A NEW CONSTRUCTION OF POISSON MANIFOLDS

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ABSTRACT

A new technique to construct Poisson manifolds inspired both in surgery ideas used to define Poisson structures on 3-manifolds and in Gompf's surgery construction for symplectic manifolds, is presented. As an application of these ideas it is proved that for all $n \geq d \geq 4$, d even, any finitely presentable group is the fundamental group of a n-dimensional orientable closed Poisson manifold of constant rank d. The unimodularity of some of the Poisson structures constructed is studied.

1. Introduction

The use of almost complex methods in symplectic geometry together with new surgery techniques have increased notably our understanding of the topology of symplectic manifolds (see for instance the foundational papers [5, 2, 4]). These results constitute an extraordinary mixture of "soft" and "hard" mathematical ideas in the sense of Gromov [6]. In spite of all this success very little is known for nontrivial families of symplectic manifolds. Families of symplectic manifolds lead naturally to the notion of Poisson manifolds.

A Poisson structure on a manifold M is a Poisson algebra structure on its sheaf of functions. That is to say, given two local functions f, g on M we define on its common domain of definition a bilinear bracket $\{f, g\}$ verifying the following properties:

(1) Skewsymmetry, $\{f, g\} = -\{g, f\}$.
(2) Leibnitz' rule, $\{f, gh\} = g\{f, h\} + \{f, g\}h$.
(3) Jacobi identity, $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$.

Alternatively, a Poisson structure on a manifold M is given by a bivector Λ such that $[\Lambda, \Lambda] = 0$, where $[\cdot, \cdot]$ denotes the Schouten bracket (see for instance [10] and references therein). The Poisson bracket $\{f, g\}$ of two functions is given in terms of Λ by $\Lambda(df, dg)$. Moreover, the Poisson tensor Λ defines a natural bundle morphism $\#\colon T^*M \to TM$ whose range defines an involutive distribution $\mathcal{S}_\Lambda$ whose integral leaves are equipped with a canonical symplectic structure. Conversely, any foliation $\mathcal{S}$ by symplectic manifolds of a manifold M such that for any smooth function the hamiltonians of the restriction of the function to each leaf glue into a smooth vector field, induces a unique Poisson structure whose symplectic foliation is precisely $\mathcal{S}$ [10]. In this sense we make precise the idea above that Poisson structures on manifolds provide the geometrical setting to describe smooth families of symplectic structures.

Looking at the known examples of Poisson manifolds, we see that in most occasions the starting point for them is an algebraic structure (a Lie algebra, a cocycle, etc.) and then we construct the manifold whose Poisson structure is related with the initial algebraic one. However it would be most interesting to explore the converse viewpoint. Given a manifold M determine the nontrivial Poisson structures that it supports. This is of course an extremely difficult task because of the intrinsic nonlinearity of Poisson structures. Some work in this direction has been already done by M. Bertelson who has studied in [1] the problem of characterizing regular foliations which arise from Poisson structures. In order to accomplish part of this task, another step which is worth taking is checking the possibility of extending the smooth topological constructions to the Poisson category. Some of them have already been carried to the symplectic setting; D. McDuff [8] has defined the blowing up of a symplectic submanifold. R. Gompf [4] has used the normal connected sum to construct symplectic manifolds with arbitrary fundamental group. In this sense symplectic geometry is "flexible" in sharp contrast with Kähler geometry. Of course, trivial families of symplectic manifolds can be constructed just by taking the product of a symplectic manifold M with an arbitrary compact manifold Q. Such trivial product Poisson structures will have the same fundamental group as M provided that $\pi_1(Q) = 0$. Hence unless we are looking for one-parameter families of symplectic manifolds, the existence of families with arbitrary fundamental group is trivial (not at all its classification). Thus the problem of constructing Poisson manifolds with arbitrary fundamental group is reduced to the particular

situation of codimension 1 symplectic foliations, and more concretely to the search of 5-dimensional compact Poisson manifolds of constant rank 4 with arbitrary fundamental group. In the process of constructing them, we will introduce a surgery operation for Poisson manifolds that naturally extends Gompf's construction. We will study in addition some properties of some of the Poisson manifolds obtained in this way which are characteristic of the Poisson category, like the behavior of its modular class [12]. We will also provide various examples exhibiting some of these properties. We will start by recalling in section 2 how one can endow any compact orientable 3-manifold with a rank 2 regular Poisson structure. This construction, which relies on the well-known results on foliations of orientable 3-manifolds, will be adapted to define a surgery technique that extends Gompf's to the Poisson category in sections 3 and 4. Section 5 is devoted to study the modular class of some Poisson manifolds constructed by surgery and in section 6 we will prove that any finitely presentable group can be realized as the fundamental group of a compact regular Poisson manifold of dimension n and even rank d with $n \geq d \geq 4$.

2. Poisson structures on 3-dimensional manifolds

A regular Poisson structure on a 3-manifold M^3 is just a foliation by surfaces with a leafwise smooth area form. In particular, if M^3 is orientable, finding such a structure turns out to be an easy problem of differential topology, whose non trivial part is to endow the manifold with the foliation. The problem of finding a codimension one foliation on an oriented 3-manifold is a classical one which is already solved. We now give an outline of a solution (in which we do not ask much to the foliation), because it essentially contains the ideas that give rise to a surgery construction for Poisson manifolds.

Recall that every orientable compact 3-manifold M can be obtained from S^3 by surgery on a link with components k_j. Moreover, the framings are of the form $(m_j \pm 1 l_j)$, where m_j, l_j are the meridian and longitude of the boundary tori. Notice that the components of the link can be chosen to be very close to the unknot and hence transversal to the Reeb foliation $\mathcal{R}$ of S^3, i.e., the knots are *submanifolds transversal to the leaves inheriting the trivial Poisson structure*. Once open tubular neighborhoods of k_j have been removed, in the solid tori $T_j = D^2 \times S^1$ to be glued, the boundary of the leaves of $\mathcal{R}$ will be non separating curves on ∂T_j (and cutting once the meridian). This curves are non-trivial in the homology of T_j, so we cannot hope to add a punctured surface to get closed leaves, but if we remove a small tubular neighborhood $\mathcal{N}_j$ of the longitude $\alpha_j = \{0\} \times S^1$ we can find a map $\phi_j : T_j - \mathcal{N}_j \to S^1 \times I \times S^1$ such that the image of the curve m_j on T_j goes to $S^1 \times \{0\} \times \{e\}$, the meridian of $S^1 \times I \times S^1 \subset D^2 \times S^1$. Hence, pulling back the Reeb foliation of $S^1 \times I \times S^1$ we get a foliation except inside a solid torus, where we again put a Reeb component. We have thus proved the following well known proposition.

Proposition 1. *Every oriented compact 3-manifold admits a regular rank 2 Poisson structure.*

Using the ideas above we see that any fibered knot of a 3-manifold gives a foliation with a Reeb component and a "modified" Reeb component, where instead of having disks approaching to the torus we have punctured oriented surfaces (the Seifert surfaces of the knot).

It is for dimensions bigger than 3 where surgery constructions are a powerful tool to construct manifolds with prescribed topology (for example arbitrary finitely presented fundamental group). Hence having one such construction compatible with

Poisson structures would allow us to conclude the existence of Poisson manifolds with a wide range of different topological properties.

3. FIBERED POISSON STRUCTURES

We have seen in the previous section that to perform surgery in an oriented Poisson 3-manifold we do not need to worry about the Poisson tensor itself but only about extending the symplectic foliation we had. It is not difficult to propose a surgery technique for Poisson manifolds which is indeed an extension of the normal connected sum for symplectic manifolds. Roughly speaking, we will use a transversal submanifold that intersects symplectically with the leaves of the symplectic foliation. This will allow us to perform the normal connected sum along the symplectic submanifolds and we will show that the resulting manifold admits a Poisson structure determined (up to sum extent) by the ones we had initially. We will see that with the appropriate setting the proofs will be natural generalizations of those of Gompf [4].

3.1. **Poisson structures compatible with fiber bundle structures.** Let $\pi\colon P \to Q$ be a fiber bundle. We call a Poisson structure Λ_P on P *compatible with the fiber bundle structure* if the symplectic leaves of Λ_P are the fibers of π (hence the fibers are connected). We will also call the triple (P, π, Λ_P) a *fibered Poisson manifold*. If P is compact this is equivalent to saying that the space of leaves is a smooth manifold Q such that the projection $\pi\colon P \to Q$ is a submersion.

