

LEFSCHETZ PENCIL STRUCTURES FOR 2-CALIBRATED MANIFOLDS

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ABSTRACT. We prove that for closed 2-calibrated manifolds there always exist Lefschetz pencil structures. This generalizes similar results for symplectic [4] and contact manifolds [10].

1. INTRODUCTION

The programme started by S. Donaldson [3] by using approximately holomorphic geometry to study the topology of symplectic manifolds has been extended recently to the wider framework of 2-calibrated manifolds (see [7] and references therein). In the same vein Donaldson was able to show the existence of Lefschetz pencil structures for closed symplectic manifolds, extending in this way well-known ideas in Hodge geometry to the symplectic category. In this note we will show the existence of the corresponding notion of Lefschetz pencils for 2-calibrated structures generalizing the construction by F. Presas [10] for contact manifolds (see also the related open book decompositions introduced in [6]).

A 2-calibrated manifold is a smooth manifold M of dimension $2n + 1$ carrying a codimension 1 distribution D , and a closed 2-form ω non-degenerate on D [7]. The notion of 2-calibrated manifold includes as particular instances those of contact manifolds, certain constant rank Poisson manifolds, positive confoliations, and as a very particular instance smooth taut foliations of 3-manifolds. Thinking precisely on the later example, we could imagine a Lefschetz pencil as a map defining on each leaf a ramified covering of the Riemann sphere by a nearly holomorphic map with singular points of index 2. Thus the ramification set would be given by a link transversal to the foliation and such that its image would be in general position. A natural generalization for 2-calibrated manifolds is (see also [10]):

Definition 1. *Let (M, D, ω) be a 2-calibrated manifold. A Lefschetz pencil structure on M is given by a triple (f, A, B) , where B is a compact real codimension four 2-calibrated submanifold, and $f: M \setminus B \rightarrow S^2$ is a smooth map whose set of singular points along D is the compact 2-calibrated one-dimensional submanifold A . The image of A (with possibly many connected components) has to be an immersed curve with transverse double points. Moreover, f has the following local structure:*

- (1) *For each point $x \in B$ there are coordinates z_1, \dots, z_n, s compatible with (ω, D) centered at x and a holomorphic chart in $\mathbb{C}\mathbb{P}^1$, such that in a small enough ball the set B is given by $z_1 = z_2 = 0$, and out of B the map is given by $f(z, s) = z_2/z_1$.*
- (2) *For each point $y \in A$ there are coordinates z_1, \dots, z_n, s compatible with (ω, D) centered at y and a holomorphic chart in $\mathbb{C}\mathbb{P}^1$, such that $f(z, s) = g(s) + z_1^2 + \dots + z_n^2$, where $g(0) = f(y)$ and $g'(0) \neq 0$.*

The inverse image of a regular point c is thus an open 2-calibrated submanifold in $M \setminus A$. Because of the local model around points in B , the closure of $f^{-1}(c)$ is the closed calibrated submanifold $f^{-1}(c) \cup B$, that will be called the fibre of f at c . We can state now the main result of this note.

Theorem 1. *Any closed 2-calibrated manifold has a Lefschetz pencil structure.*

Applying a slightly refined version of this result to a smooth taut foliation \mathcal{F} of M^3 we get:

Corollary 1. *There exist a pair (f, A) where A is a link transversal to \mathcal{F} and $f: M^3 \rightarrow S^2$ is a smooth map which (i) is a submersion along $T\mathcal{F}$ on $M \setminus A$, (ii) $f(A)$ is in general position and (iii) around any point $x \in A$ there exist coordinates z, s compatible with (ω, \mathcal{F}) , and a complex coordinate in S^2 such that $f(z, s) = g(s) + z^2$, with $g(0) = 0$, $g'(0) \neq 0$.*

In dimension 3 the foliation is by Riemann surfaces. Our techniques do not produce in general meromorphic functions. These have been constructed by E. Ghys [5] for laminations by Riemann surfaces of compact spaces (see also [2] for more general results), but they are not generic in our sense. It will be shown elsewhere that when $(M, D, \omega) \subset \mathbb{C}\mathbb{P}^N$, D is a distribution by complex subspaces and ω is the restriction of the Fubini-Study 2-form, an approximately holomorphic theory yielding holomorphic results can be developed.

2. LOCAL MODELS AND APPROXIMATELY HOLOMORPHIC COORDINATES

We will use systematically local charts that *coincide approximately* with the flat model $(\mathbb{C}^n \times \mathbb{R}, D_h, J_0, g_0)$ for a 2-calibrated manifolds, where (D_h, J_0) is the natural integrable horizontal distribution $\mathbb{C} \times \{\cdot\}$ with its canonical leafwise complex structure and g_0 the Euclidean metric.

Let g_k denote the rescaled metric kg .

