GENERIC LINEAR SYSTEMS FOR PROJECTIVE CR MANIFOLDS

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Abstract. For compact CR manifolds of hypersurface type which embed in complex projective space, we show that for all k large enough there exist linear systems of $\mathcal{O}(k)$ which when restricted to the CR manifold are generic in a suitable sense. These systems are constructed using approximately holomorphic geometry.

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1. Introduction

Approximately holomorphic geometry can be applied to compact symplectic manifolds endowed with a compatible almost complex structure to obtain "generic" approximately holomorphic maps to complex projective spaces [3]. These maps can be understood as analogs in symplectic geometry of generic linear systems for Hodge manifolds. The analogy is valid for constructions only using (pseudo-holomorphic) 1 and 2-jets [13, 2]; it breaks down for higher order jets due to the difficulty of developing a theory of normal forms in the absence of an integrable almost complex structure. However if the symplectic structure comes from a Hodge one, then according to section 7 in [12] it is possible to adjust the constructions of approximately holomorphic geometry so that the outcome are holomorphic maps, giving thus a new construction of generic linear systems.

Applications of approximately holomorphic theory to symplectic manifolds include the construction of plenty of symplectic submanifolds with control on their topology [12, 1, 26], construction of symplectic invariants [4], and a proof -via Lefschetz pencil structures [13] and making little use of elliptic theory- of the existence of symplectic curves realizing the canonical class of a 4-dimensional symplectic manifold [14].

A 2-calibrated manifold is defined as a triple $(M^{2n+1}, D^{2n}, \omega)$, where D is a codimension one distribution and ω a closed 2-form maximally non-degenerate over D (i.e. making D into a symplectic distribution). The most relevant classes of 2-calibrated manifolds are related to the two extreme behaviors of the distribution: when D is integrable we speak of 2-calibrated foliations, and when D is maximally non-integrable 2-calibrated structures include contact structures.

Approximately holomorphic geometry can be adapted to compact 2-calibrated manifolds to obtain "generic" approximately holomorphic maps to complex projective spaces [22]. Very much as in the symplectic case, due the difficulty of developing a theory of normal forms interesting applications only use (pseudo-holomorphic)

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1-jets. They include the existence of contact submanifolds whose Poincaré dual realizes any determinantal class [22], the construction of open book decompositions compatible with a contact structure [17, 16], and a description of the leaf spaces of 2-calibrated foliations [23].

When the auxiliary compatible almost complex structure on D needed to develop the approximately holomorphic theory is integrable, we also have a CR structure (of hypersurface type). In this paper we are interested in analyzing under which conditions the approximately holomorphic techniques can be refined to yield CR constructions, and which are the applications that can be obtained. To this extent our main result is the following theorem (the reader is referred to section 2 for a full explanation of its statement):

Theorem 1. *Let $(M^{2n+1}, \mathcal{F}, J)$ be a closed Levi-flat CR manifold of hypersurface type endowed with a positive CR line bundle. Fix h,r $\in \mathbb{N}$, $r \leq$ h-2. Then for any integer m there exists $\phi\colon M\backslash B \to \mathbb{C}P^m$ an r-generic map. More precisely, we obtain the following:*

1. *A Levi-flat CR submanifold B of real codimension 2m+2 and class C^h.*
2. *A CR map $\phi\colon M\backslash B \to \mathbb{C}P^m$ such that for each leaf F of $\mathcal{F}$, the holomorphic r-jet of $\phi_{|F}$ is transverse to the corresponding Thom-Boardman stratification of the bundle of holomorphic r-jets of holomorphic maps from F to $\mathbb{C}P^m$. These bundles fit into a bundle of class C^{h-r} -the bundle of CR r-jets of CR maps from M to $\mathbb{C}P^m$- and the same holds for the strata of the Thom-Boardman stratifications. The CR r-jet of ϕ is leafwise transverse to this stratification by Levi-flat manifolds of class C^{r-h}. Therefore, the pull-back of each stratum is a Levi-flat submanifold of the expected codimension and of class C^{r-h}.*

For foliated manifolds there is an obvious notion of leafwise genericity, and the existence of leafwise r-generic maps (holomorphic, smooth) is in general obstructed; such a map would give rise to a stratification of the manifold transverse to the foliation whose existence might be in general not possible for topological reasons. A simplified interpretation of theorem 1 is that for a certain class of manifolds foliated by complex leaves, one can find -away from suitable submanifolds- leafwise holomorphic r-generic maps.

There are related results in the literature addressing the embedding problem for CR manifolds into projective space [7, 28, 21] (thus only involving CR 1-jets). More generally E. Ghys [15] and B. Deroin [11] have shown that compact manifolds laminated by complex leaves -subject to some additional conditions- do admit enough (meromorphic) functions, so that leafwise immersions in projective spaces are possible. To the best of our knowledge our r-genericity result, for arbitrary r, is new.

The structure of the paper is as follows: in section 2 we recall the definitions and results from CR geometry needed to state the results of this paper. For complex projective manifolds classical generic linear systems are constructed using dual geometry. In section 3 we outline the difficulties to develop a dual geometry for arbitrary projective CR manifolds (i.e. those which have a CR embedding in projective space); we mention very briefly a family of CR manifolds for which the classical approach extends. Finally, in section 4 we sketch how to adapt the constructions of approximately holomorphic geometry for projective CR manifolds, thus proving the results stated in section 2.

2. Definitions and statements of the results

Definition 1. *A CR manifold (always of hypersurface type for us) is a triple* (M^{2n+1}, D^{2n}, J)*, where* M *is a manifold and* D *is a codimension one distribution of* TM *endowed with an almost complex structure* J*, such that either of the eigenbundles* $D^{*1,0}, D^{*0,1}$ *of* $D^* \otimes_{\mathbb{R}} \mathbb{C}$ *associated to the eigenvalues* i*,* $-i$ *respectively, are involutive.*

All CR manifolds in this paper will be smooth, closed, co-oriented and oriented, and all maps and tensors will also be smooth unless otherwise stated.

The Levi form of a CR manifold (M, D, J) is the symmetric bilinear tensor given by

$$\begin{array}{rcl} D \times D & \longrightarrow & TM/D \\ (u, v) & \longmapsto & [U, JV]/\sim \ , \end{array}$$

where U, V are local sections of D extending $u, v \in T_xM$, and we consider the class of the above Lie bracket at x in the quotient real line bundle TM/D (which is oriented, so we can make sense of positive and negative values). The Levi form keeps track of the behavior of the distribution D. Its vanishing is equivalent to D integrating into a foliation $\mathcal{F}$, in which case we speak of a Levi-flat CR manifold. The other extreme case is that of strictly pseudo-convex (resp. pseudo-concave) CR manifolds, for which the Levi form is strictly positive (resp. negative); in particular the distribution D of such CR manifolds is a contact distribution.

Let (M, D, J) be a CR manifold and (M', D', J') either a CR manifold or a complex manifold (in which case $D' = TM'$). A map $\phi\colon M \to M'$ is CR if $\phi_* D \subset D'$ and $\phi_* \circ J = J' \circ \phi_*$. A CR vector bundle is a complex vector bundle $\pi\colon E \to (M, D, J)$ defined by CR transition maps.

2.1. **Positivity of CR line bundles and CR sections.**

Our goal is finding CR line bundles $L \to M$ with plenty of CR sections, so among them we have linear systems of CR sections whose leafwise holomorphic jets solve appropriate transversality problems. These would be our generic CR linear systems. A natural condition to impose is that of positivity of L along D; in the Levi-flat case and if all leaves are compact -so from the differential viewpoint the foliated manifold $(M, \mathcal{F})$ is a mapping torus- L restricts to each leaf to a positive line bundle, and therefore large enough powers admit generic holomorphic linear systems. One expects suitable S^1-families of such generic holomorphic linear systems to fit into generic CR linear systems of $L^{\otimes k}$.

