# NON-LINEAR SYMPLECTIC GRASSMANNIANS AND HAMILTONIAN ACTIONS IN PREQUANTUM LINE BUNDLES

DAVID MARTÍNEZ TORRES

ABSTRACT. In this note we extend to the Fréchet setting the following well known fact about finite dimensional symplectic geometry: if a Lie group $G$ acts on a symplectic manifold in a Hamiltonian fashion with momentum map $\mu$, given $x \in M$ the isotropy group $G_x$ acts linearly on the tangent space in a Hamiltonian fashion, with momentum map the Taylor expansion of $\mu$ up to degree 2. We use this result to give a conceptual explanation for a formula of S. Donaldson in [6], which describes the momentum map of the Hamiltonian infinitesimal action of the Lie algebra of the group of Hamiltonian diffeomorphisms of a closed integral symplectic manifold, on sections of its prequantum line bundle.

## 1. Introduction

Among the reasons to do geometry in infinite dimensions, a prominent one is that one may relate difficult problems for some geometry in finite dimensions -typically the solving of some hard P.D.E.- to elementary constructions in the infinite dimensional setting. It is often the case that this approach gives conceptual explanations for the appearance of certain equations, thus providing new insights to solve them. A salient example of this philosophy is the program initiated by S. Donaldson [5] to attack fundamental problems in Kähler geometry by linking them with constructions in Riemannian and symplectic geometry in infinite dimensional spaces. Namely, the space of Kähler metrics in a given Kähler class on a compact complex manifold, is endowed with an symmetric space structure with respect to the group of symplectomorphisms of a fixed Kähler representative. Existence of regular enough geodesics -an elementary problem in finite dimension- is intimately linked to the very deep problem of existence and uniqueness of constant scalar curvature Kähler metrics. Similarly, special metrics -which include constant scalar curvature metrics- are the zero level of certain momentum map for the action of the group of symplectomorphisms in the space of compatible complex structures.

Another capital example of how symplectic and Riemannian geometry in infinite dimensions is behind some P.D.E.s, is Arnold's description of Euler's equations for a homogeneous incompressible fluid as a geodesic spray in the group of volume preserving transformations [2] (see also [7]). More generally, some important P.D.E.s in fluid dynamics can be formulated via Lie-Poisson reduction [14, 8], and can be seen to admit a dual pair structure [15, 12, 9].

Symplectic geometry in infinite dimensions also appears in a natural way when trying to endow certain spaces of equivalence classes with symplectic structures. Perhaps the famous example is Atiyah and Bott's construction of a symplectic structure on the space of equivalence classes of flat connections on appropriate bundles over Riemann surfaces [1]. A more recent application in this direction is the construction of symplectic structures on canonical integrations of integrable Poisson structures [4].

---

2010 *Mathematics Subject Classification.* 58B20 (53D20).
Partially supported by the Fundação para a Ciência e a Tecnologia (FCT / Portugal).

Infinite dimensional symplectic structures also arise when studying non-linear Grassmannians of distinguished submanifolds of a symplectic manifold, such as (weighted) Lagrangian [19], (weighted) coisotropic [3] and symplectic [11].

In light of the previous discussion, it becomes very relevant to extend results from finite dimensional symplectic geometry to infinite dimensional symplectic manifolds, and to find natural appearances of such constructions. In this respect it should be noted that infinite dimensional constructions are not always possible, and in trying to carry them over an important step is making the right choice among the possible settings such as formal, Banach, Sobolev, Fréchet and convenient spaces.

In this note we start by describing a rather elementary result for Hamiltonian actions in finite dimensional symplectic manifolds which extends to the Fréchet setting. We use this observation to give a conceptual explanation for a formula in [6], which describes the momentum map of the Hamiltonian infinitesimal action of the Lie algebra of the group of Hamiltonian diffeomorphisms of a closed integral symplectic manifold, on the sections of its prequantum line bundle.

More precisely, let $(M,\omega)$ be a (finite dimensional) symplectic manifold endowed with a Hamiltonian action of a Lie group $G$ with (equivariant) momentum map $\mu\colon M\to\mathfrak{g}^*$. Given $x\in M$, it is well known that the isotropy group $G_x$ acts linearly on $(T_xM,\omega_x)$ in a Hamiltonian fashion, with momentum map the Taylor expansion of $\mu$ at $0\in T_xM$ up to degree 2.

