

Toric Kähler-Sasaki Geometry

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Motivation

Toric
Kähler-Sasaki
Geometry

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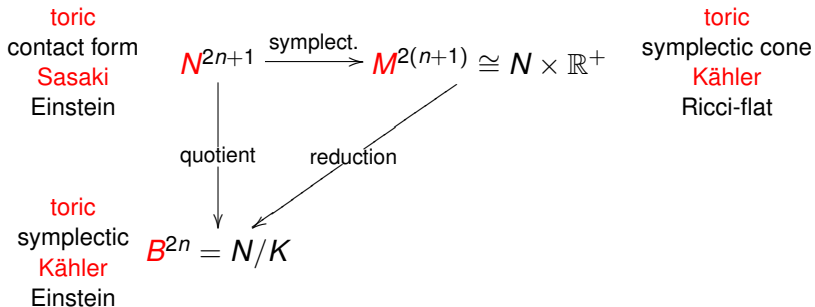
Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

Understand, through examples in **action-angle coordinates**,
the following general geometric set-up



Outline

Toric
Kähler-Sasaki
Geometry

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Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

- 1 Toric Kähler Metrics
- 2 Toric Kähler-Sasaki Metrics
- 3 Toric Kähler-Sasaki-Einstein Metrics

Definition of Toric Symplectic Manifolds

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

Definition

A **toric symplectic manifold** is a connected symplectic manifold (B^{2n}, ω) , equipped with an effective Hamiltonian action of the n -torus:

$$\tau : \mathbb{T}^n \cong \mathbb{R}^n / 2\pi\mathbb{Z}^n \hookrightarrow \text{Ham}(B, \omega).$$

The corresponding **moment map**, unique up to an additive constant, will be denoted by

$$\mu : B \rightarrow \text{Lie}^*(\mathbb{T}^n) \cong (\mathbb{R}^n)^* \cong \mathbb{R}^n.$$

Examples - $(\mathbb{R}^{2n}, \omega_{\text{st}}, \tau_{\text{st}}, \mu_{\text{st}})$

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

Consider $(\mathbb{R}^{2n}, \omega_{\text{st}})$, with linear coordinates $(u_1, \dots, u_n, v_1, \dots, v_n)$ such that

$$\omega_{\text{st}} = du \wedge dv := \sum_{j=1}^n du_j \wedge dv_j.$$

We will also use the usual identification with \mathbb{C}^n given by

$$z_j = u_j + iv_j, \quad j = 1, \dots, n.$$

The standard \mathbb{T}^n -action τ_{st} on \mathbb{R}^{2n} , given by

$$(y_1, \dots, y_n) \cdot (z_1, \dots, z_n) = (e^{-iy_1} z_1, \dots, e^{-iy_n} z_n),$$

is Hamiltonian, with moment map $\mu_{\text{st}} : \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$ given by

$$\mu_{\text{st}}(u_1, \dots, u_n, v_1, \dots, v_n) = \frac{1}{2}(u_1^2 + v_1^2, \dots, u_n^2 + v_n^2).$$

Examples - Projective Space $(\mathbb{P}^n, \omega_{\text{FS}}, \tau_{\text{FS}}, \mu_{\text{FS}})$

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

Consider projective space $(\mathbb{P}^n, \omega_{\text{FS}})$, with homogeneous coordinates $[z_0; z_1; \dots; z_n]$.

The \mathbb{T}^n -action τ_{FS} on \mathbb{P}^n given by

$$(y_1, \dots, y_n) \cdot [z_0; z_1; \dots; z_n] = [z_0; e^{-iy_1} z_1; \dots; e^{-iy_n} z_n],$$

is Hamiltonian, with moment map $\mu_{\text{FS}} : \mathbb{P}^n \rightarrow \mathbb{R}^n$ given by

$$\mu_{\text{FS}}[z_0; z_1; \dots; z_n] = \frac{1}{\|z\|^2} (\|z_1\|^2, \dots, \|z_n\|^2).$$

Hence, $(\mathbb{P}^n, \omega_{\text{FS}}, \tau_{\text{FS}}, \mu_{\text{FS}})$ is an example of a **compact** symplectic toric manifold.

Note that the **image of μ_{FS}** is the **convex hull** of the images of the $n + 1$ **fixed points** of the action, i.e. the **standard simplex** in \mathbb{R}^n .

Atiyah-Guillemin-Sternberg and Delzant Theorems (1982)

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

Atiyah-Guillemin-Sternberg'82

Let (B, ω) be a **compact** toric symplectic manifold with moment map $\mu : B \rightarrow \mathbb{R}^n$.

Then, the **image** $\mu(B)$ of the moment map is the **convex polytope** given by the convex hull of the images of the fixed points of the action.

This will be usually called the **moment polytope** and denoted by P .

Delzant'82

The moment polytope is a **complete invariant** of a compact toric symplectic manifold.

Action-Angle Coordinates

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

- $\mathbb{R}^n \supset \mu(B) = P \supset \check{P} \equiv$ interior of P

$\check{B} \equiv \mu^{-1}(\check{P}) \equiv$ {points of B where \mathbb{T}^n -action is free}

$$\cong \check{P} \times \mathbb{T}^n = \left\{ (x, y) : x \in \check{P} \subset \mathbb{R}^n, y \in \mathbb{R}^n / 2\pi\mathbb{Z}^n \right\}$$

such that $\omega|_{\check{B}} = dx \wedge dy \equiv$ standard symplectic form

Definition

$(x, y) \equiv$ symplectic/Darboux/action-angle coordinates.

- \check{B} is an open dense subset of B .
- In these coordinates, the moment map $\mu : \check{B} \rightarrow \check{P}$ is given by $\mu(x, y) = x$.

Compatible Almost Complex Structures

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Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

Definition

A **compatible almost complex structure** on a symplectic manifold (B, ω) is an almost complex structure J on B , i.e. $J \in \Gamma(\text{End}(TB))$ with $J^2 = -Id$, such that

$$g_J(\cdot, \cdot) \equiv \omega(\cdot, J\cdot)$$

is a **Riemannian metric** on B . This is equivalent to

$$\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot) \quad \text{and} \quad \omega(X, JX) > 0, \quad \forall 0 \neq X \in TB.$$

The space of all compatible almost complex structures on a symplectic manifold (B, ω) will be denoted by $\mathcal{J}(B, \omega)$.

