APPROXIMATELY HOLOMORPHIC METHODS IN SYMPLECTIC AND RELATED GEOMETRIES.

D. MARTÍNEZ TORRES

1. Introduction

In 1996 S. Donaldson proved the following remarkable result:

Theorem 1. [5] *Let (M, ω) be a compact symplectic manifold. Then it contains symplectic submanifolds of real codimension 2.*

Remark 1. *Notice that by iterating theorem 1 one can construct symplectic submanifolds of any dimension.*

Let us expand on the two notions alluded to in the theorem, submanifolds and symplectic geometry.

2. Submanifolds

In this section we recall some basic results in differential topology pertaining the (global) description of submanifolds of a manifold, and their construction. A good general reference for this section is [9]. We note that all our submanifolds, vector bundles, and sections are smooth.

Let $N \subset M$ be a submanifold. We will assume M, N to be compact, though N closed would be enough. We want to describe N using function theory. As for any closed subset of a manifold we know that there exists $F \colon M \to [0,1]$ (smooth), such that $N = f^{-1}(0)$. Of course, that does not make any distinction between a submanifold and any other closed subset of N. The distinctive feature of a submanifold is the existence of adapted coordinates about any point. That is, coordinates

$$x_1, \ldots, x_{m-d}, x_{m-d+1}, \ldots, x_m \colon (U, x) \to (\mathbb{R}^m, 0),$$

so that

$$N \cap U = \{x_{m-d+1}, \ldots, x_m = 0\}.$$

In other words, there exists $F \colon U \cap N \to \mathbb{R}^d$ so that 0 is a regular value and

$$N \cap U = F^{-1}(0).$$

An equivalent way of presenting the previous result -which makes it clear how to go to the global result- is tranforming F into a section of a trivial bundle

$$\begin{array}{rcl} s\colon U & \longrightarrow & U \times \mathbb{R}^d \\ x & \longmapsto & (x, F(x)), \end{array}$$

and therefore conclude the existence of a section $s \pitchfork \mathbf{0}$ so that $N \cap U = s^{-1}(\mathbf{0})$.

With a bit of work one can globalize to obtain the following result:

Proposition 1. *Let $N \subset M$ be a submanifold of codimension d. Then there exist a rank d vector bundle $E \to M$ together with a section s transverse to the zero section $\mathbf{0}$, and so that $N = s^{-1}(\mathbf{0})$.*

Proof. Let $\nu(N) \to N$ be the normal bundle and $\Phi\colon (\nu(N), N) \to (\mathcal{U}, N)$ a tubular neighborhood of N. Use the projection $\nu(N) \to N$ to pullback $\nu(N)$ to its total space. The result is $E' \to \nu(N)$ a vector bundle with a tautological section s' which is obviously transverse to the zero section, and has zero set N. Using the diffeomorphism furnished by the tubular neighborhood we have solved the problem over $\mathcal{U}$, and we need to extend $E'_{|\partial\mathcal{U}}$ and $s'_{|\partial\mathcal{U}}$ to a vector bundle over $M\backslash \mathrm{int}\nu(M)$ with no-where vanishing section. This problem can always be solved: the section determines a trivialized rank 1 bundle $\underline{\mathbb{R}}$ over $\partial\mathcal{U}$, and therefore and splitting

$$E'_{|\partial\mathcal{U}} = \underline{\mathbb{R}} \oplus F'.$$

Thus our problem reduces to extend F'. This is always possible. Recall that a bundle is given by a homotopy class of maps to $\mathrm{Gl}(d,\infty)$. To extend it simply take any section of the universal bundle and compose with it; we get a map to $E(d,\infty)$, which is contractible. Therefore any map to it can be extended. Compose the extension with the bundle map projection to obtain the desired extension of the map to $\mathrm{Gl}(d,\infty)$. □

Conversely, if we are given any rank d vector bundle $E \to M$, any section $s \pitchfork \mathbf{0}$ determines a codimension d submanifold defined as

$$N := s^{-1}(\mathbf{0}).$$

That we can always find sections transverse to the zero section is a classical result by Thom.

Theorem 2. *Let M, P be manifolds, and Q a submanifold of P. For simplicity we assume that all are compact. Then the subset $C_{\pitchfork}(Q; M, P)$ of maps transverse to Q is dense and open in $C^\infty(M,P)$, where the latter set carries the Whitney C^r-topology, $r \geq 1$.*

Proof. Check for example the transversality chapter in [9]. □

3. Measuring transversality

We want to apply theorem 2 to $M, P = E$ and Q the zero section of E (actually a slight variation, since we want to produce not just a map $M \to E$, but a section of the vector bundle). We find very useful for the purposes of these notes to go through the proof of (the variation of) Thom's theorem in this particular case.

The strategy is as follows: given a section s, $r \in \mathbb{N}$, and $\delta > 0$, we are going to construct a perturbation χ so that

$$|\chi|_{C^r} \leq \delta, \tag{1}$$
$$s + \chi \pitchfork \mathbf{0}. \tag{2}$$

Even though the strong/weak C^r-topology is defined without any resort to metrics, we will introduce metrics among other things to make sense of the inequality 1.

3.1. C^r-norms on $\Gamma(E)$.

We assume from now on that we fixed have a triple (∇, h, g), where ∇ is a linear connection on E, h is a bundle metric on E, and g is a metric on M. This defines in the usual fashion a metric $\hat{g}$ in E: at each $y \in E$ we write

$$T_yE = E_{\pi(y)} \oplus \mathcal{H}_y, \tag{3}$$

where $\mathcal{H}_y$ is the horizontal space of the connection ∇. We declare the sum in 3 orthogonal, use h_x in the second factor and $g_{\pi(x)}$ in the first one.

Notice as well that we can take covariant derivatives using ∇. For any section s we have

$$\nabla s \in \Gamma(TM^* \otimes E).$$

Using the Levi-Civita connection associated to g we induce a connection on $TM^* \otimes E$, and we use it to take the second covariant derivative. By similar process we can define the r-th covariant derivative. The C^r-norm referred to in 1 is the sum on $j \in \{0, \dots, r\}$ of the sup norm of

$$\nabla^j \chi \in \Gamma(TM^* \otimes \overset{(j)}{\cdots} TM^* \otimes E),$$

for the obvious induced bundle metric.

Next we want to understand how our measurements change when we change the metric. Let V be a vector space, and g, g' two inner products. We know that they define the same topology (standard one), and even better they are comparable in the following sense: we can find $C_1, C_2 > 0$ so that for all $u \neq 0$

$$g(u,u) \leq C_1 g'(u,u,), \; g'(u,u) \leq C_2 g(u,u), \tag{4}$$

and we call C_1, C_2 comparability constants. If neeeded, one can take the least constant with the above property. Notice that this is actually a property that can be studied for any two norms in a vector space.

Moving to the non-linear setting we say that two metrics g, g' in M are comparable if we have constants $C_1, C_2 > 0$ so that 4 holds in all tangent spaces.

Remark 2. *On a compact manifold any two metrics are comparable. For example on each vector space one selects the smallest comparability constant, and this defines strictly positive constinuous functions. On non-compact manifolds one can easily construct non-comparable metrics.*

Lemma 1. *Let $(\nabla, h, g), (\nabla', h', g')$ two different triples and fix $r \in \mathbb{N}$. Then there exist $C_1, C_2 > 0$ such that for any section $s \in \Gamma(E)$ one has*

$$|s|_{C^r(\hat{g})} \leq C_1 |s|_{C^r(\hat{g'})},$$

$$|s|_{C^r(\hat{g'})} \leq C_2 |s|_{C^r(\hat{g})}.$$

Otherwise said, we have on the infinite dimensional vector space $\Gamma(E)$ two norms $|\cdot|_{(\nabla,h,g)}, |\cdot|_{(\nabla',h',g')}$, which are comparable; equivalently

$$(\Gamma(E), |\cdot|_{(\nabla,h,g)}) \xrightarrow{\mathrm{Id}} (\Gamma(E), |\cdot|_{(\nabla',h',g')})$$

is a (linear) homeomorphism.

Proof. We may start by looking at the effect of putting another fibre metric h'. Comparability between h, h' gives clearly the desired result.

If we now use another connection ∇' we have

$$\nabla - \nabla' = A \in \Omega^1(M; E).$$

So $\nabla' s = \nabla s + As$, and the result follows for C^1-norms. For higher norms then the covariant derivatives of A enter into consideration, but they are bounded since we are in a compact manifold.

The third type of change affects to the metric in the base. Comparability when using both metrics in T^*M is obvious. But also the covariant derivatives on T^*M are different. In any case this is not problematic since the difference depends on the difference of Christoffel symbols (measured with either metric), and of covariant derivatives (w.r.t. either metric), and we are in a compact manifold. $\square$

3.2. Measuring transversality.

Definition 1. [5] *Let* $(E, \nabla, h) \to (M, g)$ *be a vector bundle with connection and fiber metric over a Riemannian manifold. We say that a section* s *is* η*-transverse to* $\mathbf{0}$ *at* x *if either*

(i) $|s(x)| \geq \eta$ *(i.e.* $\tau(x)$ *does not belong to the* η*-neighborhood of the zero section) or,*
(ii) $|s(x)| < \eta$ *and* $\nabla s(x) \colon T_xM \to E_x$ *is surjective and has a right inverse of norm bigger than* $1/\eta$*.*

We say that s *is* η*-transverse to* $\mathbf{0}$ *in* K *if it is* η*-transverse to* $\mathbf{0}$ *at every* $x \in K$*.*

Remark 3. *One can state the existence of a right inverse of norm bigger than* $1/\eta$ *in the following equivalent manner: the (real) vector spaces* T_xM, E_x *have inner products. We ask for the image of the unit ball in* (T_xM, g_x) *to contain the ball of radius* η *in* (E_x, h_x)*.*

Remark 4. *There is yet a more geometric way of restating what condition (ii) in the definition of estimated transversality means, which firstly appeared in* [19]*. In* $(T_{s(x)}E, \hat{g}_{s(x)})$ *we have two distinguished subspaces: the tangent space to the graph of the section* $Ts(x)$ *and the horizontal subspace w.r.t. the distribution* $\mathcal{H}_{s(x)}$*. One defines their minimal angle as the minimum of angles of two non-zero vectors each on one subspace. One can define* η*-transversality by requiring the minimum angle to be bounded from below by* η*. This definition is not exactly the one we gave, but it is comparable to it in the sense that there exist (smooth) strictly monotone functions* $C, D \colon (0, \epsilon] \to [0, \infty)$ *such that* η*-transversality implies minimum angle bounded from below by* $C(\eta)$*, and conversely minimum angle bounded from below by* η *implies* $D(\eta)$*-transversality.*

Lemma 2. *Let* $(\nabla, h, g), (\nabla', h', g')$ *two different triples and* K *a compact subset of* M*. Then there exist* $T_1, T_2 > 0$ *such that for any section* $s \in \Gamma(E)$ *and any* $\eta \leq 1$ *we have*

(1) s *is* η*-transverse to* $\mathbf{0}$ *in* K *w.r.t* (∇, h, g) *implies that* s *is* $T_1\eta$*-transverse to* $\mathbf{0}$ *in* K *w.r.t* (∇', h', g')*.*
(2) s *is* η*-transverse to* $\mathbf{0}$ *in* K *w.r.t* (∇', h', g') *implies that* s *is* $T_2\eta$*-transverse to* $\mathbf{0}$ *in* K *w.r.t* (∇, h, g)*.*

Here T_1, T_2 *depend on the comparability constants between the metrics and the norm of the difference of the connections.*

One of the advantages of measuring transversality is that we can see very neatly the effect of adding perturbations with small enough norm.