Our main observation is this paper is that the same fact holds in the Fréchet setting (see section 3 for the precise definitions):

**Theorem 1.** *Let $(\mathcal{M},\Omega)$ be a Fréchet manifold endowed with a weakly symplectic structure. Let $\mathcal{G}$ be a regular Fréchet-Lie group acting on $(\mathcal{M},\Omega)$ in a Hamiltonian fashion with momentum map*

$$\mu_\xi\colon \mathcal{M}\to\mathbb{R},\ \xi\in\mathrm{Lie}(\mathcal{G}).$$

*Let $x$ be a point in $\mathcal{M}$ and let $\mathcal{G}_x$ be its isotropy group. Then the linear action of $\mathcal{G}_x$ on $(T_x\mathcal{M},\Omega_x)$ is Hamiltonian with momentum map $\mu_\xi^{(2)}$, the terms up to degree 2 of the expansion of $\mu_\xi$ at $0\in T_x\mathcal{M}$. Moreover, the homogeneous degree 1 part of the expansion vanishes so one can write*

$$\mu_\xi^{(2)}(z)=\mu_\xi(0)+\lim_{t\to 0}\frac{\mu_\xi(tz)-\mu_\xi(0)}{t^2}. \tag{1}$$

In [11] it is shown that for any given closed symplectic manifold $(M,\omega)$, the non-linear Grassmannian of 2m-dimensional symplectic manifolds is a weakly symplectic Fréchet manifold, on which $\mathrm{Ham}(M,\omega)$ -the group of Hamiltonian diffeomorphisms of $(M,\omega)$- acts in a Hamiltonian fashion.

On the other hand given $(X^{2m},\sigma)$ a closed integral symplectic manifold, S. Donaldson in [6] shows that the group $\mathrm{Ham}(X,\sigma)$ acts infinitesimally in a Hamiltonian fashion on the vector space of sections of $(L,\nabla,|\cdot|)$, the k-th tensor power of the Hermitian prequantum line bundle of $(X,\sigma)$. The momentum map -which we reproduce below in 2 changing slightly the constants appearing in both summands- is

$$\mu_f(s)=\int_X f\left(-\frac{i}{2}\nabla s\wedge\nabla\bar{s}\wedge\frac{\sigma^{m-1}}{(m-1)!}+\frac{1}{2}k(m+1)|s|^2\frac{\sigma^m}{m!}\right), \tag{2}$$

where $s\in\Gamma(L)$ and $f$ is in the Lie algebra $\mathrm{ham}(X,\sigma)$.

A conceptual explanation for the formula in equation 2 can be given using theorem 1 together with the results of [11]: Briefly, given a Hermitian vector bundle over $(X^{2m},\sigma)$ one can use the symplectic coupling form construction ([16], section 6.4) to introduce a suitable symplectic structure in the projectivization of the bundle. The symplectic manifold $(X,\sigma)$ appears as a point in the corresponding non-linear symplectic Grassmannian $(\mathcal{M},\Omega)$. Theorem 1 provides a momentum map for the linearized action of the isotropy group of $x \in \mathcal{M}$. If the Hermitian bundle is a power the prequantum line bundle of $(X, \sigma)$, we construct a map from the Lie algebra of $\mathrm{Ham}(X, \sigma)$ into the Lie algebra of the isotropy group $\mathcal{G}_X$, that when composed with the infinitesimal linear action becomes a Lie algebra homomorphism. By applying theorem 1 it follows that the infinitesimal linear action of $\mathrm{Ham}(X, \sigma)$ in $(T_X\mathcal{M}, \Omega_X)$ is Hamiltonian, and a formula for the momentum map is obtained out of equation 1. Once the geometric data is plugged into the formula, it becomes exactly 2. Summarizing, one has:

**Theorem 2.** *Let $(X^{2m}, \sigma)$ be a closed integral symplectic manifold. Then the Hamiltonian infinitesimal action of* $\mathrm{Ham}(X, \sigma)$ *on sections of the k-th power of its prequantum line bundle described in* [6]*, is built out of the linearization at a fixed point of the Hamiltonian action in the non-linear Grassmannian as described in* [11]*. In particular the formula 2 for the momentum map can be obtained via this procedure.*

The structure of the paper is the following: in section 2 we recall the necessary results about the Hamiltonian linear action of the symplectic linear group $\mathrm{Sp}(2n)$ in $\mathbb{C}^n$ with its standard symplectic structure, and the linearization about a fixed point of Hamiltonian actions on (finite dimensional) manifolds. We explore the generalization of the previous results to the Fréchet setting in section 3; in particular theorem 1 is proved. In section 4 we specialize theorem 1 to the case of the non-linear Grassmannian of symplectic submanifolds. The description of the isotropy group is the content of subsection 4.1. Its linear action is described in subsection 4.2. The main result of this section -theorem 3- is the translation of the formula for the momentum map entirely in terms of geometric data. Finally, in subsection 4.4 we give the proof of theorem 2.

