt}[\omega_t] = 0 \Rightarrow \left[\frac{d\omega_t}{dt}\right] = 0 \Rightarrow \exists\,\text{smooth family } \alpha_t \in \Omega^1(M) \text{ s.t. } \frac{d\omega_t}{dt} = d\alpha_t\ ,\ \forall\, t \in [0,1]\,.$$

We now look for a family of vector fields $X_t \in \mathcal{X}(M)$, that will generate the required family of diffeomorphisms φ_t through their (non-autonomous) flow, i.e. the family of diffeomorphisms φ_t is uniquely determined by

$$\varphi_0 = \text{identity and } \frac{d\varphi_t}{dt} = X_t\,.$$

It turns out that the requirement $\varphi_t^*(\omega_t) = \omega_0$ and the choice we already made of the family of primitive 1-forms α_t, uniquely determines the family of vector fields X_t. In fact,

$$\varphi_t^*(\omega_t) = \omega_0 \Rightarrow \varphi_t^* \left(\mathcal{L}_{X_t}\omega_t + \frac{d\omega_t}{dt} \right) = 0 \Rightarrow \varphi_t^* \left[d(X_t \lrcorner \omega_t + \alpha_t) \right] = 0 \,,$$

where the second equality follows by differentiating the first with respect to t. The non-degeneracy of ω_t implies that there exists a unique vector field $X_t \in \mathcal{X}(M)$ such that $X_t \lrcorner \omega_t = -\alpha_t$, $\forall\, t \in [0,1]$. To finish the proof, one checks that the family of diffeomorphisms determined by the flow of this family of vector fields does have the required properties. □

Moser's theorem is also true, and can be proved in exactly the same way, if one replaces the smooth family of symplectic forms ω_t by a smooth family of volume forms σ_t. Since the space of volume forms compatible with a given orientation is always convex, one gets the following

Corollary 3.8. [Moser] *For any compact oriented manifold M with volume form σ, the group of orientation preserving diffeomorphisms $\mathrm{Diff}^+(M)$ deformation retracts to its subgroup $\mathrm{Diff}(M,\sigma)$ of volume preserving ones.*

The symplectomorphism group of S^2. Since in dimension 2 a symplectic form and a volume form are the same thing, one gets from Corollary 3.8 that the symplectomorphism group $\mathrm{Diff}(\Sigma,\sigma)$ of any symplectic surface (Σ,σ) is homotopy equivalent to the corresponding group $\mathrm{Diff}^+(\Sigma)$ of orientation preserving diffeomorphisms:

$$\mathrm{Diff}(\Sigma,\sigma) \sim \mathrm{Diff}^+(\Sigma)\,.$$

The topology of these later groups is well known. One has in particular the following

Theorem 3.9. (Smale [22]) *The group of orientation preserving diffeomorphisms of the 2-sphere S^2 is homotopy equivalente to its subgroup of standard isometries, i.e*

$$\mathrm{Diff}^+(S^2) \sim SO(3)\,.$$

Proof. The proof we present here follows one of the ideas used by Gromov in [14] for the proof of Theorem 6.2, and can be seen as a warm-up to that proof.

Let $\mathcal{J}^+(S^2)$ denote the contractible space of all complex structures on S^2 that are compatible with the given orientation. It follows from the uniformization theorem, a standard fact from complex analysis, that $\mathrm{Diff}^+(S^2)$ acts transitively on $\mathcal{J}^+(S^2)$. The isotropy of this action at any complex structure $j \in \mathcal{J}^+(S^2)$ is isomorphic to $SL(2,\mathbb{C})$. Hence

$$\mathrm{Diff}^+(S^2)/SL(2,\mathbb{C}) \cong \mathcal{J}^+(S^2) \sim \text{point} \Rightarrow \mathrm{Diff}^+(S^2) \sim SL(2,\mathbb{C}) \sim SO(3)\,.$$

□

4. Hamiltonian Torus Actions

In this section we define Hamiltonian torus actions on symplectic manifolds and present the Atiyah-Guillemin-Sternberg's Convexity Theorem for their moment maps. We also briefly discuss symplectic toric manifolds and their classification due to Delzant.

Definition and basic examples. Let (M,ω) be a symplectic manifold equipped with a symplectic action of

$$\mathbb{T}^m \equiv \mathbb{R}^m/2\pi\mathbb{Z}^m \equiv \mathbb{R}/2\pi\mathbb{Z} \times \cdots \times \mathbb{R}/2\pi\mathbb{Z} \equiv S^1 \times \cdots \times S^1\,,$$

i.e. with a homomorphism $\mathbb{T}^m \to \mathrm{Diff}(M,\omega)$. Let $X_1,\ldots,X_m \in \mathcal{X}(M)$ be the vector fields generating the action of each individual S^1-factor. Then, since the action is symplectic, we have that

$$\mathcal{L}_{X_k}\omega = 0 \Leftrightarrow X_k \lrcorner\, d\omega + d(X_k \lrcorner\, \omega) = 0 \Leftrightarrow d(X_k \lrcorner\, \omega) = 0\,,\ \text{i.e. } X_k \in \mathcal{X}(M,\omega)\,,\ \forall\, k \in \{1,\ldots,m\}\ .$$

