GLOBAL CLASSIFICATION OF GENERIC MULTI-VECTOR FIELDS OF TOP DEGREE

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Abstract. For any closed oriented manifold M, the top degree multi-vector fields transverse to the zero section of $\wedge^{\text{top}}TM$ are classified, up to orientation preserving diffeomorphism, in terms of the topology of the arrangement of its zero locus and a finite number of numerical invariants. The group governing the infinitesimal deformations of such multi-vector fields is computed, and an explicit set of generators exhibited. For the sphere S^n, a correspondence between certain isotopy classes of multi-vector fields and classes of weighted signed trees is established.

1. Introduction

The recent classification by O. Radko [5] of generic Poisson structures on oriented surfaces, raises the question of whether it is possible to extend it to higher dimensions. This classification, although stated in the language of Poisson geometry, relies on general results from differential geometry and the classification of area forms on closed surfaces. The reason is that, in dimension 2, the integrability condition that a bi-vector field must satisfy in order to be Poisson is void. So, for generic Poisson structures on an oriented surface Σ, the difficult problem of classifying solutions of a non-linear PDE reduces to the classification of (generic) sections of a trivial line bundle $\mathfrak{X}^2(\Sigma) \equiv \Gamma(\wedge^2(T\Sigma))$. Then, standard methods from differential geometry apply and the problem is greatly simplified. In this note we show how Radko's classification can be extended to higher dimensions for generic multi-vector fields of top degree.

On a surface a bi-vector field defines a Poisson structure. More generally, a multi-vector field of top degree defines a *Nambu structure* of top degree. Nambu structures are natural generalizations of Poisson structures: a *Nambu structure* of degree r, on a manifold M, is a r-multilinear, skew-symmetric bracket,

$$\{\cdot,\ldots,\cdot\}\colon \underbrace{C^\infty(M)\times\cdots\times C^\infty(M)}_{r} \to C^\infty(M),$$

which satisfies the Leibniz rule in each entry, and a Fundamental Identity [6] that naturally extends the Jacobi identity. For top degree structures the Fundamental Identity is void [7]. In spite of the formal similarities between Nambu structures and Poisson structures, for $r > 2$ the Fundamental Identity imposes much more restrictive conditions than one would expect from the Jacobi identity. That is, Nambu structures are in a sense harder to find than Poisson structures. On the other hand, Nambu structures are easier to describe.

We are interested in generic Nambu structures of top degree on a closed oriented manifold M. By Leibniz' rule, such a structure is described by a multi-vector field $\Lambda \in \mathfrak{X}^{\text{top}}(M)$. Genericity means that the graph of Λ cuts the zero section of the line bundle $\wedge^{\text{top}}TM$ transversally. In particular, the zero locus $\mathcal{H}$ of the multi-vector field Λ is a hypersurface in M. We will show how one can attach to each connected component H^i of $\mathcal{H}$ a numerical invariant, called the *modular period*, which depends only on the germ of Λ at H^i. We construct also a global invariant which measures the ratio between the volumes of the connected components of the complement of $\mathcal{H}$, called the *regularized Liouville volume.* These notions generalize corresponding notions for 2-dimensional Poisson manifolds.

Our main result is the following:

Theorem 1. *A generic Nambu structure $\Lambda \in \mathfrak{X}^{top}(M)$ is determined, up to orientation preserving diffeomorphism, by the diffeomorphism type of the oriented pair $(M, \mathcal{H})$ together with its modular periods and regularized Liouville volume.*

For dimension 2 this result recovers the classification of [5].

Using Theorem 1 we are also able to describe the Nambu cohomology group $H^2_\Lambda(M)$ which determines the infinitesimal deformations of the Nambu structure. On the other hand, we show that for dimension larger than 2, the Nambu cohomology group $H^1_\Lambda(M)$, which determines the outer automorphisms of the structure, is infinite dimensional.

The plan of the paper is as follows. In Section 1, we recall the definition of a Nambu structure of degree r and list briefly some of its main properties. In Section 2, we consider generic Nambu structures of degree n on an n-dimensional oriented manifold. We define, for each hypersurface H where the n-vector field Λ vanishes, a couple of equivalent invariants. They are the modular $(n-1)$-vector field X^H_Λ and the modular $(n-1)$-form Ω^H_Λ, which give two equivalent ways of describing the linearization of Λ along H. In Section 3 we introduce the modular period T^H_Λ, which is just the integral (or cohomology class) of the modular $(n-1)$-form, and depends only on the values of Λ on a tubular neighborhood of H. Conversely, we can recover the Nambu structure on a tubular neighborhood of the oriented hypersurface (H, Ω^H_Λ) once the modular period T^H_Λ is specified. The proof of theorem 1 is completed in Section 4, where we also introduce the regularized Liouville volume. In Section 5, among the possible cohomological structures one can attach to a Nambu structure, we consider (i) the group of infinitesimal outer automorphisms and (ii) the group of infinitesimal deformations of the structure. The later will turn out to have as many generators as the numerical invariants above and we will exhibit explicitly a set of generators, which extends that of 2-dimensional Poisson manifolds. On the other hand, we will show that the first cohomology group is infinite dimensional for $n \geq 3$, something to be expected from the local computation of this group presented in [3]. Finally, in Section 6, we observe that the correspondence between isotopy classes of generic bi-vectors on $\Sigma = S^2$ and isomorphism classes of weighted signed trees given in [5], holds for those generic Nambu structures in S^n for which the zero locus $\mathcal{H}$ only contains spheres.

2. Nambu structures

Poisson manifolds $(M, \{\cdot,\cdot\})$ are the phase spaces relevant for Hamiltonian mechanics. For a Hamiltonian system the evolution of any observable $f \in C^\infty(M)$ is obtained by solving the o.d.e.

$$\frac{df}{dt} = \{H, f\},$$

where $H \in C^\infty(M)$ is the Hamiltonian, a conserved quantity for the system (the "energy"). In 1973 Nambu [4] proposed a generalization of Hamiltonian mechanics based on an n-ary bracket. The dynamics of an observable $f \in C^\infty(M)$ would be governed by the analogous o.d.e.

$$\frac{df}{dt} = \{H_1, ..., H_{n-1}, f\},$$

associated to $n-1$ Hamiltonians $H_1, ..., H_{n-1}$, so now we would have $n-1$ conserved quantities.