Remarks

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

- For any symplectic manifold (B, ω) , the space $\mathcal{J}(B, \omega)$ is non-empty, infinite-dimensional and contractible.
- A Kähler manifold can be defined as a symplectic manifold with an integrable compatible complex structure.
- The space of integrable compatible complex structures on a symplectic manifold (B, ω) will be denoted by $\mathcal{I}(B, \omega) \subset \mathcal{J}(B, \omega)$.
- In general, $\mathcal{I}(B, \omega)$ can be empty.

Toric Compatible Complex Structures

Toric
Kähler-Sasaki
Geometry

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Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

Definition

A **toric compatible complex structure** on a toric symplectic manifold (B^{2n}, ω, τ) is a

$$\mathbb{T}^n\text{-invariant } J \in \mathcal{I}(B, \omega) \subset \mathcal{J}(B, \omega).$$

The space of all such will be denoted by

$$\mathcal{I}^{\mathbb{T}^n}(B, \omega) \subset \mathcal{J}^{\mathbb{T}^n}(B, \omega).$$

- It follows from the classification theorem of Delzant that $\mathcal{I}^{\mathbb{T}^n}(B, \omega)$ is **non-empty** for any compact toric symplectic manifold.

Local Form of Toric Compatible Complex Structures

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

For **integrable** toric compatible complex structures we have that:

$$J \in \mathcal{I}^{\mathbb{T}^n} \subset \mathcal{J}^{\mathbb{T}^n} \Leftrightarrow \frac{\partial Z_{ij}}{\partial x_k} = \frac{\partial Z_{ik}}{\partial x_j}$$

$\Leftrightarrow \exists f : \check{P} \rightarrow \mathbb{C}$, $f(x) = r(x) + is(x)$, such that

$$Z_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 r}{\partial x_i \partial x_j} + i \frac{\partial^2 s}{\partial x_i \partial x_j} = R_{ij} + iS_{ij}$$

Local Form of Toric Compatible Complex Structures

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

Any real function

$$h : \check{P} \rightarrow \mathbb{R}$$

is the **Hamiltonian** of a 1-parameter family

$$\phi_t : \check{B} \rightarrow \check{B}$$

of \mathbb{T}^n -equivariant symplectomorphisms, given in action-angle coordinates (x, y) on $\check{B} \cong \check{P} \times \mathbb{T}^n$ by

$$\phi_t(x, y) = \left(x, y - t \frac{\partial h}{\partial x}\right).$$

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Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

The natural **action** of such a ϕ_t on $\mathcal{J}^{\mathbb{T}^n}$, given by

$$\phi_t \cdot J := (d\phi_t)^{-1} \circ J \circ (d\phi_t),$$

corresponds in the Siegel Upper Half Space parametrization to

$$\phi_t \cdot (Z = R + iS) = (R + tH) + iS,$$

where

$$H = (h_{ij}) = \left(\frac{\partial^2 h}{\partial x_i \partial x_j} \right).$$

Local Form of Toric Compatible Complex Structures

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

Hence, for any **integrable** $J \in \mathcal{I}^{\mathbb{T}^n}$ there exist action-angle coordinates (x, y) on $\check{B} \cong \check{P} \times \mathbb{T}^n$ such that $R \equiv 0$ in the Siegel Upper Half Space Parametrization, i.e. such that

$$J = \begin{bmatrix} 0 & \vdots & -S^{-1} \\ \dots\dots\dots \\ S & \vdots & 0 \end{bmatrix}$$

with

$$S = S(x) = (s_{ij}(x)) = \left(\frac{\partial^2 s}{\partial x_i \partial x_j} \right)$$

for some

real potential function $s : \check{P} \rightarrow \mathbb{R}$.

Local Form of Toric Compatible Complex Structures

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

The corresponding **Riemannian (Kähler) metric**

$$g(\cdot, \cdot) \equiv \omega(\cdot, J\cdot)$$

on $\check{M} \cong \check{P} \times \mathbb{T}^n$ can be written in matrix form as

$$g = \begin{bmatrix} 0 & \vdots & I \\ \dots & \dots & \dots \\ -I & \vdots & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & \vdots & -S^{-1} \\ \dots & \dots & \dots \\ S & \vdots & 0 \end{bmatrix} = \begin{bmatrix} S & \vdots & 0 \\ \dots & \dots & \dots \\ 0 & \vdots & S^{-1} \end{bmatrix}$$

with

$$S = \left(\frac{\partial^2 s}{\partial x_i \partial x_j} \right).$$

Remarks

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

- We will call such a potential function

$$s : \check{P} \rightarrow \mathbb{R}$$

the **symplectic potential** of both the complex structure J and the metric g

- This particular way to arrive at the above form for any $J \in \mathcal{I}^{\mathbb{T}^n}$ is due to **Donaldson**, and illustrates a very particular part of his formal **general framework** for the action of the symplectomorphism group of a symplectic manifold on its space of compatible (almost) complex structures.

Examples - $(\mathbb{R}^{2n}, \omega_{\text{st}}, \tau_{\text{st}}, \mu_{\text{st}})$

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

Consider the linear complex structure $J_{\text{st}} \in \mathcal{I}^{\mathbb{T}^n}(\mathbb{R}^{2n}, \omega_{\text{st}})$ giving the standard identification between \mathbb{R}^{2n} and \mathbb{C}^n . In action-angle coordinates (x, y) on

$$\check{\mathbb{R}}^{2n} = (\mathbb{R}^2 \setminus \{(0, 0)\})^n \cong (\mathbb{R}^+)^n \times \mathbb{T}^n = \check{P} \times \mathbb{T}^n,$$

its **symplectic potential** is given by

$$s : \check{P} = (\mathbb{R}^+)^n \longrightarrow \mathbb{R}$$

$$x = (x_1, \dots, x_n) \longmapsto s(x) = \frac{1}{2} \sum_{i=1}^n x_i \log(x_i).$$

Examples - $(\mathbb{R}^{2n}, \omega_{\text{st}}, \tau_{\text{st}}, \mu_{\text{st}})$

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

Hence, in these action-angle coordinates, the **standard complex structure** has the matrix form

$$J_{\text{st}} = \begin{bmatrix} 0 & \vdots & \text{diag}(-2x_i) \\ \dots\dots\dots & & \\ \text{diag}(1/2x_i) & \vdots & 0 \end{bmatrix}$$

while the **standard flat Euclidean metric** becomes

$$g_{\text{st}} = \begin{bmatrix} \text{diag}(1/2x_i) & \vdots & 0 \\ \dots\dots\dots & & \\ 0 & \vdots & \text{diag}(2x_i) \end{bmatrix}$$