Definition 4.1. *A symplectic $\mathbb{T}^m$-action on a symplectic manifold (M,ω) is said to be* Hamiltonian *if for every $k \in \{1,\ldots,m\}$ there exists a function $h_k : M \to \mathbb{R}$ such that $X_k \lrcorner \omega = dh_k$, i.e. $X_k \equiv X_{h_k} \in \mathcal{X}_H(M,\omega)$ is the Hamiltonian vector field of h_k. In this case, the map $\mu : M \to \mathbb{R}^m$ defined by*

$$\mu(p) = (h_1(p),\ldots,h_m(p)) \,, \ \forall\, p \in M \,,$$

is called a moment map *for the action.*

Remark 4.2. *Suppose $\mu : M \to \mathbb{R}^m$ is a moment map for a Hamiltonian $\mathbb{T}^m$-action on (M,ω). Then $\mu + c$, for any given constant $c \in \mathbb{R}^m$, is also a moment map for that same action.*

Remark 4.3. *The orbits of a Hamiltonian $\mathbb{T}^m$-action on a symplectic manifold (M,ω) are always* isotropic, *i.e.*

$$\omega|_{orbit} \equiv 0 \,.$$

In fact, the tangent space to an orbit is generated by the Hamiltonian vector fields X_{h_k}, $k \in \{1,\ldots,m\}$. Using Theorem 3.5 and the fact that the torus $\mathbb{T}^m$ is abelian, we have that

$$X_{\omega(X_{h_k},X_{h_l})} = -\,[X_{h_k},X_{h_l}] \equiv 0 \Rightarrow \omega(X_{h_k},X_{h_l}) \equiv constant \,, \ \forall\, k,l \in \{1,\ldots,m\} \,.$$

Since $\mathbb{T}^m$ is compact, there is for each $k \in \{1,\ldots,m\}$ and on each $\mathbb{T}^m$-orbit a point p_k where the function $h_k|_{orbit}$ attains its maximum. Then

$$\omega(X_{h_k},X_{h_l}) = (dh_k)_{p_k}(X_{h_l}) = 0 \,.$$

Hence, the above constant is actually zero and each $\mathbb{T}^m$-orbit is indeed isotropic. This fact will be used below, in the proof of Proposition 4.8.

Example 4.4. As we already saw when discussing the inclusion $U(n) \subset Sp(2n)$, the standard $\mathbb{T}^n$-action on $(\mathbb{R}^{2n},\omega_0)$ is Hamiltonian with moment map $\mu : \mathbb{R}^{2n} \to \mathbb{R}^n$ given by

$$\mu(x_1,\ldots,x_n,y_1,\ldots,y_n) = -\frac{1}{2}(x_1^2+y_1^2,\ldots,x_n^2+y_n^2) \,.$$

Example 4.5. Consider the 2-sphere

$$S^2 = \left\{(x,y,z) \in \mathbb{R}^3 \ : \ x^2+y^2+z^2 = 1\right\} \subset \mathbb{R}^3$$

with symplectic or area form σ induced by the standard euclidean inner product in $\mathbb{R}^3$. The height function $h : S^2 \to \mathbb{R}$, given by $h(x,y,z) = z$, generates through its Hamiltonian vector field X_h the rotations of S^2 around its vertical axis. Hence, this is an example of a Hamiltonian S^1-action on (S^2,σ) with moment map $\mu \equiv h$.

In the last 25 years an incredible amount of research has been devoted to the study of moment maps and their beautiful geometric properties. We will now present two of these.

Atiyah-Guillemin-Sternberg's Convexity Theorem. Atiyah [6] and Guillemin-Sternberg [15] proved in 1982 the following Convexity Theorem.

Theorem 4.6. *Let (M,ω) be a compact, connected, symplectic manifold, equipped with a Hamiltonian $\mathbb{T}^m$-action with moment map $\mu : M \to \mathbb{R}^m$. Then*
(i) *the level sets $\mu^{-1}(\lambda)$ of the moment map are connected (for any $\lambda \in \mathbb{R}^m$);*
(ii) *the image $\mu(M) \subset \mathbb{R}^m$ of the moment map is the convex hull of the images of the fixed points of the action.*

The image $\mu(M) \subset \mathbb{R}^m$ of the moment map is called the *moment polytope*.

Example 4.7. In Example 4.5, the fixed points of the S^1-action are the poles $S = (0,0,-1)$ and $N = (0,0,1)$ of the 2-sphere $S^2 \subset \mathbb{R}^3$. The images of these fixed points under the moment map are $\mu(S) = -1$ and $\mu(N) = 1$, while the moment polytope is $\mu(S^2) = [-1,1] \subset \mathbb{R}$.

Symplectic toric manifolds. The following proposition motivates the definition of a symplectic toric manifold.

Proposition 4.8. *If a symplectic manifold* (M, ω) *has an effective Hamiltonian* $\mathbb{T}^m$*-action, then* $m \leq (\dim M)/2$.

Proof.

Effective action	$\Rightarrow$	there exist m-dimensional orbits.
Hamiltonian $\mathbb{T}^m$-action	$\Rightarrow$	orbits are isotropic (see Remark 4.3).
Linear Algebra	$\Rightarrow$	$\dim(\text{isotropic orbit}) \leq \frac{1}{2}\dim M$.

□

Definition 4.9. *A* symplectic toric manifold *is a connected symplectic manifold* (M, ω) *of dimension* $2n$ *equipped with an effective Hamiltonian* $\mathbb{T}^n$*-action.*

Example 4.10. (S^2, σ), with the S^1-action decribed in Example 4.5, is the simplest compact symplectic toric manifold.

Example 4.11. $(\mathbb{R}^{2n}, \omega_0)$, with its standard Hamiltonian $\mathbb{T}^n$-action (Example 4.4), is a non-compact symplectic toric manifold.