In order to have the "expected" dynamical properties this bracket had to satisfy certain constraints. These were clarified by Takhtajan [6], who gave the following axiomatic definition of a Nambu structure.

Definition 1. *A* ***Nambu structure*** *of degree r in a manifold M^n, where $r \leq n$, is an r-multilinear, skew-symmetric bracket,*

$$\{\cdot, \dots, \cdot\}\colon \underbrace{C^\infty(M) \times \cdots \times C^\infty(M)}_{r} \to C^\infty(M),$$

satisfying:

(i) *the Leibniz rule:*

$$\{fg, f_1, \dots, f_{r-1}\} = f\{g, f_1, \dots, f_{r-1}\} + \{f, f_1, \dots, f_{r-1}\}g,$$

(ii) *the Fundamental Identity:*

$$\{f_1, \dots, f_{r-1}, \{g_1, \dots, g_r\}\} = \sum_{i=1}^{r} \{g_1, \dots, \{f_1, \dots, f_{r-1}, g_i\}, \dots, g_r\};$$

The Liebniz rule shows that the operator $X_{f_1,\dots,f_{r-1}} : C^\infty(M) \to C^\infty(M)$ which is associated to $r-1$ functions $f_1, \dots, f_{r-1}$ by

$$X_{f_1,\dots,f_{r-1}}(g) = \{g, f_1, \dots, f_{r-1}\},$$

is a derivation and hence a vector field. This is called the *Hamiltonian vector field* associated with $f_1, \dots, f_{r-1}$. More generally, the Leibniz identity shows that we have an r-vector field $\Lambda \in \mathfrak{X}^r(M)$ such that

$$\Lambda(df_1 \wedge \cdots \wedge df_r) = \{f_1, \dots, f_r\}.$$

On the other hand, the Fundamental Identity is equivalent to the fact that the flow of any Hamiltonian vector field $X_{f_1,\dots,f_{r-1}}$ is a *canonical transformation*, i.e, preserves Nambu brackets. Its infinitesimal version reads

$$\mathcal{L}_{X_{f_1,\dots,f_{r-1}}} \Lambda = 0.$$

For Nambu structures of top degree the Fundamental Identity becomes void [7].

Example 1. *On $\mathbb{R}^n$ we have a canonical, top degree, Nambu structure which generalizes the canonical Poisson structure on $\mathbb{R}^2$. The Nambu bracket assigns to n functions $f_1, \dots, f_n$ the Jacobian of the map $\mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto (f_1(x), \dots, f_n(x))$, so that*

$$\{f_1, \dots, f_n\} = \det \left[\frac{\partial f_i}{\partial x_j}\right].$$

More generally, any volume form $\mu \in \Omega^{top}(M)$ on a manifold M determines a Nambu structure: if $(x^1, \dots, x^n)$ are coordinates on M, so that $\mu = f dx^1 \wedge \cdots \wedge dx^n$, then the Nambu tensor field is

$$\Lambda \equiv \frac{1}{\mu} = \frac{1}{f} \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^n}.$$

The Fundamental Identity for $r > 2$ is of a more restrictive nature than the case $r = 2$, where it reduces to the usual Jacobi identity: if $r > 2$ besides requiring the fulfilment of a system of first order quadratic partial differential equations, the coefficients *must also* satisfy certain system of quadratic algebraic equations. For example, for a locally constant r-vector field the system involving first derivatives is automatically satisfied, while the algebraic relations are non-trivial, and in fact coincide with the well-known Plücker equations. Hence, only decomposable r-vectors define constant Nambu structures. Another example of this rigidity is the following well-known proposition (see [6]):

Proposition 1. *Let Λ be a Nambu structure. For any function $f \in C^\infty(M)$, the contraction $i_{df}\Lambda$ is also a Nambu structure.*

This rigidity makes it harder to "find" Nambu structures than Poisson structures. On the other hand, it makes Nambu structures easier to describe. Henceforth, we will assume that $r > 2$ if $n \geq 3$.

First of all, the Hamiltonian vector fields span a generalized foliation for which the leaves are either points, called *singular points*, or have dimension equal to the degree of the structure. Around these *regular points* we have the following canonical form for a Nambu structure (see for example [7]):

Proposition 2. *Let $x_0 \in M$ be a regular point of a Nambu structure Λ of degree r. There exist local coordinates $(x^1, \dots, x^n)$ centered at x_0, such that*

$$\Lambda = \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^r}.$$

For the singular points there are some deep linearization results due to Dufour and Zung [1].

3. Generic Nambu structures of top degree

In this section we consider Nambu structures of degree n in a closed orientable n-dimensional manifold M. Notice that in this case the Fundamental Identity is void [7], so a Nambu structure is just a multi-vector field $\Lambda \in \mathfrak{X}^n(M)$. We will restrict our attention to *generic* Nambu structures:

Definition 2. *A Nambu structure $\Lambda \in \mathfrak{X}^n(M)$ is called* ***generic*** *if its graph cuts the zero section of the line bundle $\wedge^n TM$ transversally.*

The generic sections form an open dense set in the Whitney C^∞ topology.

Let us fix a generic $\Lambda \in \mathfrak{X}^n(M)$. Its set of zeros, denoted $\mathcal{H}$, is the union of a finite number of connected hypersurfaces: $\mathcal{H} = \bigcup_{i \in I} H^i$, $\#I < \infty$. Fix one of them and call it H.

Over the points of H there is some linear information attached to Λ, namely the *intrinsic derivative* $d\Lambda^H \in T^*_H M \otimes \wedge^n TM$. It can be defined as $d\Lambda^H \equiv \nabla\Lambda_{|H}$, where ∇ is any linear connection on $\wedge^n TM$. This is independent of the choice of connection. The intrinsic derivative gives the linearization of the Nambu structure at H: if we view Λ as a section, it is the tangent space to the graph of Λ. It is important to observe that $d\Lambda^H$ never vanishes due to the transversality assumption. Notice that $d\Lambda^H$ is a section of $T^*_H M \otimes \wedge^n TM$, but due to the nature of our (trivial) line bundle it has two equivalent interpretations which we shall now explain.