We begin by noticing that whenever one has a foliation, one can do the usual exterior calculus in the bundles associated to the distribution. In our case we will have a locally trivial fibration $\pi\colon P \to Q$ and the bundle we are interested in is the one of vertical vectors, i.e., the kernel of π. We will speak of vertical vector fields and k-forms, Lie derivatives in the direction of vertical vector fields and exterior derivative of vertical k-forms. We shall denote the set of vertical k-forms by $\Omega_{\mathrm{fib}}^k(P \to Q)$, and by d_π the exterior vertical derivative (or just d if there is no risk of confusion). Recall that one can pullback vertical forms by fiber bundle morphisms and that any known relation involving Lie derivatives also holds for vertical vector fields and forms (it holds fiberwise and defines a smooth section of the corresponding bundle).

Let us denote the cohomology groups of the complex $(\Omega_{\mathrm{fib}}^*(P \to Q), d_\pi)$ by $H_{\mathrm{fib}}^k(P \to Q)$. We have the corresponding forgetful maps $f\colon \Omega^k(P) \to \Omega_{\mathrm{fib}}^k(P \to Q)$, and $f\colon H^k(P) \to H_{\mathrm{fib}}^k(P \to Q)$.

It is straightforward to check that a Poisson structure Λ_P on P compatible with the fibration $\pi\colon P \to Q$ is determined by a closed non singular vertical 2–form $\omega_P \in \Omega_{\mathrm{fib}}^2(P \to Q)$ (and hence $[\omega_P] \in H_{\mathrm{fib}}^2(P \to Q)$). We will call ω_P *the Poisson 2-form* (or just *the Poisson form*) of Λ_P.

There are some results about the cohomology $H_{\mathrm{fib}}^*(P \to Q)$ that will be used later. We start by recalling that for a closed manifold with a metric, Hodge theory allows one to obtain for any k-form α a unique decomposition:

$$(1) \qquad\qquad \alpha = d\beta \oplus \delta\eta \oplus \rho$$

where β is coexact, η exact and ρ harmonic, and all three are images of α by smooth operators. Moreover we also have relative Hodge theory for a pair (N, K), where N is a compact manifold and K a closed set (i.e, for forms with support contained in $N - K$). This implies that we also have the above results for a compact manifold N with non empty boundary and forms with support in the interior on N (to show it just double the manifold and apply relative Hodge theory).

When $\pi\colon P \to Q$ is a locally trivial fibration and P closed, we can also apply Hodge theory to get the same decomposition of eq. (1) above for vertical k-forms. We observe that any metric on P restricts to a metric on each fiber and there we can apply the usual Hodge theory. After gluing what we construct in this way fiberwise, we get again smooth projection operators because in a trivialization we are just working in a fiber with a smooth family of forms and metrics. If P is compact and $\partial P \neq \emptyset$, the relative Hodge theory $(P, \partial P)$ (where we use forms whose support do not intersect ∂P) also holds because on each trivialization (that we use to see that the construction is smooth) the boundaries are setwise identified. As a consequence, we see that for $\pi\colon P \to Q$ locally trivial and P closed (resp. compact with $\partial P \neq \emptyset$), a vertical closed form (resp. a closed form whose support does not intersect ∂P and hence vanishing in a neighborhood of the boundary) is exact if and only if it is fiberwise exact (resp. exact with potential form vanishing in a neighborhood of the boundary). The result also implies that for a smooth (compact) family of exact vertical k-forms one can find a smooth family of vertical $(k-1)$-forms whose exterior derivative is the initial family (we would be working with vertical forms on the direct sum of our initial bundle and the trivial bundle with rank the number of parameters of the family). If all the k-forms of the family vanished in a neighborhood of the boundary, the $k-1$ forms will also vanish in that neighborhood.

3.2. **Transversal Poisson fibered submanifolds.** A smooth Poisson submanifold [11] of a Poisson manifold (M, Λ_M) is defined as a triple (P, Λ_P, j) where $j\colon (P, \Lambda_P) \to (M, \Lambda_M)$ is a Poisson morphism embedding P into M. Besides these, there are submanifols of a Poisson manifold which inherit a Poisson structure (the foliation induces a foliation by symplectic submanifolds which fits into a Poisson structure) and where the natural inclusion map is not a Poisson morphism. We will consider submanifolds of Poisson manifolds from this more general perspective. Thus, *a Poisson submanifold* of a given Poisson manifold (M, Λ_M) will be a submanifold intersecting the leaves in symplectic submanifolds and inheriting a Poisson structure (necessarily unique) from Λ_M (these are the natural generalization of the symplectic submanifolds of a symplectic manifold). In particular we will be dealing with a special class of Poisson submanifolds compatible with a given fibration.

Definition 1. *Let (M, Λ_M) be an n-dimensional Poisson manifold of rank d, (P, Λ_P) a Poisson manifold where P is compact and fibers over the $(n-d)$-dimensional manifold Q, and Λ_P is compatible with the fibration. An embedding $j\colon P \to (M, \Lambda_M)$ is said to embed (P, Λ_P) as a transversal Poisson fibered submanifold of (M, Λ_M) if:*

i. $j(P)$ is contained in the regular set of (M, Λ_M).

ii. $j(P)$ cuts transversally the symplectic leaves of (M, Λ_M).

iii. $j(P)$ inherits a Poisson structure from (M, Λ_M) that coincides with Λ_P.

The existence of such a submanifold implies that the symplectic leaves of (M, Λ_M) are nicely arranged in a neighborhood of the submanifold. To be more precise:

Lemma 1. *If $j\colon P \to (M, \Lambda_M)$ embeds the fibered Poisson manifold $(P, \Lambda_P) \to Q$ in M as a codimension r transversal Poisson fibered submanifold of (M, Λ_M), then its normal bundle, with the induced Poisson structure, is also a fibered Poisson manifold over Q.*

Proof. For each $x \in P$ let $S_M(j(x))$ be the symplectic leaf of Λ_M passing through the point $j(x)$, thus $\Lambda_{M|S(j(x))}$ is the inverse of a symplectic form $\omega_M(x)$ on $T_{j(x)}S(j(x))$ and $T_{j(x)}(j(P) \cap S(j(x)))^{\perp_{\omega_M}}$, the symplectic orthogonal of $T_{j(x)}(j(P) \cap$

$S(j(x)))$, is a symplectic r-plane transversal to $T_{j(x)}j(P)$, so we can take it as model for the normal bundle $\nu(P)$ of the embedding. Moreover, for each leaf $S_P \subset P$, the restriction of this model of normal bundle is the corresponding model for the embedding of that leaf $S_P \subset S_M$. In fact, one can choose any compatible almost complex structure for the regular set of (M, Λ_M) and consider the leafwise associated metric. $T_{j(x)}(j(P) \cap S(j(x)))^{\perp_{\omega_M}}$ is then the orthogonal complement of $T_{j(x)}(j(P) \cap S(j(x)))$ with respect to this metric. We can use this leafwise metric to identify the normal bundle with a small enough tubular neighborhood of $j(P)$. This open set inherits a Poisson structure which can be pulled back to the normal bundle (hence the Poisson structure depends on the almost complex structure, but different choices give isomorphic structures). The local triviality of the fiber bundle $\tilde{\pi} \colon \nu(P) \to Q$ follows from that of the associated sphere bundle, which is a compact manifold (and the projection a surjective submersion). $\qquad\square$

4. The main construction: Poisson surgery

Let (M, Λ_M) be an n-dimensional Poisson manifold of rank d and let (P, Λ_P) be a fibered compact $(n-2)$-dimensional Poisson manifold over the $(n-d)$-dimensional manifold Q (Λ_P is compatible with the fiber bundle structure $\pi \colon P \to Q$). Suppose we have two disjoint embeddings $j_a \colon P \to M$, $a = 1, 2$, that both embed (P, Λ_P) as a transversal Poisson fibered submanifold of (M, Λ_M). Assume that the normal bundles ν_a (using the model provided by Lemma 1 and considering the orientation induced by the Poisson bracket) have opposite Euler class. After identifying ν_a with a tubular neighborhood V_a of $j_a(P)$, any orientation reversing identification $\psi \colon \nu_1 \to \nu_2$ allows us to get a diffeomorphism $\varphi \colon V_1 - j_1(P) \to V_2 - j_2(P)$ preserving the orientation of the fibers (the disks) as the composition of ψ with the diffeomorphism $h(x) = x/\|x\|^2$ that turns each punctured normal fiber inside out.