Example 4.12. The $\mathbb{T}^n$-action on $(\mathbb{CP}^n, \omega_{FS})$ given in homogeneous coordinates by

$$(\theta_1, \ldots, \theta_n) \cdot [z_0; z_1; \ldots; z_n] = \left[z_0; e^{i\theta_1} z_1; \ldots; e^{i\theta_n} z_n\right]$$

is Hamiltonian, with moment map $\mu : \mathbb{CP}^n \to \mathbb{R}^n$ given by

$$\mu\left([z_0; z_1; \ldots; z_n]\right) = -\frac{1}{\|z\|^2}\left(\|z_1\|^2, \ldots, \|z_n\|^2\right)$$

where $\|z\|^2 = \|z_0\|^2 + \|z_1\|^2 + \cdots + \|z_n\|^2$. Hence, $(\mathbb{CP}^n, \omega_{FS})$ equipped with this $\mathbb{T}^n$-action is a compact symplectic toric manifold. Note that its moment polytope is a simplex in $\mathbb{R}^n$.

It follows from Theorem 4.6 that any compact symplectic toric manifold has an associated convex polytope, the moment polytope of the torus action. In 1988 Delzant [12] characterized the convex polytopes that arise as moment map images of compact symplectic toric manifolds, and showed that any such convex polytope determines a unique compact symplectic toric manifold. More precisely, if two compact symplectic toric manifolds have the same moment polytope, then there exists an equivariant symplectomorphism between them. Delzant's result can be summarized in the following theorem.

Theorem 4.13. [Delzant] *The moment polytope is a complete invariant of a compact symplectic toric manifold.*

5. Pseudo-holomorphic Curves in Symplectic Manifolds

In this section we define compatible almost complex structures on symplectic manifolds and discuss Gromov's Compactness Theorem for their pseudo-holomorphic curves. We will also present two special geometric properties that pseudo-holomorphic curves have in dimension 4. For a further understanding of pseudo-holomorphic curves and its applications to symplectic topology see [19], where you can also find a detailed list of references.

Almost complex manifolds. Recall that an *almost complex manifold* is a pair (M, J), where M is a smooth manifold and $J : TM \to TM$ is an endomorphism of its tangent bundle such that

$$J_p^2 = -\mathrm{id}_p : T_pM \to T_pM\,,\ \forall\, p \in M\,.$$

The condition $J^2 = -\mathrm{id}$ implies that the dimension of any almost complex manifold is even.

Any *complex manifold* is an almost complex manifold with an integrable almost complex structure. In real dimension 2, any almost complex manifold is a complex curve, i.e. a Riemann surface (Σ, j). In higher dimensions a generic almost complex structure is non-integrable. However, it is quite hard to decide when a given almost complex manifold does not have any integrable almost complex structure. As far as I know, this question is completely open in dimension greater or equal to six, in particular for S^6 (which is the only almost complex sphere other than S^2). In dimension four it is known for example that

$$\mathbb{CP}^2 \sharp \mathbb{CP}^2 \sharp \mathbb{CP}^2$$

is an almost complex manifold with no integrable almost complex structure, but one still needs to use Kodaira's classification of compact complex surfaces to prove it.

Pseudo-holomorphic curves.

Definition 5.1. *A* parametrized pseudo-holomorphic curve *is a map from a Riemann surface to an almost complex manifold,*

$$f : (\Sigma, j) \to (M, J)\,,$$

such that

$$df \circ j = J \circ df\,. \tag{1}$$

Its image $C = f(\Sigma)$ is an unparametrized pseudo-holomorphic curve*. We might also write J-holomorphic curve, instead of pseudo-holomorphic curve.*

Remark 5.2.
(i) *The Cauchy-Riemann equation (1) gives rise to a quasi-linear first order elliptic system of PDE's with good analytical properties: smooth solutions, removal of singularities, etc.*
(ii) *Any immersed real 2-dimensional surface $C \looparrowright (M, J)$ with complex tangent space, i.e. such that $J(TC) = TC$, is an unparametrized pseudo-holomorphic curve. For example, any complex curve in a complex manifold.*

Compatible almost complex structures.

Definition 5.3. *A* compatible almost complex structure *on a symplectic manifold (M, ω) is an almost complex structure J on M such that*

$$\langle \cdot, \cdot \rangle_J \equiv \omega(\cdot, J\cdot)$$

is a Riemannian metric on M. This is equivalent to $\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot)$ and $\omega(X, JX) > 0\,,\ \forall\, 0 \neq X \in TM$.

The space of all compatible almost complex structures on a symplectic manifold (M, ω) will be denoted by $\mathcal{J}(M, \omega)$.

Remark 5.4.
(i) *$\mathcal{J}(M, \omega)$ is non-empty, infinite-dimensional and contractible, for any symplectic manifold (M, ω). In particular, any (M, ω) has well defined Chern classes $c_k \equiv c_k(M, \omega) \in H^{2k}(M, \mathbb{Z})$.*
(ii) *A* Kähler manifold *is a symplectic manifold (M, ω) with an integrable compatible complex structure J.*

Gromov's Compactness Theorem. Let (M, ω) be a symplectic manifold and $J \in \mathcal{J}(M, \omega)$ a compatible almost complex structure. We will now illustrate how the compatibility condition provides enough apriori geometric control on pseudo-holomorphic curves for a satisfactory global theory to hold.

Fix an homology class $A \in H_2(M, \mathbb{Z})$ and let $f : (\Sigma, j) \to (M, J)$ be a closed pseudo-holomorphic curve whose image $C = f(\Sigma) \subset M$ represents A, i.e. such that

$$[C] = f_* \left([\Sigma]\right) = [A] \in H_2(M, \mathbb{Z})\,.$$

Proposition 5.5. *The area of C with respect to $\langle \cdot, \cdot \rangle_J \equiv \omega(\cdot, J\cdot)$ is given by*

$$\mathit{area}(C) = \int_C \omega = [\omega]\,(A)\,,$$

and is minimal within the class of submanifolds of M representing the homology class A.