Fix a volume form Ω in some neighborhood of H in M, so that after contracting it with the factor of $d\Lambda^H$ in $\wedge^n TM$, we obtain $d\Lambda^H \otimes \Omega \in T^*_H M$.

Definition 3. *The* ***modular*** *$(n-1)$**-vector field** of Λ along H is the unique $(n-1)$-vector field $X^H_\Lambda \in \mathfrak{X}^{n-1}(H)$ such that $i_{X^H_\Lambda}\Omega = d\Lambda^H \otimes \Omega$.*

This definition does not depend on the choice of Ω: if $\tilde{\Omega}$ is another volume form, then $\tilde{\Omega} = f\Omega$ for some non-vanishing smooth function f, and we find

$$i_{X^H_\Lambda} f\Omega = d\Lambda^H \otimes f\Omega.$$

Notice that since X^H_Λ is "tangent" to H (i.e. $X^H_\Lambda \in \mathfrak{X}^{n-1}(H)$) and no-where vanishing, we can define the *modular $(n-1)$-form* along H to be the dual $(n-1)$-form $\Omega^H_\Lambda \in \Omega^{n-1}(H)$; that is, $\Omega^H_\Lambda(X^H_\Lambda) = 1$. If we fix a vector field Y over H, which is transverse to H, the modular form along H is given by

$$\Omega_\Lambda^H = (-1)^{n-1}\frac{1}{i_Y d\Lambda^H \otimes \Omega} j^* i_Y \Omega,$$

where $j\colon H \hookrightarrow M$ is the inclusion and $i_Y d\Lambda^H \otimes \Omega \in C^\infty(H)$ the no-where vanishing function obtained by contracting Y with $d\Lambda^H \otimes \Omega$. This expression is independent of Y. The modular $(n-1)$-form along H is everywhere non-zero, and hence Λ determines an orientation in H. It is clear that to give either one of $d\Lambda^H$, X_Λ^H or Ω_Λ^H, determines the others.

Let us relate these definitions with the well-known notion of modular class of a Poisson manifold. For any Nambu structure of degree r on an oriented manifold there is a natural generalization of the modular class of a Poisson manifold [2], which we now recall. Again we fix a volume form Ω on M. Then, for any $r-1$ functions $f_1, \dots, f_{r-1}$ on M, we can compute the divergence of the corresponding Hamiltonian vector field:

$$(f_1, \dots, f_{r-1}) \mapsto \operatorname{div}^\Omega(X_{f_1,\dots,f_{r-1}}) \equiv \frac{1}{\Omega}\mathcal{L}_{X_{f_1,\dots,f_{r-1}}}\Omega.$$

It turns out that this assignment defines a $(r-1)$-vector field $\mathcal{M}_\Lambda^\Omega$ on M. If $\tilde{\Omega} = g\Omega$ is another volume form, where g is some no-where vanishing smooth function, we have

$$\mathcal{M}_\Lambda^{\tilde{\Omega}} = \mathcal{M}_\Lambda^\Omega + X_g,$$

where X_g is the $(r-1)$-vector field

$$X_g(f_1, \dots, f_{r-1}) = \{f_1, \dots, f_{r-1}, g\}.$$

One can introduce certain Nambu cohomology groups to take care of this ambiguity so that the cohomology class $[\mathcal{M}_\Lambda^\Omega]$ is well-defined and independent of Ω. This class is called the *modular class* of the Nambu manifold M and is the obstruction for the existence of a volume form on M invariant under Hamiltonian automorphisms.

Back to top degree Nambu structures, each volume form Ω determines a modular $(n-1)$-vector field $\mathcal{M}_\Lambda^\Omega$ representing the modular class, and which will depend on Ω. However, at points where the Nambu tensor vanishes all modular vector fields give the same value (see [2]). The modular $(n-1)$-vector field X_Λ^H along H, that we have introduced above, is nothing but the restriction of any modular vector field to H. In our case, however, it has the additional properties that it is non-zero and tangent to H.

4. Local characterization of a Nambu structure

In this section we study the local behavior of a generic Nambu structure $\Lambda \in \mathfrak{X}^n(M)$ on a neighborhood of its zero locus. We show that the germ of Λ around a connected component H of its zero locus is determined, up to isotopy, by the modular periods (to be introduced below).

Since the intrinsic derivative is functorial we immediately conclude that

Lemma 1. *Given two generic Nambu structures Λ_1 and Λ_2 with zero locus $\mathcal{H}_1$ and $\mathcal{H}_2$, and a diffeomorphism of Nambu structures $\psi\colon (M,\Lambda_1) \longrightarrow (M,\Lambda_2)$ then*

$$\psi_* X_{\Lambda_1}^{\mathcal{H}_1} = X_{\Lambda_2}^{\mathcal{H}_2}, \text{ and } \psi_*\Omega_{\Lambda_1}^{\mathcal{H}_1} = \Omega_{\Lambda_2}^{\mathcal{H}_2}.$$

Hence, it follows that a necessary condition for such a map to exist is that the cohomology classes $[\Omega_{\Lambda_2}^{\mathcal{H}_2}]$ and $[\Omega_{\Lambda_1}^{\mathcal{H}_1}]$ correspond to each other.

Now recall that, given a generic Nambu structure Λ, each connected component H of its zero locus $\mathcal{H}$ has an induced orientation from the n-vector field Λ. Hence, a class in $H_{dR}^{n-1}(H)$ is completely determined by its value on the fundamental cycle H.

Definition 4. *The **modular period** T^H_Λ of the component H of the zero locus of Λ is*

$$T^H_\Lambda \equiv \int_H \Omega^H_\Lambda > 0.$$

In fact, this positive number determines the Nambu structure in a neighborhood of H up to isotopy. To prove that we need the following classical result concerning the classification of volume forms.

Lemma 2 ((Moser))**.** *Let M be an orientable closed manifold, Ω_1 and Ω_2 two volume forms in M. If $[\Omega_1] = [\Omega_2] \in H^{top}(M)$, there exists a diffeomorphism isotopic to the identity which sends Ω_1 to Ω_2. Moreover, it can be chosen to have support in the closure of the complement of the closed set where the two volume forms coincide.*

The above result can be adapted to volume forms in compact manifolds with boundary which coincide in neighborhoods of the boundary components. We can now state and prove the main result in this section.