Lemma 3. *Let* $s \in \Gamma(E)$ *be* η*-transverse to* $\mathbf{0}$ *in* K*, and let* $\chi \in \Gamma(E)$ *so that* $|\chi|_{C^1(K)} \leq \delta$*. Then* $s + \chi$ *is* $\eta - \delta$*-tranverse to* $\mathbf{0}$ *in* K*.*

Proof. If $|s + \chi(x)| \leq \eta - \delta$, then must look at the image of the unit ball under

$$\nabla s(x) + \nabla \chi(x) \colon T_xM \to E_x.$$

The first summand already contains the ball of radius η. The second affects by decreasing the norm by at most δ, so the result follows. □

We will assume for simplicity that E is a line bundle. Let us prove Thom's transversality theorem as follows: we will be given $B_g(x_1, \rho), \ldots, B_g(x_l, \rho)$ a covering of M with the following properties:

(1) The bundle E triviallizes via a (perhaps local) section τ_j and so that

$$|\tau_{j|B_g(x_j,\rho)}| \geq \kappa > 0, \ |\tau|_{C^2} \leq A. \tag{5}$$

(2) We have charts $\varphi_j \colon B(0,3) \subset \mathbb{R}^m \to B_g(x_j,\rho)$ so that $|\varphi_j|_{C^2} \leq B$. We also have $B(0,3) \subset \varphi_j^{-1}(B_g(x_j,\rho))$; there both metrics are comparable because of the bound on $|\varphi_j|_{C^2}$, and because we work with finitely many charts (ultimately because M is compact) the bound is common and so the comparability constant is.
(3) Define $K_j := \varphi_j(\bar{B}(0,1))$. Then we have

$$\bigcup_{j=1}^{l} K_j = M.$$

Next we want to transfer the estimated transversality problem for sections over K_j to a similar one for functions: To do that we note that in $B_g(x_j,\rho)$ we can write

$$s = f_j \tau_j.$$

We pullback everything to $B(0,3)$ using our chart. There we use a triple different from (∇, h, g). In the base we use the Euclidean metric g_{std}; the bundle metric h_{std} is such that τ_j has norm 1; finally the connection d is the one for which τ_j is flat.

Lemma 4. *A local section $s = f_j\tau_j$ is η-transverse to $\mathbf{0}$ over $\bar{B}(0,3)$ w.r.t.* $(\mathrm{d}, h_{\mathrm{std}}, g_{\mathrm{std}})$ *iff either the Euclidean norm of f_j is greater than η, or else its derivative has (Euclidean) norm greater or equal than η.*

We now apply Sard's theorem in the following way:

Proposition 2. *Given a function $f \colon B(0, 11/10) \to \mathbb{R}$, and given $\gamma > 0$, there exists $|w| \leq \gamma$ such that $f + w$ is transverse to $\mathbf{0}$ in $\bar{B}(0,1)$. By compactness of the latter subset, $f + w$ is $\alpha(f+w)$-transverse to $\mathbf{0}$.*

Remark 5. *In principle we do not have an estimated version of Sard's theorem, so that α is a function of γ and the C^r-bounds on f.*

To prove Thom's transversality we start with our section s, and give ourselves $\delta > 0$ the bound for the size of the perturbation. In the first step we proceed as follows:

- Over $B_g(x_1,\rho)$, we write $s = f_1\tau_1$.
- We apply proposition 2 to f_1 with $\gamma = \delta/2C_2$, where C_2 is the constant granted by lemma 1. Whe fix once and for all β a bump function supported in $B(0,3)$, with $\beta_{|B(0,1)} = 1$, and $|\beta|_{C^2} \leq 1$. We define our first perturbation as

$$\chi_1 = w_1 \beta \tau_1,$$

Then it follows that by lemma 1 its norm is bounded by $\delta/2$ and makes $s + \chi_1$ and by lemma 2 $s + \chi_1$ is $T_2\alpha_1$-$\pitchfork$-to $\mathbf{0}$ over K_1.

In the second step we repeat a similar construction over $B_g(x_2,\rho)$, though this time we have two types of constraints: firstly the perturbation χ_2 should not exceed say $\delta/4$, so by induction the final perturbation χ is bounded by δ. Secondly we do not want to destroy the transversality $T_2\alpha_1$ achieved over K_1. By lemma 3 we keep half of it if we force $|\chi_2|_{C^1} \leq T_2\alpha_1/2$. Therefore we solve the local problem with a perturbation bounded by $\min(T_2\alpha_1/2C_2, \delta/4C_2)$. More generally, in the j-th step we solve the local transversality problem with a perturbation bounded by $\min(T_2\alpha_{j-1}/2C_2, \delta/2^jC_2)$.

We would like to optimize the number of steps. Observe that on each induction step we can work with several open subset at a time as long as they do not intersect, because then what we do in either of them does not interfere with what we do in the others. So we can reagroup our l subsets into N different groups of disjoint subsets, each labelled by a colour for example. Then we make induction on colours.

We said that we do not have in general an estimated version of Sard's theorem, but we do have it for a very particular class of (complex valued) functions.

Theorem 3. [22, 5, 3] *Let $f\colon B^{2n}(0, 11/10) \to \mathbb{C}$. Let δ be a constant $0 < \delta < 1/2$. Let $\eta(\delta) = \delta(P(\log(\delta^{-1}))^{-1}$, where P is a real monomial depending on n. If in the ball of radius 11/10 we have*

$$|f|_{g_{\mathrm{std}}} \leq 1,\ |\bar{\partial} f|_{g_{\mathrm{std}}} \leq \eta,\ |\mathrm{d}\bar{\partial} f|_{g_{\mathrm{std}}} \leq \eta,$$

then there exists $w \in \mathbb{C}$, $|w| \leq \delta$, such that $f + w$ is η-transverse to $\mathbf{0}$ in $\bar{B}(0, 1)$.

Assume that E is a complex line bundle, and we are given data above so that when we write $s + \chi_1 + \cdots + \chi_{j-1} = f_j \tau_j$, f_j satisfies the hypothesis in theorem 3 (for example if it is holomorphic and satisfies the first bound). Then if we have N colours we deduce that the final amount of transversality is at least

$$\eta^N(\delta') \tag{6}$$

with $\delta' = \delta/2C_2$, and the monomial P suitably rescaled.

We sumarize the ingredients we need to have certain control of the amount of transversality:

- Charts φ_j giving uniform control on the induced metrics and the standard one, and so that $K_j := \varphi_j(\bar{B}(0, 1))$ cover M.
- Bump sections which triviallize the section over each open subset of the cover, with control on their norm, and so that when writting locally a section s as a function, the latter is in the hypothesis of theorem 3 (that also is a condition on the charts).
- Theorem 3 granting a fixed amount of transversality in terms of (part of) the C^2-size of the function and the size of the perturbation.

4. Symplectic geometry

4.1. Linear symplectic geometry.

Definition 2. *Let V be a finite dimensional vector space over the reals. A symplectic form is a 2-form $\omega \in \bigwedge^2 V^*$ which is non-degenerate. The pair (V, ω) is called a symplectic vector space.*

A 2-form is a bilinear form $\omega\colon V \times V \to \mathbb{R}$ which is antisymmetric. The kernel of a 2-form is defined to be

$$\ker\omega = \{u \in V \mid \omega(u, v) = 0,\ \forall v \in V\}.$$

A 2-form ω is non-degenerate if $\ker\omega = 0$.

In a less invariant fashion, if $v_1, \ldots, v_m$ is a basis of V, then $\{v_i^* \wedge v_j^*\}_{i<j}$ is a basis of the vector space of antisymmetric bilinear forms $\bigwedge^2 V^*$, whose dimension is $\frac{m(m-1)}{2}$, $m = \dim V$. Here our convention is that for $\alpha, \beta \in V^*$,

$$\alpha \wedge \beta(u, v) = \alpha(u)\beta(v) - \alpha(v)\beta(u).$$

Then a 2-form can be written

$$\omega = \sum_{i<j} w_{ij} v_i^* \wedge v_j^*.$$

If understood as a linear map, then in the fixed basis and its dual $\omega^{\#}$ is given by an antisymmetric matrix W,

$$W_{ij} = \omega_{ij},\ i < j, W_{ij} = -\omega_{ij},\ i > j, W_{ii} = 0.$$

Notice as well that

$$W_{ij} = \omega(v_i, v_j).$$

Antisymmetry and non-degeneracy of the bilinear form associated to a given matrix W are equivalent to

$$W^t = -W, \, \mathrm{det}W \neq 0.$$

4.2. **Dimension.**

Lemma 5. *Let (V, ω) be a symplectic vector space. Then V has even dimension.*

Proof. If we let W be the matrix representing ω in some basis, then we have $W = -W^t$. Therefore

$$\mathrm{det}W = \mathrm{det}(-W^t) = (-1)^{\mathrm{dim}V}\mathrm{det}W,$$

and the result follows by the non-degeneracy. □

4.3. **Standard form.**

Example 1. *Take V^{2n} with basis $e_1, f_1, \ldots, e_n, f_n$, and define*

$$\omega_{\mathrm{std}}(e_i, e_j) = \omega_{\mathrm{std}}(f_i, f_j) = 0, \omega_{\mathrm{std}}(e_i, f_j) = -\omega_{\mathrm{std}}(f_j, e_i) = \delta_{ij}.$$

Its matrix W_{std} is diagonal with n 2×2 blocks

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

4.4. **Symplectic linear group and symplectic properties/magnitudes.**

Definition 3. *The (Lie) group of linear symplectic transformations of ω is*

$$\mathrm{Symp}(V, \omega) = \{A \in \mathrm{Gl}(V) \,|\, \omega(A, A) = \omega\}$$

Linear symplectic geometry is the study of those properties/magnitudes of V invariant under $\mathrm{Symp}(V, \omega)$.