## 2. Induced linear Hamiltonian actions about fixed points.

Let us consider $\mathbb{C}^n$ with its standard Hermitian inner product $h_{\mathrm{st}}$. By definition $\omega_{\mathrm{st}} = -\mathrm{Im} h_{\mathrm{st}}$ is the standard symplectic form.

Recall that the symplectic linear group $\mathrm{Sp}(2n)$ acts on $(\mathbb{C}^n, \omega_{\mathrm{st}})$ in a Hamiltonian fashion: $A \in \mathrm{sp}(2n)$ is equivalent to $J_{\mathrm{st}}A$ being symmetric, and thus one has the quadratic momentum map

$$\begin{array}{rcl} \mu\colon \mathbb{C}^n & \longrightarrow & \mathrm{sp}(2n)^* \\ z & \longmapsto & \langle \mu(z), A\rangle = -\dfrac{1}{2} z^t J_{\mathrm{st}} A z, \end{array} \tag{3}$$

If we restrict the action to the unitary group, we obtain the corresponding $\mathfrak{u}(n)^*$-valued momentum map; using the Killing form $(A, B) \mapsto \mathrm{tr}(A^*B)$ to identify $\mathfrak{u}(n)$ and its dual, 3 gives the $\mathfrak{u}(n)$-valued momentum map

$$\begin{array}{rcl} \mu\colon \mathbb{C}^n & \longrightarrow & \mathfrak{u}(n) \\ z & \longmapsto & \dfrac{i}{2} z z^*. \end{array} \tag{4}$$

We state as a lemma the following well known result about induced linear Hamiltonian actions:

**Lemma 1.** *Let $(M, \omega)$ be a symplectic manifold. Let $G$ be a Lie group acting on $(M, \omega)$ in a Hamiltonian fashion, with momentum map $\mu\colon M \to \mathfrak{g}^*$. Let $x \in M$ and let $G_x$ denote the isotropy group of $x$. Then $G_x$ acts linearly on $(T_xM, \omega_x)$ in a Hamiltonian fashion with momentum map*

$$\mu^{(2)}\colon T_xM \to \mathfrak{g}_x^*, \tag{5}$$

*where $\mu^{(2)}$ are the terms up to degree 2 of the expansion of $\mu$ at $0 \in T_xM$. Moreover, the homogeneous degree 1 part of the expansion vanishes, so one can write*

$$\mu^{(2)}(z) = \mu(0) + \lim_{t\to 0} \frac{\mu(tz) - \mu(0)}{t^2}.$$

**Remark 1.** *The proof follows by a simple argument on Taylor expansions: one ends up with a symplectic linear action on a symplectic vector space, which as recalled at the beginning of this section is Hamiltonian with quadratic momentum map. Note that because the homomorphism $\Theta\colon G_x \to \mathrm{Sp}(2n)$ is not necessarily onto, the induced momentum map may have non vanishing constant term (*$\mathrm{Sp}(2n)$ *is simple and thus the momentum map in 3 is unique); in any case one could always remove the constant term and still obtain a momentum map.*

## 3. Fréchet manifolds and linear Hamiltonian actions.

Let $\mathcal{V}$ be an Fréchet (vector) space. A 2-form $\Omega$ on $\mathcal{V}$ is called weakly symplectic if for any $u \in \mathcal{V}$ non-zero, there exists $v \in \mathcal{V}$ such that $\Omega(u,v) \neq 0$. The pair $(\mathcal{V},\Omega)$ is called a weakly symplectic Fréchet space.

It is not clear how to extend the theory of symplectic vector spaces and Hamiltonian linear actions to Fréchet spaces. Firstly, because in an infinite dimensional Fréchet space weakly symplectic linear forms are not in principle equivalent, so there might be different symplectic linear groups. Secondly, even considering a Hermitian Fréchet space $(\mathcal{V},h)$ and its associated weakly symplectic Fréchet space $(\mathcal{V},-\mathrm{Im}h)$, unitary representations are Hamiltonian (with quadratic momentum map as in equation 4) under the hypothesis of the Fréchet Hermitian space being Hilbert ([13], section 49.11). Anyhow, we will see that an analog of lemma 1 exists.