Among CR linear systems generic ones are open, since they are defined by transversality conditions. Roughly speaking the way in which one would think of producing them is by breaking the problem into two parts: firstly, finding linear systems which are already generic and very close to be CR (genericity can also be extended to not necessarily CR linear systems). Secondly, projecting the previous generic nearly CR linear system into a CR linear system by solving the corresponding tangential Cauchy-Riemann equation, and making sure that the correction is small enough so that genericity is preserved. The first part of the problem can be solved: the assumption of positivity along D implies that the curvature of a compatible Hermitian connection on L divided by $-2\pi i$, makes (M, D) into a 2-calibrated manifold for which J is a compatible almost complex structure. The theory developed in [22] grants the existence of generic nearly CR sections. What cannot be granted in general is the existence of a solution to the tangential Cauchy-Riemann problem close enough to the nearly CR section; for Levi-Flat manifolds this should be possible by using results of Ohsawa-Sibony [28]. We prefer to choose another approach in which we assume that M has a CR embedding into a (compact) complex manifold X, and L is the restriction of a positive holomorphic line bundle $\mathcal{L} \to X$. Moreover, rather than looking at arbitrary CR sections of $L^{\otimes k}$, we will be only interested in those which are restriction of holomorphic sections of $\mathcal{L}^{\otimes k}$, k $\gg 1$, the advantage being that for the latter line bundles we have a lot of control for the projection of a section onto its holomorphic part. Still, one needs to show that among these very particular CR sections genericity can be achieved. In the language of [22] what we explained is why we do not know how to adapt intrinsic approximately holomorphic theory to the CR setting, and we need to impose the existence of $\mathcal{L} \to X$ and explore whether relative approximately holomorphic theory can be adapted to $(L, M) \hookrightarrow (\mathcal{L}, X)$.

For simplicity we will assume X to be some projective space and $\mathcal{L}$ to be the hyperplane line bundle $\mathcal{O}(1)$.

2.2. **Projective CR manifolds.**

Definition 2. *A CR manifold (M, D, J) is projective if it admits a CR embedding into some $\mathbb{CP}^N$. It is called embeddable if it admits a CR embedding into some $\mathbb{C}^N$.*

A large number of projective CR manifolds are provided by the following embedding theorems.

Theorem 2. *(Boutet de Monvel [7]) Any (compact and oriented) strictly pseudo-convex (resp. pseudo-concave) CR manifold of dimension bigger or equal than five is embeddable.*

A recent result of Marinescu and Yeganefar [21] states that any Sasakian manifold is embeddable. Sasakian manifolds are strictly pseudo-convex. 3-dimensional ones -to which theorem 2 does not apply- are also covered by the aforementioned result of Marinescu and Yeganefar.

Theorem 3. *(Ohsawa, Sibony [28]) Let (M, D, J) be a (compact) Levi-flat CR manifold and $L \to M$ a positive line bundle. Let h be any natural number. Then there exists another natural number d(h) and a CR embedding $j \colon M \hookrightarrow \mathbb{CP}^{d(h)}$ of class C^h.*

2.3. **1-jets and Lefschetz pencil genericity condition.** Let (M, D, J) be a CR manifold (not necessarily projective). Let $(M'J')$ be either a complex or a CR manifold, and let us denote in either case by D' the maximal complex distribution of TM'.

Definition 3. *The bundle of CR 1-jets of CR maps from M to M', denoted by $\mathcal{J}^1_{\mathrm{CR}}(M, M') \to M$, is defined to be the bundle over M whose fiber over x is*

$$\mathcal{J}^1_{\mathrm{CR}}(M, M')_x := \{(y, h) \mid y \in M', h \in \mathrm{Hom}_{\mathbb{C}}(D_x, D'_y)\}.$$

Inside this bundle there is a distinguished submanifold

$$\Sigma = \{(x, y, h) \in \mathcal{J}^1_{\mathrm{CR}}(M, M') \mid h = 0\}.$$

The CR 1-jet of a CR map $\phi \colon M \to M'$ is by definition $j^1_{\mathrm{CR}}\phi := (\phi, d_D\phi)$, where $d_D\phi$ is the restriction of the differential $d\phi$ to D. It is a section of $\mathcal{J}^1_{\mathrm{CR}}(M, M')$.

If (M, D, J) is a projective CR manifold there is a second relevant complex distribution associated to M: the complex envelope of its tangent bundle, denoted by TM^J, where at each point $x \in M$ we put the smallest complex subspace of $T_x\mathbb{CP}^N$ containing T_xM . For some $\epsilon > 0$ small enough, we use the Fubini-Study metric to extend both D and TM^J by parallel transport along normal geodesics to complex distributions defined in $\mathcal{N}_\epsilon(M)$, the neighborhood of M of radius ϵ. The extended distributions will be also denoted by D and TM^J whenever there is no risk of confusion.

We have the corresponding bundle $\mathcal{J}^1_{TM^J}(\mathcal{N}_\epsilon(M), M') \to \mathcal{N}_\epsilon(M)$ of holomorphic 1-jets along TM^J of holomorphic maps from $\mathcal{N}_\epsilon(M)$ to M'. We denote by Σ^J the submanifold corresponding to the holomorphic 1-jets along TM^J whose degree 1 homogeneous component is vanishing. Similarly, the holomorphic 1-jet along TM^J of a holomorphic map $\Phi\colon \mathcal{N}_\epsilon(M) \to M'$ is by definition $j^1_{TM^J}\Phi := (\Phi, d_{TM^J}\Phi)$. It is a section of $\mathcal{J}^1_{TM^J}(\mathcal{N}_\epsilon(M), M')$.

Definition 4. *Let $j\colon (M, D, J) \hookrightarrow \mathbb{CP}^N$ be a projective CR manifold. A (CR) Lefschetz pencil structure (of degree k) on M is defined to be a pencil of (degree k) hypersurfaces of $\mathbb{CP}^N$, so that any two hypersurfaces intersect transversely in the base locus $\tilde{B}$, and the following conditions hold:*

1. *$\tilde{B}$ is transverse to M and therefore $B := \tilde{B} \cap M$ is a real codimension four CR submanifold of M.*
2. *Let $\Phi\colon \mathbb{CP}^N \backslash \tilde{B} \to \mathbb{CP}^1$ denote the holomorphic map associated to the pencil and $\phi\colon M\backslash B \to \mathbb{CP}^1$ its restriction to $M\backslash B$. Then*
$$j^1_{\mathrm{CR}}\phi\colon M\backslash B \to \mathcal{J}^1_{\mathrm{CR}}(M\backslash B, \mathbb{CP}^1)$$
is transverse along D to Σ.
3. *The holomorphic 1-jet along TM^J*
$$j^1_{TM^J}\Phi\colon \mathcal{N}_\epsilon(M)\backslash\tilde{B} \to \mathcal{J}^1_{TM^J}(\mathcal{N}_\epsilon(M)\backslash\tilde{B}, \mathbb{CP}^1)$$
is transverse along M to Σ^J.

We will also call the pencil of hypersurfaces giving a Lefschetz pencil structure a generic rank one (CR) linear system of the CR line bundle $\mathcal{O}(k)_{|M} := j^*\mathcal{O}(k)$.

We denote the Lefschetz pencil structure by the triple (ϕ, B, Δ), where Δ are the points of M where the derivative $d_D\phi$ vanishes. Any point $x \in \Delta$ is called a singular point of the pencil. Points in B are base points, and points in $M\backslash(B\cup\Delta)$ are called regular. Similarly, $a \in \mathbb{CP}^1\backslash\phi(\Delta)$ is called a regular value and singular otherwise. Notice that at a singular point of the pencil the map ϕ fails to be a submersion.

2.3.1. *Transversality along distributions.* Recall that transversality along D to Σ of a section $\sigma\colon M \to \mathcal{J}^1_{\mathrm{CR}}(M, \mathbb{CP}^1)$ is defined as follows [22]: the pullback of D by the projection $\mathcal{J}^1_{\mathrm{CR}}(M, \mathbb{CP}^1) \to M$ defines a distribution $\hat{D}$ in $\mathcal{J}^1_{\mathrm{CR}}(M, \mathbb{CP}^1)$. At every point x such that $\sigma(x) \in \Sigma$ one requires
$$\hat{D}_{\sigma(x)} = T_{\sigma(x)}\Sigma \cap \hat{D}_{\sigma(x)} + T_x\sigma \cap \hat{D}_{\sigma(x)},$$
where $T_x\sigma$ is the tangent space of the graph of σ at $\sigma(x)$.

The above definition extends in the obvious way to transversality along D to any submanifold S of a bundle $E \to M$.

Transversality along $M \subset \mathcal{N}_\epsilon(M)$ is defined at the points of M as we did for transversality along D, but with $TM \subset T\mathcal{N}_\epsilon(M)_{|M}$ playing the role of D. Since $TM \subset T\mathcal{N}_\epsilon(M)_{|M}$ is integrable, transversality along M of $j^1_{TM^J}\Phi$ is equivalent to usual transversality for the restriction
$$j^1_{TM^J}\Phi\colon M\backslash B \to j^*\mathcal{J}^1_{TM^J}(\mathcal{N}_\epsilon(M)\backslash\tilde{B}, \mathbb{CP}^1).$$

Transversality along distributions is stronger than usual transversality.