Remark 6. *A symplectic form is the antisymmetric version of an inner product, but antisymmetry makes things very different.*

Definition 4. *Let (V, ω) be a symplectic vector space, and let $U \subset V$ be a vector subspace. The symplectic annihilator/orthogonal of U is*

$$U^\omega := \{v \in V \,|\omega(u, v) = 0, \, \forall u \in U\}.$$

Equivalently:

$$U^\omega = (\omega^\#(U))^\circ.$$

Lemma 6.

- $\mathrm{dim}U + \mathrm{dim}U^\omega = 2n$.
- $(U^\omega)^\omega = U$.

Some vector subspaces are classified according to an interesting incidence relation between U and U^ω.

Definition 5. *One says that U is*

- *symplectic if $U \cap U^\omega = \{0\}$ (equivalently, $(U, \omega_{|U})$ is symplectic).*
- *isotropic if $U \subset U^\omega$ (equivalently, $\omega_{|U} \equiv 0$).*
- *coisotropic if $U^\omega \subset U$.*
- *Lagrangian if $U = U^\omega$.*

These are all properties stable under symplectic transformations.

4.5. **Symplectic/Darboux coordinates.**

Lemma 7. *Let (V,ω) be a symplectic vector space. There exist a basis $e_1, f_1, \ldots, e_n, f_n$ in which the matrix representing ω is W_{std}. Such a basis is called symplectic/Darboux basis.*

Proof. It is by induction in $\dim V/2$, and it is an analog of the Gram-Schmidt algorithm.

□

Another way to interpret the existence of Darboux coordinates is as follows:

Corollary 1 (Stability). GL(V) *acts transitively on the (open) subset of $\bigwedge^2 V^*$ of symplectic forms (recall that openness of the orbit(s) of the action being the classical notion of stability). Therefore*

$$\dim \mathrm{Symp}(V,\omega) = \dim \mathrm{GL}(V) - \dim \bigwedge^2 V^*.$$

4.6. **More on** Symp$(2n)$**. Compatible almost complex structures.** In $\mathbb{R}^{2n}$ we have $\omega_{\mathrm{std}}, J_{\mathrm{std}}, g_{\mathrm{std}}$ and their corresponding groups of symmetries. We have:

Theorem 4.

$$\mathrm{Symp}(2n) \cap \mathrm{GL}(n,\mathbb{C}) = \mathrm{Symp}(2n) \cap \mathrm{SO}(2n) = \mathrm{GL}(n,\mathbb{C}) \cap \mathrm{SO}(2n) = \mathrm{U}(n), \quad (7)$$

and Symp$(2n)$ *deformation retracts onto* U(n)*. The latter is a maximal compact subgroup.*

What we used to describe a maximal compact subgroup was the almost complex structure J_{std}. They key properties used give rise to

Definition 6. *Let (V,ω) be a symplectic vector space. An almost complex structure $J\colon V \to V$, $J^2 = -\mathrm{Id}$ (*Spec$(J) = \{i, -i\}$*), is compatible with ω if*

(1) *$J \in \mathrm{Symp}(V,\omega)$.*
(2) *The symmetric bilinear form $\omega(\cdot, J\cdot)$ is an inner product.*

We denote the space of compatible almost complex structures by $\mathcal{J}(V,\omega)$.

Lemma 8. *If $J \in \mathcal{J}(V,\omega)$ and g the associated metric. Then for any linear subspace U*

$$(JU)^{\omega} = U^{\perp_g}.$$

In particular

- *Any J-complex subspace is symplectic.*
- *The symplectic orthogonal of a J-complex subspace is J-complex.*

Given a metric g in (V,ω) and a symplectic subspace U we can measure how symplectic U is by looking at the norm of the orthogonal projection $U^{\omega} \to U^{\perp}$.

Corollary 2. *If $J \in \mathcal{J}(V,\omega)$ and g is the associated inner product, then the "most symplectic" vector spaces are the J-complex ones. Moreover, is an even dimensional vector space U is close enough to be J-complex (i.e. the orthogonal projection $J(U) \to U^{\perp}$ has small enough norm), then it is symplectic.*

Proof. If the alluded norm is small enough, then U and U^{ω} are very close to be orthogonal, and therefore transverse. In particular they have trivial intersection, so U is symplectic. □

By lemma 7 any symplectic vector space is isomorphic to $(\mathbb{R}^{2n}, \omega_{\mathrm{std}})$. We know any complex vector space is isomorphic to $(\mathbb{R}^{2n}, J_{\mathrm{std}})$.

Lemma 9. *If $J \in \mathcal{J}(V,\omega)$ then there exist an isomorphism sending (V,ω,J) to $(\mathbb{R}^{2n}, \omega_{\mathrm{std}}, J_{\mathrm{std}})$.*

4.7. **Symplectic manifolds.**

Definition 7. *A symplectic form $\omega \in \Omega^2(M)$ is a no-where degenerate 2-form (i.e. we make each tangent space into a symplectic vector space) which is closed.*

A submanifold N of M is symplectic, isotropic, coisotropic or Lagrangian, if $T_xN \subset (T_xM, \omega_x)$ is symplectic, isotropic, coisotropic or Lagrangian for all $x \in N$.

Remark 7. *Closedness is the natural P.D.E. associated to forms. Riemannian structures have O.D.E.'s associated to the variational problem seeking to minimize the energy of a path joining to fixed points. This is not the case for non-degenerate 2-forms, and its study without the rigidity condition coming from closedness is very complicated.*

Example 2.

(1) *Any linear symplectic structure on a vector space is a symplectic structure.*
(2) *Any area form on an (orientable) surface is a symplectic form. All curves are Lagrangian submanifolds.*
(3) *For any manifold M, its cotangent bundle T^*M carries the canonical symplectic form $d\lambda$, where $\lambda \in \Omega^1(M)$ is the Liouville 1-form. In coordinates $x_1, \ldots, x_n$ of M, once we complete with dual coordinates $\xi_1, \ldots, \xi_n$, then*
$$\lambda := \sum_{i=1}^{n} \xi_i dx_i.$$
This local definition gives rise to a well defined 1-form. More invariantly it is characterized by the universal property
$$\alpha^*\lambda = \alpha,\ \alpha \in \Omega^1(M).$$
(4) *For any Kahler manifold (X, J, g), the Kahler form ω_g is a symplectic form (one can also define a Kahler manifold as complex manifold (X, J) with a symplectic structure ω so that $J \in \mathcal{J}(X, \omega)$; the Kahler metric is $g := \omega(\cdot, J\cdot)$). Any complex submanifold of a Kahler manifold is Kahler (since metrics restrict to metrics), and therefore symplectic.*
(5) *The product of symplectic manifolds is symplectic in the obvious way. More generally we have some others ways to produce new symplectic structures out of given ones (symplectic structures on certain fiber bundles, blow ups, normal connected sum, reduction).*

4.8. **Moser's stability.** If a given manifold M admits symplectic structures we would like to classify them. A natural equivalence relation is equality up to diffeomorphism, that is up to global change of coordinates. It is actually more convenient to use just diffeomorphisms in the connected component of the identity, since those can be controlled by vector fields when the manifold is compact. Notice that if $\phi \in \mathrm{Diff}^0(M)$, then $[\omega] = [\phi^*\omega]$. Moser's technique gives necessary conditions for a pair of cohomologous symplectic forms, to be equivalent up to isotopy.

Theorem 5. [18] *Let M be a compact manifold and let ω_0, ω_1 symplectic forms in the same cohomology class. If there exists ω_t a family of symplectic forms joining them so that $\frac{d}{dt}\omega_t = d\alpha_t$ (this is the infinitesimal statement corresponding to the existence of a path of symplectic forms with constant cohomology class connecting them; the equivalence uses that we are in a compact manifold), then there exist an isotopy ϕ_t so that*
$$\phi_t^*\omega_t = \omega_0. \tag{8}$$

Corollary 3. *If ω_0, ω_1 are cohomologous symplectic forms on M compact, so that the segment $\omega_t := (1-t)\omega_0 + t\omega_1$, $t \in [0,1]$ is by symplectic forms, then $\phi_t^*\omega_t = \omega_0$, for a suitable isotopy.*

Proof.

$$\frac{d}{dt}\omega_t = \omega_1 - \omega_0 = d\alpha.$$

□

Corollary 4. *Let (M^{2n}, ω) be a symplectic manifold. Then for every point x there exist Darboux coordinates, i.e. open neighborhoods U_x and U_0 of x and 0 in M and $\mathbb{R}^{2n}$ respectively, and*

$$\varphi\colon (U_x, \omega) \to (U_0, \omega_{\mathrm{std}})$$

a symplectic diffeomorphism.

Proof. Fix any coordinates $x_1, \dots, x_{2n}$ about x and push the symplectic form to ω in Euclidean space. Associated to the coordinates we have the frame of $\Omega^2(\mathbb{R}^{2n})$ with sections $\{dx_i \wedge dx_j\}_{i<j}$, so that

$$\omega = \omega_{ij} dx_i \wedge dx_j.$$

Take the degree 0 part of the expansion of ω at the origin,

$$\omega^{(0)} = \omega(0) = \omega_{ij}(0) dx_i \wedge dx_j.$$

This is a constant 2-form and it is symplectic. Just notice that if W is the matrix representing $\omega^{\#}$, then

$$\det(W(0)) = \det W(0) \neq 0.$$

Consider the convex combination

$$\omega_t = (1-t)\omega^{(0)} + t\omega.$$

In a small enough neighborhood of the origin all ω_t are symplectic. Then

$$\frac{d}{dt}\omega_t = \omega - \omega^{(0)}$$

is closed and vanishing at the origin. By Poincaré lemma we can find 1-forms α_t so that

- $d\alpha_t = \frac{d}{dt}\omega_t$.
- $\alpha_t(0) = 0$.

We can now apply theorem 5, and observe that the family of vector fields we obtain vanish at the origin, so the associated (local) isotopy fixes the origin.

By the Darboux linear lemma we know that linear symplectic forms are linearly equivalent. □

Corollary 5. *The only local invariant of a symplectic manifold is its dimension.*

Remark 8. *Darboux' theorem is very useful for a number of problems in symplectic geometry, because any symplectic construction in a arbitrarily small neighborhood of the origin in $(\mathbb{C}^n, \omega_{\mathrm{std}})$ can be transported to any symplectic manifold. One may try to use this idea to build symplectic submanifolds, but we see that it does not produce anything: indeed, one may try to construct locallized symplectic submanifolds. Actually it would be enough to get compact symplectic submanifolds, which after rescaling can be as locallized as desired. It might seem natural to look for them among complex submanifolds, since they would be Kahler and therefore symplectic. But there are no interesting compact complex submanifolds in $\mathbb{C}^n$, because by the maximum principle any holomorphic function on them has to be constant on connected components. Thus the restriction of any holomorphic function in $\mathbb{C}^n$ has to be constant on connected components, and therefore connected components must be just points.*

5. Holomorphic sections for Kahler manifolds.

A symplectic structure is not defined by a sheaf of (local) "symplectic functions". That implies that we do not have in principle functions/sections whose zero subsets give rise to symplectic submanifolds. The exception is the particular case of Kahler manifolds. There we do have the sheaf of holomorphic functions; sheaf theoretic tehniques produce complex submanifolds, which inside a Kahler manifold are also Kahler, and therefore symplectic. More generally given a symplectic manifold (M,ω) and $J \in \mathcal{J}(M,\omega)$, J-holomorphic functions/sections produce J-holomorphic submanifolds which according to lemma 8 are also symplectic. In general one cannot find J-holomorphic functions/sections when M has dimension bigger than 2 [12]. But by corollary 2 functions whose zero subset is close enough to be J-holomorphic produce symplectic submanifolds.