Proof. This result follows from the following linear algebra inequality, known as Wirtinger's inequality:

$$\omega(X,Y) = \langle JX, Y\rangle \leq \|JX\|\|Y\| = \|X\|\|Y\|\,, \text{ with equality iff } Y \text{ is a multiple of } JX.$$

□

Hence, on a symplectic manifold with a compatible almost complex structure,

homology controls the area of pseudo-holomorphic curves.

In 1985, Gromov [14] realized that this fact could be used to prove a compactness theorem for pseudo-holomorphic curves. The following is a simple version of that theorem, that is however enough for the applications we will discuss in these lectures.

Theorem 5.6. [Gromov] *Let (M,ω) be a compact symplectic manifold and*

$$f_n : (\Sigma, j) \to (M, J_n)$$

a sequence of pseudo-holomorphic curves representing a fixed homology class $0 \neq A \in H_2(M,\mathbb{Z})$. Assume that Σ is closed, $J_n \in \mathcal{J}(M,\omega)$ and there exists a $J_\infty \in \mathcal{J}(M,\omega)$ such that $J_n \to J_\infty$ smoothly. Then,

(i) *either there exists a subsequence of f_n that smoothly converges to a pseudo-holomorphic curve*

$$f_\infty : (\Sigma, j) \to (M, J_\infty)$$

also representing the homology class A (if Σ contains spherical components, the non-compact reparametrization group $SL(2,\mathbb{C})$ needs to be taken into account here);

(ii) *or there exists a J_∞-holomorphic spherical "bubble", i.e. the class A can be written as*

$$A = B + A' \in H_2(M,\mathbb{Z})\,,$$

where $0 \neq B \in H_2(M,\mathbb{Z})$ can be represented by a J_∞-holomorphic sphere (the "bubble") and $0 \neq A' \in H_2(M,\mathbb{Z})$ can be represented by a J_∞-holomorphic curve.

The positivity in the compatibility condition between J and ω implies that a necessary condition for a homology class $A \in H_2(M,\mathbb{Z})$ to be represented by a J-holomorphic curve is

$$[\omega](A) > 0\,.$$

This simple fact implies the following corollary to Theorem 5.6.

Corollary 5.7. *If the homology class $A \in H_2(M,\mathbb{Z})$ cannot be written as $A = A_1 + A_2$, where $[\omega](A_i) > 0\,,\ i = 1,2$, and either A_1 or A_2 is a spherical class, then (i) in Theorem 5.6 holds and no "bubbles" can occur.*

Properties of pseudo-holomorphic curves in dimension 4**.** It turns out that in dimension four homology controls important geometric properties of pseudo-holomorphic curves. These will be used later on.

Theorem 5.8. [Positivity of Intersections] *Two distinct closed J-holomorphic curves C and C' in an almost complex 4-manifold (M,J) have only a finite number of intersection points. Each such point $x \in C \cap C'$ contributes a number $k_x \geq 1$ to the algebraic intersection number $[C]\cdot[C'] \in \mathbb{Z}$. Moreover, $k_x = 1$ if and only if the curves C and C' intersect transversally at x.*

Remark 5.9.

(i) *This Theorem implies that two J-holomorphic curves always intersect positively. This is not true for intersections of symplectic submanifolds. For example, one can easily find two symplectic planes in $(\mathbb{R}^4, \omega_0)$ with a negative transversal intersection at the origin.*

(ii) *This Theorem also implies that if C and C' are distinct J-holomorphic curves then*

$$[C]\cdot[C'] = 0 \Rightarrow C \text{ and } C' \text{ are disjoint.}$$

Let $f : (\Sigma, j) \to (M^4, J)$ be a pseudo-holomorphic curve, $g \equiv$ genus of Σ and $C = f(\Sigma) \subset M$. Assume that f is somewhere injective, i.e.

$$\text{there exists } z \in \Sigma \text{ such that } df(z) \neq 0 \text{ and } f^{-1}(f(z)) = \{z\}.$$

This condition avoids the trivial constant pseudo-holomorphic curve and multiple coverings of a fixed pseudo-holomorphic map.

Definition 5.10. *The* virtual genus *of C is defined to be the number*

$$g(C) = 1 + \frac{1}{2}\left([C] \cdot [C] - c_1([C])\right),$$

where $c_1 \in H^2(M, \mathbb{Z})$ is the first Chern class of the complex vector bundle (TM, J) over M.

Theorem 5.11. [Adjunction Formula] *The virtual genus $g(C)$ is an integer. Moreover,*

$$g(C) \geq g$$

with equality iff C is embedded.

6. The Symplectomorphism Groups of $S^2 \times S^2$

In this section we show how pseudo-holomorphic curves can be used to study the topology of the symplectomorphism groups of $S^2 \times S^2$.