Let $(\mathcal{M},\Omega)$ be a weakly symplectic Fréchet manifold. That is to say a Fréchet manifold $\mathcal{M}$ with a weakly symplectic closed two form (recall that Cartan calculus goes through to Fréchet manifolds by using Koszul formula as the definition of exterior derivative; see for example [13], section 33).

Let $\mathcal{G}$ be a regular Fréchet-Lie group. Regularity roughly means that each smooth path in the Lie algebra integrates into a path in the Lie group, with an appropriate smooth dependence ([13], definition 38.4). In particular there is an exponential map. A consequence which we will be using is that given a smooth action

$$\mathcal{G} \times \mathcal{M} \to \mathcal{M},$$

each vector $\xi \in \mathrm{Lie}(\mathcal{G})$ gives rise to a fundamental vector field $X_\xi \in \Gamma(T\mathcal{M})$ characterized by the usual equation

$$X_\xi(x) = \frac{d}{dt}\mathrm{e}^{t\xi} \cdot x_{|t=0}.$$

All the interesting Fréchet-Lie groups (diffeomorphisms of manifolds, gauge groups) are regular. As a matter of fact no example of non-regular Fréchet-Lie group is known [13]. We assume from now on that Fréchet-Lie groups are regular unless otherwise stated.

Let $(\mathcal{M},\Omega)$ be a weakly symplectic manifold acted upon by a Fréchet-Lie group $\mathcal{G}$. The action is symplectic if as usual for all $g \in \mathcal{G}$ one has

$$g^*\Omega = \Omega.$$

The action is Hamiltonian if for each $\xi \in \mathrm{Lie}(\mathcal{G})$ one has a function $\mu_\xi$ such that

$$d\mu_\xi = i_{X_\xi}\Omega, \tag{6}$$

and the following two conditions hold:

(1) (Linearity) The assignment $\xi \mapsto \mu_\xi$ is linear.

(2) (Equivariance) For all $g \in \mathcal{G}$, all $\xi \in \mathrm{Lie}(\mathcal{G})$ and all $x \in \mathcal{M}$

$$\mu_\xi(g \cdot x) = \mu_{\mathrm{Ad}_{g^{-1}} \cdot \xi}(x). \tag{7}$$

Now we have all the ingredients to extend lemma 1 to the Fréchet setting:

*Proof of theorem 1.* We start by remarking that we do not make any consideration about $\mathcal{G}_x$ being a regular Fréchet-Lie group (we do have fundamental vector fields for the action of $\mathcal{G}_x$, so we can make sense of the momentum map condition). In any case in our application in theorem 3 the isotropy groups will be seen to be regular Fréchet-Lie groups.

We work locally in a neighborhood $\mathcal{U}$ of 0 in $T_x\mathcal{M} = \mathcal{V}$. Equation 6 becomes an equality of functions defined in $\mathcal{U} \times \mathcal{V}$, and linear in the second factor. The equality still holds if we take the derivative at point $u \in \mathcal{U}$ in the direction of a vector $v \in \mathcal{V}$, and we need to analyze what we obtain on each side. The right hand side is the composition of the weakly symplectic form $\Omega \colon \mathcal{U} \times \mathcal{V} \times \mathcal{V}$ with the fundamental vector field $X_\xi \colon \mathcal{U} \to \mathcal{U} \times \mathcal{V}$. By applying the chain rule and the derivation law for functions which are linear in some variables (theorem 3.3.4 and 3.4.4. in [10]), one obtains the bilinear form

$$D i_{X_\xi} \Omega(0, u)(v) = \Omega(0)(X^{(1)}(v), u),$$

where $X^{(1)}(v)$ is the linearization of $X$ at 0 evaluated at $v$, which is the second component of the derivative $DX(0)(v)$.

Regarding the l.h.s. of equation 6, recall that since $X_\xi(0) = 0$ we deduce that $D\mu_\xi(0)(v) = 0$. The result of taking derivative of the l.h.s. of 6 at $u$ in the direction of $v$ is

$$D^2\mu_\xi(0)(v, u),$$

which is symmetric on $u, v$. Let $\Delta \colon \mathcal{V} \to \mathcal{V} \times \mathcal{V}$ be the diagonal map and consider

$$\frac{1}{2} D^2\mu_\xi(0)(z, z) := \frac{1}{2} D^2\mu_\xi(0) \circ \Delta(z)$$

the quadratic form associated to $D^2\mu_\xi(0)$. Because it is the composition of a linear and a bilinear map, linear algebra easily implies

$$d\left(\frac{1}{2} D^2\mu_\xi(0)\right)(u)(v) := D\left(\frac{1}{2} D^2\mu_\xi(0)\right)(u)(v) = D^2\mu_\xi(0)(u, v),$$

and thus

$$\mu_\xi(0) + \frac{1}{2} D^2\mu_\xi(0)(z, z),\ z \in \mathcal{V}$$

is a momentum map for the linear action of $\mathrm{e}^{t\xi}$.