From the transversality assumption along D on $j^1_{\mathrm{CR}}\phi$ we conclude that Δ is a 1-dimensional submanifold transverse to D. A dimension count implies that transversality of $j^1_{TM^J}\Phi$ along M to Σ^J is equivalent to the section not intersecting Σ^J. Therefore in the points of Δ the derivative $d\phi\colon TM \to T\mathbb{CP}^1$ cannot vanish, because being Φ holomorphic that would imply the vanishing of $d_{TM^J}\Phi$. Thus $\phi(\Delta) \subset \mathbb{CP}^1$ is an immersed curve, which is why condition 3 in definition 4 is required.

Remark 1. *If the CR embedding $j\colon M \hookrightarrow \mathbb{CP}^N$ is of class C^h, $h \geq 2$, definition 4 also makes sense because transversality (along any distribution) is a C^1-notion.*

Our first main result is the following:

Theorem 4. *Let (M^{2n+1}, D, J) be a projective CR manifold of class C^h, $h \geq 3$. Then for any k large enough M admits degree k Lefschetz pencil structures. Let W be the fiber of ϕ over any regular value, compactified by adding B. Then the morphisms at the level of homotopy (and homology groups) $l_*\colon \pi_i(W) \to \pi_i(M)$ induced by the inclusion are isomorphisms for $0 \leq i \leq$ n-2, and an epimorphism for $i =$ n-1.*

Theorem 4 is a result about the existence of generic rank one linear systems of the bundles $\mathcal{O}(k)_{|M} \to M$, for k $\gg 1$.

It is not possible in general to give a normal form for ϕ in neighborhoods of points of Δ and B. In the Levi-flat setting normal forms exist.

Proposition 1. *(see also* [20],[23]*) Let (ϕ, B, Δ) be a Lefschetz pencil structure on a projective Levi-flat CR manifold. Then we have the following:*

(1) *For any $x \in B$ there exist CR coordinates $z^1, \ldots, z^n, s$ centered at x and a holomorphic chart of $\mathbb{CP}^1$, such that*
$$B \equiv z^1 = z^2 = 0$$
and
$$\phi(z^1, \ldots, z^n, s) = z^2/z^1.$$
(2) *For any $x \in \Delta$ there exists CR coordinates $z^1, \ldots, z^n, s$ centered at x, a holomorphic chart of $\mathbb{CP}^1$ and a real map t of class C^{h-1} such that*
$$\phi(z^1, \ldots, z^n, s) = (z^1)^2 + \cdots + (z^n)^2 + t(s),\ t(0) = 0,\ t'(0) \neq 0.$$

Due to the existence of normal forms it is possible to adapt results of [23] and know more about the intersection of a regular fiber of the pencil with a leaf of the foliation.

Theorem 5. *Let $(M, \mathcal{F}, J)$ be a projective Levi-flat CR manifold of dimension bigger or equal than five, and let (ϕ, B, Δ) be one of the pencils provided by theorem 4. Then for any regular fiber W of ϕ (compactified adding the base points) and for every leaf F of $\mathcal{F}$, the intersection $F \cap W$ is connected. As a consequence the inclusion $l\colon (W, \mathcal{F}_W) \hookrightarrow (M, \mathcal{F})$, where $\mathcal{F}_W$ is the induced foliation, descends to a homeomorphism of leaf spaces.*

Theorem 5 can also be understood as a leafwise Lefschetz hyperplane theorem for the π_0, and for sections (degree k hypersurfaces) which fit into a Lefschetz pencil for $(M, \mathcal{F}, J)$. Notice as well how unexpected this result is for non-compact leaves (some of which can be dense!).

We would like to single out theorem 4 together with proposition 1 for 3-dimensional projective Levi-flat CR manifolds.

Let M^3 be a closed, orientable 3-manifold endowed with a smooth co-orientable foliation $\mathcal{F}$ by surfaces. We recall the following elementary result.

Lemma 1. *$(M^3, \mathcal{F})$ admits the structure of a Levi-flat CR manifold (with CR foliation $\mathcal{F}$) with a positive CR line bundle, if and only if $\mathcal{F}$ is a taut foliation.*

Proof. The existence of a Levi-flat CR structure with a positive CR Hermitian line bundle (L, ∇) with compatible connection clearly implies tautness. Following Sullivan [29] tautness is equivalent to the existence of a closed 2-form restricting to a leafwise area form. In our case the curvature of ∇ divided by $-2\pi i$ has this property.

Conversely, if the foliation is taut we do have an integral closed 2-form ω which is non-degenerate when restricted to each leaf. Let J be a leafwise compatible almost complex structure. Then J is integrable, for the leaves are 2-dimensional, and ω is of type (1,1) with respect to J. Therefore we can construct (L, ∇), the corresponding pre-quantum line bundle, which is a Hermitian line bundle with connection whose curvature satisfies $F_\nabla = -2\pi i\omega$. Being ω of type (1,1) the bundle admits a leafwise holomorphic structure, and hence a CR one. Positivity is also clear. □

Corollary 1. *Let $(M^3, \mathcal{F})$ be a closed, orientable 3-manifold endowed with a co-orientable taut foliation, and fix $h \in \mathbb{N}$, $h \geq 3$. Then there exists a CR structure in $(M^3, \mathcal{F})$ and maps $\phi\colon M^3 \to \mathbb{CP}^1$ with the following properties:*

(1) *ϕ is of class C^h and leafwise holomorphic.*
(2) *The restriction of ϕ to each leaf is a branched cover with index 2 singular points.*
(3) *The leafwise singular sets fit into a transverse link Δ of class C^{h-1}.*
(4) *Around each point $a \in \Delta$ there exist local CR coordinates (z, s) and a complex coordinate in $\mathbb{CP}^1$, so that $\phi(z, s) = z^2 + t(s)$, where t is C^{h-1} and $t(0) = 0$, $t'(0) \neq 0$.*

Corollary 2. *Let $(M^3, \mathcal{F})$ be a closed and orientable 3-manifold endowed with a co-orientable foliation. Then $\mathcal{F}$ is taut is and only if there exist a leafwise complex structure for which $(M^3, \mathcal{F}, J)$ admits a (CR) Lefschetz pencil structure.*

Also observe that by applying theorem 5 inductively we conclude that any projective Levi-flat CR manifold contains a taut 3-dimensional foliation whose leaves are complex, and such that the inclusion descends to a homeomorphism of leaf spaces. Thus leaf spaces of projective Levi-flat CR manifolds are no more complicated than those of 3-dimensional taut foliations.

2.4. **Higher order jets.** Regarding higher order genericity, note that if (M, D, J) is a CR manifold, M' either a CR or complex manifold and $\phi\colon M \to M'$ a CR map, it is not possible in general to define the second CR jet of ϕ; there is no intrinsic way of taking a second derivative along a general distribution D. Therefore, to consider higher order jets we must assume that $(M, \mathcal{F}, J)$ is Levi-flat.

On the bundle of foliated holomorphic r-jets of CR maps from $(M, \mathcal{F}, J)$ to M' one has the CR Thom-Boardman stratification $\mathcal{T}^{\mathcal{F}}$, which is the union of the usual holomorphic Thom-Boardman stratifications [6] over each leaf $F \in \mathcal{F}$. A CR map ϕ is defined to be r-generic if its foliated holomorphic r-jet (i.e. its CR r-jet) is leafwise transverse to $\mathcal{T}^{\mathcal{F}}$. For such a map the pullback of each stratum $\Sigma_I^{\mathcal{F}}(\phi)$ would be transverse to the leaves; its intersection with each leaf F would be the holomorphic Thom-Boardman stratum $\Sigma_I(\phi_{|F})$.

Let us for simplicity forget about holomorphic functions and consider the foliated/leafwise genericity problem in the smooth setting. A strategy to solve transversality problems for foliated smooth jets is to use the canonical submersion from the bundle of r-jets to the bundle of foliated r-jets [5], to pull back the leafwise Thom-Boardman stratification to the bundle of r-jets. Leafwise r-genericity is equivalent to transversality along the leaves of $\mathcal{F}$ of the (full) r-jet to the pulled back stratification.

To develop a similar strategy to solve the leafwise holomorphic genericity problem, we would need to embed M inside a complex manifold X, and transfer the problems for leafwise holomorphic jets to problems for full holomorphic jets. The next step would be to solve the foliated (strong) transversality problem in the bundle of full holomorphic r-jets. This is always possible locally, but there is no reason why a global solution should exist.

If $(M, \mathcal{F}, J)$ is Levi-flat and possess a positive CR line bundle, theorem 3 implies that for any natural number h we obtain a leafwise holomorphic embedding in $\mathbb{CP}^{d(h)}$ of class C^h. This projective space plays the role of the ambient complex space X.

The target space of the r-generic maps we will construct will not be an arbitrary complex manifold, but complex projective space, for our maps will come again from linear systems.