Proposition 3. *Let Λ_1 and Λ_2 be generic Nambu structures in M (oriented) which share a common component H of their zero locus with equal induced orientation, and for which the modular periods coincide: $T^H_{\Lambda_1} = T^H_{\Lambda_2}$. Then, there exists a diffeomorphism $\varphi : M \to M$, isotopic to the identity, and neighborhoods U_1 and U_2 of H, such that φ sends (U_1, Λ_1) to (U_2, Λ_2).*

Proof. First we can use Moser's lemma to construct a diffeomorphism $\phi : M \to M$ isotopic to the identity, which maps H to itself and sends $\Omega^H_{\Lambda_1}$ to $\Omega^H_{\Lambda_2}$. Hence, we can assume that $\Omega^H_{\Lambda_1} = \Omega^H_{\Lambda_2}$, and the problem reduces to a global linearization one.

We fix a collar $U = [-1, 1] \times H$ of the hypersurface H, with transverse coordinate r. Denoting the Nambu structure by Λ we define $\Lambda_0 = (-1)^{n-1}\frac{\partial}{\partial r} \wedge X^H_\Lambda$. We can write $\Lambda = f\Lambda_0$ for some $f \in C^\infty(U)$ and the linearization of Λ is $\Lambda_1 = r\Lambda_0$. The o.d.e. for a change of coordinates $\phi(r, x) = (g(r, x), x)$ (x a coordinate in H) so that $\phi_* f\Lambda_0 = r\Lambda_0$ is:

$$\frac{\partial g}{\partial r} = \frac{g}{f} \tag{1}$$

Since f has for each x an expansion along the radial direction of the form $r + a_2(x)r^2 + \cdots$, any solution $g(r, x) = ke^{\int \frac{1}{f} dr}$, $k > 0$, defines a smooth orientation preserving change of coordinates fixing H. □

Remark 1. *The existence of a one parameter family of solutions for equation 1 reflects the fact that for any linear Nambu structure $cr\frac{\partial}{\partial r} \wedge X^H_\Lambda$, $c \in \mathbb{R} \setminus \{0\}$, rescaling the radial coordinate is a transformation preserving the Nambu structure (canonical transformation). Note also that the reflection along H is a canonical transformation, but it changes the orientation of the tubular neighborhood. We are interested in transformations isotopic to the identity so our choice of g above is with $k > 0$.*

5. The global description of Nambu structures

In order to have a diffeomorphism between two Nambu manifolds it is necessary to have a diffeomorphism sending the zero locus of one structure to the zero locus of the other, preserving their induced orientations. Assuming this condition to hold, our problem is that of transforming a generic structure Λ_1 into another Λ_2, with common oriented zero set $\mathcal{H} = \bigcup_{i \in I} H^i$.

First of all, in the previous section we proved that if the modular periods of each component coincide, we can find collars U^i_1 and U^i_2 of the hypersurfaces H^i

and a diffeomorphism isotopic to the identity φ sending $(\mathcal{U}_1, \Lambda_1)$ to $(\mathcal{U}_2, \Lambda_2)$, where $\mathcal{U}_j = \bigcup_{i\in I} U_j^i$, $j = 1, 2$.

The restriction of the Nambu structures to the connected components of $M \setminus \mathcal{H}$ define volume forms (the dual volume forms). However, their volumes are infinite so one cannot require them to match. Instead, we could try to define finite ratios of the volumes between the various components. This raises some accounting problems, so instead we observe that for a component H, a volume form Ω defined in a neighborhood of H and the volume form Λ^{-1} define orientations on the complement of H, which match on one side of H and are opposite on the other side. Given any function $h \in C^\infty(M)$ only vanishing linearly in the components of $\mathcal{H}$ (its graph is transverse to the zero section and vanishes exactly at $\mathcal{H}$), we let $M^\epsilon(h) = h^{-1}(\mathbb{R} \setminus (-\epsilon, \epsilon))$, with $\epsilon > 0$ small enough so that $M^\epsilon(h)$ contains the complement of the union of collars of the H^i, and we set

$$V^\epsilon_\Lambda(h) = \int_{M^\epsilon(h)} \Lambda^{-1}.$$

Here Λ^{-1} denotes the volume form dual to Λ, and to integrate we use the given orientation of M. The following definition generalizes the one given in [5] for the case of 2-dimensional Poisson manifolds.

Definition 5. *The **regularized Liouville volume** of Λ is defined as*

$$V_\Lambda = \lim_{\epsilon\to 0} V^\epsilon_\Lambda(h),$$

where h is any function only vanishing linearly at $\mathcal{H}$.

We only need to prove the independence on the choice of function h, because for a function that locally coincides with a radial coordinate in which the n-vector field is linear the existence of the limit is straightforward.

Independence on the choice of h can be checked by adapting Radko's proof for surfaces [5] to our higher dimensional setting. We fix coordinates (r, x) around each component H such that $\Lambda = (-1)^{n-1} r \frac{\partial}{\partial r} \wedge X^H_\Lambda$, and we consider two cases:

(1) Assume $h(r, x) = g(x)r$, with $g(x) \neq 0$ for all $x \in H$. Let us denote by $H^{>1}(|g|)$ (resp. $H^{<1}(|g|)$) the points of H where the absolute value of g is greater (resp. smaller) than 1. Then

$$\begin{aligned} V^\epsilon_\Lambda(h) - V^\epsilon_\Lambda(r) \;&=\; \pm \int_{H^{>1}(|g|)} \left(\int_{[-\epsilon,-\epsilon/|g(x)|]\cup[\epsilon/|g(x)|,\epsilon]} (-1)^{n-1} \frac{dr}{r} \right) \Omega^H_\Lambda \mp \\ &\mp \int_{H^{<1}(|g|)} \left(\int_{[-\epsilon/|g(x)|,-\epsilon]\cup[\epsilon,\epsilon/|g(x)|]} (-1)^{n-1} \frac{dr}{r} \right) \Omega^H_\Lambda, \end{aligned}$$

and each summand vanishes (for every $\epsilon > 0$ small enough).