Pseudo-holomorphic spheres in $S^2 \times S^2$. Consider the symplectic manifold $(S^2 \times S^2, \omega = \sigma \times \sigma)$. Let A and B denote the homology classes

$$A = \left[S^2 \times \{p\}\right] \quad \text{and} \quad B = \left[\{p\} \times S^2\right] \quad \text{in} \quad H_2(S^2 \times S^2, \mathbb{Z}).$$

Proposition 6.1. *For any compatible almost complex structure $J \in \mathcal{J}(S^2 \times S^2, \omega)$ and any point $p \in S^2 \times S^2$, there exist a J-holomorphic sphere representing the homology class A and a J-holomorphic sphere representing the homology class B, both passing through the point p.*

Proof. The subset of $\mathcal{J}(S^2 \times S^2, \omega)$ formed by compatible almost complex structures for which the statement of the proposition is true is:

(i) non-empty, since it contains the standard split complex structure $J_0 = j_0 \times j_0$;
(ii) open, because the pseudo-holomorphic equation (1) is elliptic;
(iii) closed, by Gromov's compactness Theorem 5.6 and Corollary 5.7.

□

If we consider on $S^2 \times S^2$ the split symplectic form

$$\omega_\lambda = \lambda\sigma \times \sigma \quad \text{with} \quad 1 < \lambda \in \mathbb{R},$$

the homology class A no longer satisfies the conditions of Corollary 5.7 (although the smaller class B still does). In fact, A can be written as

$$A = (A - B) + B, \text{ with } [\omega_\lambda](A - B) = \lambda - 1 > 0 \text{ and } [\omega_\lambda](B) = 1 > 0.$$

Moreover, the anti-diagonal

$$\overline{D} = \left\{(p, -p) : p \in S^2\right\} \subset S^2 \times S^2$$

is an embedded symplectic sphere representing the homology class $(A - B)$ and one can construct compatible almost complex structures $J \in \mathcal{J}(S^2 \times S^2, \omega_\lambda)$ that make it J-holomorphic. Hence, there are compatible almost complex structures $J \in \mathcal{J}(S^2 \times S^2, \omega_\lambda)$ for which the homology class $(A - B)$ is represented by a J-holomorphic sphere. Note that by positivity of intersections, Theorem 5.8, and since

$$(A - B) \cdot A = A \cdot A - B \cdot A = 0 - 1 = -1 < 0,$$

whenever there is a J-holomorphic sphere representing the class $(A-B)$ there is no J-holomorphic sphere representing the class A. This means that Proposition 6.1 does not hold as stated for the symplectic manifold $(S^2 \times S^2, \omega_\lambda = \lambda\sigma \times \sigma)$, $1 < \lambda \in \mathbb{R}$, and the homology class A (although it does hold for the smaller class B).

As we will see, this change in the structure of pseudo-holomorphic spheres, that arises from a variation of the symplectic form, is related to a change in the topology of the corresponding symplectomorphism groups.

Gromov's Theorem. Let G denote the subgroup of the symplectomorphism group $\mathrm{Diff}(S^2 \times S^2, \omega = \sigma \times \sigma)$ that consists of symplectomorphisms that act as the identity in $H_2(S^2 \times S^2, \mathbb{Z})$. Note that the full symplectomorphism group $\mathrm{Diff}(S^2 \times S^2, \omega)$ is a simple $\mathbb{Z}_2$-extension of G.

Theorem 6.2. (Gromov [14]) *G is homotopy equivalent to its subgroup of standard isometries of $S^2 \times S^2$, i.e.*

$$G \sim SO(3) \times SO(3)\,.$$

Proof. To simplify notation, denote by $\mathcal{J}$ the contractible space of compatible almost complex structures $\mathcal{J}(S^2 \times S^2, \omega)$. As in the proof of Smale's Theorem 3.9, G acts on $\mathcal{J}$ by conjugation, with isotropy at $J_0 = j_0 \times j_0$ given by $SO(3) \times SO(3)$, but now this action is no longer transitive. For example, one cannot send an integrable compatible almost complex structure to a non-integrable one. This means that $G/SO(3) \times SO(3) \neq \mathcal{J}$.

However, this natural action of G on $\mathcal{J}$ gives us a map

$$\begin{array}{rcl} \beta : G/SO(3) \times SO(3) & \to & \mathcal{J} \\ {[\varphi]} & \mapsto & \varphi_*(J_0)\,, \end{array}$$

which is still an homotopy equivalence. To prove that, one uses pseudo-holomorphic spheres to construct an homotopy inverse

$$\begin{array}{rcl} \alpha : \mathcal{J} & \to & G/SO(3) \times SO(3) \\ J & \mapsto & [\varphi_J] \end{array}$$

in the following way.

It follows from Proposition 6.1 that, for any $J \in \mathcal{J}$, there exist two foliations $\mathcal{F}_J^A$ and $\mathcal{F}_J^B$ of $S^2 \times S^2$ whose leaves are J-holomorphic spheres representing the homology classes $A = [S^2 \times \{p\}]$ and $B = [\{p\} \times S^2]$. Note that

(i) the positivity of intersections Theorem 5.8 implies that any two spheres in the same foliation are disjoint, since $A \cdot A = 0 = B \cdot B$;

(ii) positivity of intersections also implies that each sphere in $\mathcal{F}_J^A$ intersects each sphere in $\mathcal{F}_J^B$ at exactly one point and in a transverse way, since $A \cdot B = 1$;

(iii) the adjunction formula of Theorem 5.11 implies that all spheres in both foliations are embedded, since in this case

$$\text{virtual genus } = 1 + \frac{1}{2}(0-2) = 0 = \text{ genus of } S^2.$$

(iv) the leaves of $\mathcal{F}_{J_0}^A$ and $\mathcal{F}_{J_0}^B$ are exactly the spheres $S^2 \times \{p\}$ and $\{p\} \times S^2$, with $p \in S^2$.