Theorem 6. *Let $(M^{2n+1}, \mathcal{F}, J)$ be a Levi-flat CR manifold endowed with a positive CR line bundle $L \to M$. Fix h,r $\in \mathbb{N}$, r $\leq$ h-2. Then for any integer m we can find the following:*

1. *A Levi-flat CR submanifold B of real codimension 2m+2 and class C^h.*
2. *A CR map $\phi\colon M\backslash B \to \mathbb{CP}^m$ -which is the restriction of a holomorphic map $\Phi\colon \mathbb{CP}^N\backslash\mathcal{B} \to \mathbb{CP}^m$- such that its CR r-jet is leafwise transverse to the Thom-Boardman stratification of the bundle of CR r-jets (which is by Levi-flat strata of class C^{r-h}). Therefore the pullback of each stratum is a Levi-flat submanifold of the expected codimension and of class C^{r-h}.*
3. *The 1-jet of Φ along TM^J is transverse along M to the stratification of $\mathcal{J}^1_{TM^J}(\mathcal{N}_\epsilon(M), \mathbb{CP}^m)$, whose strata are defined according to the rank of the degree 1 homogeneous component of $\sigma \in \mathcal{J}^1_{TM^J}(\mathcal{N}_\epsilon(M), \mathbb{CP}^m)$.*

Observe that the outcome of theorem 6 is a generic CR map ϕ defined away from a CR submanifold (properties (1) and (2)), together with a requirement on the full derivative of Φ analogous to the third one in the definition of a Lefschetz pencil structure. The latter condition gives a lower bound -depending on the dimensions m,n- on the rank $d\phi$, this providing supplementary information about the differential of ϕ in the direction transverse to D in those points where the rank along D drops enough.

Notice as well that theorem 1 in the introduction is a simplified version of theorem 6.

3. Dual geometry of projective CR manifolds

For any projective manifold $X \hookrightarrow \mathbb{CP}^N$ generic linear systems of $\mathcal{O}(1)$ can be constructed by applying basic transversality results to its dual variety $X^* \hookrightarrow \mathbb{CP}^{N*}$. It might happen that the dual variety is not a divisor, but it is indeed a divisor if we twist the embedding (compose with the k-th Veronese embedding, for any k $\geq$ 2). So in the worst case there exist always degree 2 Lefschetz pencils [10].

Let $(M, D, J) \hookrightarrow \mathbb{CP}^N$ be a projective CR variety. We define its dual set

$$M^* = \{H \in \mathbb{CP}^{N*} \mid D_x \subset H_x \text{ for some } x \in M\}.$$

By pulling back the usual diagram of flag varieties, it can be shown that there exists $\pi\colon S_D \to M$ a smooth (resp. C^{h-1} for C^h-embeddings of Levi-flat manifolds) fiber bundle of real dimension 2N-1, and a smooth (resp. C^{h-1}) map $\nu\colon S_D \to \mathbb{CP}^{N*}$ -the dual map- such that $M^* = \nu(S_D)$.

The difference with respect to the complex setting -where X^* is known to be a (singular) complex variety because it is the image of a complex manifold by a complex map- stems from the fact that S_D has only the structure of smooth manifold, and ν is just a smooth map. When $(M, \mathcal{F}, J)$ is Levi-flat the situation is slightly better, but still not good enough to do geometry with the dual variety.

Indeed, the fiber bundle S_D is a Levi-flat manifold and the dual map $\nu\colon S_D \to \mathbb{CP}^{N*}$ is a CR map. If $y \in S_D$ is a regular point for the leafwise dual map, then it is easy to check that it is a regular point for the full dual map. Hence, ν fails to be regular where the leafwise dual map is degenerate. That might lead to some control on the singular points S^* of M^*, which in turn might imply the existence of pencils of hyperplanes avoiding S^* and intersecting the regular part transversely, i.e. pencils fulfilling conditions (1) and (2) in definition 4. If $(M, \mathcal{F}, J)$ is a Levi-flat manifold of class at least C^2 it cannot have codimension one in $\mathbb{CP}^N$, $N \geq$ (or Lipschitz if $N \geq 3$) [8, 9]. Then inside of S_D there is a real codimension 2 sub-bundle S_0 corresponding to those hyperplanes, that as well as containing some D_x, also contain T_xM^J. Let $S_0^* := \nu(S_0)$. In order to construct degree 1 Lefschetz pencil structures as defined in 4 we would further need to ask the pencils fulfilling conditions (1) and (2), to miss S_0^*. The difficulty comes from the fact that it is not clear that S_0 is a Levi-flat manifold (it is not clear that the leaves of the obvious codimension one foliation are complex), so once more we cannot say much about its image S_0^*. In particular we cannot argue that the set of pencils not intersecting S_0^* is non-empty.

We can just use Sard's theorem to argue that M^* has measure zero and hence that hyperplane sections do exist. We recall that the topology of the hyperplane section can only be related to that of M when the Levi form has some degeneracy, in which case the results of Ni and Wolfson apply [27].

One might think heuristically of theorems 4 and 6 as a manifestation of the existence some sort of dual geometry for the re-embeddings provided by the k-th Veronese maps for k very large. That is, the corresponding dual sets M_k^* are in "most of its points" close to be stratified varieties; similarly whenever M has real codimension bigger than one, and for k $\gg 1$, the image of $S_{0,k}$ should be thought of being close to be a stratified variety with complex strata of complex codimension at least two. Therefore, a generic pencil of hyperplanes should be able to avoid both the closed strata of M_k^* and the image of $S_{0,k}$. However, we were not able to come up with a proof of this fact and hence with a geometric proof of theorems 4 and 6.

In a similar vein the results about the topology of the smooth fiber of a Lefschetz pencil should not come as a surprise. They coincide with the results of [27] for Levi-flat manifolds. And one should bear in mind that approximately holomorphic theory is based on the study of the CR manifold at a very small scale, where it looks like a Levi-flat one.

As we just discussed dual geometry fails in general to produce Lefschetz pencils for projective CR manifolds, due to the lack of structure on the dual set M^*. In [24] we studied real hypersurfaces M of projective space for which the dual map $\nu\colon S_D \cong M \to \mathbb{CP}^{N*}$ is an immersion. It is proved that injectivity of the differential of ν at any $x \in M$ is equivalent to the real shape operator restricted to $D_x = T_xM \cap JT_xM$ being non-degenerate. We also showed that for such hypersurfaces a pencil of hyperplanes $\mathbb{L} \subset \mathbb{CP}^{N*}$ was a Lefschetz pencil as in definition 4 (for real hypersurfaces condition 3 is void), if and only if $\mathbb{L}$ was transverse to M^* (the later being a compact immersed hypersurface, transversality at each point is understood as transversality to all branches through the point). A dimension count and Sard's theorem easily gives the existence of complex lines transverse to M^*, therefore dual geometry provides Lefschetz pencils for these hypersurfaces.

A real hypersurface $M \hookrightarrow \mathbb{CP}^N$ for which the shape operator restricted to the CR distribution D is strictly positive at each point, is called an strictly $\mathbb{C}$-convex hypersurface. These hypersurfaces are embeddable, since they miss hyperplanes. In particular one can always find pencils of hyperplanes whose base misses M, and among them Lefschetz ones. The corresponding CR function ϕ does not have base points and it is 1-generic, so we can talk of a CR Morse function. For strictly convex real hypersurfaces (inside Euclidean space) pencils of real hyperplanes whose base misses the hypersurface provide the simplest Morse function, i.e. one with two critical points, a maximum and a minimum; in particular they give a homeomorphism from the hypersurface to the sphere. Our CR Morse functions on strictly $\mathbb{C}$-convex hypersurfaces are analogs of the former: the singular set Δ is just a copy of S^1, and it is mapped diffeomorphically by ϕ to the boundary of the image of ϕ, which is diffeomorphic to a closed disk (in the real case the image is a closed interval bounded by the image of the two critical points). It is also possible to reconstruct M from the CR Morse function to conclude that it is diffeomorphic to S^{2N-1} [24] (a result already known but just at the topological level [18]).

4. Approximately holomorphic geometry for projective CR manifolds

Let us fix some notation. Given $s_k\colon \mathbb{CP}^N \to \underline{\mathbb{C}}^{m+1} \otimes \mathcal{O}(k)$, the restriction of s_k to M will be denoted by τ_k, and its zero set is the base locus B. The projectivization of τ_k to $M \backslash B$ will be denoted by ϕ_k.

The holomorphic vector bundles $\underline{\mathbb{C}}^{m+1} \otimes \mathcal{O}(k)$ -which will be also denoted by E_k- carry a natural connection ∇_k coming from the flat one in $\underline{\mathbb{C}}^{m+1}$ and the connection in $\mathcal{O}(k)$ associated to the Fubini-Study form. We will use the same notation for the restriction of ∇_k to $E_{k|M}$ if there is no risk of confusion.