One is going to proceed by analogy with Kahler geometry. Let (X,J,ω) be a Kahler manifold so that $\omega/2\pi$ is of integral type, i.e. $[\omega/2\pi] \in \mathrm{im}(H^2(X;\mathbb{Z}))$. Then according to a theorem of Weil we can form its prequantum line bundle (L,∇,h). This is a holomorphic hermitian line bundle with compatible connection and so that

$$F_\nabla = -iw. \tag{9}$$

As k grows large, the k-th tensor powers have many holomorphic sections. By Riemann-Roch the space of holomorphic sections has dimension given by a polynomial whose leading term is $\frac{c_1(L)}{n!}k^n$, n the complex dimension (so it is the volume of the Liouville volume form associated to $k\omega$, or equialently the Riemannian volume associated to kg).

One can give a "local explanation" for this situation. We follow the exposition by Donaldson [5] based on earlier work on "peak sections" by Tian [21]: Recall that about any $x \in X$ we can use the complex structure and symplectic structure to get either complex or Darboux coordinates. In general we do not have coordinates which are both complex and symplectic, but let us assume for the moment that this is the case. Then we have $z_1,\ldots,z_n\colon B_g(x,\rho) \to \mathbb{C}$ so we can write

$$w = \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j. \tag{10}$$

Over $B_g(x,\rho)$ we are going to do something which will allow to see holomorphic functions from a different perspective: rather than taking a holomorphic local trivialization τ, so that any holomorpic section $s = f\tau$ with f holomorphic, we select τ a unitary trivialization whose associated connection form in the fixed holomorphic coordinates is the primitive of the curvature

$$A = \frac{1}{4}\sum_{j=1}^{n}(z_j d\bar{z}_j - \bar{z}_j dz_j). \tag{11}$$

Now if s is holomorphic, $s = f\tau$, f will not be a holomorphic function. We will see a very important example of such f.

The covariant derivative of any section

$$\nabla s \in T^*X \otimes L$$

is at each point a linear map between complex vector spaces, so we can split it into its linear and antilinear components

$$\nabla s = \partial_\nabla s + \bar{\partial}_\nabla s,$$

$$\partial_\nabla s = \frac{1}{2}(\nabla s - i\nabla_J s) \in T^{*1,0}X \otimes L,\ \bar{\partial}_\nabla s = \frac{1}{2}(\nabla s + i\nabla_J s) \in T^{*0,1}X \otimes L.$$

After our choice of local trivialization we can write $s = f\tau$ and

$$\bar{\partial}_\nabla s = (\bar{\partial}_{\mathrm{std}} f + A^{0,1} f)\tau,$$

where J_{std} is the standard (almost) complex structure, $\mathrm{d} = \partial_{\mathrm{std}} + \bar{\partial}_{\mathrm{std}}$, and $A^{0,1} = \frac{1}{4}\sum_{j=1}^n z_j d\bar{z}_j$.

Lemma 10. *The function*

$$u(z) = \exp^{-z\bar{z}/4}$$

is a solution of the model Cauchy-Riemann equations, i.e.

$$\bar{\partial}_{\mathrm{std}} u + A^{0,1} u = 0. \tag{12}$$

Because the trivialization we used is unitary, we have found a (local) holomorphic section $\tau'_x = u\tau$ whose norm behaves like a Gaussian, thus having a bump-function like behaviour. Of course, we do need to extend it to a global holomorphic section, and we would still like to have the same behavior on its norm. To do that the standard way to proceed is as follows:

(1) Extend τ'_x to τ_x multiplying times a bump function β with suitable support.
(2) Use the projection $p\colon L^2(L) \to \mathcal{H}(L)$ of L^2-sections into holomorphic ones. According to [4] (see [10]) we know that we have a constant $C_{\mathcal{H}}$ so that

$$|s - p(s)|_{L^2(X,g)} \leq C_{\mathcal{H}} |\bar{\partial}_\nabla s|_{L^2(X,g)}. \tag{13}$$

Notice as well that in balls the C^r-norm is controlled by the L^2-norm and viceversa (it follows from the Cauchy integral formula for holomorphic fnctions, and also for sections by choosing arbitrary local holomorphic trivializations). Because these metrics are comparable a controlled variation (over balls) of one is equivalent to a controlled variation of the other.
(3) One realizes that to keep the Gaussian decay in τ_x the bump function β needs its slope to be very small. That can be achieved by thinking of $B(0,1) \subset \mathbb{C}^n$ has having much smaller radius in X. We zoom in by a dilation

$$\begin{array}{ccc} (L^{\otimes k}, \nabla_k, h_k) & \xrightarrow{\rho^*_{k^{1/2}}} & (L, \nabla, h) \\ \downarrow & & \downarrow \\ (B(0,1), k g_{\mathrm{std}})) & \xrightarrow{\rho_{k^{1/2}}} & (B(0,1), g_{\mathrm{std}})), \end{array} \tag{14}$$

and we let $z_{k,1}, \ldots, z_{k,n}$ be the rescaled coordinates on the r.h.s. of diagram 14. If we think of $u(z_k, \bar{z}_k)$ solving equation 12 as giving a local section of the r.h.s. of diagram 14, then its pullback to the l.h.s. with respect to the k-th tensor power of the local trivialization is the function

$$u_k(z) = \exp^{-kz\bar{z}/4},$$

which solves

$$\bar{\partial}_{\mathrm{std}} u_k + k A^{0,1} u_k = 0. \tag{15}$$

If one takes $\beta\colon \mathbb{R} \to [0,1]$ a bump function supported in the ball of radius one and so that $\beta_{[-1/2,1/2]}$, then

$$\tau_{k,x} = \beta(k^{1/3} z) u_k(z,\bar{z})\tau^{\otimes k} = \rho^*_{k^{1/2}}(\beta(k^{-1/6} z_k) u(z_k,\bar{z}_k)\tau) \tag{16}$$

is such that

$$|\bar{\partial}_\nabla \tau_{k,x}|_{C^r(X,kg)} \leq B k^{-1/2}. \tag{17}$$

Then using equations 17, 13, and comparability (in g_k-balls of radius ρ) between the C^r and L^2-metrics for holomorphic functions/sections, one checks that $p(\tau_{k,x})$ is a holomorphic section whose norm behaves like a Gaussian function concentrated in $B_{g_k}(x,\rho')$, where ρ' does not depend on k, x.

(4) Here we made an assumption about the existence of symplectic coordinates which are holomorphic. This is in general not true, but it is up to first order about x. That means that in those coordinates $z_1, \ldots, z_n$ we have

$$|\bar{\partial}_J u(z, \bar{z})|_{C^r(B(0,1), g_{\mathrm{std}})} \leq B_1 |z|.$$

But when we rescale we easily deduce

$$|\bar{\partial}_J u_k(z_k, \bar{z}_k)|_{C^r(B(0,1), k g_{\mathrm{std}})} \leq B_2 |z_k| k^{-1/2},$$

and thus we do have

$$|\bar{\partial}_\nabla \tau_{k,x}|_{C^r(M,kg)} \leq B_3 k^{-1/3}.$$

Therefore when projecting over the holomorphic functions and recalling the comparability between C^r and L^2 metrics, for all $y \in X$ we have

$$|p(\tau_{k,x}) - \tau_{k,x}|_{C^r(B_{kg}(y,\rho'))} \leq B_4 k^{-1/3}.$$

Thus we construct concentrated holomorphic functions with Gausian decay w.r.t. any $x \in X$, and for all $k \gg 1$.

Notice that compactness implies that all the constants above can be arranged to be independent of x. Likewise, they do not depend on k. We call such constants uniform.

Remark 9. *From the existence of holomorphic peak sections one can deduce that the order of holomorphic sections is the one given by R-R ($O(k^n)$). Indeed, if one fixes $B_g(x, \rho)$, one can work via a chart in Euclidean space and pack inside k^n pairwise disjoints balls of g_k-radius $C\rho$. For k-large each contains a peak section concentrated at the ball. That implies linear independence of those sections.*

6. Approximately holomorphic geometry

We are going to transfer as much as we can of the previuos constructions to symplectic manifolds.

6.1. The line bundles, almost complex structure, and metric.

Let (M, ω) be a symplectic manifold with $\omega/2\pi$ an integral cohomology class. The pre-quantum line bundle $L_\omega := L$ is the unique -up to diffeomorphism- hermitian line bundle with compatible connection (L, ∇, h) so that

$$F_\nabla = -iw.$$

The powers $L^{\otimes k}$ inherit connections and hermitian metrics (∇_k, h_k). We will omit the subindex k for them when no confussion can arise.

Any symplectic vector space admmits c.a.c.s. Because the latter space is contractible homotopy theory arguments show that $\mathcal{J}(M, \omega)$ -the space of a.c. compatible with ω is non-empty. As a matter of fact the polar decomposition process ([15], chapter I.4) produces one such c.a.c.s starting with an arbitrary metric, an it is smooth on the metric.

We fix and $J \in \mathcal{J}(M, \omega)$ and g the associated metric. We denote $g_k := kg$ and by d_k the induced metric. Using the connection the total space of L gets $\hat{J}$ an almost complex structure. The same happens for $L^{\otimes k}$, $k \in \mathbb{N}$.

6.2. Approximately holomorphic sections.

Given $s_k \in \Gamma(L^{\otimes k})$ we can use the almost complex structure J and the complex structure of the fibers to split

$$\nabla s_k = \partial_\nabla s_k + \bar{\partial}_\nabla s_k,$$

$$\partial_\nabla s_k = \frac{1}{2}(\nabla s_k - i \nabla_J s_k) \in T^{*1,0} M \otimes L^{\otimes k}, \ \bar{\partial}_\nabla s_k = \frac{1}{2}(\nabla s_k + i \nabla_J s_k) \in T^{*0,1} M \otimes L^{\otimes k}.$$

Definition 8. [5] *Let* $s_k \in \Gamma(L^{\otimes k})$*,* $k \in \mathbb{N}$*. We say that* s_k *is a sequence of approximately* J*-holomorphic sections (or just A.H. sections) if there exist* A *a uniform constant so that for all* $k \gg 1$ *the following inequalities hold.*

$$|\nabla^j s_k|_{g_k} \leq A,\ j = 0,1,2, \tag{18}$$

$$|\nabla^j \bar{\partial} s_k|_{g_k} \leq Ak^{-1/2},\ j = 0,1. \tag{19}$$

Definition 9. *A sequence* $s_k \in \Gamma(L^{\otimes k})$ *is uniformly* η*-transverse to* $\mathbf{0}$ *if for all* $k \gg 1$ *the section* $s_k\colon (M, g_k) \to (L^{\otimes k}, \nabla_k, h_k)$ *is* η*-transverse to* $\mathbf{0}$.