Examples - Compact Toric Symplectic Manifolds

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

- To any bounded, convex, simple, integral polytope $P \subset \mathbb{R}^n$, a canonical **symplectic reduction construction** of Delzant associates a compact Kähler toric manifold

$$(B_P^{2n}, \omega_P, \tau_P, \mu_P, J_P) \text{ such that } \mu_P(B_P) = P.$$

- Let F_i denote the i -th **facet** of the polytope. The **affine defining function** of F_i is the function

$$\begin{aligned} \ell_i : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ x &\longmapsto \ell_i(x) = \langle x, \nu_i \rangle - \lambda_i, \end{aligned}$$

where $\nu_i \in \mathbb{Z}^n$ is a **primitive** inward pointing **normal** to F_i and $\lambda_i \in \mathbb{R}$ is such that $\ell_i|_{F_i} \equiv 0$. Note that $\ell_i|_{\check{P}} > 0$.

Examples - Compact Toric Symplectic Manifolds

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

Guillemin'94

In appropriate action-angle coordinates (x, y) , the canonical symplectic potential $s_P : \check{P} \rightarrow \mathbb{R}$ for $J_P|_{\check{P}}$ is given by

$$s_P(x) = \frac{1}{2} \sum_{i=1}^d l_i(x) \log l_i(x),$$

where d is the number of facets of P .

Examples - Projective Space $(\mathbb{P}^n, \omega_{FS}, \tau_{FS}, \mu_{FS})$

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

For **projective spaces** \mathbb{P}^n the polytope $P \subset \mathbb{R}^n$ can be taken to be the **standard simplex**, with defining affine functions

$$\ell_i(x) = x_i, \quad i = 1, \dots, n, \quad \text{and} \quad \ell_{n+1}(x) = 1 - r,$$

where $r = \sum_i x_i$. The canonical symplectic potential $s_P : \check{P} \rightarrow \mathbb{R}$, given by

$$s_P(x) = \frac{1}{2} \sum_{i=1}^n x_i \log x_i + \frac{1}{2} (1 - r) \log(1 - r),$$

defines the **standard complex structure** J_{FS} and **Fubini-Study metric** g_{FS} on \mathbb{P}^n .

Toric $\partial\bar{\partial}$ -Lemma in Action-Angle Coordinates

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

A.'01

Let $J \in \mathcal{I}^{\mathbb{T}^n}(B_P, \omega_P)$. Then, in suitable action-angle coordinates (x, y) on $\check{B} \cong \check{P} \times \mathbb{T}^n$, J is given by a symplectic potential $s : \check{P} \rightarrow \mathbb{R}$ of the form

$$s(x) = s_P(x) + h(x),$$

where $h \in C^\infty(P)$, $\text{Hess}_x(s) > 0$ in \check{P} and $\det(\text{Hess}_x(s)) = (\delta(x) \prod_i \ell_i)^{-1}$, with $\delta \in C^\infty(P)$ and $\delta(x) > 0$, $\forall x \in P$.

Conversely, any such s is the **symplectic potential** of a $J \in \mathcal{I}^{\mathbb{T}^n}(\check{P} \times \mathbb{T}^n)$ that compactifies to a **well defined** $J \in \mathcal{I}^{\mathbb{T}^n}(B_P, \omega_P)$.

Toric Kähler Metrics and Scalar Curvature

- Toric Kähler metric

$$g = \begin{bmatrix} (s_{ij}) & \vdots & 0 \\ \dots\dots\dots & & \\ 0 & \vdots & (s^{jj}) \end{bmatrix}$$

where $(s_{ij}) = \text{Hess}_x(s)$ for $s : \check{P} \subset \mathbb{R}^n \rightarrow \mathbb{R}$.

- Formula for its **scalar curvature** [A.'98]:

$$S \equiv - \sum_{j,k} \frac{\partial}{\partial x_j} \left[s^{jk} \frac{\partial \log(\det \text{Hess}_x(s))}{\partial x_k} \right] = - \sum_{j,k} \frac{\partial^2 s^{jk}}{\partial x_j \partial x_k}$$

- (**Donaldson'02**) Appropriate interpretation of this formula by viewing the scalar curvature as a **moment map** for the action of the symplectomorphism group of a symplectic manifold on its space of compatible complex structures.

Examples - Toric Constant Scalar Curvature Metrics in Real Dimension 2

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

$$-\left(\frac{1}{s''(x)}\right)'' = 2k \Rightarrow s''(x) = -\frac{1}{kx^2 - 2bx - c}, \quad k, b, c \in \mathbb{R}$$

$[k = 0, b = 0]$ (need $c > 0$)

$$s''(x) = \frac{1}{c} \Rightarrow \left\| \frac{\partial}{\partial y} \right\|^2 = c, \quad s(x) = \frac{x^2}{2c} \Rightarrow \text{cylinder of radius } \sqrt{c}$$

$[k = 0, b > 0]$ (can assume $c = 0$, need $x > 0$)

$$s''(x) = \frac{1}{2bx} \Rightarrow s(x) = \frac{1}{b} \cdot \frac{1}{2} x \log x \Rightarrow \text{cone of angle } \pi b$$

If $b = 1/m$, $m \in \mathbb{N}$, get orbifold flat metric on \mathbb{C}/\mathbb{Z}_m .

Examples - Toric Constant Scalar Curvature Metrics in Real Dimension 2

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

$[k \neq 0]$ (can assume $b = 0$)

$$s''(x) = \frac{1}{c - kx^2} > 0$$

$[k > 0]$ (need $c > 0$ and $-\sqrt{c/k} < x < \sqrt{c/k}$)

$$s(x) = \frac{1}{\sqrt{ck}} \cdot \frac{1}{2} \left[(x + \sqrt{c/k}) \log(x + \sqrt{c/k}) \right. \\ \left. + (-x + \sqrt{c/k}) \log(-x + \sqrt{c/k}) \right]$$

Singular american football metric. **Smooth** european football metric of Gauss curvature k iff $c = 1/k$.