(2) Since h vanishes linearly at $\mathcal{H}$, the function $h - \frac{\partial h}{\partial r}(0, x)r$ vanishes in the radial direction at least to second order at H. The compactness of H implies that for all $x \in H$ and all $\epsilon > 0$ small enough, positive constants k_1 and k_2 exists (independent of ϵ and x), with $a_\epsilon > k_1\epsilon$ and $b_\epsilon - a_\epsilon < k_2\epsilon^2$, such that:

$$|V^\epsilon_\Lambda(h) - V^\epsilon_\Lambda(\tfrac{\partial h}{\partial r}(0, x)r)| \leq \int_H \left(\int_{[-b_\epsilon,-a_\epsilon]\cup[a_\epsilon,b_\epsilon]} \left|\frac{1}{r}\right| dr \right) \Omega^H_\Lambda.$$

Thus

$$|V^\epsilon_\Lambda(h) - V^\epsilon_\Lambda(\tfrac{\partial h}{\partial r}(0, x)r)| \leq \int_H k\epsilon\Omega^H_\Lambda = k\epsilon T^H_\Lambda,$$

for some $k > 0$ and hence when $\epsilon \to 0$ the difference vanishes.

The modular periods and the regularized volume determine the Nambu structure:

Theorem 2. *For $j = 1, 2$, let M_j be oriented closed manifolds with generic Nambu structures Λ_j having zero locus $\mathcal{H}_j = \bigcup_{i\in I} H_j^i$. Assume that there exists a diffeomorphism ψ sending $(M_1, \mathcal{H}_1)$ to $(M_2, \mathcal{H}_2)$ and preserving the induced orientations of the zero locus. Then there exists an isomorphism between the two Nambu structures, isotopic to ψ, if and only if the following conditions are satisfied:*

(i) *the modular periods coincide, i.e, $T_{\Lambda_1}^{H_1^i} = T_{\Lambda_2}^{\psi H_1^i}$, $\forall i \in I$,*
(ii) *the regularized volumes match, i.e, $V_{\Lambda_1} = \varepsilon V_{\Lambda_2}$, where $\varepsilon = 1$ if ψ is orientation preserving and $\varepsilon = -1$ if it reverses the orientations on the M_i.*

Proof. Because of proposition 3 if the modular periods of each component coincide, we can find collars U_1^i and U_2^i of the hypersurfaces H^i, and a diffeomorphism isotopic to the identity φ which sends $(\mathcal{U}_1, \Lambda_1)$ to $(\mathcal{U}_2, \Lambda_2)$, where $\mathcal{U}_j = \bigcup_{i\in I} U_j^i$.

The volume of each connected component of $M \setminus \mathcal{H}$ with respect to the duals Λ_i^{-1} of any of the n-vectors is infinite. For any such connected component L, we can select a hypersurface H^{i_0} on its boundary and shrink accordingly the size of either $U_1^{i_0}$ or $U_2^{i_0}$ (recall we have canonical transformations doing that) such that one can find compact submanifolds $W_j \subset L$ which are the result of removing from L the corresponding side of the collars of radius $1/2$ say (the original radius is 1), verifying:

(i) the Λ_1^{-1}-volume of W_1 coincides with the Λ_2^{-1}-volume of W_2.
(ii) φ sends W_1 to W_2.

Finally, we apply Moser theorem to conclude the existence of a diffeomorphism isotopic to the identity also matching the Nambu structures at L.

Observe that when we modify the size of $U_1^{i_0}$ say, we are changing the volume of both $L \setminus \mathcal{U}_1$ and $L' \setminus \mathcal{U}_1$, where L and L' are the connected components of $M \setminus \mathcal{H}$ whose boundary contains H^{i_0}. It follows that without further assumptions we can make our n-vector fields coincide, as well as in the collars of $\mathcal{H}$, in all the connected components but possibly one. To see this we can take the graph dual to the splitting given by $\mathcal{H}$, where each vertex represents a connected component of $M \setminus \mathcal{H}$ and an edge joining two vertices stands for a connected hypersurface on its common boundary, and consider a maximal tree (a contractible subgraph containing all the vertices, which always exists); we fix a vertex v_0 on this tree and consider the graph distance (of vertices) with respect to v_0. We can then proceed by stages, where at each stage we consider all the vertices at the same distance of v_0, starting from the furthest way vertices. For those vertices, i.e, connected components of $M \setminus \mathcal{H}$, we apply the above reasoning to the hypersurface representing the only edge reaching them (if a furthest way vertex is reached by two different edges, we could join the two vertices in the boundary of the edges with v_0 and then form a loop). When we are done we erase those vertices and edges connecting to them, so we obtain a smaller tree. We keep on doing that until we reach the vertices at distance one (notice that each step does not affect the connected components corresponding to further vertices, since that would imply the existence of a loop in the tree). The same process is applied to all but one edge. The fact that the regularized volumes match, grants us the matching of the volumes of the two remaining components of $M \setminus \mathcal{H}$ for both Nambu structures, once an appropriate collar of the hypersurface representing the last edge has been removed, and this finishes the proof. □

The set of generic Nambu structures has an action of $\mathrm{Diff}_0(M)$ (resp. $\mathrm{Diff}^+(M)$). Its space of orbits has as many connected components as isotopy classes (resp. oriented diffeomorphism classes) of oriented hypersurfaces $\mathcal{H} = \bigcup_{i\in I} H^i$. Theorem

2 gives an explicit parametrization of each connected component of this moduli space.

6. Nambu cohomology

There are several cohomology theories one can associate to a Nambu manifold (see [2, 3]). Here we will be interested in the cohomology associated with the complex

$$0 \longrightarrow \wedge^{n-1} C^\infty(M) \longrightarrow \mathfrak{X}(M) \longrightarrow \mathfrak{X}^n(M) \longrightarrow 0\,,$$

where the first map is $f_1 \wedge \cdots \wedge f_{n-1} \mapsto X_{f_1,\dots,f_{n-1}}$, while the second map is $X \mapsto \mathcal{L}_X \Lambda$. Notice that the associated cohomology groups have simple geometrical meanings:

(i) $H^0_\Lambda(M)$ is the space of Casimirs of the Nambu structure;
(ii) $H^1_\Lambda(M)$ is the space of infinitesimal outer automorphisms of the Nambu structure;
(iii) $H^2_\Lambda(M)$ is the space of infinitesimal deformations of the Nambu structure.