Proposition 3. *Let* $s_k \in \Gamma(L^{\otimes k})$ *be an A.H. sequence uniformly* η*-transverse to* $\mathbf{0}$*. Then for all* $k \gg 1$ *the submanifold* $W_k := s_k^{-1}(\mathbf{0})$ *is symplectic.*

Proof. We write

$$\nabla s_k = \bar{\partial} s_k + \partial s_k.$$

For all $x \in W_k$ we have

$$\lim_{k\to\infty} \frac{\bar{\partial} s_k}{\partial s_k} = 0,$$

the convergence being uniform on x. That implies that for any $x \in W_k$ the kernel of the linear map

$$\nabla s_k\colon (T_x M, J, g_k) \to (L_x^{\otimes k}, J_{\mathrm{std}}, h_k)$$

is arbitrarily close to be J-complex. By corollary 2 we deduce that W_k is symplectic. □

6.3. **The approximately holomorphic charts and reference sections.** In approximately holomorphic geometry there is no operator projecting into holomorphic sections. Rather one has to work with A.H. sections and use differential topology tools to construct new A.H. sections which are transverse enough. That requires the use of A.H. sections playing the role of bump functions; they will have enough norm in certan ball and then decay as a Gaussian. They are the analogs of the holomorphic sections $p(\tau_{k,x})$ constructed in the previous section.

Definition 10. [1] *Let* $x \in M$ *be a point. A sequence* $s_k \in \Gamma(L^{\otimes k})$ *has Gaussian decay w.r.t.* x *if there exist a uniform constant* $\lambda > 0$ *and polynomials* P_j*,* $j = 0,1,2$*, such that for all* $k \gg 1$ *the following inequalities hold:*

$$|\nabla^j s_k(y)|_{g_k} \leq P_j(d_k(x,y)) \exp^{-\lambda d_k(x,y)^2},\ j = 0,1,2. \tag{20}$$

The sections we look for are as in equation 16. They are written in coordinates which are both holomorphic and Darboux. We need Darboux coordinates whose defect from being holomorphic grows smaller with k.

Definition 11. *Let* $\varphi_{k,x}\colon (\mathbb{C}^n, 0) \to (U_{k,x}, x)$*, for all* $x \in M$ *and all* $k \gg 1$*, be a family of charts with coordinates* $z_{k,1}, \dots, z_{k,n}$*. We call them a family of approximately holomorphic coordinates if there exist uniform constants (independent of* k, x*) so that the following estimates hold for all* $k \gg 1$ *at the points of* $B(0, 3k^{1/2})$*,* $\rho > 0$*:*

1. *The Euclidean and the induced metric are comparable to order 2, i.e.*

   $$g_k \leq C_1 g_{\mathrm{std}},\ g_{\mathrm{std}} \leq C_2 g_k \text{ and } |\nabla^j \varphi_{k,x}^{-1}|_{g_k} \leq O(k^{-1/2}),\ j = 1,2,$$

   where ∇ *denotes the Levi-Civita connection with respect to* g.
2. *Regarding the antiholomorphic components,*

   $$|\bar{\partial}\varphi_{k,x}^{-1}(z_k)|_{g_k} \leq |(z_k)| O(k^{-1/2}),$$

   $$|\nabla^j \bar{\partial}\varphi_{k,x}^{-1}(z_k)|_{g_k} \leq O(k^{-1/2}),\ j = 0,1,$$

   where $\bar{\partial}\varphi_{k,x}^{-1}$ *is the antiholomorphic component of* $\nabla\varphi_{k,x}^{-1}$.

We speak of Darboux coordinates when the additional condition $\varphi_{k,x}^* k\omega = \omega_0$ *holds.*

Lemma 11. *We can always find a family of A.H. Darboux coordinates.*

Proof. Start with Darboux charts for (M, g) centred at a point x. By lemma 9 we can compose on the left with a symplectic linear transformation so that the induced almost complex structure at the origin is J_{std}. In that way we have constructed a chart

$$\varphi_{1,x} \colon B_{g_{\mathrm{std}}}(0,3) \to B_g(x,\rho)$$

almost complex at x. The construction depends smoothly on x; beacause M is compact we get a family $\varphi_{1,x}$ which satisfies all the equations for $k = 1$. To do it for all k we simply compose with the appropriate dilation

$$\varphi_{k,x} := \varphi_{1,x} \circ \rho_{k^{-1/2}}.$$

□

Now we use the Darboux condition: over the Darboux coordinates for $k = 1$ we choose a unitary trivialization of L so that the connection form is an is equation 11. Its k-th tensor power also triviallizes $L^{\otimes k}$.

Proposition 4. *Let* $\tau_{k,x}^{\mathrm{ref}} \colon (M, J, g_k) \to L^{\otimes k}$ *be the sections given locally over A.H. coordinates with the above trivialization by*

$$u(z_k) = \beta(k^{-1/6} z_k) \exp^{-z_k \bar{z}_k/4}.$$

Then the following holds:

1. *The sections* $\tau_{k,x}^{\mathrm{ref}}$ *are A.H. (constant* A *is uniform on* k, x*).*
2. *The sections* $\tau_{k,x}^{\mathrm{ref}}$ *have Gaussian decay w.r.t.* x *(constant* λ *and polynomials* P_j *are uniform on* k, x*).*
3. *There exists* $\kappa > 0$ *so that* $|\tau_{k,x}^{\mathrm{ref}}| \geq \kappa$ *in* $\varphi_{k,x}(\bar{B}(0, 11/10))$*.*

6.4. Getting local uniform estimated transversality.

Proposition 5. *Let* $s_k \in \Gamma(L^{\otimes k})$ *be a sequence of A.H. sections with* C^2*-uniform constant* $A > 0$*. Fix* $\delta > 0$*. Then there exist* $k_0 \in \mathbb{N}$*,* T_2, C_1, C_2 *uniform on* k, x*, such that for any* $x \in M$ *and every* $k \geq k_0$ *one can find* $\chi_{k,x}$ *an A.H. perturbation with the following properties:*

1. $|\chi_{k,x}|_{C^2(g_k)} \leq \delta$*.*
2. $s_k + \chi_{k,x}$ *is* $\frac{T_1 A C_2}{\kappa}\eta(\frac{\kappa\delta}{AC_1C_2})$*-transverse to* $\mathbf{0}$ *in* $K_{k,x} := \varphi_{k,x}(\bar{B}(0,1))$*; by rescaling the monomial defining* η *and setting* $\delta' = \frac{\delta\kappa}{AC_1C_2}$*, we get* $\eta(\delta')$*-transversality.*

Proof. We fix A.H. Darboux coordinates, this giving rise to reference sections $\tau_{k,x}^{\mathrm{ref}}$; they have norm on $\varphi_{k,x}(\bar{B}(0, 11/10))$ bounded from below by κ. Then lemma 1 gives uniform constants C_1, C_2. The constant being uniform is clear for the C^0-norm. For higher ones we must use that Christoffel symbols have norm bounded by $Bk^{-1/2}$, again B uniform.

We write on $\varphi_{k,x}(\bar{B}(0, 11/10))$

$$s_k = f_{k,x} \tau_{k,x}^{\mathrm{ref}}$$

so we obtain for the functions $f_{k,x} \colon B(0, 11/10) \to \mathbb{C}$. Clearly

$$|f_{k,x}|_{C^1(B(0,11/10)), g_{\mathrm{std}})} \leq \frac{AC_2}{\kappa}. \tag{21}$$

Consider the equation

$$\bar{\partial}_{\nabla_{\mathrm{std}}} s_k = \bar{\partial}_{\mathrm{std}} f_{k,x} \tau_{k,x}^{\mathrm{ref}} + f_{k,x} \bar{\partial}_{\nabla_{\mathrm{std}}} \tau_{k,x}^{\mathrm{ref}}.$$

Because we are using A.H. charts we have

$$|\bar{\partial}_{\nabla_{\mathrm{std}}} s_k|_{C^1(B(11/10),g_{\mathrm{std}})} \leq Bk^{-1/2}.$$

Because $f_{k,x}$ is C^1-bounded and $\tau_{k,x}^{\mathrm{ref}}$ is A.H., the C^1-norm of the second summand of the r.h.s. is bounded by $A'k^{-1/2}$. So the same conclusion holds for the first summand. Because $\tau_{k,x}^{\mathrm{ref}}$ has norm bounded from below, we deduce

$$|\bar{\partial}_{\mathrm{std}} f_{k,x}|_{C^1(B(0,11/10),g_{\mathrm{std}})} \leq B'k^{-1/2}. \tag{22}$$

By equations 21 and 22 for k bigger than some k_0 the function $\frac{\kappa}{AC_2} f_{k,x}$ is in the hypothesis of theorem 3. Therefore we can find $w_{k,x} \in \mathbb{C}$ with norm bounded by $\frac{\kappa\delta}{AC_1C_2}$ so that $f_{k,x} + \frac{AC_2}{\kappa} w_{k,x}$ is $\frac{AC_2}{\kappa}\eta(\frac{\kappa\delta}{AC_1C_2})$-transverse to $\mathbf{0}$ on $\bar{B}(0,1)$. The consequence is that

(i) $\chi_{k,x} := \frac{AC_2}{\kappa} w_{k,x}\tau_{k,x}^{\mathrm{ref}}$ has C^2-norm bounded by δ;

(ii) $s_k + \chi_{k,x}$ is $\frac{T_1AC_2}{\kappa}\eta(\frac{\kappa\delta}{AC_1C_2})$-transverse to $\mathbf{0}$ over $K_{k,x}$.

□

6.5. **From local to global transversality.** Given $s_k \in \Gamma(L^{\otimes k})$ an A.H. sequence it is very far from being trivial how to obtain global transversality for some s_k by succesively applying proposition 5. If we fix A.H. Darboux coordinates, and single out some $k' \in \mathbb{N}$ so that the innequalities defining the A.H. property (and eventually the Gaussian decay) already hold, if we try to apply induction to a cover K_{k',x_i}, $i \in I$, we run into trouble: indeed if we order the open subsets $B_{g_{k'}}(x_1,\rho), \dots B_{g_{k'}}(x_{\#I},\rho)$ and want to achieve transversality by adding a perturbation of size at most δ, then in the j-th step we get $s_{k'} + \sum_{i=1}^{j} \chi_{k',i}$ which is $\eta^j(\delta')$-transverse to $\mathbf{0}$ over

$$\bigcup_{i=1}^{j} K_{k',i},$$

where $\delta = \delta/2AC_1C_2$.

