APPROXIMATELY HOLOMORPHIC GEOMETRY AND ESTIMATED TRANSVERSALITY ON 2-CALIBRATED MANIFOLDS

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ABSTRACT. The notion of 2-calibrated structure, generalizing contact structures, smooth taut foliations, etc, is defined. Approximately holomorphic geometry as introduced by S. Donaldson in [4] for symplectic manifolds (see also [2]) is extended to 2-calibrated manifolds. An estimated transversality result that enables to study the geometry of such manifolds is presented.

1. INTRODUCTION

Approximately holomorphic geometry was introduced by S. Donaldson [4], [5] to investigate the topology of symplectic manifolds. After Donaldson's insight, a number of results along similar lines have been obtained ([1, 2, 10, 15, 13, 12]). Instead of considering holomorphic sections of a complex bundle as a basic tool to study the topology of almost complex manifolds, that will not exist in general, approximately holomorphic geometry uses sequences of sections of very positive bundles that behave asymptotically as holomorphic sections. Then, if they satisfy an estimated transversality condition with respect to a well behaved stratification, we can pull it back to obtain a finer geometric picture of the underlying manifold (see [7] for an instance of this idea). In this note we will introduce a new setting where such approximately holomorphic geometry can be constructed and where an uniform transversality principle that covers instances previously considered in the literature holds.

The background space will be a 2-calibrated manifold, which is a smooth manifold M of dimension $2n + 1$ carrying a codimension 1 distribution D, and ω a closed 2-form non-degenerate on D. This notion includes as particular instances those of contact manifolds, certain constant rank Poisson manifolds, smooth taut (con)foliations, etc. If $[\omega/2\pi]$ is of integer class, once an integer lift Ω has been fixed, we shall denote by (L, ∇) the unique, up to isomorphism, hermitian line bundle with connection over M with Chern class Ω and curvature $-i\omega$. We shall fix once for all an almost complex structure J compatible with the 2-form ω on the bundle defined by the distribution D. We shall also fix a metric g on M such that $g_{|D}(\cdot, \cdot) = \omega(\cdot, J\cdot)$ and $\ker \omega = D^\perp$. The rescaled metrics kg will be consistently denoted by g_k.

If $E \to M$ is a fixed hermitian bundle with connection, we denote by E_k the sequence $E \otimes L^{\otimes k}$.

Theorem 1. *Let $E_k \to (M, D, J, g)$ and $\mathcal{S}_k = (S_k^a)_{a \in A_k}$ an approximately holomorphic sequence of finite Whitney quasi-stratifications of the space of pseudo-holomorphic r-jets along D, $\mathcal{J}_D^r E_k$. Let δ be a positive constant and h a natural number. Then for any approximately holomorphic sequence of sections τ_k of E_k there exist a constant $\eta > 0$ and an approximately holomorphic sequence of sections σ_k of E_k so that for k large enough,*

(1) $|\nabla_D^j(\tau_k - \sigma_k)|_{g_k} < \delta, \ j = 0, ..., r + h.$

(2) $j_D^r \sigma_k$, *the* r-*jet prolongation along* D *of* σ_k, *is* η-*transversal to* $\mathcal{S}_k$.

A submanifold $i \colon N \hookrightarrow (M, D, \omega)$ is 2-calibrated if $(i^*D, i^*\omega)$ defines a 2-calibrated structure. A simple application of the previous theorem is the following corollary.

Corollary 1. *Let* (M^{2n+1}, D, ω) *be a closed oriented 2-calibrated manifold of integer class. For any fixed point* $y \in M$ *it is possible to find 2-calibrated submanifolds* W_k *of* M *of real codimension* $2m$ *through* y, *such that for* k *large enough the inclusion* $i \colon W_k \hookrightarrow M$ *induces maps* $i_* \colon \pi_j(W_k) \to \pi_j(M)$ *(resp.* $i_* \colon H_j(W_k; \mathbb{Z}) \to H_j(M; \mathbb{Z})$) *which are isomorphisms for* $j = 0, ..., n - m - 1$ *and epimorphisms for* $j = n - m$. *In addition, for* $m = 1$, *the Poincaré dual of* $[W_k]$ *is* $[k\Omega]$.

When D is integrable we speak of 2-calibrated foliations. In dimension 3 these are smooth taut foliations.