As for a more precise formula for the momentum map, by theorem 3.5.6. in [10] due to the vanishing of the derivative at zero we have

$$\mu_\xi(tz) = \int_0^1 (1 - s) D^2\mu_\xi(stz)(tz, tz) ds,$$

this giving by bilinearity of the second derivative

$$\frac{\mu_\xi(tz)}{t^2} = \int_0^1 (1 - s) D^2\mu_\xi(stz)(tz, tz) ds. \tag{8}$$

If we take limits in 8, by continuity of the second derivative the limit in the r.h.s. commutes with the integral ([10], theorem 2.1.5.) and we have

$$\mathrm{lim}_{t \to 0} \frac{\mu_\xi(tz)}{t^2} = D^2\mu_\xi(0)(z, z) \int_0^1 (1 - s) ds = \frac{1}{2} D^2\mu_\xi(0)(z, z),$$

so

$$D^2\mu_\xi(0)(z, z) = \lim_{t \to 0} \frac{\mu_\xi(tz) - \mu_\xi(0)}{t^2},$$

and therefore equation 1 holds.

Because the assignment $\xi \mapsto \mu_\xi$ is linear, it easily follows that the assignment

$$\xi \mapsto \mu_\xi^{(2)}$$

is also linear.

As for equivariance, for any $g \in \mathcal{G}$ equation 7 holds by hypothesis. We just need to analyze what appears when we keep the terms up to degree 2 on both sides. The r.h.s. is

$$\mu^{(2)}_{\mathrm{Ad}_{g^{-1}}\cdot\xi}(z).$$

The l.h.s. is

$$\mu_\xi^{(2)}(g\cdot z) = \mu_\xi^{(2)}(0) + \frac{1}{2}D^2\mu_\xi^{(2)}(g\cdot z),$$

and we need to prove

$$D^2\mu_\xi^{(2)}(g\cdot z) = D^2\mu_\xi^{(2)}(Dg(0)\cdot z).$$

But this is a straightforward computation using theorem 3.5.5. in [10] together with the vanishing of $D\mu_\xi(0)$, and this proves the theorem. □

**Remark 2.** *Observe that we may as well take as momentum map the quadratic homogeneous part of the original momentum map (see remark 1).*

## 4. The non-linear symplectic Grassmannian and prequantum line bundles.

We want to apply theorem 1 to the non-linear symplectic Grassmannians, a family of Fréchet Hamiltonian manifolds associated to any closed symplectic manifold described in [11]. We recall their construction.

Let $(M^{2n},\omega)$ be a closed symplectic manifold. Fix any $m \in \{1,\dots,n-1\}$ and consider $\mathcal{M}$ the Fréchet manifold of symplectic submanifolds of $(M,\omega)$ of dimension 2m (and open subset of the Fréchet manifold of submanifolds of dimension 2m). This is a weakly symplectic Fréchet manifold: for any $S \in \mathcal{M}$ a vector in $T_S\mathcal{M}$ corresponds to a section $s$ of the normal bundle $\nu(S)$, which we identify with the symplectic orthogonal in $TM_{|S}$ to $TS$. The 2-form

$$\Omega_S(s_1,s_2) := \int_S \omega(s_1,s_2)\frac{\omega^m}{m!}$$

defines a weakly symplectic structure [11].

The regular Fréchet-Lie group of Hamiltonian diffeomorphisms $\mathrm{Ham}(M,\omega)$ acts on $\mathcal{M}$ in an obvious way. The action is Hamiltonian with momentum map

$$\mu_f(S) = \int_S f\frac{\omega^m}{m!}, \tag{9}$$

where $f \in \mathrm{ham}(M,\omega)$, i.e. $\int_M f\omega^n = 0$ (see also [5]).

Therefore by theorem 1 we have a Hamiltonian linear action

$$\Theta\colon \mathrm{Ham}(M,\omega)_S \to \mathrm{Sp}(T_S\mathcal{M},\Omega_S)$$

We would like to identify the isotropy subgroup $\mathrm{Ham}(M,\omega)_S$, its linearized action and its infinitesimal counterpart, and finally the momentum map formula in 1 in terms of geometric data.