Examples - Toric Constant Scalar Curvature Metrics in Real Dimension 2

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

$$[k < 0, c > 0] (x \in \mathbb{R})$$

$$s(x) = \sqrt{\frac{-1}{ck}} \arctan \left(\sqrt{\frac{-k}{c}} x \right) \Rightarrow \text{hyperboloid}$$

$$[k < 0, c < 0] (\text{need } x > \sqrt{c/k})$$

$$s(x) = \frac{1}{\sqrt{ck}} \cdot \frac{1}{2} \left[(x - \sqrt{c/k}) \log(x - \sqrt{c/k}) \right. \\ \left. - (x + \sqrt{c/k}) \log(x + \sqrt{c/k}) \right]$$

Singular hyperbolic planes. **Smooth** hyperbolic planes of Gauss curvature k iff $c = 1/k$.

Definition of Symplectic Cone

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

Definition

A **symplectic cone** is a triple (M, ω, X) , where (M, ω) is a connected symplectic manifold and $X \in \mathcal{X}(M)$ is a vector field generating a proper \mathbb{R} -action $\rho_t : M \rightarrow M$, $t \in \mathbb{R}$, such that $\rho_t^*(\omega) = e^{2t}\omega$. Note that the **Liouville vector field** X satisfies $\mathcal{L}_X\omega = 2\omega$, or equivalently

$$\omega = \frac{1}{2}d(\iota(X)\omega).$$

symplectic cones $\xleftrightarrow{1:1}$ co-oriented contact manifolds

In particular, (M, ω, X) is the **symplectization** of $(N := M/\mathbb{R}, \xi := \pi_*(\ker(\iota(X)\omega)))$, where $\pi : M \rightarrow M/\mathbb{R}$.

Definition of Kähler-Sasaki Cone

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

Definition

A **Kähler-Sasaki cone** is a symplectic cone (M, ω, X) equipped with a **compatible complex structure** $J \in \mathcal{I}(M, \omega)$ such that the **Reeb vector field** $K := JX$ is **Kähler**, i.e.

$$\mathcal{L}_K \omega = 0 \quad \text{and} \quad \mathcal{L}_K J = 0.$$

Note that K is then also a **Killing** vector field for the Riemannian metric

$$g_J(\cdot, \cdot) := \omega(\cdot, J\cdot).$$

Any such J will be called a **Sasaki complex structure** on the symplectic cone (M, ω, X) . The space of all Sasaki complex structures will be denoted by $\mathcal{I}_S(M, \omega, X)$.

Properties of Kähler-Sasaki Cones

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

Define $r := \|K\| = \|X\| : M \rightarrow \mathbb{R}^+$. Then

- K is the **Hamiltonian** vector field of $-r^2/2$;
- X is the **gradient** vector field of $r^2/2$.

Define $\alpha := \iota(X)\omega/r^2 \in \Omega^1(M)$. Then

$$\omega = d(r^2\alpha)/2, \quad \alpha(K) \equiv 1 \quad \text{and} \quad \mathcal{L}_X\alpha = 0.$$

Define $N := \{r = 1\} \subset M$ and let $\xi := \ker \alpha|_N$. Then

$(N, \xi, \alpha|_N, g_J|_N)$ is a **Sasaki manifold**.

Define $B := N/K$. Then $TB \cong \xi$ and

$(B, d\alpha|_\xi, J|_\xi)$ is a **Kähler space**,

the **Kähler reduction** of (M, ω, X, J) by the action of K .

Regular, Quasi-Regular and Irregular Kähler-Sasaki Cones

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

A KS cone (M, ω, X, J) , with Reeb vector field $K = JX$, is said to be:

- **regular** if K generates a **free S^1 -action**.
- **quasi-regular** if K generates a **locally free S^1 -action**.
- **irregular** if K generates an **effective \mathbb{R} -action**.

Note that the Kähler reduction $B = M//K$ is

- a **smooth** Kähler manifold if the KS cone is **regular**.
- a Kähler **orbifold** if the KS cone is **quasi-regular**.
- only a Kähler **quasifold** if the KS cone is **irregular**.

Note that the **Sasaki manifold** determined by a KS cone is always **smooth**.

Toric Symplectic Cones

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

Lemma

Let G be a Lie group. Any X -preserving symplectic G -action on a symplectic cone (M, ω, X) is Hamiltonian. Moreover, its moment map $\mu : M \rightarrow \mathfrak{g}^*$ can be chosen so that

$$\mu(\rho_t(m)) = e^{2t}\mu(m), \quad \forall m \in M, t \in \mathbb{R}.$$

Definition

A **toric symplectic cone** is a symplectic cone (M, ω, X) of dimension $2(n+1)$ equipped with an **effective X -preserving \mathbb{T}^{n+1} -action**, with moment map $\mu : M \rightarrow \mathfrak{t}^* \cong \mathbb{R}^{n+1}$ such that $\mu(\rho_t(m)) = e^{2t}\mu(m)$, $\forall m \in M, t \in \mathbb{R}$. Its **moment cone** is defined to be the set $C := \mu(M) \cup \{0\} \subset \mathbb{R}^{n+1}$.

Example - $(\mathbb{R}^{2(n+1)} \setminus \{0\}, \omega_{\text{st}}, X_{\text{st}})$

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

$(\mathbb{R}^{2(n+1)} \setminus \{0\}, \omega_{\text{st}}, X_{\text{st}})$, with linear coordinates $(u_1, \dots, u_{n+1}, v_1, \dots, v_{n+1})$ such that

$$\omega_{\text{st}} = du \wedge dv := \sum_{j=1}^{n+1} du_j \wedge dv_j$$

and

$$X_{\text{st}} = u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v} := \sum_{j=1}^{n+1} \left(u_j \frac{\partial}{\partial u_j} + v_j \frac{\partial}{\partial v_j} \right),$$

equipped with the **standard \mathbb{T}^{n+1} -action**, is a toric symplectic cone with moment map $\mu_{\text{st}} : \mathbb{R}^{2(n+1)} \setminus \{0\} \rightarrow \mathbb{R}^{n+1}$ given by

$$\mu_{\text{st}}(u_1, \dots, u_{n+1}, v_1, \dots, v_{n+1}) = \frac{1}{2}(u_1^2 + v_1^2, \dots, u_{n+1}^2 + v_{n+1}^2).$$

Its moment cone is $C = (\mathbb{R}_0^+)^{n+1} \subset \mathbb{R}^{n+1}$.