Corollary 2. *Let* (M^{2n+1}, D, ω) *be a closed 2-calibrated foliation of integer class. Then for* k *large enough it is possible to find maps* $\phi_k \colon M \to \mathbb{CP}^{2n}$ *verifying:*

(1) ϕ_k *is a leafwise immersion.*
(2) *The restriction of* $\phi_k^* \omega_{FS}$ *to the leaves of* D *coincides with* $k\omega$, *where* ω_{FS} *is the Fubini-Study 2-form. In other words, the Poisson structure defined by* $\omega_{|D}$ *is induced from the symplectic structure* $(\mathbb{CP}^{2n}, \omega_{FS})$ *via* ϕ_k.

Corollary 1 can be understood as a generalization in one direction of Sullivan's theory for surface foliations of M^3 [16], relating leafwise positive closed 2-forms and transverse cycles (via foliated cycles). Our results rely on the existence of plenty of sections very close to be holomorphic. If D is a codimension 1 foliation by complex leaves and L a positive hermitian line bundle, Ohsawa and Sibony [14] have shown the existence of enough holomorphic sections to construct a leafwise holomorphic embedding in some projective space (of arbitrarily high, though finite, order of regularity by working with high powers of L).

More generally for laminations by riemann surfaces of a compact space, E. Ghys [6] has given conditions according to the type of the leaves granting the existence of enough meromorphic functions. In a similar spirit, if M is a compact lamination by complex leaves without vanishing cycle, and L is a positive hermitian line bundle, B. Deroin [3] in his thesis has shown the existence of plenty of holomorphic sections that produce holomorphic embeddings in projective spaces (continuous in the transverse direction). Interestingly enough, the sections are obtained by patching certain holomorphic sections defined in the universal covers of the leaves (as done by Ghys), each of which is constructed by refining the approximately holomorphic theory for compact hermitian manifolds to hermitian manifolds with bounded geometry (see also related ideas in [8]).

2. Approximately holomorphic geometry

Let (M, D, ω, J) be a 2-calibrated manifold of integer type. The sequence $\pi_k \colon L^{\otimes k} \to M$ of powers of the pre-quantum line bundle is a *very ample sequence of line bundles*: its curvature F_k verifies (i) $|F_{k|D} - F_{k|D}^{1,1}|_{g_k} \leq \delta k^{-1/2}$, (ii) $|\nabla^j F_k|_{g_k} \leq C_j, \forall j \geq 0$ and (iii) $\| iF_k(v, Jv) \| \geq \gamma g_k(v, v), \forall v \in D$, with δ, γ, C_j independent of k.

For k large and suitable charts centered around any point x, $(B_{g_k}(x, c), D, J, g_k)$ looks very much like the euclidean unit ball of $\mathbb{C}^n \times \mathbb{R}$ with its canonical CR structure (the foliation by complex hyperplanes). There, conditions (i) and (ii) ensure that $(L^{\otimes k}, \pi_k^* D)$ –with the almost complex structure induced by J, the connection and the complex structure of the fibers– is very close to be a CR structure. The positivity condition (iii) will imply the existence of many (sequences of) sections very close to be holomorphic.

Definition 1. *A sequence τ_k of E_k is approximately holomorphic with bounds $C_j^D, C_j > 0$ (A.H.(C_j^D, C_j) or just A.H.) if $\sum_{l=0}^{j} |\nabla_D^l \tau_k|_{g_k} \leq C_j^D$, $\sum_{l=0}^{j} |\nabla^l \tau_k|_{g_k} \leq C_j$, $\sum_{l=0}^{j} |\nabla^{l-1} \bar{\partial} \tau_k|_{g_k} \leq C_j k^{-1/2}$, for all $j \in \mathbb{N}$.*

Definition 2. *A section τ of $\pi \colon E \to M$ is η-transverse along D to the zero section if $|\nabla \tau_D(x)| \geq \eta$ for all x with $|\tau(x)| \leq \eta$. More geometrically, for those x such that $|\tau(x)| \leq \eta$, the minimum angle (see [13]) at $\tau(x)$ between the subspaces obtained by intersecting $\pi^* D$ with the graph of τ and the horizontal subspace of the connection ∇, has to be greater than some η'.*

This geometrical definition easily extends to other subspaces of E and more generally to suitable stratifications (see [11, 9]).

If τ_k is an A.H. sequence of sections of $\mathbb{C}^{m+1} \otimes L^{\otimes k}$ uniformly transverse along D (η-transverse for k large enough) to the zero section, its pullback defines a 2-calibrated submanifold B_k. "Genericity" for the projectivizations $\phi_k \colon M \backslash B_k \to \mathbb{CP}^m$ is obtained by solving certain uniform transversality problems for its *pseudo-holomorphic jets*. In turn, those problems can be "pulled back" to uniform transversality problems for the *pseudo-holomorphic jets* of τ_k.