Hence, given any $J \in \mathcal{J}$ we can construct a diffeomorphism

$$\psi_J : S^2 \times S^2 \to S^2 \times S^2$$

that maps the J_0-foliations to the corresponding J-foliations. One checks that, for a diffeomorphism ψ_J with this property, the symplectic form $\omega_J \equiv \psi_J^*(\omega)$ is linearly isotopic to ω, i.e.

$$\omega_t = t\omega_J + (1-t)\omega\,,\ t \in [0,1]\,,\ \text{is an isotopy of symplectic forms.}$$

Moser's method can then be used to deform ψ_J to a symplectomorphism $\varphi_J \in G$.

With appropriate care, this whole construction can be made canonical modulo $SO(3) \times SO(3)$ and the map $\alpha : \mathcal{J} \to G/SO(3) \times SO(3)$ obtained this way can be checked to be indeed an homotopy inverse to the map β. □

Topology of G_λ. As was already explained, the structure of pseudo-holomorphic spheres on $(S^2 \times S^2, \omega_\lambda = \lambda\sigma \times \sigma)$ changes as $\lambda \in \mathbb{R}$ increases from $\lambda = 1$ to any value $\lambda > 1$. In fact, this structure changes every time λ increases past a positive integer value. These changes and the way they relate to the symplectomorphism groups

$$G_\lambda \equiv \mathrm{Diff}(S^2 \times S^2, \omega = \lambda\sigma \times \sigma)$$

can be summarized in the following way (see [1], [4] and [17] for further details).

Given $1 < \lambda \in \mathbb{R}$, let $\mathcal{J}_\lambda$ denote the contractible space of all almost complex structures on $S^2 \times S^2$ compatible with ω_λ and $\ell \in \mathbb{N}$ be the unique positive integer such that $\ell < \lambda \leq \ell + 1$. Then, there is a stratification $\mathcal{J}_\lambda$ of the form

$$\mathcal{J}_\lambda = U_0 \sqcup U_1 \sqcup \cdots \sqcup U_\ell \,, \tag{2}$$

where:

(i)

$$U_k \equiv \{J \in \mathcal{J}_\lambda : \ (A - kB) \in H_2(S^2 \times S^2; \mathbb{Z}) \text{ is represented by a } J\text{-holomorphic sphere}\} \,.$$

Note that $[\omega_\lambda](A - kB) > 0 \Leftrightarrow k \leq \ell$.

(ii) U_0 is open and dense in $\mathcal{J}_\lambda$. For $k \geq 1$, U_k has codimension $4k - 2$ in $\mathcal{J}_\lambda$.

(iii) $\overline{U_k} = U_k \sqcup U_{k+1} \sqcup \cdots \sqcup U_\ell$.

(iv) Each stratum U_k has an integrable element $J_k \in U_k$ (such that $(S^2 \times S^2, J_k) \cong (2k)$-Hirzebruch surface), for which the Kähler isometry group

$$K_k \equiv \mathrm{Isom}(S^2 \times S^2, \langle \cdot, \cdot \rangle_{\lambda,k} \equiv \omega_\lambda(\cdot, J_k \cdot))$$

is such that

$$K_k \cong \begin{cases} SO(3) \times SO(3) \,, & \text{if } k = 0; \\ S^1 \times SO(3) \,, & \text{if } k \geq 1. \end{cases}$$

(v) The inclusion

$$\begin{aligned} (G_\lambda / K_k) &\longrightarrow U_k \\ [\varphi] &\longmapsto \varphi_*(J_k) \end{aligned}$$

is a weak homotopy equivalence.

Although we do not know apriori the topology of the strata U_k, the fact that each is homotopy equivalent to a quotient of G_λ and that their union is the contractible space $\mathcal{J}_\lambda$ can be used to obtain important information regarding the topology of G_λ, in particular its rational cohomology ring.

Theorem 6.3. [4] *When $\lambda > 1$,*

$$H^*(G_\lambda; \mathbb{Q}) = \Lambda(a, x, y) \otimes S(w_\ell) \,,$$

where $\Lambda(a, x, y)$ denotes the exterior algebra over $\mathbb{Q}$ on generators a, x and y of degrees $\deg(a) = 1$, $\deg(x) = \deg(y) = 3$*, and $S(w_\ell)$ denotes the polynomial algebra over $\mathbb{Q}$ on the generator w_ℓ of degree 4ℓ.*

7. A Moment Map Approach to the Topology of Symplectomorphism Groups

In this section, after introducing non-abelian Hamiltonian actions and moment maps, we describe very briefly an infinite dimensional moment map approach to the study of the topology of symplectomorphism groups. This approach can be combined with the one described in section 6 to give a better geometric understanding of the topology of the symplectomorphism groups of $S^2 \times S^2$, based on a result of independent interest: the space of compatible integrable complex structures on $(S^2 \times S^2, \omega_\lambda)$ is (weakly) contractible. See [2] and [3] for further details.

Non-abelian Hamiltonian actions and moment maps.

Definition 7.1. *Let (M, ω) be a symplectic manifold and G a Lie group. A* symplectic action *of G on (M, ω) is a homomorphism*

$$\tau : G \to \mathrm{Diff}(M, \omega)\,.$$

Note that the derivative of a symplectic action $\tau : G \to \mathrm{Diff}(M, \omega)$ gives rise to a Lie algebra homomorphism between the Lie algebras $\mathcal{G}$ of G and $\mathcal{X}(M, \omega)$ of $\mathrm{Diff}(M, \omega)$:

$$\begin{array}{rcl} d\tau : \mathcal{G} & \to & \mathcal{X}(M, \omega) \\ \xi & \mapsto & d\tau(\xi) \equiv X_\xi\,. \end{array}$$