For a projective Levi-flat CR manifold of class C^h, we let $\mathcal{J}^r_{\mathrm{CR}}(M, \mathbb{CP}^m)$ denote the bundle of CR r-jets of CR maps to $\mathbb{CP}^m$. This is a bundle of class C^{h-r} and it inherits an obvious CR-structure. There is a leafwise Thom-Boardman stratification $\mathbb{P}\mathcal{T}^{\mathcal{F}}$ whose strata are Levi-flat CR submanifolds of class C^{h-r}. For any CR map ϕ to $\mathbb{CP}^m$ of class C^h, its CR r-jet prolongation $j^r_{\mathrm{CR}}\phi$ is a CR section of class C^{h-r}.

In order to prove points (1) and (2) of theorem 6, we need to find suitable sequences of sections s_k of E_k such that (1) the base locus of τ_k is a CR submanifold of the expected dimension, and (2) $j^r_{\mathrm{CR}}\phi$ is transverse along $\mathcal{F}$ to $\mathbb{P}\mathcal{T}^{\mathcal{F}}$. This two conditions -as we will see- can actually be stated in a more compact manner as the solution to a single transversality problem for τ_k. What we will do is breaking the problem into three parts:

(A) The aforementioned transversality problem can be linearized, i.e. the bundle $\mathcal{J}^r_{\mathrm{CR}}(M, \mathbb{CP}^m)$, the Thom-Boardman stratification and the notion of CR jet.
(B) The linearized problem has a CR solution τ_k -with projectivization ϕ_k- which will be provided by a suitable version of approximately holomorphic geometry.
(C) The CR solution of the linearized problem also solves the original problem.

We need to recall a number of notions and results from approximately holomorphic geometry. Let $s_k \in \Gamma(E_k)$. Using J the complex structure of $\mathbb{CP}^N$ we can write

$$\nabla s_k = \partial s_k + \bar{\partial} s_k,\ \partial s_k \in \Gamma(T^{*1,0}\mathbb{CP}^N \otimes E_k),\ \bar{\partial} s_k \in \Gamma(T^{*0,1}\mathbb{CP}^N \otimes E_k).$$

Similarly, given $\tau_k \in \Gamma(E_{k|M})$ the restriction of $\nabla \tau_k$ to D can be written

$$\nabla_D \tau_k = \partial \tau_k + \bar{\partial} \tau_k,\ \partial \tau_k \in \Gamma(D^{*1,0} \otimes E_{k|M}),\ \bar{\partial} \tau_k \in \Gamma(D^{*0,1} \otimes E_{k|M}).$$

Let g denote the Fubini-Study metric and let g_k denote the rescaled metric kg. We use the same notation for the restriction of these metrics to M.

Definition 5. *A sequence of sections s_k of E_k is approximately J-holomorphic (or approximately holomorphic or simply A.H.), if positive constants $(C_j)_{j\geq 0}$ exist such that for all $k \gg 1$*

$$|\nabla^j s_k|_{g_k} \leq C_j, \quad |\nabla^{j-1}\bar{\partial} s_k|_{g_k} \leq C_j k^{-1/2}.$$

If in an A.H. sequence the sections s_k are holomorphic we speak of a uniformly bounded sequence of holomorphic sections.

Similarly, a sequence of sections τ_k of $E_{k|M}$ is approximately J-holomorphic (or approximately holomorphic or simply A.H.), if positive constants $(C_j)_{j\geq 0}$ exist such that for all $k \gg 1$

$$|\nabla^j \tau_k|_{g_k} \leq C_j, \quad |\nabla^{j-1}\bar{\partial} \tau_k|_{g_k} \leq C_j k^{-1/2}.$$

If the sections in the sequence are CR we say that it is a uniformly bounded sequence of CR sections.

The previous definition can also be given requiring control on a finite number of covariant derivatives, so we have C^h-A.H. sequences of sections when inequalities hold for $j = 0, \ldots, h$.

4.1. **Linearization of the bundles of CR jets, the Thom-Boardman stratification, and the notion of CR r-jet.** In this subsection we address point (A) in our strategy by presenting a linearized version of r-genericity for a sequence of sections of $E_{k|M}$ which are not necessarily CR.

Over the CR manifold M we define the sequence of vector bundles of pseudo-holomorphic r-jets as

$$\mathcal{J}^r_D E_{k|M} := (\sum_{j=0}^{r} D^{*1,0} \odot \cdots^{(j)} \cdots \odot D^{*1,0}) \otimes E_{k|M}, \tag{1}$$

where $\odot$ denotes the symmetric product. These bundles carry a natural connection $\nabla_{k,r}$ and a metric (see subsection 5.2 in [22] for more details).

The pseudo-holomorphic r-jet of τ_k will be a section of $\mathcal{J}^r_D E_{k|M}$ defined by induction: let $j^{r-1}_D \tau_k \in \mathcal{J}^{r-1}_D E_{k|M}$ be the pseudo-holomorphic (r-1)-jet of τ_k. It has homogeneous components of degrees $0, 1, \ldots, r-1$. We will denote the homogeneous component of degree $j \in \{0, \ldots, r-1\}$ by $\partial^j_{\mathrm{sym}} \tau_k \in \Gamma((D^{*1,0})^{\odot j} \otimes E_{k|M})$. The connection $\nabla_{k,r-1}$ is actually a direct sum of connections defined on the direct summands $(D^{*1,0})^{\odot j} \otimes E_{k|M}$, $j = 0, \ldots, r-1$. For simplicity and if there is no risk of confusion, we will use the same notation for the restriction of $\nabla_{k,r-1}$ to each of the summands. The restriction of $\nabla_{k,r-1}\partial^{r-1}_{\mathrm{sym}}\tau_k$ to D defines a section

$$\nabla_{k,r-1,D}\partial^{r-1}_{\mathrm{sym}}\tau_k \in \Gamma(D^* \otimes (D^{*1,0})^{\odot r-1} \otimes E_{k|M}).$$

For each $x \in M$ it is a form on D with values in the complex vector space $(D^{*1,0})^{\odot r-1} \otimes E_{k|M}$. Therefore, we can consider its (1,0)-component

$$\partial\partial^{r-1}_{\mathrm{sym}}\tau_k \in \Gamma(D^{*1,0} \otimes (D^{*1,0})^{\odot r-1} \otimes E_{k|M}).$$

By applying the symmetrization map

$$\mathrm{sym}_j \colon (D^{*1,0})^{\otimes j} \to (D^{*1,0})^{\odot j}$$

we obtain $\partial^r_{\mathrm{sym}}\tau_k \in \Gamma((D^{*1,0})^{\odot r} \otimes E_{k|M})$.

Definition 6. *Let τ_k be a section of $(E_{k|M}, \nabla_k)$. The pseudo-holomorphic r-jet $j^r_D\tau_k$ is a section of the bundle $\mathcal{J}^r_D E_{k|M}$ defined out of the pseudo-holomorphic (r-1)-jet by the formula*

$$j^r_D\tau_k := (j^{r-1}_D\tau_k, \partial^r_{\mathrm{sym}}\tau_k).$$

Now we want to define the linearization of $\mathcal{J}^r_{\mathrm{CR}}(M,\mathbb{CP}^m)$ so that the "projectivization" of $j^r_D\tau_k$ -which is going to be the linearized r-jet of ϕ_k- is a section of it. We will do it by gluing pieces defined very much as in equation 1.

Let $Z^0,\dots,Z^m$ be the complex coordinates associated to the trivialization of $\underline{\mathbb{C}}^{m+1}$ and let $\pi\colon \mathbb{C}^{m+1}\backslash\{0\}\to\mathbb{CP}^m$ be the canonical projection. Consider the canonical affine charts

$$\begin{array}{rcl}\varphi_i^{-1}\colon U_i & \longrightarrow & \mathbb{C}^m \\ [Z_0:\cdots:Z_m] & \longmapsto & \left(\dfrac{Z^1}{Z^0},\dots,\dfrac{Z^{i-1}}{Z^0},\dfrac{Z^{i+1}}{Z^0},\dots,\dfrac{Z^m}{Z^0}\right).\end{array}$$

For each chart φ_i, $i=0,\dots,m$, we consider the bundle

$$\mathcal{J}^r_D(M,\mathbb{C}^m)_i := \left(\sum_{j=0}^r (D^{*1,0})^{\odot j}\right)\otimes\underline{\mathbb{C}}^m. \tag{2}$$

On each bundle $\mathcal{J}^r_D(M,\mathbb{C}^m)_i$ we have a notion of pseudo-holomorphic r-jet as given in definition 6, where we use instead of $E_{k|M}$ the trivial bundle $\underline{\mathbb{C}}^m$ with trivial connection and standard Hermitian metric associated to the frame $\xi_{i,1},\dots,\xi_{i,m}$ given by the above affine coordinates.