Computations of Nambu cohomology for *germs* of Nambu structures defined by quasipolynomials (functions vanishing at the origin with finite codimension) were done by Monnier in [3]. Here we are interested in *global* Nambu structures with the simplest singularity. These computations can be thought of as a infinitesimal version on the classification theorem; in particular, $H^2_\Lambda(M)$ will turn out to be the tangent space of the class of $[\Lambda]$ in the moduli space of generic Nambu structures.

The main result of this section is the following

Theorem 3. *Let Λ be a generic Nambu structure on an oriented closed manifold M with zero locus $\mathcal{H} = \bigcup_{i\in I} H^i$. The group $H^2_\Lambda(M)$ has dimension $\#I + 1$ and a set of generators is given by*

$$\beta_1(-1)^{n-1} r \frac{\partial}{\partial r} \wedge X_\Lambda^{H^1}, \dots, \beta_{\#I}(-1)^{n-1} r \frac{\partial}{\partial r} \wedge X_\Lambda^{H^{\#I}},\ \Omega,$$

where Ω is a volume form, and each β_i is a bump function supported in a collar of the hypersurface H^i.

We can give a geometric description of the isomorphism $H^2_\Lambda(M) \simeq \mathbb{R}^{\#I+1}$ as follows. Each $\Theta \in \mathfrak{X}^n(M)$ is cohomologous to an n-vector field whose vanishing set contains $\mathcal{H}$ and is generic in a neighborhood of $\mathcal{H}$. Then we can write $[\Theta] = [g\Lambda]$ where g is some smooth function which assumes a constant value c_i in the collar of each U^i. The isomorphism is

$$[\Theta] \longmapsto \left(\frac{T_\Lambda^{H^1}}{T_\Theta^{H^1}}, \dots, \frac{T_\Lambda^{H^{\#I}}}{T_\Theta^{H^{\#I}}}, V_\Theta^{\mathcal{H},\Lambda} \right),$$

where:

(a) $T_\Lambda^{H^i} / T_\Theta^{H^i} = c_i$,
(b) $V_\Theta^{\mathcal{H},\Lambda}$ is the regularized integral of $g\frac{1}{\Lambda}$.

The rest of this section is dedicated to the proof of this result, which consists of a Mayer-Vietoris argument: we first compute the groups in the collars and then we glue them using information about the infinitesimal automorphisms in those neighborhoods.

6.1. **Computation of $H^2_\Lambda(U)$.** Let us fix $H \subset \mathcal{H}$ and $U = (-1,1) \times H$ a collar in which Λ is linear.

Proposition 4. *$H^2_\Lambda(U) \simeq \mathbb{R}$ and a generator is given by the linearization $(-1)^{n-1} r \frac{\partial}{\partial r} \wedge X_\Lambda^H$.*

Proof. Any vector field X can be written $X = A\frac{\partial}{\partial r} + X_H$, where $A \in C^\infty(U)$, $X_H \in \pi_2^* TH$, $\pi_2\colon (-1,1)\times H \to H$ the projection onto the second factor. Defining $\Lambda_0 = (-1)^{n-1}\frac{\partial}{\partial r}\wedge X_\Lambda^H$, one has:

$$\begin{aligned}\mathcal{L}_X\Lambda &= \mathcal{L}_X r\Lambda_0 = A\Lambda_0 + r\mathcal{L}_X\Lambda_0 = \\ &= \left(A - r\frac{\partial A}{\partial r}\right)\Lambda_0 + (-1)^{n-1} r\frac{\partial}{\partial r}\wedge \mathcal{L}_{X_H}X_\Lambda^H = \\ &= \left(A - r\frac{\partial A}{\partial r} + r\mathrm{div}^{\,\Omega_\Lambda^H}(X_H)\right)\Lambda_0, \qquad (2)\end{aligned}$$

where $\mathrm{div}^{\,\Omega_\Lambda^H}(X_H)$ is the divergence of X_H with respect to Ω_Λ^H (for each r).

Any n-vector field $f\Lambda_0$ in U is equivalent to a unique linear one: we observe that if f is at least quadratically vanishing at the origin, it is a coboundary; write $f = r^2 g$ and apply equation 2 to obtain

$$\mathcal{L}_{(-r\int gdr)\frac{\partial}{\partial r}}\Lambda = (-r\int gdr + r\int gdr + r^2 g)\Lambda_0 = f\Lambda_0.$$

We first make f vanish at the origin by adding the coboundary $\mathcal{L}_{-f\frac{\partial}{\partial r}}\Lambda$ (so it becomes $r\frac{\partial f}{\partial r}\Lambda_0$). Writing $f = r\hat{f}$, the coefficient of the linear representative we look for is $c = \int_{\{0\}\times H}\hat{f}\Omega_\Lambda^H \in \mathbb{R}$, because if we choose any $Y \in \mathfrak{X}(H)$ such that $\hat{f}_{|H} + div^{\Omega_\Lambda^H}(Y) = c$, from equation 2 again we deduce that $\Lambda_1 = f\Lambda_0 + \mathcal{L}_Y\Lambda$ has $cr\Lambda_0$ as constant linear part at H. Finally, $\Lambda_1 - cr\Lambda_0$ is at least quadratically vanishing at H and hence it is a coboundary.

The uniqueness of the linear representative will follow from the non existence of solutions for the equation $E_c \equiv A - r\frac{\partial A}{\partial r} - r\mathrm{div}^{\,\Omega_\Lambda^H}(X_H) = cr$, for $c = 1$. We need also to study the equation E_0 of Nambu infinitesimal automorphisms.