4.1. **The isotropy group** $\mathrm{Ham}(M,\omega)_S$**.** Let $\mathrm{Diff}(M)_S$ be the group of diffeomorphisms of $M$ preserving $S$ setwise. Then the isotropy group at $S$ of the action of $\mathrm{Ham}(M,\omega)$ in $\mathcal{M}$ is

$$\mathrm{Ham}(M,\omega)_S = \mathrm{Ham}(M,\omega) \cap \mathrm{Diff}(M)_S.$$

Its main properties are the content of the following lemma.

**Lemma 2.** *The group* $\mathrm{Ham}(M,\omega)_S$ *of Hamiltonian diffeomorphisms of* $(M,\omega)$ *preserving a symplectic submanifold* $S$ *of* $(M,\omega)$ *setwise, is a Fréchet-Lie subgroup of* $\mathrm{Ham}(M,\omega)$*. Its Lie algebra can be identified with the closed Fréchet subspace of* $\mathrm{ham}(M,\omega)$ *of functions which vanish linearly along* $\nu(S)$*, or equivalently with Hamiltonian vector fields tangent to* $S$*. Moreover, it is a regular Fréchet-Lie group.*

*Proof.* That $\mathrm{Diff}(M)_S$ is a Fréchet-Lie subgroup of $\mathrm{Diff}(M)$ is a result of Ebin and Marsden [7]. Also $\mathrm{Ham}(M,\omega)$ is a regular Fréchet-Lie group [13]. We need to modify the proof of the latter result to prove our relative statement.

Recall that for the group of diffeomorphisms ([13], section 43) the construction of Fréchet charts requires a *local addition*

$$\alpha \colon U \subset TM \to M \times M,$$

which is a local diffeomorphism defined in a neighborhood of the zero section, which identifies the latter with the diagonal in $M \times M$. For the relative construction we fix a local addition such that it restricts to a local addition

$$\alpha_S \colon U \cap TS \to S \times S.$$

This is always possible using tubular neighborhood theorems for $S$ or using a metric for which $N$ is totally geodesic. A computation shows that for $\varphi \in \mathrm{Diff}(M)_S$ the chart constructed using the relative addition sends $\mathrm{Diff}(M)_S$ to the closed Fréchet subspace

$$\Gamma(TM)_S = \{X \in \Gamma(TM) \mid X_{|S} \in \Gamma(TS)\}.$$

Therefore $\mathrm{Diff}(M)_S$ is a Fréchet submanifold of $\mathrm{Diff}(M)$, and thus a Fréchet-Lie subgroup since the smooth structural maps for $\mathrm{Diff}(M)$ restrict to smooth structural maps for $\mathrm{Diff}(M)_S$ [7].

We want to include into the picture the symplectic structure. Recall that to find charts for the groups of Hamiltonian and symplectic diffeomorphisms, one further fixes a bundle isomorphism

$$\beta \colon T^*M \to TM$$

and corrects with a diffeomorphism

$$\gamma \colon V \subset T^*M \to W \subset T^*M$$

which is the identity in the zero section and such that

$$(\alpha \circ \beta \circ \gamma)^*(\omega \oplus -\omega) = d\lambda,$$

where $\lambda$ is the canonical 1-form. The universal covering space of $\mathrm{Ham}(M,\omega)$ is defined as a subgroup of the universal covering of $\mathrm{Diff}(M,\omega)$, the group of symplectic diffeomorphisms, by identifying it with the kernel of the flux homomorphism defined in the latter. The Fréchet spaces parameterizing points in $\widetilde{\mathrm{Diff}(M,\omega)}$ and $\widetilde{\mathrm{Ham}(M,\omega)}$ near the identity correspond to closed and exact 1-forms respectively, the latter identified with zero symplectic mean functions ([13], 43.12 and 43.13). We claim that such chart can be understood as a local chart near the identity for $\mathrm{Diff}(M,\omega)$ and $\mathrm{Ham}(M,\omega)$, and we will use it in this sense. The claim holds because the flux conjecture it is true [17]. That means that is the open subset is small enough, we will only have Hamiltonians given by exact 1-forms.