Definition 2. *Let $\#_\psi M$ denote the smooth, foliated manifold, obtained from $M - (j_1(P) \cup j_2(P))$ by identifying $V_1 - j_1(P)$ with $V_2 - j_2(P)$ via the composition $h \circ \psi$. If M is a disjoint union $M_1 \coprod M_2$ and j_a maps P into M_a, the manifold will be called the normal connected sum of M_1 and M_2 along P (via $h \circ \psi$) and will be also denoted by $M_1 \#_\psi M_2$.*

It is easy to check that the diffeomorphism type (as a foliated manifold) it is determined by (j_1, j_2) and the orientation reversing identification $\psi \colon \nu_1 \to \nu_2$ (up to fiber preserving isotopy). Once one of these identifications has been chosen, the remaining possibilities are $[P, S^1] \cong H^1(P; \mathbb{Z})$.

4.1. **Topological remarks.** If we are given an orientation μ_M on (M, Λ_M), it determines in a neighborhood of $j_a(P)$, together with the Poisson structure, an orientation on P, and this one together with the restricted Poisson form ω_P, an orientation on Q. It is clear that if the orientations on Q obtained in this way from each neighborhood V_a are the same, then μ_M induces an orientation on $\#_\psi M$.

There are some very well know results about the topology of $M_1 \#_\psi M_2$ (see the remarks by Gompf [4]). First of all, $\#_\psi M$ is (oriented) cobordant to M. This is seen after identifying in the cobordism $M \times [0, 1]$ neighborhoods of $j_1(P)$ and $j_2(P)$ in the level $\{1\}$, and then rounding corners to get the cobordism manifold X.

Hence, the Pontrjagin numbers ($\#_\psi M$ oriented) behave additively, and in the even dimensional case the formuli for the Euler characteristic and signature are, respectively,

$$(2) \qquad \chi(M_1 \#_\psi M_2) = \chi(M_1) + \chi(M_2) - 2\chi(P).$$

(3) $$\sigma(M_1 \#_\psi M_2) = \sigma(M_1) + \sigma(M_2), \ (\#_\psi M \text{ oriented})$$

As it is the case for symplectic manifolds, if $\#_\psi M$ is oriented the surgery construction is compatible (choosing an appropriate framing) with spin structures, and we can conclude:

Lemma 2. *If M admits a spin structure and $H^2(P; \mathbb{Z})$ has no $\mathbb{Z}_2$–torsion, then there is a choice of ψ such that $\#_\psi M$ admits a spin structure extending the one on M.*

Proof. See [4], proposition 1.2. $\square$

4.2. **Remarks regarding the foliation of $\#_\psi M$.** If we start from a regular transversally orientable manifold M, then $\#_\psi M$ is also regular transversally orientable (orientability in regular Poisson manifolds is equivalent to transversal orientability), and its Godbillon-Vey class $GV(\#_\psi M, \Lambda)$ can be computed in terms of the one of M. In particular:

Lemma 3. *Let M be transversally orientable. Then $GV(M, \Lambda_M) = 0$ if and only if $GV(\#_\psi M, \Lambda) = 0$.*

Proof. We know that be can remove disjoint fibered neighborhoods (closed) W_a of $j_a(P)$ such that we have an inclusion $i: M - (W_1 \cup W_2) \to \#_\psi M$ and both ends of $M - (W_1 \cup W_2)$ fiber over P (with fiber an annulus). The condition $GV(M, \Lambda_M) = 0$ implies, by naturallity, the vanishing of the Godbillon-Vey class of $M - (W_1 \cup W_2)$. Since its ends are fibered, one can choose a representant β of the class vanishing on these ends and conclude the existence of a form γ vanishing on the ends and whose exterior derivative is β. Finally, extending β and γ to forms $\tilde{\beta}$ and $\tilde{\gamma}$ defined on M we obtain $d\tilde{\gamma} = \tilde{\beta}$, where $[\tilde{\beta}] = GV(M, \Lambda_M)$. The other direction is proven similarly. $\square$

4.3. **Constructing the Poisson form on $\#_\psi M$.** Our final aim is to put a Poisson structure on $\#_\psi M$. To do that, we have to modify slightly the previous construction. Since we have to construct a symplectic structure on each resulting leaf, it is more convenient to use instead of the normal bundles (whose fibers have infinite area), the bundles ν_a^0 of disks of radius $\pi^{-1/2}$. We will compose ψ (that can be assumed to preserve the area of each fiber) with the map

(4) $$i(x) = \left(\frac{1}{\pi \|x\|^2} - 1 \right)^{1/2} x \,,$$

which turns each punctured disk inside out.

We notice that V_1, V_2 and Y, the image of $(V_1 \cup V_2) \times [0, 1]$ in X (the cobordism between M and $\#_\psi M$), are locally trivial fiber bundles over Q. Any closed form $\omega \in \Omega_{\text{fib}}^k(V_1 \cup V_2 \to Q)$ satisfying $j_1^* \omega = j_2^* \omega$ induces a form $\Omega_{\tilde{V}} \in \Omega_{\text{fib}}^k(\tilde{V} \to Q)$, up to a choice of a compactly supported exact k-form $d\alpha, \alpha \in \Omega_{\text{fib}}^{k-1}(\tilde{V} \to Q)$, where $\tilde{V} \subset \#_\psi M$ is the image of $V_1 \cup V_2$ in $\#_\psi M$. The way to get a representant $\Omega_{\tilde{V}}$ of this family is by retracting disjoint neighborhoods of $j_a(P)$ (containing V_a^0, the image of ν_a^0) onto $j_a(P)$ and extending this map to a smooth retraction $\rho: M \to M$ isotopic to the identity, which coincides with the identity out of a compact set of $V_1 \cup V_2$, preserves the fibers of V_a and commutes with $\hat{j}_2 \circ \psi \circ \hat{j}_1^{-1}$ on V_1 and V_2). This k-form is the restriction to $\tilde{V}$ of the one induced on Y by $\rho^* \omega$. Two different choices of the retraction will give raise to two k-forms whose difference

will be a compactly supported element of $\Omega^k_{\text{fib}}(\tilde{V} \to Q)$. To see that this compactly supported closed form is exact, it is enough to check it fiberwise. The procedure is the one described by Gompf. We recall that when we round corners to get the cobordism manifold Y, we can think of having added some new levels (i.e., we now have a map $p_2 \colon Y \to [0, 1 + \epsilon]$) so that the level set $1 + \epsilon$ is $\tilde{V}$, where the circumferences of radius $\pi^{-1/2}$ are identified. As we go from $1 + \epsilon$ to 1 we identify circumferences of smaller and smaller radius until we reach the level set 1 where $j_1(P)$ and $j_2(P)$ are identified. The level sets corresponding to the values smaller than 1 are diffeomorphic to $V_1 \cup V_2$. Given another retraction ρ', to evaluate the difference of the k-forms $\rho^* \omega_{|\tilde{V}} - \rho'^* \omega_{|\tilde{V}}$, we homotope ("pushing down") the corresponding smooth oriented k-manifold $M_k \subset p_2^{-1}(1 + \epsilon)$ (possibly singular) such that it lies in $p_2^{-1}([0, 1])$ and in the level 1 is contained in $j_1(P) \times \{1\}$, cut open Y and project $(V_1 \cup V_2) \times [0, 1] \to V_1 \cup V_2 \times \{0\}$, and in the zero level set integrate $\rho^* \omega - \rho'^* \omega$ over the correspondent manifold with boundary M'_k. But since the retractions were homotopic to the identity, both $\rho^* \omega$ and $\rho'^* \omega$ represent the same homology class as ω. This, together with the fact that $j_1^* \omega = j_2^* \omega$, implies that $\int_{M'_k} \rho^* \omega - \omega = 0 = \int_{M'_k} \rho'^* \omega - \omega$. Hence their difference integrates to 0.