We gather a few key results concerning these vector bundles:

- By point (1) in proposition 6.1 in [22], the vector bundles $\mathcal{J}^r_D(M,\mathbb{C}^m)_i$ can be glued to define the fiber bundles $\mathcal{J}^r_D(M,\mathbb{CP}^m)$ of pseudo-holomorphic r-jets of maps from M to $\mathbb{CP}^m$. One can put global metrics coming from g_k in the base, though it is not necessary. The computations one needs to do are local so one can always assume that the sections belong to some $\mathcal{J}^r_D(M,\mathbb{C}^m)_i$, where we have metrics induced by g_k, the flat connection, and the standard Hermitian metric.
- According to point (2) in proposition 6.1 in [22], given $\phi_k\colon M\to\mathbb{CP}^m$ there exist a (unique) notion of pseudo-holomorphic r-jet extension

$$j^r_D\phi_k\colon M\to\mathcal{J}^r_D(M,\mathbb{CP}^m)$$

  which is compatible with the notion of pseudo-holomorphic r-jet of definition 6 for the sections $\varphi_i^{-1}\circ\phi_k\colon M\to\mathbb{C}^m$. This is our linearized version of CR r-jet. Note that neither D nor J have to be integrable. If J is integrable the linearized notion of CR r-jet does not require ϕ_k to be a CR map.
- Define $\mathcal{J}^r_D E^*_{k|M} := \mathcal{J}^r_D E_{k|M}\backslash Z_k$, where Z_k denotes the sequence of strata of $\mathcal{J}^r_D E_{k|M}$ of r-jets whose degree 0 component vanishes. Then according to proposition 6.2 in [22], $\pi\colon\mathbb{C}^{m+1}\backslash\{0\}\to\mathbb{CP}^m$ induces bundle maps $j^r\pi\colon \mathcal{J}^r_D E^*_{k|M}\to\mathcal{J}^r_D(M,\mathbb{CP}^m)$ such that for any section τ_k of $E_{k|M}$, in the points where it does not vanish and its projectivization ϕ_k is defined the following relation holds:

$$j^r\pi(j^r_D\tau_k) = j^r_D\phi_k. \tag{3}$$

We will move onto explaining why the sequence of bundles $\mathcal{J}^r_D(M,\mathbb{CP}^m)$ are the right linearization of the bundles $\mathcal{J}^r_{\mathrm{CR}}(M,\mathbb{CP}^m)$, and we will also introduce the analog of the Thom-Boardman stratification in the linearized setting.

We fix a family of so called approximately holomorphic charts $\varphi_{k,x}\colon(\mathbb{C}^n\times\mathbb{R},0)\to(M,x)$ (definition 3.1 in [22]; see also section 3 in [22] for more details on their construction). This is a notion which only uses that D is a codimension one distribution and J an almost complex structure on it. We also demand the charts to be given by CR maps as well. It is easy to see that this is always possible since our coordinates can be chosen to be restriction of holomorphic coordinates in the ambient projective space. Note that in the Levi-flat case D_h -the canonical foliation of $\mathbb{C}^n \times \mathbb{R}$ by complex hyperplanes- is sent to $\mathcal{F}$.

We treat the Levi-flat case in more detail since it is the one for which our main theorem applies: for each point $x \in M$, k $\gg 1$, and associated to the CR coordinates $z_k^1, \ldots, z_k^n, s_k$ defined by $\varphi_{k,x}$ over the ball $B_{g_k}(x, \rho)$, $\rho > 0$ independent of k, x, we have the local bundles $\mathcal{J}^r_{D_h,n,m}$ of CR r-jets with a canonical bundle map

$$\Psi^{\mathrm{lin}}_{k,x,i} \colon \mathcal{J}^r_D(M, \mathbb{C}^m)_i \to \mathcal{J}^r_{D_h,n,m} \tag{4}$$

obtained as follows: the basis $dz_k^1, \ldots, dz_k^n \in \Gamma(D^{*1,0})$ identifies $D^{*1,0}$ with $T^{*1,0}\mathbb{C}^n$; let I be an (N+2)-tuple $I = (i_0, i_1, \ldots, i_n, i)$, $1 \leq i_0 \leq m$, $0 \leq i_j \leq r$, $i = 0, \ldots, m$, $i_1 + \cdots + i_n = r$. The frame

$$\mu_{k,x,I} := dz_k^{1 \odot i_1} \odot \cdots \odot dz_k^{n \odot i_n} \otimes \xi_{i,i_0} \tag{5}$$

defines the bundle map of equation 4.

The local bundles $\mathcal{J}^r_{D_h,n,m}$ glue into the non-linear bundle $\mathcal{J}^r_{\mathrm{CR}}(M, \mathbb{C}^m)_i$: let $y \in M$ be a point belonging to two different charts centered at x_0 and x_1 respectively. If we send y in both charts to the origin via a translation, then the change of coordinates restricts to the leaf through the origin to a bi-holomorphic map fixing the origin. The fibers over y are related by the action of the holomorphic r-jet of the bi-holomorphism. If we only take the linear part of the action, there is an induced vector bundle map

$$\Psi^{\mathrm{lin}}_{k,x_0,x_1,i} \colon \mathcal{J}^r_{D_h,n,m} \to \mathcal{J}^r_{D_h,n,m} \tag{6}$$

which defines a vector bundle, for the cocycle condition still holds. This vector bundle is $\mathcal{J}^r_D(M, \mathbb{C}^m)_i$ as defined in equation 2 (it is rather a sequence of bundles in which the metric in the $D^{*1,0}$ factors is induced from g_k). Thus for Levi-flat CR manifolds the vector bundles $\mathcal{J}^r_D(M, \mathbb{C}^m)_i$ are "linear approximations" for k $\gg 1$ of the non-linear bundles $\mathcal{J}^r_{\mathrm{CR}}(M, \mathbb{C}^m)_i$.

Notice that to make sense of equations 4, 5, and 6 one does not quite need (M, D, J) to be a CR manifold, just D a codimension one distribution endowed with an almost complex structure J is enough.

The third piece of data to be linearized is the CR Thom-Boardman stratification. We have seen that the vector bundles $\mathcal{J}^r_D(M, \mathbb{C}^m)_i$ and the fiber bundles $\mathcal{J}^r_{\mathrm{CR}}(M, \mathbb{C}^m)_i$ use the same building blocks $\mathcal{J}^r_{D_h,n,m}$, but different transition maps. We will apply the same idea for the stratifications. Each local bundle $\mathcal{J}^r_{D_h,n,m}$ carries a corresponding CR Thom-Boardman stratification $\mathcal{T}^{\mathcal{F}}_{n,m}$ (or rather a refinement which is a Whitney (A) stratification [25], which is invariant by r-jet extensions of bi-holomorphic transformations). The CR Thom-Boardman stratification $\mathbb{P}\mathcal{T}^{\mathcal{F}}_i$ of $\mathcal{J}^r_{\mathrm{CR}}(M, \mathbb{C}^m)_i$ is the result of gluing the local stratifications $\mathcal{T}^{\mathcal{F}}_{n,m}$, for as mentioned these are preserved by the transition maps; the stratifications $\mathbb{P}\mathcal{T}^{\mathcal{F}}_i$ in turn are used to build $\mathbb{P}\mathcal{T}^{\mathcal{F}}$ the Thom-Boardman stratification of $\mathcal{J}^r_{\mathrm{CR}}(M, \mathbb{CP}^m)$. The relevant observation is that $\mathcal{T}^{\mathcal{F}}_{n,m}$ is also preserved by $\Psi^{\mathrm{lin}}_{k,x_0,x_1,i}$, thus giving rise to the Thom-Boardman-Auroux stratifications $\mathbb{P}\mathcal{T}_{k,i}$ of $\mathcal{J}^r_D(M, \mathbb{C}^m)_i$; these stratifications in turn are also compatible with the gluing that defines $\mathcal{J}^r_D(M, \mathbb{CP}^m)$ giving rise to the Thom-Boardman-Auroux stratifications $\mathbb{P}\mathcal{T}_k$ ([22], definition 6.4).

For general projective CR manifolds we just work with CR 1-jets. The linearization process does not change neither the bundle nor the 1-jet. The only thing to bear in mind is that we work with sequences of bundles k $\gg 1$, for which the total space of the bundle is the same but the metric changes (or rather the metric defined in the vector bundles associated to the canonical affine charts of projective space).