We define the bundles $\mathcal{J}_D^r E_k := \sum_{l=0}^{r} ((D^{*1,0})^{\odot l}) \otimes E_k$ where $(D^{*1,0})^{\odot l}$ denotes the symmetric tensor product of l copies of the bundle $D^{*1,0}$. These bundles inherit natural metrics and connections $\nabla_{k,j}$, and there is an obvious notion of approximately holomorphic sequence of sections of $\mathcal{J}_D^r E_k$.

Definition 3. *Let τ_k be a sequence of sections of (E_k, ∇_k). The pseudo-holomorphic r-jet prolongation $j_D^r \tau_k$ along D is a section of the bundle $\mathcal{J}_D^r E_k$ defined recursively by taking the holomorphic 1-jet associated to $\nabla_{k,j}$ to obtain an element of $D^{*1,0} \otimes (\sum_{l=0}^{r} ((D^{*1,0})^{\odot l}) \otimes E_k)$, and then symmetrizing to obtain –after adding the section τ_k itself (the degree 0 component)– a section of $\mathcal{J}_D^{r+1} E_k$.*

If $\tau_k \in \Gamma(E_k)$ is A.H, the presence of curvature prevents $j_D^r \tau_k \in \Gamma(\mathcal{J}_D^r E_k)$ from being A.H. Still, it is possible to modify the connection on $\mathcal{J}_D^r E_k$ so that $j_D^r \tau_k$ becomes A.H. and thus uniform transversality can be achieved (see [11, 9] for more details on the construction and proofs).

3. A few words about the proof

The proof of theorem 1 follows the same pattern as that of D. Auroux in [2]. The strata S_k^a of approximately holomorphic quasi-stratifications $(S_k^a)_{a \in A_k}$ (see [11]) are locally defined by functions $f_k \colon U_k \to \mathbb{C}^p$ so that if $\xi_k \colon B_{g_k}(x, c) \approx \mathbb{C}^n \times \mathbb{R} \to U_k$ is A.H, then $f_k \circ \xi_k \colon \mathbb{C}^n \times \mathbb{R} \to \mathbb{C}^p$ is also A.H. Moreover, η-transversality along D of ξ_k to S_k^a in $B_{g_k}(x, c)$ is equivalent to $c_a \eta$-transversality along D of $f_k \circ \xi_k$ to zero in the unit ball. By using the standard globalization procedure of approximately holomorphic geometry ([2, 4]), one finds that uniform transversality to $(S_k^a)_{a \in A_k}$ follows from the following local estimated transversality result which refines the one given in [15].

Proposition 1. *([11]) Let B^+ the unit ball of $\mathbb{C}^n$ and $F \colon B^+ \times [0,1] \to \mathbb{C}^p$, $0 < \delta < \frac{1}{2}$ a constant and $\sigma = \delta(\log(\delta^{-1})^{-e}$, where e is a fixed integer depending only on the dimensions n, p. Assume that for any $s \in [0,1]$, the estimates $|F_s| \leq 1$, $|\bar{\partial} F_s| \leq \sigma$, $|\nabla \bar{\partial} F_s| \leq \sigma$ hold for F_s in B^+ (using the leafwise canonical complex structure). Then, a smooth curve $w \colon [0,1] \to \mathbb{C}^p$ exists such that $|w| < \delta$ and the function $F - w$ is σ-transverse to $\mathbf{0}$ over $B(0, 1/2)$ along the directions of $\mathbb{C}^n$. Moreover, if $|\nabla^j \partial F / \partial s| < C_j$ for all $j \in \mathbb{N}$, then w can be chosen so that $|d^j w / ds^j| < \Phi_j(\delta)$, for all $j \in \mathbb{N}$ and $d^j w / ds^j(0) = 0$ y $d^j w / ds^j(1) = 0$ for all $j \in \mathbb{N}$, where Φ_j is a function only depending on the dimensions n, p.*

The improvement obtained here is the control of all the derivatives of the perturbation, which is needed to solve strong transversality problems in $\mathcal{J}_D^r E_k$, $r > 0$ (see [11]).

Corollary 1 (resp. corollary 2) is the result of applying theorem 1 to the sequence of zero sections of $E \otimes L^{\otimes k}$ (resp. to an approximately holomorphic quasi-stratification of $\mathcal{J}_D^1(\mathbb{C}^{2n+1} \otimes L^{\otimes k})$ generalizing the Thom-Boardmann stratification, together with the leafwise version of Moser's theorem).

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