Now we will see that this construction works in the Poisson category.

Theorem 1. *Let (M, Λ_M) be an n-dimensional Poisson manifold of rank $d \geq 2$ and let (P, Λ_P) be a compact $(n-2)$-dimensional Poisson manifold such that Λ_P is compatible with the fiber bundle structure $\pi \colon P \to Q$, where Q is a $(n - d)$-dimensional manifold. Let $j_a \colon (P, \Lambda_P) \to (M, \Lambda_M)$, $a = 1, 2$, be two disjoint embeddings of (P, Λ_P) as a transversal Poisson fibered submanifold of (M, Λ_M). Suppose that there is an orientation reversing isomorphism of the normal bundles $\psi \colon \nu_1 \to \nu_2$. Then $\#_\psi M$, the normal connected sum along the normal bundles of $j_a(P)$, can be given a canonical Poisson structure Λ, characterized as follows:*

Given disjoint identifications $\hat{\jmath}_a \colon \nu_a \to V_a$ of normal bundles with tubular neighborhoods V_a of $j_a(P)$ that send fibers into leaves, if we denote by $\tilde{V}$ the image of $V_1 \cup V_2$ in $\#_\psi M$, $\tilde{V}$ is a locally trivial fiber bundle with base space Q. Then, there exists a unique fiber isotopy class of Poisson forms on $\tilde{V}$ containing elements ω satisfying the following characterization:

(1) *Let $\Omega_{\tilde{V}}$ be any of the 2-forms induced in $\tilde{V}$ by ω_M (as shown in the previous paragraph). Then $\omega - \Omega_{\tilde{V}} \in \Omega^2_{\text{fib}}(\tilde{V} \to Q)$ (which is closed) has compact support and is exact (it does not depend on the representative).*

(2) *The identification $\hat{\jmath}_1 \colon \nu_1 \to V_1 \subset M$ can be chosen in such a way (i.e., isotopic (rel $j_1(P)$) by an isotopy with compact support) that the Poisson 2-form ω_M is $SO(2)$-invariant on $V_1^0 = \hat{\jmath}_1(\nu_1^0)$, with ν_1^0 the open disk bundle of radius $\pi^{-1/2}$, and on the closure of each fiber of V_1^0 it is symplectic with area t_0 independent of the fiber (we can isotope the initial embedding into the second one fixing the complement of each disk of radius $r > \pi^{-1/2}$). The forms $(1 - s)\omega_M + s\pi^* \omega_P$, $0 \leq s < 1$, are all Poisson on the closure of V_1^0.*

(3) *There is a closed vertical 2-form ζ with compact support in $V_2^0 = \hat{\jmath}_2(\nu_2^0)$, with ν_2^0 the open disk bundle of radius $\pi^{-1/2}$, such that for all $t \in [0, t_0]$ the form $\omega_M + t\zeta$ is Poisson on both $V_1 \cup V_2$ and $j_2(P)$.*

(4) *There is a map $\chi \colon \nu_2 \to \nu_2$ (preserving the disks) isotopic to the identity by an isotopy with support in ν_2^0, such that outside of a compact subset K of V_1^0, the map $\varphi = \hat{\jmath}_1 \circ \psi \circ i \circ \chi \circ \hat{\jmath}_2 \colon V_1^0 - j_1(P) \to V_2^0 - j_2(P)$ (where $\hat{\jmath}_1$ is as in point 2) is Poisson with respect to the Poisson form $\tilde{\omega}_M = \omega_M + t_0 \zeta$ on M (i.e, we modify the embedding $\hat{\jmath}_2$ to $\chi \circ \hat{\jmath}_2$). The manifold $\#_\psi M$ is*

obtained from $(M - (K \cup j_2(P)), \tilde{\omega}_M)$ by gluing via φ (it follows that ω equals ω_M on the image on $\#_\psi M$ of the complement of $V_1^0 \cup V_2^0$).

Moreover, different choices of embeddings of the normal bundles are connected by an isotopy that preserves the isotopy class described above.

Finally, the form ω depends smoothly on ω_M, ω_P (and hence on j_1, j_2) and it can be constructed with each $V_a, a = 1, 2$ lying inside any preassigned neighborhood of $j_a(P)$.

Now we will devote the next paragraphs to prove Theorem 1.

4.4. **The contraction operator.** We recall that $\nu(P)$ is an $SO(2)$-bundle. Let $\tau_s : \nu(P) \to \nu(P)$, $0 \le s \le 1$ denote the multiplication by s on each disk and let X_s denote the corresponding vector field. Since X_s is a vertical vector field with respect to the fiber bundle structure $\nu(P) \to Q$, we can define the operator $I : \Omega_{\text{fib}}^k(\nu(P) \to Q) \to \Omega_{\text{fib}}^{k-1}(\nu(P) \to Q)$ by

$$(5) \qquad I(\rho) = \int_0^1 \tau_s^*(i_{X_s}\rho)ds$$

As usual, if ρ is closed and $j^*\rho = 0$, then $dI(\rho) = \rho$. It is also true that I commutes with any action preserving the $SO(2)$-bundle structure.

Corollary 1. *Let ω_1, ω_2 be two Poisson forms on $\nu(P)$ compatible with the fiber bundle structure $\nu(P) \to Q$ verifying $j^*\omega_1 = j^*\omega_2$ and inducing the same orientation on $\nu(P)$. There exist U_1, U_2 neighborhoods of P in $\nu(P)$ and an isomorphism $\phi : \nu(P) \to \nu(P)$ isotopic (rel P) to the identity, by an isotopy with compact support, such that $\phi : U_1 \to U_2$ verifies $\phi^*\omega_2 = \omega_1$. If both forms already coincide over a compact subset C of P, we may assume the isotopy to have support on a preassigned neighborhood of the closure of $P - C$.*

The isomorphism ϕ can be chosen to depend smoothly on ω_1 and ω_2. In fact, if we are given smooth families $\omega_{1,r}, \omega_{2,r}$, $b \le r \le c$, coinciding on a fixed neighborhood of a given compact set C, and construct isomorphisms (as in the proof that follows) ϕ_b, ϕ_c verifying $\phi_b^*\omega_{2,b} = \omega_{1,b}$ and $\phi_c^*\omega_{2,c} = \omega_{1,c}$, then there exists a smooth family ϕ_r that verifies $\phi_r^*\omega_{2,r} = \omega_{1,r}$ on a fixed neighborhood of P and equals the identity in the chosen neighborhood of the closure of $P - C$.

Proof. As in the Darboux-Weinstein theorem proof, we consider the vertical closed 2-form $\eta = \omega_1 - \omega_0$ and the family $\omega_t = \omega_0 + t\eta$ (also vertical closed 2-forms). We can find a small neighborhood of P in which the ω_t are non-degenerate (because on P both forms induce the same orientation on the normal disk and because of the compactness of P). There, we know that $\eta = d\alpha$, with $\alpha = I(\eta)$, and we can find a family of vertical vector fields Y_t characterized by the equation $i_{Y_t}\omega_t = -\alpha$. After using a suitable bump function, this 1-parameter family defines a global flow Ψ_t on $\nu(P)$, leaving P stationary. Computing $\frac{d}{dt}(\Psi_t^*\omega_t)$ we conclude that $\Psi_t^*\omega_t$ does not depend on t near P. If the forms coincided in a neighborhood of C, η vanishes on that neighborhood. Regarding families, we see that in the procedure we made a choice of a bump function, and we can smoothly join two such choices. $\qquad \square$

Corollary 2. *Let (M, Λ_M) be an n-dimensional Poisson manifold of rank d. Let (P, Λ_P) be a regular compact Poisson manifold of dimension $n - 2$ which fibers over the $(n - d)$-dimensional manifold Q and such that Λ_P is compatible with the fiber bundle structure. Assume that $j_a : (P, \Lambda_P) \to (M, \Lambda_M)$, $a = 1, 2$, embeds (P, Λ_P) as a transversal Poisson fibered submanifold of (M, Λ_M). Suppose that both normal bundles are trivial and let $\psi : \nu_1(P) \to \nu_2(P)$ be a bundle isomorphism identifying*

them and preserving the orientation of the fibers. Then $\#_\psi M$ can be given a Poisson structure Λ.