Definition 2. *A family of adapted charts to the sequence (M, D, g_k) is a set of maps $\psi_{k,x}: (\mathbb{C}^n \times \mathbb{R}, 0) \rightarrow (U_{k,x}, x)$, $x \in M$, such that $\psi_{k,x}^* D_x = D_h(0)$ and, for k large enough, there are constants $\gamma, \rho_0 > 0$, with $\gamma^{-1}g_0 \leq g_k \leq \gamma g_0$ such that $|\nabla^j \psi_{k,x}^{-1}|_{g_k} \leq O(1)$ in $B_g(x, \rho_0)$.*

Definition 3. *Let E_k be a sequence of either hermitian or $O(n)$ -vector (or unitary) bundles with compatible connection ∇_k over the almost CR manifold (M, D, J, g) and let $T_k \in \Gamma(E_k)$. We will say that T_k approximately vanishes up to order r , and we will denote it by $T_k \approx_r 0$, if for every k large enough the inequalities $|\nabla^j T_k|_{g_k} \leq O(k^{-1/2})$, $j = 0, \dots, r$, hold.*

If the inequalities hold for all r , we will say that we have an approximated equality and we will denote it simply by $T_k \approx 0$.

If $E_k = E$ is a constant sequence, in local adapted coordinates x_k^1, \dots, x_k^{2n+1} definition 3 requires

$$|\partial^{|p|} \partial x_k^p (\psi_{k,x}^* T_k)|_{g_0} \leq O(k^{-1/2}), \quad p = (p_1, \dots, p_{2n+1}), \quad |p| = p_1 + \dots + p_{2n+1}, \quad |p| \leq r,$$

at points in $B_{g_k}(0, O(1))$ and independently of k and x .

The choice of adapted charts provides trivializations of the bundles associated to TM . Therefore, by choosing a fixed tensor T_0 of the corresponding trivial bundle we are selecting a local tensor for each k and x (for example J_0). In this situation, we say that a given sequence of tensors T_k of the corresponding bundle coincides with T_0 up to order r , if at the points of $B_{g_k}(0, O(1))$ (and independently of k and x)

$$|\partial^{|p|} / \partial x_k^p (\psi_{k,x}^* T_k - T_0)|_{g_0} \leq O(k^{-1/2}), \quad p = (p_1, \dots, p_{2n+1}), \quad |p| = p_1 + \dots + p_{2n+1}, \quad |p| \leq r$$

Definition 4. *We will say that a family $\psi_{k,x}: \mathbb{C}^n \times \mathbb{R} \rightarrow U_{k,x} \subset M$ of local adapted charts for (M, D, J, g_k) is approximately holomorphic if $\psi_{k,x}^* D \approx D_h$, $V_{k,x}$ -defined to be the unique local vector field such that $\partial / \partial s_k + V_k \in \psi_{k,x}^*(D^\perp)$ - is of size $O(1)$*

and all its derivatives of size $O(k^{-1/2})$, and the corresponding coordinate functions $z_k^j: U_{k,x} \rightarrow \mathbb{C}$ are approximately holomorphic (see [7]).

Approximately holomorphic coordinates as defined in this note always available. We will assume that we have fixed one such a family. The important property is that a sequence of sections τ_k of E_k is approximately holomorphic (see [7]) if and only if its pullback by the approximately holomorphic charts is a sequence of approximately holomorphic sections (defined in $B(0, O(1))$) of the almost CR manifold $(\mathbb{C}^n \times \mathbb{R}, D_h, J_0, g_0)$ (see [9, 1] for a more general definition of approximately holomorphic coordinates with takes care of what happens in the full domain $B_{g_k}(0, O(k^{1/2}))$).

If we do not take into account a chosen almost CR structure associated to a 2-calibrated manifold, we have the following notion.

Definition 5. *Let (M, D, ω) be a 2-calibrated manifold. We will say that a local chart $\varphi: \mathbb{C}^n \times \mathbb{R} \rightarrow M$ is compatible with ω if $\varphi_*(D_h)$ coincides with D at the origin and, with respect to the canonical complex structure on D_h , the restriction of ω is positive and $(1, 1)$.*

3. PROOF OF THE MAIN THEOREM

We shall denote by E_k as above the sequence of very ample hermitian bundles $\mathbb{C}^2 \otimes L^{\otimes k} \rightarrow (M, D, J, g)$. The bundle of pseudo-holomorphic 1-jets along $D - \mathcal{J}_D^1 E_k$ has an approximately holomorphic quasi-stratification with strata Z_k and $\Sigma_{k,n}$ (see [9, 7]). The main theorem of [7] asserts the existence of approximately holomorphic sequences of sections τ_k of E_k uniformly transversal along D to these quasi-stratification. Quotienting the components of τ_k we will obtain an approximately holomorphic sequence of maps $\phi_k: M \setminus B_k \rightarrow \mathbb{C}\mathbb{P}^1$ verifying:

- (1) The set of base points $B_k = \tau_k^{-1}(Z_k)$ is a compact submanifold of real codimension 4 uniformly transversal to D .
- (2) The set $\Sigma_n(\phi_k)$ given by the points where $\partial\phi_k$ is singular is a compact submanifold of codimension $2n$ uniformly transversal to D .