We say that a sequence of A.H. sections τ_k of $E_{k|M}$ is r-generic if the following conditions hold:

(1') τ_k is uniformly transverse along $\mathcal{F}$ to Z_k.
(2') $j^r_D\phi_k$ is uniformly transverse along $\mathcal{F}$ to $\mathbb{P}\mathcal{T}_k$.

We refer the reader to [22], section 4, for the notion of uniform transversality along distributions to stratifications, but we just note that it uses a quantification of transversality, and requires its independence on k, for k $\gg 1$.

4.2. **Existence of CR solutions to the linearized problem.** The existence of r-generic sequences of A.H. sections of $E_{k|M}$ is the content of theorem 8.4 and proposition 6.3 in [22]:

Theorem 7. *Fix any $\delta > 0$ and $r, h \in \mathbb{N}$, $h - r \geq 2$. Then a constant $\eta > 0$ and a natural number k_0 exist such that for any C^h-A.H. sequence σ_k of E_k it is possible to find a C^h-A.H. sequence s_k of E_k, so that for any k bigger than k_0*

- $|\nabla^j(s_k - \sigma_k)|_{g_k} < \delta, j = 0, \ldots, h.$
- *Let τ_k denote the restriction of s_k to M and ϕ_k its projectivization. Then τ_k is η-transverse along $\mathcal{F}$ to Z_k and $j^r_D\phi_k$ is η-transverse along $\mathcal{F}$ to $\mathbb{P}\mathcal{T}_k$.*

Our aim in this subsection is to make sure that the solution s_k in theorem 7 can be chosen to be a uniformly bounded sequence of holomorphic sections of E_k. For that we need to briefly recall how theorem 7 is proved: the Thom-Boardman-Auroux stratification $\mathbb{P}\mathcal{T}_k$ is pulled back to a stratification $j^r\pi^*\mathbb{P}\mathcal{T}_k \cup Z_k$ of $\mathcal{J}^r_D E_{k|M}$. Theorem 7.2 in [22] grants the existence of a small enough perturbation s_k of σ_k, so that $j^r_D\tau_k$ is uniformly transverse to $j^r\pi^*\mathbb{P}\mathcal{T}_k \cup Z_k$. This encompasses both conditions (1') and (2'): condition (1') is fulfilled by adding Z_k to the stratification of $\mathcal{J}^r_D E_{k|M}$. That $j^r_D\phi_k$ is uniformly transverse to $\mathbb{P}\mathcal{T}_k$ is mostly a consequence of equation 3. Then proposition 6.3 in [22] implies that because the sections τ_k are A. H., uniform transversality gives as well the seemingly stronger uniform transversality along $\mathcal{F}$. Summarizing, the uniform transversality problem for B_k and maps $\phi_k \colon M\backslash B_k \to \mathbb{CP}^m$ is reduced to a uniform transversality problem for sections $\tau_k \colon M \to E_{k|M}$, whose solution is given as the restriction of sections $s_k \colon \mathbb{CP}^N \to E_k$. Thus we need to make sure that theorem 7.2 in [22] can produce a solution s_k which is a uniformly bounded sequence of holomorphic sections of E_k.

Theorem 7.2 in [22] gives in general a solution of a uniform transversality problem in $\mathcal{J}^r_D E_{k|M} \to M$ by applying relative approximately holomorphic theory as follows: in the complex manifold $\mathcal{N}_\epsilon(M) \hookrightarrow \mathbb{CP}^N$ one defines as in equation 1 the bundle of pseudo-holomorphic r-jets

$$\mathcal{J}^r E_k := (\sum_{j=0}^{r} T^{*1,0}\mathcal{N}_\epsilon(M) \odot \cdots^{(j)} \cdots \odot T^{*1,0}\mathcal{N}_\epsilon(M)) \otimes E_k,$$

and the pseudo-holomorphic r-jet $j^r\sigma_k$ of a section $\sigma_k \colon \mathcal{N}_\epsilon(M) \to E_k$. This is done exactly as we did for M but replacing the almost complex distribution (D, J) by the almost complex distribution $T\mathcal{N}_\epsilon(M)$. It is also necessary to "thicken" $j^r\pi^*\mathbb{P}\mathcal{T}_k \cup Z_k$ to an appropriate stratification $\mathcal{T}_k$ of $\mathcal{J}^r E_k$. Approximately holomorphic theory then produces for any given A.H. sequence σ_k of E_k an arbitrary small perturbation s_k so that $j^r s_k$ is uniformly transverse along M to $\mathcal{T}_k$, and one checks that this implies that the restriction τ_k to M is uniformly transverse to $j^r\pi^*\mathbb{P}\mathcal{T}_k \cup Z_k$. In other words, relative transversality theory associates to a suitable uniform transversality problem in the CR manifold (M, D, J), a uniform transversality problem in the complex manifold $\mathcal{N}_\epsilon(M)$ so that a solution to the latter restricts to a solution to the former.

So everything is reduced to show that in a complex manifold, when one starts with a uniformly bounded sequence of holomorphic functions σ_k -for example the sequence identically zero- the perturbations produced by approximately holomorphic theory to solve admissible uniform transversality problems can be chosen to be holomorphic. But this is essentially the content of [12], section 7 (a result which is already present in [30]): if $s^{\mathrm{ref}}_{k,x,i_0}$, $0 \le i_0 \le m$, $x \in M$, k $\gg 1$, is an appropriate family of so called reference frames of $E_k = \underline{\mathbb{C}}^{m+1} \otimes \mathcal{O}(k)$ (see section 2.3 in [3]), then their L^2-projection onto the holomorphic sections defines a family of holomorphic reference frames. Exactly the same ideas show that for $I = (i_1, \dots, i_N, i)$, $0 \le i \le m$, $0 \le i_j \le r$, $i_1 + \cdots + i_N = r$, the L^2-projection of

$$\nu^{\mathrm{ref}}_{k,x,I} := (z^1_k)^{i_1} \cdots (z^N_k)^{i_N} s^{\mathrm{ref}}_{k,x,i}$$

are holomorphic sections whose pseudo-holomorphic r-jets define reference frames of $\mathcal{J}^r E_k$. The perturbation provided by approximately holomorphic theory can be arranged to be a finite complex linear combination of the holomorphic part of the $\nu^{\mathrm{ref}}_{k,x,I}$ (see for example the original construction in [12], section 3), therefore it is also holomorphic. Thus, we can always find a uniformly bounded sequence of holomorphic sections of E_k whose restriction to M is r-generic, and this proves part (B).

4.3. **Comparison between pseudo-holomorphic jets and CR jets.** Let s_k be a uniformly bounded sequence of holomorphic sections of $\underline{\mathbb{C}}^{m+1} \otimes \mathcal{O}(k) \to \mathbb{CP}^N$ provided by theorem 7 (for example a perturbation of the trivial sequence $\sigma_k = 0$). We want to check part (C) of our strategy.

The restriction of s_k to M is a sequence τ_k of CR sections. Because τ_k is transverse to Z_k along $\mathcal{F}$ for k $\gg 1$, its zero set B_k is a CR submanifold of the expected dimension, and this proves point (1) in theorem 6.

Let $\phi_k \colon M \backslash B_k \to \mathbb{CP}^m$ be the corresponding sequence of CR maps. By hypothesis $j^r_D \phi_k \in \Gamma(\mathcal{J}^r_D(M\backslash B_k, \mathbb{CP}^m))$ is uniformly transverse along $\mathcal{F}$ to $\mathbb{P}\mathcal{T}_k$ for k $\gg 1$. The CR r-jet $j^r_{\mathrm{CR}}\phi_k$ is a section of $\mathcal{J}^r_{\mathrm{CR}}(M\backslash B_k, \mathbb{CP}^m)$ (the same bundle for all k, apart from the submanifold of base points). The key observation is again that for k $\gg 1$ the pairs $(\mathcal{J}^r_D(M, \mathbb{CP}^m), \mathbb{P}\mathcal{T}_k)$ and $(\mathcal{J}^r_{\mathrm{CR}}(M, \mathbb{CP}^m), \mathbb{P}\mathcal{T}^{\mathcal{F}})$ are constructed using the same building blocks $(\mathcal{J}^r_{D_h,n,m}, \mathcal{T}^{\mathcal{F}}_{n,m})$, but with different transition functions.

For each $x \in M\backslash B_k$, fix i such that $j^r_D\phi_k \colon B_{g_k}(x,\rho)\backslash B_k \to \mathcal{J}^r_D(M\backslash B_k, \mathbb{C}^m)_i$. Over $B_{g_k}(x,\rho)$ we use the bundle map of equation 4 to see $j^r_D\phi_k$ as a section of $\mathcal{J}^r_{D_h,n,m}$. Similarly, we can see $j^r_{\mathrm{CR}}\phi_k$ over $B_{g_k}(x,\rho)\backslash B_k$ as a section of $\mathcal{J}^r_{D_h,n,m}$.