Proof. We can identify each normal bundle with $P \times \mathbb{R}^2$ in such a way that each disk $\{z\} \times D^2$ has area form $dx \wedge dy$. We also have isomorphisms $\hat{\jmath}_a \colon P \times D_\epsilon^2 \to V_a$, $a = 1, 2$, and $\tilde{\psi} \colon P \times D_\epsilon^2 \to P \times D_\epsilon^2$. The main point is that since the normal bundles are trivial, $j_a^* \omega_a + dx \wedge dy$ are Poisson structures that restrict to $j_a^* \omega_a$ on P. Hence, we can find a real number $\delta > 0$, and diffeomorphisms $\tilde{\jmath}_a \colon P \times D_\delta^2 \to U_a$ with $\tilde{\jmath}_1^* \omega_1 = j_1^* \omega_1 + dx \wedge dy$, $\tilde{\psi}^* \tilde{\jmath}_2^* \omega_2 = j_1^* \omega_1 + dx \wedge dy$, $U_a \subset V_a$ neighborhoods of $j_a(P)$. Composing $\tilde{\psi}$ with the area preserving map $(r, \theta) \mapsto (\sqrt{\delta^2 - r^2}, -\theta)$ preserves the Poisson structure and hence allows us to define a Poisson structure on $\#_\psi M$. We notice that we could have equally asked our initial fiber bundle morhism ψ to reverse the orientation of the fibers, because by composing with the leafwise reflection $(r, \theta) \mapsto (r, -\theta)$ one can always reverse the orientation of a trivial bundle . $\qquad \square$

In the above construction, the Poisson structure coincides with Λ_M on $M - (j_1(P) \cup j_2(P))$. But we have to allow perturbations in a neighborhood of one of the embeddings to have uniqueness up to isotopy.

The main obstruction to finally solve the problem posed in Thm. 1 in general is that one cannot put a global Poisson structure on ν_a induced by $j_a^* \omega_a$ and the symplectic structure on the symplectic orthogonals, unless the normal bundle is trivial. We can overcome this difficulty in the following way. We consider ν_a^0, the bundles of open disks of radius $\pi^{-1/2}$, and identify the punctured disks by composing i with ψ to get $\mathcal{B}$, an S^2-bundle with structural group $SO(2)$ whose fibers have an $SO(2)$-invariant area form ω_{S^2} that integrates into 1 on each of them. We have two embeddings $i_0 \colon P \to \mathcal{B}$, $i_\infty \colon P \to \mathcal{B}$ with $\hat{\jmath}_1 i_0 = j_1$, $\hat{\jmath}_2 i_\infty = j_2$. Let us denote $E^0 = \mathcal{B} - P_\infty$ (resp. $E^\infty = \mathcal{B} - P_0$). Using Thurston's ideas (see [9], Thm. 6.3) we can construct a vertical 2-form η restricting to the above defined area form on each fiber: we consider a form β on $q \colon \mathcal{B} \to P$ representing the Poincare dual of P_0 so that it integrates to 1 on each fiber (sphere transverse to P_0). It can be chosen to have support in a small neighborhood of P_0, so that it vanishes on P_∞. We take trivializations $h_k \colon q^{-1}(\mathcal{U}_k) \to \mathcal{U}_k \times S^2$ of $\mathcal{B}$ and a partition of the unity ρ_k subordinated to $\{\mathcal{U}_k\}$. Since $h_k^* \pi_{S^2}^* \omega_{S^2} - \beta = d\alpha_k$ on $q^{-1}(\mathcal{U}_k)$, $\eta = f(\beta + d\sum_k (\rho_k \circ q)\alpha_k)$, where f is the forgetful map $f \colon \Omega^2(\mathcal{B}) \to \Omega_{\text{fib}}^2(\mathcal{B} \to Q)$, satisfies the requirements. The result of averaging $\eta - q^* i_0^* \eta$ (both q, i_0 are maps lifting $\text{id} \colon Q \to Q$) under the $SO(2)$-action is a vertical $SO(2)$-invariant 2-form, that we will still call η, such that it restricts to the canonical volume form on each sphere and $i_0^* \eta = 0$. We can even choose η so that $\eta_{|E^0}$ extends over ν_1 to a closed vertical form that is symplectic on the planes (fibers). We only need to pick β with support away of P_∞, so that on the intersection of that neighborhood with $q^{-1}(\mathcal{U}_i)$ ($\mathcal{U}_i$ contractible) α_k can be chosen to be $h_k^* \pi_{S^2}^* \alpha'$, for any α' with $d\alpha' = \omega_{S^2}$ on that neighborhood. In particular, the restriction of the 1-form $\alpha = 1/2(r^2 - \frac{1}{\pi})d\theta \in \Omega^1(\mathbb{R}^2 - \{0\})$ (given in polar coordinates) to the disk of radius $\pi^{-1/2}$ admits an extension to a form α' on $S^2 - \{0\}$ with $d\alpha' = \omega_{S^2}$.

The forms $\omega_t = q^* j_1^* \omega_1 + t\eta$ are non-degenerate for $0 < t \leq t_1$ because, as Thurston observed, $q^* j_1^* \omega_1$ is non-degenerate on the orthogonal to the tangent space of the spheres (which does not depend on t because it is determined by η). For a choice of η extending to ν_1 as described above, the forms ω_t will be symplectic near the closure of $E^0 \cong \nu_i^0$ in ν_1 for $t_1 \leq t$ small enough.

4.5. **Comparing the Poisson structures on $\mathcal{B}, E^0$ and E^∞.** Now that we have a family of closed non-degenerate 2-forms on $\mathcal{B}$, we would like to compare one of them with the ones defined on $E^0 \cong \nu_1^0$ and $E^\infty \cong \nu_2^0$ that come from ω_1 and ω_2. Following Cor. 1, for each t we could find neighborhoods $(\mathcal{W}_0^t, \omega_t)$ of P_0 and $(\mathcal{W}_\infty^t, \omega_t)$ of P_∞ which are Poisson equivalent to some neighborhoods (depending on t) of $(j_1(P), \omega_1)$ and $(j_2(P), \omega_2)$. But nothing guarantees that $\mathcal{B} = \mathcal{W}_0^t \cup \mathcal{W}_\infty^t$, for some t.

Instead, we use Gompf's construction again. On E^0, let $\varphi = I(\eta)$ and define the vertical vector fields Y_t, $0 < t \le t_1$ by the condition $i_{Y_t}\omega_t = -\varphi$ (also defined in a neighborhood of the closure of E^0, if η was chosen to extend to ν_1). The key property is that these vector fields are $SO(2)$-invariant. For a fixed t_0, the flow Ψ_t, required to be the identity for $t = t_0$, is $SO(2)$-invariant and of course verifies $\Psi_t^* \omega_t = \omega_{t_0}$. In principle, we know that for any $SO(2)$-invariant compact set $K \in E^0$ there exists an interval J of t_0 in $(0, t_1]$ where the flow $\Psi \colon K \times J \to E^0$ is defined. But in can be shown that Ψ is defined on $E^0 \times [t_0, t_1]$. Given any point x on E^0, it determines an $SO(2)$-orbit on its fiber and hence a disk $D(x)$. We define:

$$A(x) = \int_{D(x)} \eta$$

and,

$$A_t(x) = \int_{D(x)} \omega_t \,,$$

where the forms are pulled back to the disk. The map $A \colon E^0 \to [0, 1)$ is a smooth $SO(2)$-invariant proper surjection and it is clear that $A_t(x) = tA(x)$. Given $x \in E^0$, $t_0 \in (0, t_1]$ and $K = D(x)$ we obtain a flow as above on $D(x)$. Let $D(\Psi_t(x))$ be the disk whose boundary is the $SO(2)$-orbit of $\Psi_t(x)$ (it is also $\Psi_t(\partial D(x))$). Then we have:

$$
\begin{aligned}
tA(\Psi_t(x)) &= A_t((\Psi_t(x))) = \int_{D(\Psi_t(x))} \omega_t \\
&= \int_{\Psi_t(D(x))} \omega_t = \int_{D(x)} \Psi_t^* \omega_t = \int_{D(x)} \Psi_{t_0}^* \omega_{t_0} = t_0 A(x),
\end{aligned}
$$