Lemma 3. *The equation E_1 has no solutions, and $Z_\Lambda^1(U)$ –the space of solutions of E_0– can be identified with the vector space:*

$$Z_\Lambda^1(U) \cong span\langle r\frac{\partial}{\partial r}, X_H \in \pi_2^*TH \,|\, \mathrm{div}^{\,\Omega_\Lambda^H}(X_H(0)) = 0\rangle.$$

of Lemma 3. In the equations E_c we can write the term $\mathrm{div}^{\,\Omega_\Lambda^H}(X_H)$ in the form ψ_r, where ψ_r is a smooth family in $r \in (-1,1)$ of functions in $C^\infty(H)$ which satisfy $\int_H \psi_r\Omega_\Lambda^H = 0$, $\forall r \in (-1,1)$. The solutions of E_1 can be explicitly written as:

$$A = kr + r\int\frac{\psi_r - 1}{r}dr,\ k \in \mathbb{R}$$

Any solution has to be a smooth continuation of the above expression, but it cannot exist. Indeed, since ψ_0 has vanishing integral, we can find a point x in H, such that $\psi_0(x) = 0$. Hence, in a small segment $[-\epsilon,\epsilon]\times\{x\}$ the real valued function $r\int\frac{\psi_r(x)-1}{r}dr$ is, up to a smooth function, $r\log r$ (not even C^1). □

With this we have finished the computation of $H_\Lambda^2(U)$. □

Remark 2. *Regarding E_0, its solutions are of the form*

$$A = kr + r\int\frac{\psi_r}{r}dr,$$

which will be smooth if and only if $\psi_0 = 0$, or the corresponding vector field $X_H(0)$ is divergence free with respect to Ω_Λ^H. Hence,

$$Z_\Lambda^1(U) \cong span < r\frac{\partial}{\partial r}, X_H \in \pi_2^*TH \,|\, \mathrm{div}^{\,\Omega_\Lambda^H}(X_H(0)) = 0 >$$

6.2. **From $H^2_\Lambda(U)$ to $H^2_\Lambda(M)$.** The remaining step is to piece all the local information. We just showed that in the same radial coordinate in which Λ is linearized in $U^i = (-1,1)\times H^i$, we can find a representant Θ of the cohomology class such that $\Theta_{|U^i} = c_i(-1)^{n-1} r\frac{\partial}{\partial r}\wedge X_\Lambda^{H^i} = c_i\Lambda$, where the relative period $\frac{T_\Lambda^{H^i}}{T_\Theta^{H^i}}$ is c_i, which might be zero. In particular, for a no-where vanishing Nambu structure all the local invariants vanish because by looking at its dual form it is clear that since it does not vanishes, we can push its graph down (or up) to the zero section to make it vanish in the U^i's. This operation can be made without changing the area (do it randomly and multiply by the ratio of both areas). We also saw that we can restrict our attention to coboundaries X such that $X_{|U^i}$ is a solution of E_0 in the radial coordinates.

The global regularized volume with respect to Λ is well defined because the regularized volume of $\mathcal{L}_X\Lambda$ vanishes, for X infinitesimal automorphism of Λ in U^i. To see that we choose a function h coinciding with the radial coordinate in each U^i, $M^r(h) = M\smallsetminus\bigcup_{i\in I}(-r,r)\times H^i$. We have:

$$\int_{M^r(h)}\mathcal{L}_X\Lambda = \int_{M^r(h)} di_X\frac{1}{\Lambda} = \pm\sum_{i\in I}\left(\int_{\{r\}\times H^i} i_X\frac{1}{\Lambda} - \int_{\{-r\}\times H^i} i_X\frac{1}{\Lambda}\right) \tag{3}$$

And for a fixed component H and $X = kr + r\int\frac{\operatorname{div}^{\Omega_\Lambda^H}(X_H)}{r}dr\frac{\partial}{\partial r} + X_H$, the function
$I(r) = \int_{\{r\}\times H} i_X\frac{1}{\Lambda}$ equals:

$$\begin{aligned} I(r) &= (-1)^{n-1}\int_{\{r\}\times H}\frac{1}{r}\left(kr + r\int\frac{\operatorname{div}^{\Omega_\Lambda^H}(X_H)}{r}dr\right)\Omega_\Lambda^H = \\ &= (-1)^{n-1}\int_{\{r\}\times H}\left(k + \int\frac{\operatorname{div}^{\Omega_\Lambda^H}(X_H)}{r}dr\right)\Omega_\Lambda^H \end{aligned}$$

Due to the fact that $\operatorname{div}^{\Omega_\Lambda^H}(X_H)(0) = 0$, the above formula defines a smooth function for all $r\in[-1,1]$. Its derivative is easily computed:

$$\begin{aligned} \frac{dI}{dr} &= (-1)^{n-1}\frac{d}{dr}\int_{\{r\}\times H}\left(k + \int\frac{\operatorname{div}^{\Omega_\Lambda^H}(X_H)}{r}dr\right)\Omega_\Lambda^H = \\ &= (-1)^{n-1}\int_{\{r\}\times H}\frac{\operatorname{div}^{\Omega_\Lambda^H}(X_H)}{r}\Omega_\Lambda^H = 0 \end{aligned}$$

The vanishing is clear for $r\neq 0$ and follows by continuity. Hence $I(r)$ is constant and $V_\Theta^{\mathcal{H},\Lambda}$ is well defined.

It only remains to show that two n-vectors Θ_1 and Θ_2 with equal linearizations and regularized volume are in the same class. Its difference has a representative $\tilde\Theta$ vanishing in a neighborhood of the boundary of $M-\mathring{\mathcal{U}}$. Then the form $\frac{1}{\Lambda}(\tilde\Theta)\cdot\frac{1}{\Lambda}$ has compact support (shrinking a bit the collars if necessary) and vanishing integral, so we can find a compactly supported vector field Y whose divergence is $\frac{1}{\Lambda}(\tilde\Theta)\cdot\frac{1}{\Lambda}$. It follows that $\mathcal{L}_{\tilde Y}\Lambda = \tilde\Theta$, where $\tilde Y$ extends Y trivially.

The assertion about the basis of $H^2_\Lambda(M)$ follows easily.

6.3. **Some comments about $H^1_\Lambda(M)$ and $H^0_\Lambda(M)$.** We will focus our attention in what happens in a collar U. Given $f\in C^\infty(U)$ we write $df = \frac{\partial f}{\partial r}dr + d_H f$. We can express the vector space $B^1_\Lambda(U)$ of Hamiltonian vector fields as follows:

$$\begin{aligned} B^1_\Lambda(U) &= \{(-1)^{n-1} r X^H_\Lambda(d_H f_1,\ldots,d_H f_{n-1})\frac{\partial}{\partial r} + \\ &+ \sum_{j=1}^{n-1}(-1)^{n-i} r\frac{\partial f_j}{\partial r} X^H_\Lambda(d_H f_1,\ldots,\widehat{d_H f_j},\ldots,d_H f_{n-1})\}, \end{aligned}$$

with $f_1,\ldots,f_{n-1}\in C^\infty(U)$.