The checking we have to make requires different ideas depending on whether we stay uniformly away from B_k -i.e. at some small fixed distance independently of k- or not.

Because ϕ_k is a projectivization if we stay uniformly away from B_k then ϕ_k is a sequence of uniformly bounded CR sections (for either g_k or the local standard metric g_0 in $B_{g_k}(x,\rho)\backslash B_k$, the flat connection and the standard Hermitian metric). In such a situation on easily checks

$$|j^r_D\phi_k - j^r_{\mathrm{CR}}\phi_k|_{C^1, g_k} \le O(k^{-1/2}), \tag{7}$$

for all k $\gg 1$.

Uniform transversality along $\mathcal{F}$ is an open condition (see [22], section 7), meaning that if $j^r_D\phi_k$ is δ-transverse along $\mathcal{F}$ to $\mathbb{P}\mathcal{T}_k$ and $|j^r_D\phi_k - \xi_k|_{C^1} \ll \delta$, then ξ_k is $\delta/2$-transverse along $\mathcal{F}$ to $\mathbb{P}\mathcal{T}_k$. But this is exactly what equation 7 grants in the points uniformly away from B_k. The conclusion is that on each $B_{g_k}(x,\rho)\backslash U_{k,x,\epsilon}$, where $U_{k,x,\epsilon}$ is the neighborhood of $B_k \cap B_{g_k}(x,\rho)$ of uniform radius ϵ independent of x, k, the local section $j^r_{\mathrm{CR}}\phi_k \in \Gamma(\mathcal{J}^r_{D_h,n,m})$ is uniformly transverse to the linearized Thom-Boardman stratification $\mathcal{T}^{\mathcal{F}}_{n,m}$, which is also the CR Thom-Boardman stratification.

As for the points in $U_{k,x,\epsilon}$, due to the definition of $j^r\pi^*\mathbb{P}\mathcal{T}_k \cup Z_k$ the graph of $j^r_D\tau_k$ stays uniformly away from the points in Z_k to which the other strata of $j^r\pi^*\mathbb{P}\mathcal{T}_k \cup Z_k$ converge, and this is something that just have to do with the 1-jet prolongation $j^1_D\tau_k$. This implies that in $U_{k,x,\epsilon}$ $j^1_D\phi_k$ stays uniformly away from all the strata of $\mathbb{P}\mathcal{T}_k$ of strictly positive codimension ([22], remark 6.2). By the construction of the Thom-Boardman-Auroux stratification any section of $\mathcal{J}^r_D(M, \mathbb{CP}^m)$ whose degree 1 part coincides with $j^1_D\phi_k$ will also stay uniformly away from these strata. Thus inside $U_{k,x,\epsilon}$ $j^r_{\mathrm{CR}}\phi_k \in \Gamma(\mathcal{J}^r_{D_h,n,m})$ -whose degree 1 part is $j^1_D\phi_k \in \Gamma(\mathcal{J}^1_{D_h,n,m})$- will also stay uniformly away from the strata of strictly positive codimension. Therefore by its very definition uniform transversality for $j^r_{\mathrm{CR}}\phi_k$ to $\mathbb{P}\mathcal{T}^{\mathcal{F}}$ holds as well in $U_{k,x,\epsilon}$, and this proves part (C).

4.4. **Proof of theorem 6.** Given $h \in \mathbb{N}$ we apply theorem 3 to embed (M, D, J) as a C^h-Levi-flat submanifold of some projective space. Then the previous three subsections imply that we can always construct s_k uniformly bounded sequences of holomorphic sections of $\underline{\mathbb{C}}^{m+1} \otimes \mathcal{O}(k)$, such that (1) the set of base points B_k of the restriction to M is a CR submanifold of codimension 2m+2 and class C^h, and (2) the projectivization of the restriction $\phi_k \colon M \backslash B_k \to \mathbb{CP}^m$ is a r-generic CR map.

To prove point (3) in the statement of the theorem 6 we have to solve yet another uniform transversality problem: one considers the bundles $\mathcal{J}^1_{TM^J}(\mathcal{N}_\epsilon(M), \mathbb{CP}^m)$ of pseudo-holomorphic 1-jets along TM^J for maps to $\mathbb{CP}^m$, defined in section 1. We can equally define them using affine coordinates of $\mathbb{CP}^m$ and then gluing the vector bundles $\mathcal{J}^1_{TM^J}(\mathcal{N}_\epsilon(M), \mathbb{C}^m)_i$. These are sequences of vector bundles with metric depending on k. Inside we have the submanifolds $\Sigma_{r,k}$ of 1-jets along TM^J whose degree 1 homogeneous component has rank $r \in \{0, \ldots, m\}$. It is possible to pull them back to strata $\Sigma^J_{r,k}$ of the bundles $\mathcal{J}^1 E_k \to \mathcal{N}_\epsilon(M)$ of pseudo-holomorphic 1-jets of E_k.

The stratification $Z_k \cup \Sigma^J_{r,k}$ of $\mathcal{J}^1 E_k$ is such that there is a version of theorem 7 asserting that for any $\delta' > 0$ there exist $\eta' > 0$ and s'_k a uniformly bounded sequence of holomorphic section of E_k, such that for all k $\gg 1$

- $|\nabla^j(s_k - s'_k)|_{g_k} < \delta', j = 0, \ldots, h$
- $j^1_{TM^j}\Phi'_k$ is η'-transverse along M to $\Sigma_{r,k}$, where Φ'_k is the projectivization of s'_k.

By choosing δ'' small enough the sequence s'_k will be such that ϕ'_k will still have properties (1) and (2) at the beginning of this subsection, and this proves theorem 6.

4.5. **Existence of Lefschetz pencil structures and proofs of theorems 4 and 5.** In the Lefschetz pencil condition we ask the holomorphic sections s_k to have components with zero locus intersecting in $\tilde{B}_k$ a real codimension 4 holomorphic submanifold, which intersects M transversaly in B_k. This is yet another transversality problem for 0-jets. Namely, one asks that each summand of s_k intersects the zero section of the corresponding summand of E_k transversaly, that s_k intersects the zero section of E_k transversaly, and that that s_k intersects the zero section of E_k transversaly along M.

Property (2) it attained with a small perturbation following mostly from the proof of theorem 6. Indeed, as we mentioned at the end of subsection 4.1, for 1-jets the linearized version of the CR jets coincides with the CR 1-jet as introduced in definition 3. The linearized version of the bundles of CR 1-jets is the same bundle of CR 1-jets, but having in mind that in the sequence the only piece of data that varies is the metric.

Subsection 4.2 is an account in first place how to solve the r-genericity problem in the framework of approximately holomorphic geometry, and no assumption on the integrability of either D or J is needed. Here we just use the result granting the existence of sequences of 1-generic A.H. sections of E_k. Then there is a sketch of how to get solutions which are uniformly bounded sequences of holomorphic sections. The only necessary ingredient is the fact that the ambient symplectic manifold admits a compatible complex structure.

Subsection 4.3 explains how for a Levi-flat CR manifold, the pseudo-holomorphic r-jets and the CR r-jets are as close as desired as k grows. Since for 1-jets there is no difference between both concepts, we do not need to use it here and we can conclude the existence holomorphic sections whose restriction to our arbitrary projective CR manifold is 1-generic.

Point (3) in theorem 6 coincides with condition (3) in the definition of Lefschetz pencil, so we can proceed as we did in the proof of theorem 6 to get it (which again requires no integrability of D).

The part concerning the "Lefschetz hyperplane theorem" for homotopy (homology) groups of a hyperplane section is a standard result valid for Lefschetz pencil structures for 2-calibrated structures ([19], corollary 1.2), and this finishes the proof of theorem 4.

Back to Levi-flat CR manidfolds, proposition 1 is also elementary: coordinates about B_k are again constructed as for Lefschetz pencil structures for 2-calibrated structures ([20], theorem 1.2), but we obtain CR coordinates because the two components of $\tau_k \in \Gamma(\underline{\mathbb{C}^2} \otimes \mathcal{O}(k)_{|M})$ are already CR; CR Morse coordinates about Δ_k are obtained by applying the complex Morse lemma with parameters.

Theorem 5 is proved in [23] for general 2-calibrated foliations and for Lefschetz pencils so that the normal forms of proposition 1 hold and the curve $\phi_k(\Delta)$ is an immersed curve with generic self-intersections. One observes that the proof easily generalizes for $\phi_k(\Delta)$ an immersed curve, which is what theorem 4 grants.

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