To apply again proposition 5 we write over $K_{k',x_{j+1}}$

$$s_{k'} + \sum_{i=1}^{j} \chi_{k',i} = f_{j+1}\tau_{k,x_{j+1}}^{\mathrm{ref}},$$

and because the size of the perturbation has to be at most one half of the attained transversality, we need to make sure that

$$|\bar{\partial}_{\mathrm{std}}(f_{j+1}/AC_2)|_{C^1(B(0,11/10),g_{\mathrm{std}})} \leq \eta^{j+1}(\delta'). \tag{23}$$

From the A.H. condition we have

$$|\bar{\partial}_{\mathrm{std}}(f_{j+1}/AC_2)|_{C^1(B(0,11/10),g_{\mathrm{std}})} \leq Bk^{'-1/2},$$

so we must have

$$Bk^{'-1/2} \leq \eta^{j+1}(\delta'),\ j \in \{0,\dots,\#I-1\}. \tag{24}$$

The complication is that $\#I$ depends on k', and in principle if we are simple minded is of order $= O(k^{'2n})$. Which such a large number equation 24 will not hold. Indeed, for $\delta < 1$ we have

$$\eta(\delta)^{k^{2n}} < \delta^{2nk} < Bk^{-1/2}$$

for $k \gg 1$.

We must reduce the number of induction steps by taking several open subsets at a time. In $\mathbb{C}^n$ if we fix D a distance and take the rectangular lattice $\Lambda(D)$ associated to it. Then we may take all balls of radius 1 say centred at points of the lattice, and label them with a color. By translating the lattice we can cover $\mathbb{C}^n$ with $N := O(D^{2n})$ colours, and the same holds for (M,g). For each k we can use the same lattice in the domain of $\varphi_{k,x}$, and end up with the same number of colours. Two observations are in order: firstly as k grows the points in the lattice and in the domain of the chart tend to be the whole lattice; secondly and more delicate for a fixed colour, that is the balls centred at points of $\Lambda(D)$ say, the corresponding reference sections have support which extends beyond $B_{g_k}(x,\rho)$; rather the support vanishes only at d_k-distance $O(k^{1/6})$ of x. That eventually makes the support to be contained in the domain of the chart.

Notice that N is independent of k, so we are free to increase the latter so that proposition 5 can be applied on each induction step. The new difficulty is that on each induction step -i.e. for a given colour- we have as many compacts $K_{k,x}$ as points in $\Lambda(D)$. Proposition 5 for a perturbation of size α provides $\eta(\alpha')$-transversality on each $K_{k,x}$. But for each compact in the lattice one has the perturbations coming from the other compacts, which in principle may destroy the $\eta^{j+1}(\delta')$-transversality. To avoid that one must assure for any $y \in \Lambda(D)$

$$|\sum_{x\in\Lambda(D),\, x\neq y} \chi_{k,x}|_{C^1(B_{g_k}(y,\rho))} \leq \eta^{j+1}(\delta')/2,\ j\in\{1,\dots,N\}. \tag{25}$$

For each $x' \in B_{g_k}(y,\rho)$ and $x \in \Lambda(D)\backslash\{y\}$ we use the Gaussian decay to obtain

$$|\chi_{k,x}(x')|_{C^1(g_k)} = |w_{k,x}\tau^{\mathrm{ref}}_{k,x}(x')|_{C^1(g_k)} \leq \eta^j(\delta')P(n(x)D/2)\exp^{-\lambda n(x)^2D^2/4}, \tag{26}$$

where $n(x) > 0$ is a natural number.

If D is large enough one can absorbe the polymonial by a negative exponential, so we get

$$|\chi_{k,x}(x')|_{C^1(g_k)} \leq \eta^j(\delta')\exp^{-\lambda n(x)^2D^2/5}, \tag{27}$$

Again, because for D big enough we have for any r a fixed natural number and for very $n>1$ natural number

$$\exp^{-n^2D^2} < rn\exp^{-D^2}/2^n$$

it is easy to deduce

$$|\sum_{x\in\Lambda(D),\, x\neq y} \chi_{k,x}(x')|_{C^1(g_k)} \leq \eta^j(\delta')\exp^{-\lambda D^2/6} = \eta^j(\delta')\exp^{-\lambda' D^2}, \tag{28}$$

with $\lambda' = \lambda/6$.

Therefore, the inequality we need to check is

$$\eta^j(\delta')\exp^{-\lambda' D^2} \leq \eta^{j+1}(\delta')/2,\ j\in\{1,\dots,N\}, \tag{29}$$

which is equivalent to

$$2\exp^{-\lambda' D^2} \leq \frac{\eta^{j+1}}{\eta^j}\ j\in\{1,\dots,N\}. \tag{30}$$

Analisys of the function η implies the existence of a constant C only depending on the initial δ so that

$$\frac{\eta^{j+1}}{\eta^j} \geq \frac{C}{((j+1)\log(j+1))^p}. \tag{31}$$

Because $j+1 \leq C'D^{2n}$, we conclude

$$\frac{\eta^{j+1}}{\eta^j} \geq \frac{C}{D^{2np+1}}. \tag{32}$$

We choose D so that

$$\exp^{-\lambda' D^2} \leq \frac{2C}{D^{2np+1}},$$

and by equations 29, 30, and 32 this concludes the proof.

7. Topology of the symplectic divisors

Theorem 6. [5] *Let $s_k \in \Gamma(L^{\otimes k})$ be an A.H. sequence of sections uniformly transverse to $\mathbf{0}$. Let $W_k := s_k^{-1}(\mathbf{0})$ and let $i\colon W_k \to M$ be the inclusion. Then for $k \gg 1$ we have:*

1. *$[W_k] = P.D.[k\omega] \in H^{2n-2}(M;\mathbb{Z})$.*
2. *$i_*\colon H_j(W_k;\mathbb{Z}) \to H_j(M,\mathbb{Z})$ is an isomorphism for $j \leq n-1$ and an epimorphism for $j = n$. The same result holds for homotopy groups.*

Proof. It is a standard result that W_k is Poincaré dual to $c_1(L^{\otimes k})$ [16], which by construction is $k\omega$.

In the Kahler case if $s \in \Gamma(L)$ is a holomorphic section transverse to $\mathbf{0}$, then the second result is the famous Lefschetz hyperplane theorem. The proof is based on the fact that the function

$$f := s\bar{s}\colon X \to [0,b]$$

-after perhaps a small perturbation so that it becomes Morse- presents X as the results of adding some handles to $W = f^{-1}(0)$. Those handles are of index at least n and therefore standard homology/homotopy theory arguments give the result.

The assertion about the index of the handles is seen as follows: rather than building X starting with W, we think the other way around, that is we look at the Morse function $-\log f\colon X\backslash W \to [a,\infty)$

By Poincare Duality we need to check that the index of critical points of $-f$ do not exceed n. But $-f$ is strictly plurisubharmonic. Indeed, the Hessian is the quadratic form whose associated $(1,1)$ form is

$$\mathrm{Hess} f := 2i\partial\bar{\partial} f = -2i\partial\frac{\bar{\partial}(s\bar{s})}{s\bar{s}} = -2i\partial\bar{A} = 2iF_\nabla = 4\pi\omega. \tag{33}$$

In the symplectic case the situation is the same: the function f has critical points with index no bigger than n. Indeed, due to estimated transversality critical points can only occur for $s\bar{s} \geq \delta$, where $\delta > 0$ is as usual uniform: if x is a critical point then

$$0 = (\nabla s)\bar{s} + \overline{(\nabla s)\bar{s}},$$

so $(\nabla s)\bar{s}$ is real. But because $s(x) \neq 0$ it means that the image of $\nabla s(x)$ is contained in the line spanned by $s(x)$. But if $s(x)$ is smaller than η, then $\nabla s = \partial s + \partial s$ is nearly almost complex, so in particular it is onto. This is a contradiction. Therefore for $k \gg 1$ critical points of the norm of s are outside the tubular neighborhood of radius η of $\mathbf{0}$.

If x is a critical point we can use in $B_{g_k}(x,\rho)$ the complex structure J_{std}. Then s is A.H. and we have

$$\mathrm{Hess} f = 2i\bar{\partial}\partial f = 2i\bar{\partial}\left(\frac{\partial s}{s} + \frac{\bar{\partial} s}{\bar{s}}\right).$$

Because s is A.H. and its norm is bounded from below we can write

$$\mathrm{Hess} f = 2i\bar{\partial}\left(\frac{\partial s}{s} + \frac{\bar{\partial} s}{s}\right) + O(k^{-1/2}) = 2i\bar{\partial}\frac{\nabla s}{s} + O(k^{-1/2}).$$

Because the curvature of $L^{\otimes k}$ is of type $(1,1)$, and the $(0,1)$ part of $\nabla s/s$ is of size at most $O(k^{-1/2})$, we have

$$\left|\partial\left(\frac{\nabla s}{s}\right)\right|_{g_k} \leq O(k^{-1/2}).$$

Thus the consequence is

$$\mathrm{Hess} f = 2iF_\nabla + O(k^{-1/2}) = 4\pi\omega + O(k^{-1/2}). \tag{34}$$

□

8. Further results.

8.1. Higher rank bundles.

Theorem 7. [1] *Let E be a hermitian vector bundle with connection of (complex) rank d, and let $s_k \in \Gamma(E \otimes L^{\otimes k})$ be an A.H. sequence of sections uniformly transverse to $\mathbf{0}$, which always exists. Let $W_k := s_k^{-1}(\mathbf{0})$ and let $i\colon W_k \to M$ be the inclusion. Then for $k \gg 1$ we have:*

1. *W_k and symplectic submanifolds of (real) codimension $2d$.*
2. *$[W_k] = P.D.[c_d(E)] \in H^{2n-2d}(M;\mathbb{Z})$.*
3. *$i_*\colon H_j(W_k;\mathbb{Z}) \to H_j(M,\mathbb{Z})$ is an isomorphism for $j \leq n-d$ and an epimorphism for $j = n-d+1$. The same result holds for homotopy groups.*

The definition of A.H. sequence is clear, once we endow the total space of $E \otimes L^{\otimes k}$ with and almost complex structure and bundle metric using the tensor product of the data.

Locally, rather than using reference sections we use reference frames that we build by tensoring reference sections of $L^{\otimes k}$ and local reference frames of E.