Definition 7.2. *A symplectic action $\tau : G \to \mathrm{Diff}(M, \omega)$ is said to be* Hamiltonian *if there exists a map*

$$\mu : M \to \mathcal{G}^*$$

such that:

(i) *for each $\xi \in \mathcal{G}$, the function $h_\xi : M \to \mathbb{R}$ defined by $h_\xi(p) = \mu(p)(\xi)\,,\ \forall p \in M$, satisfies*

$$dh_\xi = X_\xi \lrcorner\, \omega$$

(i.e. $X_\xi \in \mathcal{X}_H(M, \omega) \subset \mathcal{X}(M, \omega)$ is the Hamiltonian vector field of the function h_ξ);

(ii) *μ is equivariant with respect to the given action τ of G on (M, ω) and the coadjoint action of G on $\mathcal{G}^*$.*

Such a map μ is called a moment map *for the Hamiltonian action.*

Remark 7.3. *It is easy to check that this definition agrees with Definition 4.1 when $G \cong \mathbb{T}^m$ is a connected compact abelian Lie group.*

As we have seen in the abelian case, moment maps of Hamiltonian actions are special maps with beautiful geometric properties. The following general principle formulates a particular one.

General Principle. The norm square of a moment map,

$$\|\mu\|^2 : M \to \mathbb{R}\,,$$

behaves like a G-invariant Morse-Bott function, whose critical manifolds compute the equivariant cohomology $H^*_G(M) \equiv H^*(M \times_G EG)$. Here, the norm is taken with respect to an inner product on $\mathcal{G}^*$ invariant under the coadjoint action of G.

This General Principle has been formulated and proved in a rigorous and precise manner by Kirwan [16], when G and M are finite dimensional. It has also proven to be very useful in infinite dimensional contexts, such as in [7], where one should apply it with appropriate care. The rest of this section presents another example of such an infinite dimensional context.

Hamiltonian action of symplectomorphism groups on compatible complex structures. Let (M, ω) be a compact symplectic manifold of dimension $2n$ and assume that $H^1(M, \mathbb{R}) = 0$. Let $G \equiv \mathrm{Diff}(M, \omega)$ be the symplectomorphism group of (M, ω). This is an infinite dimensional Lie group whose Lie algebra $\mathcal{G}$ can be identified with the space of functions on M with integral zero (see Remark 3.6):

$$\mathcal{G} = C^\infty_0(M) \equiv \left\{ f : M \to \mathbb{R}\ :\ \int_M f\, \frac{\omega^n}{n!} = 0 \right\}\,.$$

$\mathcal{G}$ has a natural invariant inner product $\langle\cdot,\cdot\rangle$, given by

$$\langle f, g \rangle \equiv \int_M f \cdot g\, \frac{\omega^n}{n!}\,,$$

which will be used to formally identify $\mathcal{G}^*$ with $\mathcal{G}$.

Consider again the space $\mathcal{J}(M, \omega)$ of almost complex structures J on M which are compatible with ω. This can be seen as the space of sections of a bundle over M with fiber the contractible symmetric Kähler manifold $Sp(2n, \mathbb{R})/U(n) \equiv$ Siegel upper half space [21]. This fiberwise symmetric Kähler structure, together with the volume form induced by ω on M, turns $\mathcal{J}(M, \omega)$ into an infinite dimensional (contractible) Kähler manifold. We will denote the symplectic form of this Kähler structure by Ω. The space of compatible integrable complex structures

$$X \equiv \mathcal{J}^{\text{int}}(M,\omega) \subset \mathcal{J}(M,\omega)\,,$$

determined by the vanishing of the Nijenhuis tensor, is a complex submanifold of $\mathcal{J}(M,\omega)$. Hence, $(X, \Omega|_X)$ is also a (possibly empty) infinite dimensional symplectic manifold.

The natural action of the symplectomorphism group G on $\mathcal{J}(M,\omega)$, given by conjugation

$$\phi \cdot J \equiv \phi_*(J) = d\phi \circ J \circ d\phi^{-1}\,,\ \forall\, \phi \in G\,,\ J \in \mathcal{J}(M,\omega)\,,$$

preserves the Kähler structure on $\mathcal{J}(M,\omega)$, in particular its symplectic form Ω. Donaldson [13] shows that this symplectic action is always Hamiltonian, with moment map

$$\mu : \mathcal{J}(M,\omega) \to \mathcal{G}^* \cong C_0^\infty(M)$$

given by

$$\mu(J) = (\text{Hermitian scalar curvature } S(J) \text{ of the metric } \langle\cdot,\cdot\rangle_J \equiv \omega(\cdot, J\cdot)) - d\,,$$

where d is the constant defined by

$$d \cdot \int_M \frac{\omega^n}{n!} \equiv 2\pi c_1(M) \wedge [\omega]^{n-1}(M) = \int_M S(J)\ \frac{\omega^n}{n!}\,.$$

Note that $X \subset \mathcal{J}(M,\omega)$ is invariant under the action of G and for integrable $J \in X$ the Hermitian scalar curvature $S(J)$ coincides with the usual scalar curvature of the Riemannian metric $\langle\cdot,\cdot\rangle_J$. Hence

$$\begin{array}{rcl} \mu : X & \to & \mathcal{G}^* \cong C_0^\infty(M) \\ J & \mapsto & S(J) - d \end{array}$$

is a moment map for the Hamiltonian action of G on $(X, \Omega|_X)$.

The previous General Principle says that the critical points of

$$\|\mu\|^2 : X = \mathcal{J}^{\text{int}}(M,\omega) \to \mathbb{R}\,,\ \|\mu\|^2(J) = \int_M S^2(J)\ \frac{\omega^n}{n!}\ +\ \text{constant},$$

should determine the equivariant cohomology $H_G^*(X)$. These critical points are, in particular, extremal Kähler metrics in the sense of Calabi ([8] and [9]).