Classification of Good Toric Symplectic Cones

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

Definition (Lerman)

A cone $C \subset \mathbb{R}^{n+1}$ is **good** if there exists a non-empty **minimal** set of **primitive** vectors $\nu_1, \dots, \nu_d \in \mathbb{Z}^{n+1}$ such that

- (i) $C = \bigcap_{a=1}^d \{x \in \mathbb{R}^{n+1} : \ell_a(x) := \langle x, \nu_a \rangle \geq 0\}$.
- (ii) any codimension- k face F of C , $1 \leq k \leq n$, is the intersection of exactly k facets whose **set of normals** can be **completed** to an **integral base** of \mathbb{Z}^{n+1} .

Theorem (Banyaga-Molino, Boyer-Galicki, Lerman)

For each good cone $C \subset \mathbb{R}^{n+1}$ there **exists a unique** toric symplectic cone $(M_C, \omega_C, X_C, \mu_C)$ with moment cone C .

Like for compact symplectic toric manifolds, **existence follows from a symplectic reduction construction** starting from a symplectic vector space.

Boothby-Wang Cones (Lerman)

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

Let $P \subset \mathbb{R}^n$ be an **integral Delzant polytope**. Then, its standard cone

$$C := \{z(x, 1) \in \mathbb{R}^n \times \mathbb{R} : x \in P, z \geq 0\} \subset \mathbb{R}^{n+1}$$

is a **good cone**. Moreover

- (i) the toric symplectic manifold (B_P, ω_P, μ_P) is the $S^1 \cong \{1\} \times S^1 \subset \mathbb{T}^{n+1}$ **symplectic reduction** of the toric symplectic cone $(M_C, \omega_C, X_C, \mu_C)$ (at level one).
- (ii) $(N_C := \mu_C^{-1}(\mathbb{R}^n \times \{1\}), \alpha_C := (\iota(X_C)\omega_C)|_{N_C})$ is the **Boothby-Wang** manifold of (B_P, ω_P) . The restricted \mathbb{T}^{n+1} -action makes it a **toric contact manifold**.
- (iii) (M_C, ω_C) is the **symplectization** of (N_C, α_C) .

Example: Boothby-Wang Cone of the Standard Simplex

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

Let $P \subset \mathbb{R}^n$ be the **standard simplex**, i.e. $B_P = \mathbb{P}^n$. Its standard cone $C \subset \mathbb{R}^{n+1}$ is the moment cone of $(M_C = \mathbb{C}^{n+1} \setminus \{0\}, \omega_{\text{st}}, X_{\text{st}})$ equipped with the \mathbb{T}^{n+1} -action given by

$$\begin{aligned} & (y_1, \dots, y_n, y_{n+1}) \cdot (z_1, \dots, z_n, z_{n+1}) \\ &= (e^{-i(y_1 + y_{n+1})} z_1, \dots, e^{-i(y_n + y_{n+1})} z_n, e^{-iy_{n+1}} z_{n+1}) \end{aligned}$$

The moment map $\mu_C : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{R}^{n+1}$ is given by

$$\mu_C(z) = \frac{1}{2}(|z_1|^2, \dots, |z_n|^2, |z_1|^2 + \dots + |z_n|^2 + |z_{n+1}|^2).$$

and

$$N_C := \mu_C^{-1}(\mathbb{R}^n \times \{1\}) = \left\{ z \in \mathbb{C}^{n+1} : \|z\|^2 = 2 \right\} \cong \mathbb{S}^{2n+1}.$$

Toric Kähler-Sasaki Cones

Definition

A **toric Kähler-Sasaki cone** is a toric symplectic cone (M, ω, X, μ) equipped with a **toric Sasaki complex structure** $J \in \mathcal{I}_S^{\mathbb{T}}(M, \omega)$.

- It follows from the classification theorem that **any good toric symplectic cone has toric Sasaki complex structures**.
- On a toric Kähler-Sasaki cone (M, ω, X, μ, J) , the **Kähler action** generated by the Reeb vector field $K = JX$ is **part of the torus action**.
- The Kähler reduction $B = M//K$ is a **toric Kähler space**: manifold (regular case), orbifold (quasi-regular case) or quasifold (irregular case).

Cone Action-Angle Coordinates on (M, ω, X, μ)

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

$$\mathbb{R}^{n+1} \supset \mu(M) = C \setminus \{0\} \supset \check{C} \equiv \text{interior of } C.$$

$$\begin{aligned} \check{M} &\equiv \mu^{-1}(\check{C}) \equiv \{\text{points of } M \text{ where } \mathbb{T}^{n+1}\text{-action is free}\} \\ &\cong \check{C} \times \mathbb{T}^{n+1} = \left\{ (x, y) : x \in \check{C}, y \in \mathbb{T}^{n+1} \equiv \mathbb{R}^{n+1}/2\pi\mathbb{Z}^{n+1} \right\} \end{aligned}$$

such that $\omega|_{\check{M}} = dx \wedge dy \equiv$ standard symplectic form,

$$\mu(x, y) = x \quad \text{and} \quad X|_{\check{M}} = 2x \frac{\partial}{\partial x} = 2 \sum_{i=1}^{n+1} x_i \frac{\partial}{\partial x_i}.$$

Definition

$(x, y) \equiv$ cone symplectic/Darboux/action-angle coordinates.

\check{M} is an open dense subset of M .

Toric Sasaki Complex Structures in Cone Action-Angle Coordinates

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

Any toric **Sasaki** complex structure $J \in \mathcal{I}_S^{\mathbb{T}}(\check{M}, \omega, X)$ can be written in suitable cone action-angle coordinates (x, y) on $\check{M} \cong \check{C} \times \mathbb{T}^{n+1}$ as

$$J = \begin{bmatrix} 0 & -S^{-1} \\ S & 0 \end{bmatrix}$$

with

$$S = S(x) = (s_{ij}(x)) = \left(\frac{\partial^2 s}{\partial x_i \partial x_j} \right) > 0$$

for some **symplectic potential** $s : \check{C} \rightarrow \mathbb{R}$, such that

$$S(e^t x) = e^{-t} S(x), \quad \forall t \in \mathbb{R}, \quad x \in \check{C},$$

i.e. $S(x) = \text{Hess}_x(s)$ is **homogeneous** of degree -1 .

Reeb Vector Fields in Cone Action-Angle Coordinates

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

The Reeb vector field $K := JX$ of such a toric Sasaki complex structure $J \in \mathcal{I}_S^{\mathbb{T}}(\check{M}, \omega, X)$ is given by

$$K = \sum_{i=1}^{n+1} b_i \frac{\partial}{\partial y_i} \quad \text{with} \quad b_i = 2 \sum_{j=1}^{n+1} s_{ij} x_j.$$

Lemma (Martelli-Sparks-Yau)

If $S(x) = (s_{ij}(x))$ is homogeneous of degree -1 , then

$K_S := (b_1, \dots, b_{n+1})$ is a constant vector.