If we use the relative local addition described above, it follows that Hamiltonian diffeomorphisms preserving $S$ setwise correspond to exact 1-forms whose image by $(\beta\circ\gamma)^{-1}$ is tangent to $S$. The slight complication is that the latter condition is not linear on exact 1-forms and does not define a closed Fréchet subspace to be taken as the modeling one for our Fréchet submanifold. To correct that, we first assume that $\beta$ is given by the sharp diffeomorphism associated to the symplectic form $\omega$. Let $E$ be the subbundle of $T^*M_{|S}$ of 1-forms vanishing in the symplectic orthogonal $\nu(S)$. The sharp diffeomorphism sends vector fields tangent to $S$ to 1-forms whose restriction to $S$ belongs to $E$. As remarked above, this is a closed Fréchet subspace of $\Omega^1(M)$, but when composed with $\gamma$ is not linear anymore, since $\gamma(E)$ is not necessarily a vector bundle over $S$. In any case, near the zero section $E$ is a symplectic submanifold of $(T^*M, d\lambda)$ which contains $S$ as a Lagrangian submanifold. Moreover, $\gamma(E)$ is tangent to $E$ along $S$. Both $\gamma(E)$ and $E$ are symplectic submanifolds of $(T^*M, d\lambda)$ whose symplectic forms coincide along $S$. Due to the transitivity of the action of Hamiltonian diffeomorphisms on connected components of non-linear symplectic Grassmannians, it is not hard to find $\delta\colon (T^*M, d\lambda) \to (T^*M, d\lambda)$ a Hamiltonian diffeomorphism which fixes the zero section and sends $E$ to $\gamma(E)$.

For the relative construction we use the chart centered at the identity

$$\alpha\circ\beta\circ\gamma\circ\delta\colon Y\subset T^*M \to M\times M,$$

and the induced induced ones for any $\varphi\in \mathrm{Ham}(M,\omega)_S$. By construction, they send Hamiltonian diffeomorphism preserving $S$ setwise to exact 1-forms whose restriction to $S$ lies in $E$. Thus they are submanifold charts and define a Fréchet-Lie subgroup of $\mathrm{Ham}(M,\omega)$.

As for the description of the Lie algebra of $\mathrm{Ham}(M,\omega)_S$, one uses that the evolution operator for the group of diffeomorphisms

$$\mathrm{Evol}\colon C^\infty(\mathbb{R},\Gamma(TM)) \to C^\infty(\mathbb{R},\mathrm{Diff}(M))$$

is explicitly given by solving the O.D.E.

$$\frac{d}{dt}\varphi_t = X_t(\varphi_t).$$

Thus, paths in the Fréchet-Lie subgroup $\mathrm{Ham}(M,\omega)_S$ are identified with Hamiltonian vector fields tangent to $S$, which we canonically identify with zero symplectic mean functions with linear vanishing along $\nu(S)$.

Thus we have

$$\mathrm{ham}(M,\omega)_S = \{f\in C^*(M)\mid \int_M f\omega^n = 0,\, df_{|\nu(S)}=0\}.$$

Similarly, because the restriction of the evolution operator

$$\mathrm{Evol}\colon C^\infty(\mathbb{R},\Gamma(TM)) \to C^\infty(\mathbb{R},\mathrm{Diff}(M))$$

to $\mathrm{ham}(M,\omega)_S$ takes values in $\mathrm{Ham}(M,\omega)_S$, it is smooth and hence $\mathrm{Ham}(M,\omega)_S$ is a regular Fréchet-Lie group. □

4.2. **The linear action.** As for the linear action

$$\Theta\colon \mathrm{Ham}(M,\omega)_S \to (T_S\mathcal{M},\Omega_S),$$

given $\varphi\in\mathrm{Ham}(M,\omega)_S$, we let $\varphi^{(1)}$ be the induced bundle isomorphism on $\nu(S)$. Then one has

$$\Theta(\varphi)\cdot s = \varphi^{(1)}\circ s\circ\varphi^{-1}.$$

If $X\in\Gamma(TM)$ by [11] the induced vector field $\mathcal{X}\in\Gamma(T\mathcal{M})$ is

$$\mathcal{X}(N) = X_{|\nu(N)}.$$

If the vector field is tangent to $S$, in a chart given by sections of $\nu(S)$ for any $s \in \Gamma(\nu(S))$ one can take as common model of the normal bundle $\nu(S)$, and write

$$\mathcal{X}(s) = X_{|\nu(S)}(s(S)),$$

where the restriction to $\nu(S)$ at $s(S)$ uses in the splitting given by the tangent space to the graph of the section and $\nu(S)$.