Kähler geometry of $S^2 \times S^2$: the space X_λ. Let us now go back to $(S^2 \times S^2, \omega_\lambda)$. One can use results from complex and Kähler geometry to study its space $X_\lambda \subset \mathcal{J}_\lambda$ of compatible integrable complex structures. The geometric picture one gets from this study matches perfectly the one suggested by the moment map General Principle. In fact, given $1 < \lambda \in \mathbb{R}$, there is a stratification of X_λ of the form

$$X_\lambda = V_0 \sqcup V_1 \sqcup \cdots \sqcup V_\ell\,, \tag{3}$$

with $\ell \in \mathbb{N}$ such that $\ell < \lambda \le \ell + 1$ and where:

(i)

$$V_k \equiv \{J \in X_\lambda :\ (A - kB) \in H_2(S^2 \times S^2; \mathbb{Z}) \text{ is represented by a } J\text{-holomorphic sphere}\}\,.$$

(ii) V_0 is open and dense in X_λ. For $k \ge 1$, V_k has codimension $4k - 2$ in X_λ.

(iii) $\overline{V_k} = V_k \sqcup V_{k+1} \sqcup \cdots \sqcup V_\ell$.

(iv) For each $k \in \mathbb{N} \cup \{0\}$, there is a complex structure $J_k \in V_k$, unique up to the action of G_λ, such that $\langle\cdot,\cdot\rangle_{\lambda,k} \equiv \omega_\lambda(\cdot, J_k\cdot)$ is an extremal Kähler metric. Note that, at least formally, V_k is the unstable manifold of the invariant critical manifold of $\|\mu\|^2$ determined by J_k and G_λ.

(v) The Kähler isometry group

$$K_k \equiv \text{Isom}(S^2 \times S^2, \langle\cdot,\cdot\rangle_{\lambda,k})$$

is such that

$$K_k \cong \begin{cases} SO(3) \times SO(3)\,, & \text{if } k = 0; \\ S^1 \times SO(3)\,, & \text{if } k \ge 1. \end{cases}$$

(vi) The inclusion

$$\begin{aligned}(G_\lambda/K_k) &\longrightarrow V_k \\ [\varphi] &\longmapsto \varphi_*(J_k)\end{aligned}$$

is a weak homotopy equivalence.

(vii) Each V_k has a tubular neighborhood $NV_k \subset X_\lambda$, with normal slice given by

$$H^1(H_{2k}, \Theta) \cong \mathbb{C}^{2k-1} \,,$$

where Θ = sheaf of holomorphic vector fields on the $(2k)$-Hirzebruch surface H_{2k}.

(viii) For $k \geq 1$, the representation of $K_k \cong S^1 \times SO(3)$ on the normal slice $\mathbb{C}^{2k-1}$ at $J_k \in V_k$ is the following: S^1 acts diagonally and $SO(3)$ acts irreducibly with highest weight $2(k-1)$.

This detailed information implies, by standard equivariant cohomology theory (see [3]), the following proposition.

Proposition 7.4. *Given $\ell \in \mathbb{N}$ and $\lambda \in \,]\ell, \ell+1]$, we have that*

$$H^*_{G_\lambda}(X_\lambda; \mathbb{Z}) \cong H^*(BSO(3) \times BSO(3); \mathbb{Z}) \oplus \oplus_{k=1}^{\ell} \Sigma^{4k-2} H^*(BS^1 \times BSO(3); \mathbb{Z}) \,,$$

where Σ denotes the suspension operator and $\cong$ indicates just a group isomorphism (not a ring isomorphism).

Contractibility of X_λ and the topology of BG_λ. Comparing the stratification (2) of $\mathcal{J}_\lambda$, obtained from pseudo-holomorphic curves, with the stratification (3) of X_λ, obtained from Kähler geometry, one is immediately led to the following theorem.

Theorem 7.5. [3] *Given $1 \leq \lambda \in \mathbb{R}$, the space X_λ of compatible integrable complex structures on $(S^2 \times S^2, \omega_\lambda)$ is weakly homotopy equivalent to the contractible space $\mathcal{J}_\lambda$ of compatible almost complex structures. In particular, X_λ is weakly contractible.*

This theorem implies that

$$H^*_{G_\lambda}(X_\lambda; \mathbb{Z}) \cong H^*(BG_\lambda; \mathbb{Z}) \,.$$

Combining this isomorphism with Proposition 7.4, we get the following theorem.

Theorem 7.6. [3] *Given $\ell \in \mathbb{N} \cup \{0\}$ and $\lambda \in \,]\ell, \ell+1]$, we have that*

$$H^*(BG_\lambda; \mathbb{Z}) \cong H^*(BSO(3) \times BSO(3); \mathbb{Z}) \oplus \oplus_{k=1}^{\ell} \Sigma^{4k-2} H^*(BS^1 \times BSO(3); \mathbb{Z}) \,,$$

where $\cong$ indicates again just a group isomorphism.

Remark 7.7. *When $\lambda = 1$ this result follows directly from Gromov's Theorem 6.2, while for $1 < \lambda \leq 2$ it also follows from the work of Anjos and Granja in [5], where the ring structure of $H^*(BG_\lambda; \mathbb{Z})$, $1 < \lambda \leq 2$, is determined. The ring structure of $H^*(BG_\lambda; \mathbb{Z})$, for any $1 \leq \lambda \in \mathbb{R}$, is also discussed in [3].*

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