In other words, the action generated by K is part of the torus action. Moreover,

regularity of toric KS cone \Leftrightarrow rationality of $K_S \in \mathbb{R}^{n+1}$.

Characteristic Hyperplane and Polytope

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

The **norm** of the Reeb vector field is then given by

$$\|K\|^2 = \|(\mathbf{0}, K_S)\|^2 = b_i s^{ij} b_j = b_i s^{ij} (2s_{jk} x_k) = 2b_i x_i = 2\langle x, K_S \rangle.$$

Hence

$$\|K\| > 0 \Leftrightarrow \langle x, K_S \rangle > 0 \quad \text{and} \quad \|K\| = 1 \Leftrightarrow \langle x, K_S \rangle = 1/2.$$

Definition (Martelli-Sparks-Yau)

The **characteristic hyperplane** H_K and **polytope** P_K of a toric Kähler-Sasaki cone (M, ω, X, μ, J) , with moment cone $C \subset \mathbb{R}^{n+1}$, are defined as

$$H_K := \{x \in \mathbb{R}^{n+1} : \langle x, K_S \rangle = 1/2\} \quad \text{and} \quad P_K := H_K \cap C.$$

Note that $N := \mu^{-1}(H_K)$ is a **toric Sasaki manifold** and P_K is the **moment polytope** of $B = M//K$.

Canonical Symplectic Potentials

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

Let $P \subset \mathbb{R}^{n+1}$ be a **polyhedral set** defined by

$$P = \bigcap_{a=1}^d \{x \in \mathbb{R}^{n+1} : \ell_a(x) := \langle x, \nu_a \rangle + \lambda_a \geq 0\}$$

where $\nu_1, \dots, \nu_d \in \mathbb{Z}^{n+1}$ are **primitive** integral vectors and $\lambda_1, \dots, \lambda_d \in \mathbb{R}$. Assume that P has a **non-empty interior** and the above set of defining inequalities is **minimal**.

Burns-Guillemin-Lerman extended Delzant's **symplectic reduction construction**, associating to each such polyhedral set a toric Kähler space of dimension $2(n+1)$

$$(M_P, \omega_P, \mu_P, J_P)$$

such that

$$\mu_P(M_P) = P \quad \text{and} \quad J_P \in \mathcal{I}^{\mathbb{T}}(M_P, \omega_P).$$

Canonical Symplectic Potentials

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

Guillemin'94, Burns-Guillemin-Lerman'05

In appropriate action-angle coordinates (x, y) , the canonical symplectic potential $s_P : \check{P} \rightarrow \mathbb{R}$ for $J_P|_{\check{P}}$ is given by

$$s_P(x) = \frac{1}{2} \sum_{a=1}^d \ell_a(x) \log \ell_a(x).$$

Note that $C := P$ is a **cone** iff $\lambda_1 = \cdots = \lambda_d = 0$. In this case, $s_C := s_P$ is the symplectic potential of a toric **Sasaki** complex structure $J_C \in \mathcal{I}_S^{\mathbb{T}}(M_C, \omega_C)$, since $S_C(x) = \text{Hess}_x(s_C)$ is then **homogeneous** of degree -1 . The corresponding **Reeb** vector field $K = (\mathbf{0}, K_C)$ is given by

$$K_C = \sum_{a=1}^d \nu_a.$$

Example: canonical symplectic potential of the standard cone over the standard simplex

The standard cone over the standard simplex is given by

$$C = \bigcap_{i=1}^{n+1} \{x \in \mathbb{R}^{n+1} : \ell_i(x) := \langle x, \nu_i \rangle \geq 0\},$$

where

$$\nu_i = e_i, \quad i = 1, \dots, n, \quad \text{and} \quad \nu_{n+1} = (-1, \dots, -1, 1).$$

Hence, using $r = \sum_{i=1}^n x_i$, we have that

$$s_C(x) = \frac{1}{2} \left(\sum_{i=1}^n x_i \log x_i + (x_{n+1} - r) \log(x_{n+1} - r) \right)$$

and

$$K_C = \sum_{i=1}^{n+1} \nu_i = (0, \dots, 0, 1).$$

General Cone Symplectic Potentials (Martelli-Sparks-Yau)

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

Let $s, s' : \check{C} \rightarrow \mathbb{R}$ be two symplectic potentials defined on the interior of a cone $C \subset \mathbb{R}^{n+1}$. Then

$K_s = K_{s'} \Leftrightarrow (s - s') + \text{const.}$ is homogeneous of degree 1.

Given $b \in \mathbb{R}^{n+1}$, define

$$s_b(x) := \frac{1}{2} (\langle x, b \rangle \log \langle x, b \rangle - \langle x, K_C \rangle \log \langle x, K_C \rangle).$$

Then $s := s_C + s_b$ is such that $K_s = b$. If C is good, this symplectic potential s defines a smooth Sasaki complex structure on the symplectic cone $(M_C, \omega_C, X_C, \mu_C)$ iff

$$\langle x, b \rangle > 0, \forall x \in C \setminus \{0\}, \text{ i.e. } b \in \check{C}^*$$

where $C^* \subset \mathbb{R}^{n+1}$ is the dual cone

$$C^* := \{x \in \mathbb{R}^{n+1} : \langle v, x \rangle \geq 0, \forall v \in C\}.$$

General Cone Symplectic Potentials (Martelli-Sparks-Yau)

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

Martelli-Sparks-Yau'05

Any toric Sasaki complex structure $J \in \mathcal{I}_S^T$ on a toric symplectic cone $(M_C, \omega_C, X_C, \mu_C)$, associated to a good moment cone $C \in \mathbb{R}^{n+1}$, is given by a symplectic potential $s : \check{C} \rightarrow \mathbb{R}$ of the form

$$s = s_C + s_b + h,$$

where s_C is the **canonical** potential, $b \in \check{C}^*$ and $h : C \rightarrow \mathbb{R}$ is **homogeneous of degree 1** and **smooth on $C \setminus \{0\}$** .