So we can conclude that $A(\Psi_t(x)) = \frac{t_0}{t} A(x)$. Since A, which is proper, decreases with the flow lines (with t increasing), these flow lines cannot abandon E^0 and hence Ψ is defined in $E^0 \times [t_0, t_1]$. The inequality $A(\Psi_{t_1}(x)) < \frac{t_0}{t_1}$ implies that choosing t_0 small enough, Ψ_{t_1} sends E^0 into any initially fixed tubular neighborhood of P_0. In particular, we choose t_0 so that $\Psi_{t_1}(E^0) \subset \mathcal{W}_0^{t_1}$. Hence $\hat{\jmath}_1 \Psi_{t_1}$ sends (E^0, ω_{t_0}) into $(\hat{\jmath}_1 \Psi_{t_1}(E^0), \omega_1)$. Actually, the Poisson morphism Ψ_{t_1} extends to a neighborhood of the closure of E^0, for suitably chosen η and thus it can further be extended to a diffeomorphism $\Psi_{t_1} \colon \nu_1 \to \nu_1$ isotopic to the identity by an isotopy with compact support (but only Poisson in a neighborhood of the closure of $E^0 \cong \nu_1^0$) .

The restriction of each ω_t to P_∞ induces also a Poisson structure, but in general $i_\infty^* \omega_t \ne j_2^* \omega_2$. But we can modify ω_2 in a neighborhood of $j_2(P)$ (ω_2 has not been involved in all the previous work) so that the above equality holds: we choose $\mu \colon \mathcal{B} \to \mathcal{B}$ an $SO(2)$-equivariant map lifting $id \colon P \to P$ such that μ fixes a neighborhood of P_∞ and collapses a neighborhood of P_0 to P_0. The composition of the restriction of $\hat{\jmath}_2^{-1}$ to V_2^0 with μ can be extended to a map λ from a closed neighborhood U_2 of V_2^0 in V_2 (a neighborhood of ∂U_2 is sent to P_0). We can then modify the Poisson structure of $(U_2, \omega_2) \subset (V_2, \omega_2)$ (without modifying the symplectic foliation), by adding to ω_2 a closed vertical 2-form ζ such that $\omega_2 + \zeta$ is non degenerate (and hence Poisson) and ζ vanishes in a neighborhood of ∂U_2 in U_2. If we call $\zeta = \lambda^* \eta$, then there exists $t_2 > 0$ (by the compactness of P) such that for

all $0 \le t \le t_2$, $\tilde{\omega}_M = \omega_2 + t\zeta$ is non degenerate. To solve the problem we just need to pick our previous t_0 smaller than t_2 (and use of course $\tilde{\omega}_M = \omega_M + t_0\zeta$). So we can glue to define a Poisson form ω on $\tilde{V}$ that satisfies all the requirements of theorem 1. To be more precise, we can find a map $\chi\colon E^\infty \to E^\infty$ isotopic to the identity by an isotopy (rel P_∞) with support in ν_2^0 and Poisson, with respect to the forms ω_{t_0} and $\omega_M + t_0\zeta$, in a neighborhood U_∞ of P_∞ (the map can actually be extended to a diffeomorphism of ν_2 isotopic to the identity). We glue using the map $\hat{\jmath}_2 \circ \chi \circ i \circ \psi \circ \Psi_{t_1}^{-1} \circ \hat{\jmath}_1^{-1}\colon V_1^0 \to V_2^0$, where Ψ_{t_1} and χ are thought as diffeomorphisms of the normal bundles (instead of having domain in the sphere bundle $\mathcal{B}$). The embeddings we finally use are $\hat{\jmath}_1 \circ \Psi_{t_1}$ and we modify $\hat{\jmath}_2$ by composing on the right with $\chi\colon \nu_2 \to \nu_2$. The only condition that needs to be checked is that the difference $[\omega - \Omega_{\tilde{V}}]$ (which by construction has compact support) is exact. As we saw ,it can be checked fiberwise. Thus, it is enough to show that

$$(6) \qquad\qquad \langle \omega - \Omega_{\tilde{V}}, F \rangle = 0, \forall F \in H^2(\tilde{N}, \mathbb{Z}),$$

for all the fibers $\tilde{N}$ of $\tilde{V} \to Q$. This time we will not write the proof of equation 6 because it is, word by word, what Gompf showed ([4] pag. 547-548).

Concerning uniqueness, for any smooth family of Poisson forms $\omega_t \in H_{\mathrm{fib}}^2(\tilde{V} \to Q)$, $t \in [0,1]$, such that the forms $\omega_t - \Omega_{\tilde{V}}$ are exact and compactly supported, the forms $\omega_t - \omega_0$ are exact in compactly supported cohomology (we can find common compact set W of $\tilde{V}$ containing all the supports). Hence we can find a family of compactly supported 1-forms α_t with $\frac{d}{dt}\omega_t = \frac{d}{dt}(\omega_t - \omega_0) = d\alpha_t$ and apply Moser's theorem to show that there is an isotopy with support in $W \subset \tilde{V}$ pulling back all the forms of the family to ω_0. The isotopy class of the form constructed using the described procedure is fixed. A different choice of $t \le t_0$ can be absorbed using the parametrized version of Corollary 1. Equally, for any other choice $\hat{\eta}$ the family $\eta_s = s\eta + (1-s)\hat{\eta}$ is valid for the construction and we can again apply the same corollary to the family $\Psi_{s,t}$. Any other choices can be connected by smooth families, and the same happens when we change the embeddings of the normal bundles (preserving the foliations) and the choice of ψ (preserving the fiber bundle structure) by isotopic identifications.

Any Poisson 2-form ω verifying the four conditions of theorem 1 is isotopic to one constructed using the described procedure. We use ψ to recover the sphere bundle $\mathcal{B}$ and the modified embeddings to put in $\mathcal{B}$ a $SO(2)$-invariant Poisson form ω_{t_0} that agrees with ω_M on V_1^0 and with $\tilde{\omega}_M$ near $j_2(P)$, and that is also the result of applying the construction of the theorem with $\eta = \frac{1}{t_0}(\omega_{t_0} - q^*\omega_P)$ and $t_1 = t_0$. $SO(2)$-invariance implies that the fibers are ω_{t_0}-orthogonal to P_∞, so η is actually non-degenerate on the fibers at P_∞. Non degeneracy of ω_t at P_∞ , $(t \le t_0)$ follows from condition 3, applied first to TP_∞. We can extend η to ν_1 after shrinking the embedding $\hat{\jmath}_1\colon \nu_1 \to M$ (rel E^0)(non-degeneracy is an open condition). If we apply the construction to the embedding of condition 2 (shrinked (rel E^0), if necessary), when $t = t_0$ we get the same embedding ($\Psi_{t_0} = id$) because it was already Poisson. The same happens for the second embedding (the correction χ equals the identity), provided we chose the given ζ defining $\tilde{\omega}_M$, rather than setting $\zeta = \lambda^*\eta$. Hence, the gluing map equals φ^{-1} near $j_2(P)$. The only price to pay is that ζ may not be $\lambda^*\eta$, for λ extending the restriction of $\hat{\jmath}_2^{-1}$ to V_2^0 (but we have that $j_2^*\zeta = i_\infty\eta$, and ζ can be assumed to vanish outside $\hat{\jmath}_2(\mathcal{B} - P_0) = V_2^0$). We will show that ω and ω', constructed using $\zeta' = \lambda^*\eta$, are isotopic (by an isotopy fixing the complementary of a compact set in $\tilde{V}$). It will be enough to show that the the Poisson forms constructed using $\zeta_s = s\zeta' + (1-s)\zeta$, satisfy the condition 1 of the theorem. But

that can be proven using the ideas that proved $\langle \Omega_{\tilde{V}} - \omega, F \rangle = 0 \ \forall F \in H^2(\tilde{N}; \mathbb{Z})$ (see [4] pag. 549). $\qquad \square$

5. The modular class of $\#_\psi M$

Let (M, Λ_M) a Poisson manifold that we assume for simplicity to be orientable. An important invariant of the Poisson structure is the *modular class* [12]. Roughly speaking, it measures up to which extent the Poisson manifold admits a measure transverse to the leaves invariant by all the hamiltonian vector fields. The modular class belongs to the first group of Poisson cohomology of (M, Λ_M) (see [10]). For each volume form μ, a vector field (derivation) representing the modular class is defined by the formula

$$(7) \qquad\qquad \phi_\mu \colon f \mapsto div_\mu X_f,$$

where X_f is the Hamiltonian vector field associated to f and div_μ the divergence with respect to μ.