The local transversality result is an extension of theorem 3 [1, 6] for functions

$$f\colon B^{2n}(0, 11/10) \to \mathbb{C}^m.$$

The topological properties of the submanifolds W_k are non-difficult extension of the line bundle case.

8.2. Higher jets: Lefschetz pencils and immersions.

Transversality to the zero section is the analog of the search of a regular point for a function. We can ask for analogs of generic mappings in the sense of Thom, that is mappings whose jets of certain orders have some genericity properties. The easiest example is that of Morse functions. For $f\colon M \to \mathbb{R}$ a function is Morse/1-generic if we have $df \pitchfork \mathbf{0}$. This is a transversality condition on the first jet. Similarly, for functions $f\colon M \to \mathbb{R}^m$ if $m \geq 2n$ then $df \pitchfork \mathbf{0}$ is equivalent to $df \cap \mathbf{0} = \emptyset$. That is f is an immersion. In full generality r-generic functions $f\colon M \to \mathbb{R}^m$ are those such that

$$j^r f \pitchfork \mathcal{S},$$

where $\jmath^r f$ is the r-jet (extension) of f, and $\mathcal{S} \subset \mathcal{J}^r\mathbb{R}^m$ is the Thom-Boardman stratification of the bundle of r-jets $\mathcal{J}^r\mathbb{R}^m$ [2].

The subset of r-generic functions -M compact- is open and dense. The key result is Thom's strong transversality lemma that says that given a suitable stratification of $\mathcal{J}^r\mathbb{R}^m$ we can find arbitrarily small perturbations of f so that $j^r f$ is transverse to the stratification (see for example [8]).

If we have $s_1, s_2\colon M \to L$, then away from the base locus $B := s_1^{-1}(\mathbf{0}) \cap s_2^{-1}(\mathbf{0})$ we have a projectivization

$$\begin{array}{rcl} \phi\colon M \backslash B & \longrightarrow & \mathbb{C} \\ z & \longmapsto & [s_1(z)\colon s_2(z)]. \end{array} \tag{35}$$

If the sections are holomorphic, then we can ask ϕ to be Morse in the complex case, that is we require

$$d\phi = \partial\phi \pitchfork \mathbf{0}.$$

If x is a critical point, then by the complex Morse lemma we have complex coordinates $z_1, \ldots, z_n$ about x so that

$$\phi(z) = {z_1}^2 + \cdots + {z_n}^2. \tag{36}$$

Also if $x \in B$, there exist complex coordinates $z_1, \ldots, z_n$ so that near x

$$B = \{z_1 = z_2 = 0\},\ \phi(z) = z_1/z_2. \tag{37}$$

In the symplectic case we have the analogous result:

Given $x \in (M,\omega)$ we say that complex coordinates $z_1,\dots,z_n$ centred at x are compatible with ω if the complex and symplectic orientations agree.

Theorem 8. [6] *Let (M,ω) be a compact symplectic manifold of integral type. Then one can find pairs (ϕ,B) with the following properties:*

(1) *B is codimension 4 symplectic submanifold.*
(2) *$f\colon M\backslash B \to \mathbb{C}$ has the following properties:*
  - *For any point $x \in B$ one can find coordinates compatible with ω so that*
$$\phi(z) = z_1/z_2.$$
  - *Critical points are isolated and around any such there exist coordinates compatible with ω so that*
$$\phi(z) = {z_1}^2 + \cdots + {z_n}^2.$$
  - *The closure of the fibers of ϕ amounts to adding B, and the result is a symplectic codimension 2 submanifold (away form the isolated singular points). Regular fibres have the properties of the submanifolds in theorem 6.*

Proof. One can find pairs of A.H sequences (s_k^1, s_k^2) of $L^{\otimes k}$ or rather an A.H. sequecne of sections of $\underline{\mathbb{C}}^2 \otimes L^{\otimes k}$ with the following properties:

Firstly $s_k := (s_k^1, s_k^2)$ is (uniformly) transverse to $\mathbf{0}$, which follows from theorem 7. We define the base locus $B_k := s_k^{-1}(\mathbf{0})$. It is not hard to see that near a point of B_k if we write $s_k^i = f_k^i \tau_{k,x}^{\mathrm{ref}}$, then f_k^1, f_k^2 can be extended to coordinates $z_{k,1}, z_{k,2}$ compatible with ω, and so that
$$\phi(z_k) = z_{k,1}/z_{k,2}.$$

Once this has been done we can define $\phi_k\colon M\backslash B_k \to \mathbb{C}$ the projectivization. It is possible to arrange the sections so that
$$\partial\phi \pitchfork \mathbf{0} \subset T^{*1,0}M.$$
This is indeed the analog of Thom's strong transversality theorem.

Because
$$d\phi_k = \partial\phi_k + \bar{\partial}\phi_k = \partial\phi_k + O(k^{-1/2}),$$
for $k \gg 1$ uniform transversality implies that critical points are isolated. Next near one such critical point we may replace on $B_{g_k}(x,\rho)$ the given J by J_{std}. The section is still A.H. w.r.t. the latter. In holomorphic coordinates $z_{k,1},\dots,z_{k,n}$ we can perturb the section by just taking the quadratic part w.r.t. these coordinates (or in a different way, we can project onto the holomorphic part), and the use a suitable bump function to interpolate between the A.H. section and its holomorphic quadratic part. □

Corollary 6. [1] *Donaldson's divisors in theorem 1 are for $k \gg 1$ symplectomorphic.*

Proof. For a given J, we may take 2 sequences and perturb them using theorem 8 so that they become a pencil. As a result we are comparing the fiber over $0,\infty \in \mathbb{CP}^1$. Note that because the modification is a small as desired, the initial fibers over these two points are isotopic to the ones for the pencil, and therefore symplectomorphic by Moser's theorem. Now because we have 2 regular fibers and the submersion is with symplectic fibers, we can use the symplectic orthogonal to the fibers to parallel transport one fiber to another over a given path in $\mathbb{CP}^1$ joining these two values, and only taking regular ones.

It is also possible to prove independence of the chosen c.a.c.s., and not just for divisors but for the submanifolds in theorem 7 □

Theorem 9. [19] *For $m \geq 2n$ we can find symplectic embeddings*

$$\phi\colon (M, \omega) \to (\mathbb{CP}^m, \omega_{FS}).$$

Proof. We take $s_k\colon M \to \underline{\mathbb{C}}^{m+1} \otimes L^{\otimes k}$ with the following properties:

- The section is transverse to $\mathbf{0}$. By dimension count this implies that it actually avoids the zero section, so we can projectivize
$$\phi_k\colon M \to \mathbb{CP}^m.$$
- We can arrange the section so that the projectivization is an immersion. The cotangent bundle stratifies according to the rank of the differential, and one can make sure (strong transversality) that $d\phi_k$ is transverse to this Thom-Boardman stratification. Again dimension count implies that it stays uniformly away from the non-open strata, hence it is (uniformly an immersion).

By construction $\phi_k^*\omega_{FS}$ is cohomologous to $k\omega$. Even more, because the construction is nearly Kahler, $\phi_k^*\omega_{FS}$ is nearly $(1,1)$. More precisely the convex combination

$$(1-t)\phi_k^*\omega_{FS} + tk\omega$$

is by symplectic forms, so we can apply Moser's theorem and right compose ϕ_k with it to achieve the symplectic immersion. □

8.3. **Removing the integrality assumption.** Theorem 1 is stated for compact symplectic manifolds, but our proofs so far only work for compact symplectic manifolds of integral type. We can use an approximation argument to remove the integrality assumption.

Given ω a symplectic form on M compact, we can find $\{\omega_n\}, n \in \mathbb{N}$, a sequence of rational 2-forms converging (just in C^0-norm is enough) to ω. Because the symplectic condition is open among closed 2-forms, the $\{\omega_n\}$ can be assumed to be symplectic.

Next fix J a c.a.c.s. for ω. Then use $g = \omega(\cdot, J\cdot)$ as initial metric in the polar decomposition construction of J_n a c.a.c.s. for ω_n, $n \in \mathbb{N}$. The result is that $J_n \to J$ and $g_n \to g$.

Because ω_n is rational taking appropriate powers we can apply all our theorems. The submanifolds we get are approximately J_n-complex. Because of the convergence $g_n \to g$, for all n, k we have

$$kg \leq C_1 k g_n,\; kg_n \leq C_2 kg.$$

Therefore we can use the metrics $g_k = kg$ in all statements. In particular the submanifolds we construct with different $n \in \mathbb{N}$ and for $k \gg 1$ (depending on n) are all approximatelly J_n-holomomorphic measuring with kg.

Because $J_n \to J$, for $n \gg 1$, $k \gg 1$ (depending on n) the submanifolds we construct are arbitrarily close to be J-holomorphic, and therefore they are symplectic w.r.t. ω.

8.4. **Relative transversality: applications to contat geometry and 2-calibrated geometry.** Let C be a compact manifold of dimension $2n+1$, and α a contact form. That is, a 1-form which is maximally non-integrable, or equivalently

$$\alpha \wedge d\alpha^n$$

is a volume form.

(C, α) can be symplectized to construct

$$(M, \omega) = (C \times \mathbb{R}^{>}, d(t\alpha)).$$

The symplectic form is exact, and hence the prequantum line bundle is trivial. Actually, it can be taken to be

$$(L, \nabla) = (\underline{\mathbb{C}}, -i\alpha, h_{\text{std}}).$$

The symplectization M is not compact but we can do A.H. geometry in the compact region $(C \times [1/2, 3/2], \omega)$. As a consequence we can construct s_k A.H. sections of

$$(\underline{\mathbb{C}}^{\otimes k}, -i\alpha, h_{\text{std}}) \to (C \times [1/2, 3/2], d(t\alpha), J).$$

We will assume that s_k has certain relative transversality property:

Definition 12. *Let $f\colon \mathbb{R}^d \to \mathbb{R}^m$ be a linear map and $D < \mathbb{R}^d$ a linear subspace. We say that f is transverse to $\mathbf{0}$ along D is the restriction $f_{|D}$ is surjective. If we have metrics we can define estimated transversality along D by restricting the inner product to D. If now D is a distribution of TM (perhaps defined in a submanifold) for $s\colon (M, g) \to (E, \nabla, h)$ we can in a straightforward manner define estimated transversality of s along D to $\mathbf{0}$.*

A special case we will deal with is when D is the tangent bundle of a submanifold.

It is clear that estimated transversality along D is stronger that estimated transversality.