The required local model for ϕ_k around the points of B_k follows from the following result.

Proposition 1. *(see [10, 9]) For every $x \in B_k$ there exist approximately holomorphic coordinates z_k^1, \dots, z_k^n, s_k , centered at x and a holomorphic chart on $\mathbb{C}\mathbb{P}^1$ such that in $B_{g_k}(x, O(1))$, B_k is given by $z_k^1 = z_k^2 = 0$, and $\phi_k(z_k, s_k) = z_k^2/z_k^1$.*

Regarding the points of the singular link, the following result holds (see [9]):

Proposition 2. *There exists a perturbation τ'_k of the sequence τ_k with $\tau'_k \cong \tau_k$, approximately holomorphic coordinates around any point $y \in \Sigma_n(\phi'_k)$ compatible with ω , and holomorphic charts in $\mathbb{C}\mathbb{P}^1$, such that the formula for the maps $\phi'_k = (\tau'_k)^2/(\tau'_k)^1$ is $\phi'_k(z_k, s_k) = \phi'_k(0, s_k) + (z_k^1)^2 + \dots + (z_k^n)^2$.*

Being more precise, we can prove that it is possible to find radius $\rho_2 > \rho_1 > 0$ and new distributions D_k , almost-complex structures J_k and functions ϕ'_k , such that:

- (1) $D_k \cong D$ and $J_k \cong J$, where $D_k = D$, $J_k = J$ out of a tubular neighborhood of $\Sigma_n(\phi_k)$ or radius ρ_2 , and both are integrable in the tubular neighborhood of radius ρ_1 .
- (2) $\phi_k \cong \phi'_k$, with $\phi_k = \phi'_k$ out of a tubular neighborhood of $\Sigma_n(\phi_k)$ or radius ρ_2 , and ϕ'_k holomorphic in the tubular neighborhood of radius ρ_1 .

For k large enough, the triple $(A_k, \phi'_k, \Sigma_n(\phi'_k))$ is the Lefschetz pencil structure of theorem 1. The genericity of $\phi'_k(\Sigma_n(\phi'_k))$ is a direct consequence of Prop. 2 by considering a perturbation independent of the holomorphic coordinates. We also

notice that the coordinates obtained by using Prop. 1 are not in general compatible with ω . In any case an easy adaptation of the argument in [4, 10] allows us to modify the function in such a way that we have the desired coordinates.

If the distribution of the 2-calibrated manifold (M, D, J, g) integrates into a foliation \mathcal{F} it is not necessary to choose a retraction r for the projection $p: T^*M \rightarrow D^*$ to define higher covariant derivatives along D (and its ∂ and $\bar{\partial}$ components). We can use on each leaf the Levi-Civita connection defined by $g|_{\mathcal{F}}$. In this way we obtain a natural (foliated) approximately holomorphic theory on M . The approximately holomorphic charts and the charts compatible with ω can be chosen to send \mathcal{F} into the horizontal distribution D_h . In this foliated theory we will obtain again the same estimated transversality theorems and normal forms that were obtained in [7] and [9]. We will attach a subindex \mathcal{F} for objects constructed in the foliated theory. It is easy to show that for E_k very ample, $\tau_k \in \Gamma(E_k)$ is approximately holomorphic if and only if it is approximately holomorphic for the foliated theory. Moreover, we have bundle maps $r_j: (D^{*1,0})^{\odot p} \otimes E_k \rightarrow (r(D)^{*1,0})^{\odot p} \otimes E_k$, induced by the retraction r associated to the metric. The important observation at this point is that the image of $j_{\mathcal{F}}^p \tau_k$ coincides approximately with $j_D^p \tau_k$ and that the corresponding Thom-Boardmann quasi-stratifications are related by the bundle morphism $\mathcal{J}_{\mathcal{F}}^p E_k \rightarrow \mathcal{J}_D^p E_k$ induced by r . Thus, if $j_D^p \tau_k$ is uniformly transversal to the Thom-Boardmann quasi-stratification, the same will happen for $j_{\mathcal{F}}^p \tau_k$. Thus we conclude that the results obtained in [7] and Thm. 1 (using the foliated analogs of propositions 1 and 2) also hold for foliated almost-complex manifolds and its corresponding foliated almost holomorphic geometry discussed here. In particular, this proves corollary 1.

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