The dual cone C^* can be equivalently defined as

$$C^* = \cap_{\alpha} \{x \in \mathbb{R}^{n+1} : \langle \eta_{\alpha}, x \rangle \geq 0\},$$

where $\eta_{\alpha} \in \mathbb{Z}^{n+1}$ are the **primitive generating edges** of C .

Boothby-Wang Symplectic Potentials

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

Definition

Let $P \subset \mathbb{R}^n$ be an integral Delzant polytope and $C \subset \mathbb{R}^{n+1}$ its standard good cone. Given a symplectic potential $s : \check{P} \rightarrow \mathbb{R}$, define its **Boothby-Wang** symplectic potential $\tilde{s} : \check{C} \rightarrow \mathbb{R}$ by

$$\tilde{s}(x, z) := z s(x/z) + \frac{1}{2} z \log z, \quad \forall x \in \check{P}, z \in \mathbb{R}^+.$$

If $P = \bigcap_{a=1}^d \{x \in \mathbb{R}^n : l_a(x) := \langle x, \nu_a \rangle + \lambda_a \geq 0\}$ and

$$s(x) = \frac{1}{2} \sum_{a=1}^d l_a(x) \log l_a(x) - \frac{1}{2} l_\infty(x) \log l_\infty(x),$$

where $l_\infty(x) := \sum_a l_a(x) = \langle x, \nu_\infty \rangle + \lambda_\infty$, then

$$\tilde{s}(x, z) = s_C(x, z) + s_b(x, z) \quad \text{with} \quad b = (0, \dots, 0, 1).$$

Example: Boothby-Wang symplectic potential of the standard cone over the standard simplex

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

If $P \subset \mathbb{R}^n$ is the standard simplex and $r = \sum_i x_i$, then

$$s_P(x) = \frac{1}{2} \left(\sum_{i=1}^n x_i \log x_i + (1 - r) \log(1 - r) \right)$$

and

$$\begin{aligned} 2\tilde{s}_P(x, z) &= 2 \left(z s_P(x/z) + \frac{1}{2} z \log z \right) \\ &= \sum_{i=1}^n x_i \log(x_i/z) + (z - r) \log((z - r)/z) + z \log z \\ &= \sum_{i=1}^n x_i \log x_i + (z - r) \log(z - r) = 2s_C(x, z), \end{aligned}$$

where C is the standard cone over P .

Symplectic Potentials and Reduction

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

Calderbank-David-Gauduchon'02

Symplectic potentials **restrict** naturally under toric symplectic **reduction**.

Suppose (B_P, ω_P, μ_P) is a **toric** symplectic **reduction** of (M_C, ω_C, μ_C) . Then there is an **affine** inclusion $P \subset C$ and

any $\tilde{J} \in \mathcal{I}^{\mathbb{T}}(M_C, \omega_C)$ induces a reduced $J \in \mathcal{I}^{\mathbb{T}}(B_P, \omega_P)$.

This theorem says that if

$\tilde{s} : \check{C} \rightarrow \mathbb{R}$ is a symplectic potential for \tilde{J}

then

$s := \tilde{s}|_{\check{P}} : \check{P} \rightarrow \mathbb{R}$ is a symplectic potential for J .

Symplectic Potentials and Affine Transformations

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

Proposition

Symplectic potentials **transform** naturally under **affine** transformations.

Let $T \in GL(n)$ and consider the linear symplectic change of action-angle coordinates $x' := T^{-1}x$ and $y' := T^t y$.

Then $P = \bigcap_{a=1}^d \{x \in \mathbb{R}^n : \ell_a(x) := \langle x, \nu_a \rangle + \lambda_a \geq 0\}$ becomes

$$P' := T^{-1}(P) = \bigcap_{a=1}^d \{x' \in \mathbb{R}^n : \ell'_a(x') := \langle x', \nu'_a \rangle + \lambda'_a \geq 0\}$$

with $\nu'_a = T^t \nu_a$ and $\lambda'_a = \lambda_a$, and symplectic potentials transform by $s' = s \circ T$ (in particular, $s_{P'} = s_P \circ T$). The corresponding Hessians are related by $S' = T^t(S \circ T)T$.

Symplectic Potentials and Affine Transformations - Hirzebruch Surfaces

Toric
Kähler-Sasaki
Geometry

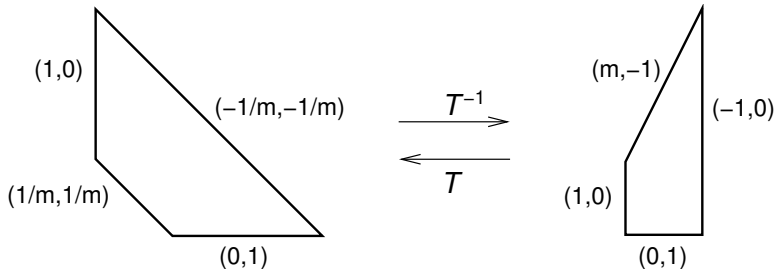
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Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics



$$T = \begin{bmatrix} m & -1 \\ 0 & 1 \end{bmatrix}$$

Symplectic Potentials and Scalar Curvature

Toric
Kähler-Sasaki
Geometry

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Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

- Toric **Kähler metric**

$$\begin{bmatrix} \mathbf{S} & 0 \\ 0 & \mathbf{S}^{-1} \end{bmatrix}$$

where $\mathbf{S} = (s_{ij}) = \text{Hess}_x(\mathbf{s})$ for a symplectic potential $\mathbf{s} : \check{P} \subset \mathbb{R}^n \rightarrow \mathbb{R}$.

- Formula for its **scalar curvature** [A.'98]:

$$Sc \equiv - \sum_{j,k} \frac{\partial}{\partial x_j} \left[s^{jk} \frac{\partial \log(\det \mathbf{S})}{\partial x_k} \right] = - \sum_{j,k} \frac{\partial^2 s^{jk}}{\partial x_j \partial x_k}$$

- **Donaldson'02** - appropriate interpretation for this formula: view **scalar curvature** as **moment map** for **action** of $\text{Ham}(M, \omega)$ on $\mathcal{I}(M, \omega)$.