Theorem 10. *Let (M, ω) be a symplectic manifold, and let N be a (closed) submanifold. Then one can find $s_k\colon M \to E \otimes L^{\otimes k}$ an A.H. sequence which is uniformly transverse along N (TN) to $\mathbf{0}$.*

We just mention that the proof of theorem 10 reduces to a relative local transversality theorem:

Theorem 11. [17] *Let $f\colon B^{2n}(0, 11/10) \to \mathbb{C}$ and let V be a linear subspace of $\mathbb{R}^{2n}$. Let δ be a constant $0 < \delta < 1/2$. Let $\eta(\delta) = \delta(P(\log(\delta^{-1}))^{-1}$, where P is a real monomial depending on n. If in the ball of radius 11/10 we have*

$$|f|_{g_{\text{std}}} \leq 1,\ |\bar{\partial} f|_{g_{\text{std}}} \leq \eta,\ |\mathrm{d}\bar{\partial} f|_{g_{\text{std}}} \leq \eta,$$

then there exists $w \in \mathbb{C}$, $|w| \leq \delta$, such that $f + w$ is η-transverse along V to $\mathbf{0}$ in $\bar{B}(0, 1)$.

We want to apply theorem 10 to $C = C \times \{1\} \hookrightarrow (C \times [1/2, 3/2], \omega)$, where we have chosen J making the contact distribution $\xi \subset TC$ into a complex subbundle. In that way we obtain A.H. sections whose restriction

$$s_{k|C}\colon C \to L^{\otimes k}$$

is uniformly transverse to $\mathbf{0}$. We claim that $s_{k|C}$ is also uniformly transverse along ξ to $\mathbf{0}$. We just prove the claim for transversality (not estimated); we further assume that s_k is actually J-holomorphic. Everything reduces to a linear checking. We do have

$$\nabla s_{k|M}\colon T_x C \to \mathbb{C}$$

a linear map which is J-complex along the real codimension one subspace ξ_x. The restriction to ξ_x must be onto. If not, the kernel of the linear map

$$\nabla s_{k|M}\colon \xi_x \to \mathbb{C}$$

has real dimension at least 2n-1. Notice that the map cannot be zero, because its extension to $T_x C$ is onto. Now because the restriction to ξ_x is complex linear, the kernel is a complex subspace. Therefore, if it has real dimension at least 2n-1, it must have real dimenson at least 2n, and that is a contradiction.

We now have the following information:

- Estimated transversality for $x \in C \cap s_k^{-1}(\mathbf{0})$ implies that the linear subspace $\ker\nabla s_k \subset (T_xM, \omega)$ is nearly J-complex and hence symplectic for $k \gg 1$.
- Estimated transversality along C for $x \in C \cap s_k^{-1}(\mathbf{0})$ implies that nearly J-complex linear subspace $\ker\nabla s_k \subset (T_xM, \omega)$ is uniformly transverse to T_xC for $k \gg 1$.
- Even more, estimated transversality along C for $x \in C \cap s_k^{-1}(\mathbf{0})$ implies that the nearly J-complex linear subspace $\ker\nabla s_k \subset (T_xM, \omega)$ is uniformly transverse to ξ_x for $k \gg 1$.

The third point above implies that $U_k := s_k^{-1}(\mathbf{0}) \cap C$ is uniformly transverse to ξ. It is easy to check that U_k is a contact submanifold of C iff $TU_k \cap \xi$ is a symplectic distribution of $(\xi, d\alpha)$.

For $x \in U_k$, we have $TU_k = TC \cap \ker\nabla s_k$ and therefore

$$TU_k \cap \xi = \ker\nabla s_k \cap \xi. \tag{38}$$

The two subspaces in the r.h.s. of 38 are symplectic, and in principle its intersection need not be symplectic. But the extra property that we have is that both are nearly J-complex. We know that transverse J-complex subspaces intersect in a J-complex subspaces. If the minimum angle between the subspaces is much bigger that the defect from being J-complex, then the intersection is nearly J-complex, and therefore symplectic. But this is the geometric translation of uniform transversality.

As a consequence we have proven the following result:

Theorem 12. [11] *Let (C, α) be a compact contact manifold. Then it contains contact submanifolds of real codimension 2.*

And we have the analogs of the results for symplectic manifolds.

Theorem 13. *Let (C, α) be an exact contact manifold and let $E \to C$ be a hermitian vector bundle with connection of (complex) rank d. Then there exist $s_k \colon C \to E \otimes L^{\otimes k}$ an A.H. sequence of sections uniformly transverse to $\mathbf{0}$. Let $U_k := s_k^{-1}(\mathbf{0})$ and let $i \colon U_k \to C$ be the inclusion. Then for $k \gg 1$ we have:*

1. *U_k are contact submanifolds of (real) codimension 2d.*
2. *$[U_k] = P.D.[c_d(E)] \in H^{2n-2d}(C; \mathbb{Z})$.*
3. *$i_* \colon H_j(U_k; \mathbb{Z}) \to H_j(C, \mathbb{Z})$ is an isomorphism for $j \leq n - d - 1$ and an epimorphism for $j = n - d$. The same result holds for homotopy groups.*

There is an analog result about the existence of Lefschetz pencils. In this case rather than a finite number of critical points there is a finite number of curves transverse to ξ of critical points. Their images -the critical values- are curves γ_i in $\mathbb{CP}^1$ in general position. The complement has a finite number of connected components. Two (regular) values in the same component correspond to contactomorphic fibres. But each time that we cross a curve γ_i the 2n-1 contact fiber changes by the attaching of an symplectic n-handle along a legendrian sphere [20].

We also have

Theorem 14. [14] *Let (C, α) be a contact manifold of dimension 2n+1. Then there exists isometric contact embeddings $\phi \colon (C, \alpha) \to (S^{4n+3}, \xi_{\mathrm{std}})$.*

The most spectacular application of A.H. geometry to contact geometry is the construction of open book decompositions: in $\mathbb{R}^2$ we have $\mathcal{B}_{std}$ the standard open book decomposition. This is the stratification given by the origin -the binding- and all half lines -the pages-. An open book decomposition on a manifold is given by a map $f \colon X \to \mathbb{R}^2$ so that $f \pitchfork \mathcal{B}_{\mathrm{std}}$. We can use the map to pullback the standard open book decomposition to X, constructing the open book decomposition $\mathcal{B}$. If (C, α) is an exact contact manifold then $\mathcal{B}$ is said to support α if the binding $K = f^{-1}(0)$ is a contact submanifold and all pages $(F, d\alpha)$ are Liouville manifolds, i.e.

we have $X \in \mathfrak{X}(F)$ so that $L_X d\alpha = d\alpha$. Roughly, if and open book decomposition supports α the Reeb vector field R_α is transverse to the pages; for a fixed page its flow defines a first return map which is a symplectomorphism. This map can be isotoped to a compactly supported one ψ, so that up to contact isotopy (C, α) is equivalent to $(F, d\alpha, \psi)$. Giroux and Mohsen [7] proved the existence of open book decompositions supporting α.

One can develop (relative) A.H. geometry not just to contact structures, but to slightly more general ones:

Definition 13. *A 2-calibrated structure is a triple (N, D, ω), with N a (closed) manifold of dimension 2n, D a codimension 1 distribution and ω a closed 2-form such that (D, ω) is a symplectic distribution.*

Exact contact structures (C, α) are an example of 2-calibrated structures (the triple is $(C, \xi, d\alpha)$). Here the distribution is maximally non-integrable. In the opposite side i.e. when D integrates into a foliation we have 2-calibrated foliations, which are a generalization of 3-dimensional taut foliations.

One has existence of 2-calibrated submanifolds, existence of Lefschetz pencil structures, and some embedding theorems [13].

References

[1] Auroux, D. Asymptotically holomorphic families of symplectic submanifolds. Geom. Funct. Anal. 7 (1997), no. 6, 971–995.

[2] Boardman, J. M. Singularities of differentiable maps. Inst. Hautes tudes Sci. Publ. Math. No. 33 1967 21–57.

[3] Yomdin, Yosef; Comte, Georges Tame geometry with application in smooth analysis. Lecture Notes in Mathematics, 1834. Springer-Verlag, Berlin, 2004.

[4] Demailly, J.-P., Complex analytic and differential geometry. Available at http://www-fourier.ujf-grenoble.fr/ demailly/books.html.

[5] Donaldson, S. K. Symplectic submanifolds and almost-complex geometry. J. Differential Geom. 44 (1996), no. 4, 666–705.

[6] Donaldson, S. K. Lefschetz pencils on symplectic manifolds. J. Differential Geom. 53 (1999), no. 2, 205–236.

[7] Giroux, Emmanuel Gomtrie de contact: de la dimension trois vers les dimensions suprieures. (French) [Contact geometry: from dimension three to higher dimensions] Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), 405–414,

[8] Golubitsky, M.; Guillemin, V. Stable mappings and their singularities. Graduate Texts in Mathematics, Vol. 14. Springer-Verlag, New York-Heidelberg, 1973. x+209 pp.

[9] Hirsch, Morris W. Differential topology. Corrected reprint of the 1976 original. Graduate Texts in Mathematics, 33. Springer-Verlag, New York, 1994.

[10] Hrmander, Lars An introduction to complex analysis in several variables. Third edition. North-Holland Mathematical Library, 7. North-Holland Publishing Co., Amsterdam, 1990.

[11] A. Ibort, D. Martínez Torres, F. Presas, On the construction of contact submanifolds with prescribed topology. J. Differential Geom. 56 (2000), no. 2, 235–283.

[12] Kruglikov, Boris S. Non-existence of higher-dimensional pseudoholomorphic submanifolds. Manuscripta Math. 111 (2003), no. 1, 51–69.

[13] Martínez Torres, D., The geometry of 2-calibrated structures. Portugal. Math. (N.S.) Vol 66, Fasc. 4, 2009, 427–512.

[14] Martínez Torres, D., Contact isometric embeddings in standard contact spheres. Preprint.

[15] McDuff, Dusa; Salamon, Dietmar Introduction to symplectic topology. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995. viii+425 pp.

[16] Milnor, John W.; Stasheff, James D. Characteristic classes. Annals of Mathematics Studies, No. 76. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974.

[17] J. P. Mohsen, Thesis. École Norm. Sup. Lyon (2001).

[18] Moser, Jrgen On the volume elements on a manifold. Trans. Amer. Math. Soc. 120 1965 286–294.

[19] Muoz, V.; Presas, F.; Sols, I. Almost holomorphic embeddings in Grassmannians with applications to singular symplectic submanifolds. J. Reine Angew. Math. 547 (2002), 149–189.
[20] F. Presas, Lefschetz type pencils on contact manifolds. Asian J. Math. 6 (2002), no. 2, 277–302.
[21] Tian, Gang On a set of polarized Khler metrics on algebraic manifolds. J. Differential Geom. 32 (1990), no. 1, 99–130.
[22] Yomdin, Y. The geometry of critical and near-critical values of differentiable mappings. Math. Ann. 264 (1983), no. 4, 495–515.

Depart. of Math., Instituto Superior Técnico, 1049-001 Lisbon, Portugal
E-mail address: martinez@math.ist.utl.pt