A Poisson manifold with vanishing modular class is called unimodular. It is clear from what we said that a orientable Poisson manifold is unimodular if and only if there exists a volume form invariant by all the hamiltonian vector fields. Since (at least in the regular set) a volume form is the wedge product of the leafwise Liouville volume form (which is invariant by the hamiltonian vector fields) and a transverse volume form, the invariance of this transverse volume form is equivalent to the invariance of the whole form (and that is why we spoke about measuring the existence of an invariant transverse volume form).

Now let us assume that $\#_\psi M$ is oriented.

Proposition 2. *If $(\#_\psi M, \Lambda)$ is unimodular then (M, Λ_M) is also unimodular, but the converse is not true.*

Proof. We first notice that if we have an oriented Poisson manifold (N, Λ_N) and an open set U such that $(U, \Lambda_{N|U})$ is unimodular, then (N, Λ_N) will be unimodular if any of the invariant volumes on $(U, \Lambda_{N|U})$ can be extended to an invariant volume on (N, Λ_N). We will see that there are cases where (N, Λ_N) is unimodular but not all the invariant volumes on a certain open set can be extended to be invariant on (N, Λ_N). It is worth noticing that when (N, Λ_N) is a Poisson fibered manifold and U cuts each leaf in an open connected set (non-empty), then any invariant volume form in $(U, \Lambda_{N|U})$ extends to a unique invariant form on (N, Λ_N) [12]. It follows easily that in a general Poisson manifold (N, Λ_N), if we take a closed set V contained in an open one U, such that U (connected) is fibered and V intersects each fiber in a non-empty set whose complement (in the fiber) is connected, then (N, Λ_N) is unimodular if and only if $(N - V, \Lambda_{N|N-V})$ is unimodular. As a consequence, any perturbation of the Poisson bivector on V that preserves the foliation does not affect the unimodularity (resp. non-unimodularity) of (N, Λ_N). Hence, the unimodularity of $(\#_\psi M, \Lambda)$ implies the unimodularity of (M, Λ_M). If we start with (M, Λ_M) unimodular, since V_a fibers over Q, any invariant volume on (M, Λ_M) will determine a couple of volume forms on Q. It is clear that $(\#_\psi M, \Lambda)$ will be unimodular if and only if we are able to find an invariant volume form such that the induced volume forms on Q are the same. Though in general this not true (and we will end up the proof of the proposition constructing counterexamples), we will describe now some situations where this occurs.

Definition 3. *Let (M, Λ_M), (P, Λ_P) and $j_1 \colon (P, \Lambda_P) \to (M, \Lambda_M)$ be as in Theorem 1. Assume that $j_1(P)$ has trivial normal bundle. Then once we have fixed a trivialization ψ of the normal bundle, we can apply our construction to the disjoint union of (M, Λ_M) with (M, Λ_M). We denote the resulting manifold by $(M \#_\psi M, \Lambda_M \# \Lambda_M)$*

Corollary 3. *Let (M, Λ_M), (P, Λ_P) be as in the above definition . Then (M, Λ_M) is unimodular if and only if $(M \#_\psi M, \Lambda_M \# \Lambda_M)$ is unimodular.*

To construct counterexamples we begin by proving the following lemma:

Lemma 4. *There exist Poisson fibered manifolds (actually symplectic bundles) with open sets having invariant volume forms which do not extend to invariant volume forms on the whole manifold.*

Proof. The idea is to start with our fibered open set, and then glue some of the fibers into a single one (so we are putting restrictions on the volume form we pull back from the base space). We consider the Poisson fibered manifold $S^{2n-1} \times D^2 \to S^{2n-1}$, where D^2 is the corresponding closed unit disk with its usual symplectic form (have in mind the case $n = 1$). For each point of S^{2n-1} we consider its image by the antipodal map and identify the boundaries of the corresponding fibers via a reflection (say, on the y-axis) $r_y \colon S^1 \to S^1$. The resulting manifold is a symplectic bundle over $\mathbb{R}P^{2n-1}$ with fiber the sphere with the usual area form (it can also be constructed by considering $S^{2n-1} \subset \mathbb{R}^{2n} \subset \mathbb{R}^{2n+1}$, taking a closed tubular neighborhood of fixed radius of $S^{2n-1} \subset \mathbb{R}^{2n+1}$ and identifying its boundary using the antipodal map and then rescaling the area form). If we remove all the equators we obtain the initial open disk bundle. In this open set, the invariant volume forms come from volume forms on S^{2n-1}, but only the ones invariant under the action of the antipodal map on S^{2n-1} extend to invariant volume forms on the whole manifold.

There is a third way of constructing these manifolds, starting from the final Poisson manifold, which gives much more examples. We choose $(Q, G, (F, \omega), \rho)$ where Q is a compact manifold, G is a normal subgroup of $\pi_1(Q)$ of finite index and ρ is a representation of $K = \pi_1(Q)/G$ in the group of symplectomorphisms of (F, ω) such that there are points in F with trivial stabilizers. Q_G, the cover of Q associated to the subgroup G is a principal K-bundle, so we can construct the associated bundle to the chosen representation ρ by symplectomorphisms. Our resulting manifold M is a symplectic bundle and hence a Poisson manifold, but as a bundle, since it has discrete structural group, it has the unique lifting property. Thus, if on the fiber over the base point x_0 of Q, we pick a point z with trivial stabilizer, the lifting to z of all the homotopy classes of paths based on x_0 gives us an embedding of Q_G in M (transverse to the fibers). On the fiber over x_0, the points close to z have trivial stabilizer which implies that the normal bundle to Q_G is trivial. We can even take as a tubular neighborhood the result of pushing a small disk around z using the unique lifting property, which gives us a symplectic subbundle. It is clear that the invariant volumes on a small tubular neighborhood of Q_G that extend to invariant volume forms on the whole manifold are those which come from K-invariant volume forms on Q_G. $\square$

Now we are ready to finish the proof of proposition 2:

To construct the counterexample we take two copies of any of the symplectic fibrations $(Q, G, (F, \omega), \rho) \to Q$ of lemma 4 (with F a surface) and consider in both the same embedding of Q_G. Now we fix a volume form μ on Q_G that descends to Q. Then we pick a point $z \in Q_G$ and consider a diffeomorphism $f : Q_G \to Q_G$

homotopic to the identity (rel z) which is the identity in a neighborhood of the remainig points of the orbit of z and which does not preserve μ in z. We identify both embeddings of Q_G in M via f and perform the fibered connected sum using any framing ψ to obtain manifold which is non unimodular. If it was, an invariant volume form would induce a volume form $e^h \mu$ on Q_G both invariant by the action of K and the action of K conjugated by f, but this cannot happen at the point z. $\square$

6. Poisson manifolds with arbitrary fundamental groups

Using the previous results we can prove the theorem that extends Gompf's existence result to regular Poisson manifolds.

Theorem 2. *Let G be any finitely presentable group. Then for any integers $n, d \geq 4$, d even, there exist an oriented closed regular Poisson manifold of dimension n and constant rank d $(M^{n,d}, \Lambda)$ such that $\pi_1(M^{n,d}) \cong G$. These manifolds have vanishing Godbillon-Vey class but the ones with a codimension 1 symplectic foliation are non-unimodular. Moreover, they can be chosen to be spin.*

Proof. As it was remarked in the introduction, we only need to prove the case $n = 5$, $d = 4$ because Gompf already showed it for n even and $d \neq n - 1$ (multiplying one of its manifolds by an sphere of the appropriate dimension), and the odd higher dimensional cases follow from the 5-dimensional one (by multiplying by simply connected symplectic manifolds of the appropriate dimension in the case of a codimension 1 symplectic foliation).