Scalar Curvature of Boothby-Wang Symplectic Potentials

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

Proposition

Let $P \subset \mathbb{R}^n$ be a polyhedral set and $C \subset \mathbb{R}^{n+1}$ its standard cone. Given a symplectic potential $s : \check{P} \rightarrow \mathbb{R}$, let $\tilde{s} : \check{C} \rightarrow \mathbb{R}$ be its **Boothby-Wang** symplectic potential:

$$\tilde{s}(x, z) := z s(x/z) + \frac{1}{2} z \log z, \quad \forall x \in \check{P}, z \in \mathbb{R}^+.$$

Then

$$\tilde{Sc}(x, z) = \frac{Sc(x/z) - 2n(n+1)}{z}.$$

In particular

$$\tilde{Sc} \equiv 0 \Leftrightarrow Sc \equiv 2n(n+1).$$

Boothby-Wang Symplectic Potentials and Toric Kähler-Sasaki-Einstein metrics

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

In compact **Kähler** geometry, **constant scalar curvature** metric **plus** cohomological condition $c_1 = \lambda[\omega]$ implies **Kähler-Einstein** metric.

Proposition

Let $P \subset \mathbb{R}^n$ be a polyhedral set and $C \subset \mathbb{R}^{n+1}$ its standard cone. Given a symplectic potential $s : \check{P} \rightarrow \mathbb{R}$, let $\check{s} : \check{C} \rightarrow \mathbb{R}$ be its **Boothby-Wang** symplectic potential. Then,

s defines a toric **Kähler-Einstein** metric with $Sc \equiv 2n(n+1)$

iff

\check{s} defines a toric **Ricci-flat Kähler** metric.

When this happens, the corresponding toric **Sasaki** metric is **Einstein**.

Calabi's Family of Extremal Kähler Metrics

Toric
Kähler-Sasaki
Geometry

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Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

- Calabi constructed in **1982** a general **4**-parameter family of $U(n)$ -invariant **extremal** Kähler metrics, which he used to put extremal Kähler metrics on

$$H_m^n := \mathbb{P}(\mathcal{O}(-m) \oplus \mathbb{C}) \longrightarrow \mathbb{P}^{n-1}, \quad n, m \in \mathbb{N},$$

in **any** possible cohomology **class**. In particular, when $n = 2$, on all **Hirzebruch** surfaces.

- When written in **action-angle** coordinates, using symplectic potentials, Calabi's **family** can be seen to **contain** many **other** interesting Kähler **metrics** [A.'01].
- In particular, it contains a **1**-parameter family of **Kähler-Einstein** metrics directly **related** to the **Sasaki-Einstein** metrics constructed in **2004** by Gauntlett-Martelli-Sparks-Waldram [A.'08].

Calabi's Family in Action-Angle Coordinates

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

Consider symplectic potentials $s : (\mathbb{R}^+)^n \rightarrow \mathbb{R}$ of the form

$$s(x) = \frac{1}{2} \left(\sum_{a=1}^n x_a \log x_a + h(r) \right), \text{ where } r = x_1 + \cdots + x_n.$$

Then

$$Sc(x) = Sc(r) = 2r^2 f''(r) + 4(n+1)rf'(r) + 2n(n+1)f(r),$$

where $f = h''/(1 + rh')$. Moreover, **extremal** is equivalent to Sc being an **affine** function, which is then equivalent to

$$h''(r) = -\frac{1}{r} + \frac{r^{n-1}}{r^n - A - Br - Cr^{n+1} - Dr^{n+2}},$$

where $A, B, C, D \in \mathbb{R}$ are the 4 **parameters** of the family.

Interesting Particular Cases of Calabi's Family

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

$$[C = D = 0]$$

$$h''(r) = -\frac{1}{r} + \frac{r^{n-1}}{r^n - A - Br}$$

$$[C = D = 0, B = 0, A > 0]$$

Complete **Ricci flat** Kähler metrics on total space of $\mathcal{O}(-n) \rightarrow \mathbb{P}^{n-1}$, for **any** possible cohomology **class**.
(Calabi'79)

$$[C = D = 0, A, B > 0]$$

Complete **scalar flat** Kähler metrics on total space of $\mathcal{O}(-m) \rightarrow \mathbb{P}^{n-1}$, for any $m > 0$ and **any** possible cohomology **class**. (LeBrun'88 ($n = 2$), Pedersen-Poon'88 and Simanca'91 ($n > 2$))

Kähler-Sasaki-Einstein Cases in Calabi's Family

Toric
Kähler-Sasaki
Geometry

Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics

$$[B = D = 0, C = 1, 0 < A < n^n / (n + 1)^{n+1}]$$

$$h''(r) = -\frac{1}{r} + \frac{r^{n-1}}{r^n(1-r) - A}$$

Kähler-Einstein quasifold metrics with $Sc = 2n(n + 1)$ on certain $H_m^n := \mathbb{P}(\mathcal{O}(-m) \oplus \mathbb{C}) \rightarrow \mathbb{P}^{n-1}$.

For a countably infinite set of values for the variable parameter A , the corresponding Boothby-Wang cones are $GL(n + 1)$ equivalent to good cones - precisely the ones corresponding to the Sasaki-Einstein metrics of Gauntlett-Martelli-Sparks-Waldram'04 (at least when $n = 2$).

Kähler-Sasaki-Einstein Cases in Calabi's Family - $n = 2$ examples

Toric
Kähler-Sasaki
Geometry

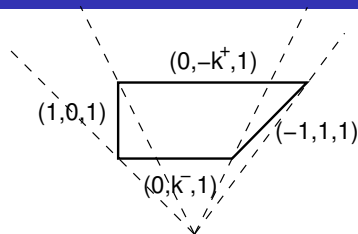
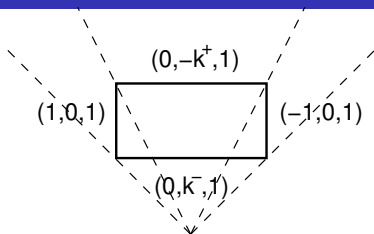
Miguel Abreu

Introduction

Toric K
Metrics

Toric KS
Metrics

Toric KSE
Metrics



$$p, q \in \mathbb{N}, \quad 0 < q < p, \quad \gcd(q, p) = 1$$

$$\text{even } p \pm q$$

$$k^\pm = (p \pm q)/2$$

$$K_s = (0, -\ell/2, 3)$$

$$\text{odd } p \pm q$$

$$k^\pm = (p \pm (q - 1))/2$$

$$K_s = (0, (3 - \ell)/2, 3)$$

$$\ell = \left(3q^2 - 2p^2 + p\sqrt{4p^2 - 3q^2} \right) / q$$

All these **Sasaki-Einstein** manifolds are diffeo. to $S^2 \times S^3$.