Hence all Hamiltonian vector fields must vanish along H. For each H^i let us denote by $\mathfrak{X}_{free}(H^i)$ the vector space of divergence free vector fields in H^i with respect to the volume form $\Omega^{H^i}_\Lambda$. Denoting by r_i to the corresponding radial coordinate, we have the following

Corollary 1.

(1) $\langle r_i\frac{\partial}{\partial r_i}\rangle \oplus \mathfrak{X}_{free}(H^i)\subset H^1_\Lambda(U^i)$.

(2) $\bigoplus_{i\in I}\left(\langle \phi_i\cdot r_i\frac{\partial}{\partial r_i}\rangle \oplus \phi_i\cdot\mathfrak{X}_{free}(H^i)\right)\subset H^1_\Lambda(M)$, *where* ϕ_i *are bump functions supported in the collars. For* $n\geq 3$ *this space is clearly infinite dimensional.*

Proof. The assertion about the divergence free vector fields is clear. Regarding the size of the space we notice that it can be identified with closed $(n-2)$-forms in H^i containing the exact ones. From the description of $B^1_\Lambda(U)$ we see that the coefficient of $r\frac{\partial}{\partial r}$ contains the factor $X_\Lambda(d_H f_1,\ldots,d_H f_{n-1})$ which cannot be everywhere nonvanishing on each $\{r\}\times H$ by compactness. □

We see that the case $n=2$ is quite special and in fact one can easily compute $H^1_\Lambda((-1,1)\times S^1)$.

Corollary 2. $H^1_\Lambda((-1,1)\times S^1)$ *is spanned by the modular vector field* $X^{S^1}_\Lambda$ *and* $r\frac{\partial}{\partial r}$.

Proof. The vector field $X^{S^1}_\Lambda$ trivializes TS^1 so any vector field can be written as $X=A\frac{\partial}{\partial r}+gX^{S^1}_\Lambda$, $A,g\in C^\infty((-1,1)\times S^1)$. One checks that

$$B^1_\Lambda=\{-rX^{S^1}_\Lambda(d_{S^1}f)\frac{\partial}{\partial r}-r\frac{\partial f}{\partial r}X^{S^1}_\Lambda \mid f\in C^\infty((-1,1)\times S^1)\}, \tag{4}$$

and

$$Z^1_\Lambda=\{\left(kr+r\int\frac{X^{S^1}_\Lambda(d_{S^1}g)}{r}dr\right)\frac{\partial}{\partial r}+gX^{S^1}_\Lambda \mid g\in C^\infty((-1,1)\times S^1),\ g_{|S^1}=k_1,\ k,k_1\in\mathbb{R}\} \tag{5}$$

And any g in the above description of a cocycle can be assumed to be the constant k_1; just add the coboundary defined by the function $f=\int\frac{g-k_1}{r}dr$ (as described in 4). □

Determining the group $H^1_\Lambda(U)$ seems in general a very difficult problem.

Also there seems to be little hope to compute $H^0_\Lambda(M)$ easily. For example for $n=3$ we see that $X_{f,g}=0$ implies that df and dg have to be proportional. If we assume f to be a Morse function, g has to be constant on its leaves. So we have as many choices for g as the ring of smooth functions of the leaf space M^3/f. This is a one dimensional space that can be very different for the same manifold (one can construct them from a handle decomposition of the manifold just looking at how the homotopy group π_0 changes when we add handles).

7. Some special families of Nambu structures

As we have seen, the problem of classifying generic Nambu structures on a given manifold includes that of the classification of certain arrangements of connected oriented hypersurfaces (those arrangements that come from the zeros of a function). For M^n one can consider the dual graph to $(M, \mathcal{H})$ and put a plus sign if the orientation of the n-tensor in the connected component coincides with that of M, and minus otherwise. Giving the signs is equivalent to giving the orientation of the hypersurfaces.

For S^2, Radko [5] defines $\mathcal{G}_k(S^2)$ as the set of generic Poisson structures on S^2 with k vanishing curves. She observes that the associated dual graphs are trees (each circle disconnects the 2-sphere). A weighted signed tree is defined as a tree with a plus or minus sign attached to each vertex so that for each vertex, those belonging to the boundary of its star have opposite sign; each edge is weighted with a positive number (the modular period), and a real number (the regularized volume) is assigned to the whole graph. She proves the following:

Theorem 4 (([5])). *The set $\mathcal{G}_k(S^2)$, up to orientation preserving isomorphisms, coincides with the isomorphism classes of weighted signed trees with $k+1$ vertices (the isomorphism has to preserve the real number attached to the graph).*

The result relies on the fact that there is a one to one correspondence between arrangements of k circles in S^2 (in fact up to isotopy) and isomorphism classes of trees with $k+1$ vertices (observe also that every tree can be signed in two ways). One can isotope two arrangements with equivalent tree because, up to isotopy, the circle sits in S^2 in a unique way splitting S^2 in two disks, and that results admits a well-known generalization.

Theorem 5 ((Smooth Schoenflies theorem)). *Any smooth embedding $j\colon S^{n-1} \hookrightarrow S^n$ bounds an n-dimensional ball and hence splits the sphere into two n-dimensional balls. In particular it is isotopic to the standard one where S^{n-1} sits inside $\mathbb{R}^n \subset \mathbb{R}^n \cup \{\infty\} = S^n$ as the boundary of the Euclidean ball of radius one. It also holds for embeddings in $\mathbb{R}^n$.*

As consequence of this result one easily proves the following:

Lemma 4. *There is a one to one correspondence between arrangements of k $(n-1)$-spheres in S^n and isomorphism classes of trees with $k+1$ vertices.*

Definition 6. *Let us define $\mathcal{G}_k(S^n)$ to be the set of generic Nambu structures in S^n whose vanishing set consist of k $(n-1)$-spheres.*

Giving S^n the usual orientation we can put signs in the dual trees. Thus we have just proved the following

Proposition 5. *The set $\mathcal{G}_k(S^n)$ is, up to isotopy, the same as the equivalence classes of weighted signed trees with $k+1$ vertices (and hence the set of isotopy classes is the same for every $n \geq 2$).*

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