We first recall Gompf's proof: one starts with a closed symplectic manifold $T^2 \times \Sigma_g$ such that G can be obtained by collapsing some elements of its fundamental group. The symplectic form is chosen so that these elements are the simple curves of some trivially embedded symplectic tori. The key step is that the manifold which is glued along each one of this tori is a rational elliptic surface (along one of its regular fibers), and the resulting fundamental group, which does not depend on the chosen framing, is the old one with the homotopy of these tori killed. It is worth recalling the topology of this rational elliptic surfaces. They are diffeomorphic to $\mathbb{C}P^2 \overset{9}{\#} (-\mathbb{C}P^2)$ and an example can be constructed by blowing up the nine points of $\mathbb{C}P^2$ where two generic cubics intersect. We get in this way a fibration $p \colon \mathbb{C}P^2 \overset{9}{\#} (-\mathbb{C}P^2) \to \mathbb{C}P^1$ whose fibers are the pencil of cubics generated by the two given ones. The general fiber is a smooth cubic (topologically a torus) and we also have 12 singular fibers which topologically are a sphere with a self intersection point (the result of collapsing a non-separating regular curve of the generic fiber). It is easy to check that the complement of a regular fiber is simply connected. Roughly speaking, the complement fibers over a disk so we only have to care about the fiber. Following [7], we see that this complement can be constructed starting from $D^2 \times T^2$, $T^2 = < a > \times < b >$. Extending the fibration to a bigger disk (in $\mathbb{C}P^1$) containing a singular fiber, amounts to gluing a two handle (with some framing) over either a or b (we have 12 singular fibers, and 6 of the disks go over a and 6 over b). The last step is to glue a neighborhood of the regular fiber over ∞. Hence, any curve contained in a fiber is trivial in $p^{-1}(\mathbb{C}P^1 - \{0, \infty\})$.

To get our Poisson 5-manifold M with $\pi_1(M) \cong G$, we consider one of Gompf's manifolds (M_G, ω_{M_G}) with $\pi_1(M_G) = G$. We can also assume that $M_G = N_G \# \mathbb{C}P^2 \overset{9}{\#} (-\mathbb{C}P^2)$ and that the fiber removed is $p^{-1}(\infty)$. Let $M_1 = M_G \times S^1$ with the product Poisson structure (the vertical 2-form $p_1^* \omega_{M_G}$, that we rename as ω_{M_G}). In M_G, the fiber $p^{-1}(0) = T$ is a trivially embedded symplectic torus with symplectic form

ω_0. Now let $M_2 = T \times S^3$ with the product Poisson structure coming for ω_0 and a Poisson structure of S^3 determined by the Reeb foliation and the usual volume form, and let $k \subset S^3$ be the unknot, which is a Poisson submanifold of S^3 transverse to the foliation. We consider the Poisson submanifolds $P_1 = T \times S^1 \subset M_1$, $P_2 = T \times k \subset M_2$. It is clear that both are transversal Poisson fibered submanifolds. Moreover, they are trivially embedded and any identification of k with the factor S^1 of P_1 identifies P_1 and P_2 as Poisson manifolds. Any identification between normal bundles will allow us to construct the corresponding connected sum along the normal directions. In this case, we have canonical framings; the one in P_1 comes from the projection $p \colon \mathbb{C}P^2 \overset{9}{\#} (-\mathbb{C}P^2) \to \mathbb{C}P^1$ and the one in P_2 from the zero-framing of the unknot. Using this framing and $<a, b, s>$ as base of $H_3(T \times S^1; \mathbb{Z})$ (the choice of s depends on the orientation we pick for M_1), any other framing is given by a triple $(l_1, l_2, l_3) \in \mathbb{Z}^3$. We will denote the obtained Poisson manifold by $M_1 \#_{(l_1, l_2, l_3)} M_2$. The computation of its fundamental group is mere routine, but we will do it anyway because this is not quite the manifold we are looking for. As usual, we apply Seifert-Van Kampen's theorem:

Let D_1 be the unit disk contained in $\mathbb{C}P^1$ and $W_2 = k \times D_2$ be a small tubular neighborhood of k in S^3. Let us call $V_1 = p^{-1}(D^1) \times S^1$, $V_2 = T \times W_2$. $M_1 - V_1 = (M_G - p^{-1}(D_1)) \times S^1$ and $\pi_1(M_G - p^{-1}(D_1))$ has the same generators as $\pi_1(M_G)$ and the same relations except from the one that assures that the loop $\hat{\alpha}$, a lift of $\alpha = \partial D_1$, is vanishing. $\pi_1(M_2 - V_2)$ is the free group generated a, b and by the loop $\beta = \partial \bar{D}_2$ generating the homotopy of $S^3 - W_2$. Now we see that the loop s generating the homotopy of S^1 in $(M_G - V_1) \times S^1$ goes to a curve isotopic to $k + l_3 \beta$. The curves $a, b \subset T \times \{x\} \subset M_2 - V_2$ are seen as the correspondent simple curves generating the homology of a fiber over a point in ∂D_1 plus some multiple of $\hat{\alpha}$. Finally, the loops $\hat{\alpha}$ and β are the same.

We probably did not get the desired manifold because we cannot conclude that $\hat{\alpha}$ is contractible, but we turned our initial problem of killing the generator of the homotopy of S^1 in $M_G \times S^1$ into a problem that amounts to kill a curve in a manifold whose topology we know quite well.

In M_G, we consider T_2, be the torus generated by the loops $\hat{\alpha} + a, b$. T_2 is a symplectic torus trivially embedded (the symplectic structure on $p^{-1}(0) \times D^2_{1+\epsilon}$ can be assumed to be the product symplectic structure). Applying Gompf's construction to M_G and a rational elliptic surface along the normal directions of T_2 and a regular fiber we get a symplectic manifold $\tilde{M}_G$. It is clear that $\pi_1(\tilde{M}_G) = \pi_1(M_G)$, but in $\tilde{M}_G$ we have a disk that bounds $\hat{\alpha}$ lying in $\tilde{M}_G - p^{-1}(D_1)$. Thus, if we do the fiber connected sum of $\tilde{M}_G \times S^1$ and $T \times S^3$ along P_1 and P_2 ($T = p^{-1}(0)$ is of course in $\tilde{M}_G$), we get a Poisson manifold $\tilde{M}_1 \#_{(l_1, l_2, l_3)} M_2$ such that $\pi_1(\tilde{M}_1 \#_{(l_1, l_2, l_3)} M_2) \cong G$. It is worth noticing that the diffeomorphism type of $\tilde{M}_1 \#_{(l_1, l_2, l_3)} M_2$ depends at most on l_3. To see that we observe that $M_2 - V_2$ is a tubular neighborhood of $T \times \hat{\beta}$, where $\hat{\beta}$ is a loop in the interior of $M_2 - V_2$ isotopic to β and thus $\partial(M_2 - V_2)$ has an S^1-bundle structure (over $T \times \hat{\beta}$). Hence the diffeomorphism type of the connected sum is totally determined by the image in $\partial(\tilde{M}_1 - V_1)$ of the S^1-bundle structure of $\partial(M_2 - V_2)$ (because $\tilde{M}_1 \#_{(l_1, l_2, l_3)} M_2$ is the result of collapsing to a point the fibers of the described fibration), and these fibrations are classified by the value of l_3 (the authors do not know whether different values of l_3 yield different diffeomorphism types).

As we already observed, if we use Kummer surfaces instead of rational elliptic ones to construct $\tilde{M}_G$, both $\tilde{M}_1$ and M_2 can be given spin structures. For any such structures, since $H^2(P_i; \mathbb{Z})$ has no torsion, one can find integers $\bar{l}_1, \bar{l}_2, \bar{l}_3$ with $\tilde{M}_1 \#_{(\bar{l}_1, \bar{l}_2, \bar{l}_3)} M_2$ admitting a spin structure extending any given ones. $\qquad \square$

Remark: In the examples above (dimension 5) there are three kinds of symplectic leaves. We have a family parametrized by S^1 which are diffeomorphic to $\tilde{M}_G - T_1$ and hence have G as fundamental group; we have another S^1-family of leaves diffeomorphic to $\mathbb{R}^2 \times T_1$ and both families fill open connected sets separated by a compact leaf $T_1 \times T$, where T is the closed torus of the Reeb foliation of S^3. Any of the non-closed leaves has the closed one as set of accumulation points.

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