Its linearization is

$$\mathcal{X}^{(1)}(s) = \frac{d}{dt} X_{|\nu(N)}(ts(S))_{|t=0}.$$

This implies that if we have two vector fields tangent to $S$ whose difference is at least quadratic at any point in $S$ (for example for any connection in $\nu(S)$ the difference has quadratic vertical and horizontal components), then they have the same linearization. In particular all functions $f \in \mathrm{ham}(M,\omega)_S$ vanishing at order at least three belong to the kernel of the Lie algebra homomorphism associated to the infinitesimal linear action

$$\theta \colon \mathrm{ham}(M,\omega)_S \to \mathrm{sp}(T_S\mathcal{M}, \Omega_S).$$

4.3. **The momentum map formula.** To write the formula for the momentum map in 1 in terms of geometric data we will use a suitable normal form for the symplectic form near $S$. We identify a neighborhood of $S$ with a neighborhood of the zero section in $\pi \colon \nu(S) \to S$. The latter is a symplectic bundle. We can reduce its structural group to the unitary one fixing a Hermitian metric $h$, and we also pick a Hermitian connection $\nabla$. According to [18], the total space of $\nu(S)$ carries a closed 2-form $\omega_c$ -the coupling form- which restricts to the fibers to the standard symplectic form associated to the fiberwise Hermitian metric $-\mathrm{Im}h$, and whose kernel is the horizontal distribution of $\nabla$. If we let $\omega_S$ denote the restriction of $\omega$ to $S$, then $\pi^*\omega_S + \omega_c$ is a closed to form which is symplectic near the zero section, and agrees with $\omega$ in the tangent space to points in the zero section. By the symplectic tubular neighborhood theorems, we can assume than $\omega$ matches $\pi^*\omega_S + \omega_c$ near the zero section. Since the computation of the Taylor expansion of the momentum map about $S \in \mathcal{M}$ only depends in arbitrarily small neighborhoods of $S$, we may assume without loss of generality that our symplectic manifold is $(\nu(S), \pi^*\omega_S + \omega_c)$.

We need a final piece of notation. Given $f \in \mathrm{ham}(M,\omega)_S$ we will look at its quadratic expansion along directions normal to $S$, and write it as a sum of two homogeneous terms

$$f^{(2)} = f_S + H^\nu f,$$

the restriction to $S$ and the Hessian along normal directions.

**Theorem 3.** *Let $(M,\omega)$ be a symplectic a closed symplectic manifold, $(\mathcal{M},\Omega)$ the Grassmannian of 2m-dimensional symplectic submanifolds, and $S$ a point in $\mathcal{M}$. Let us identify a neighborhood of $S$ in $(M,\omega)$ with a neighborhood of the zero section in $(\nu(S), \pi^*\omega_S + \omega_c)$. Then the momentum map formula in 1 for the linear Hamiltonian action of* $\mathrm{Ham}(M,\omega)_S$ *in $(T_S\mathcal{M}, \Omega_S)$ can be written*

$$\mu_f^{(2)}(s) = \int_S f^{(2)} \frac{\omega^m}{m!} - \frac{i}{2} f \left( (\nabla s \wedge \nabla \bar{s} - \mathrm{tr}(F_\nabla s s^*) \wedge \frac{\omega^{m-1}}{(m-1)!} \right), \qquad (10)$$

*where $f \in \mathrm{ham}(M,\omega)_S$, the complex conjugation and adjoint operators are defined after identifying the fibers of $\nu(S)$ with $\mathbb{C}^m$ via the Hermitian metric $h$, and $F_\nabla$ is the curvature of the connection induced by $\nabla$ in the corresponding principal unitary bundle,* *Proof.* Let us fix $f \in \mathrm{ham}(M,\omega)_S$. For any $s \in \Gamma(\nu(S))$ the first summand of the momentum map in 3 is always

$$\mu_f(S) = \int_S f\frac{\omega^m}{m!} = \int_S f_S\frac{\omega^m}{m!}. \tag{11}$$

Given $s \in \Gamma(\nu(S))$, we let $\phi_s\colon S \to s(S)$ denote the diffeomorphism given by the section itself. We can express the second summand in 3 as

$$\lim_{t\to 0}\frac{1}{t^2}\left(\int_{ts(M)} f\frac{\omega^m}{m!} - \int_S f\frac{\omega^m}{m!}\right) = \lim_{t\to 0}\frac{1}{t^2}\left(\int_S \phi_{ts}^*(f\frac{\omega^m}{m!}) - f\frac{\omega^m}{m!}\right). \tag{12}$$

In the Taylor expansion in $t$ of the terms inside in the integral of the r.h.s. of 12, we can neglect terms of order bigger or equal than three. Therefore, the r.h.s. of 12 equals

$$\lim_{t\to 0}\frac{1}{t^2}\left(\int_S (f_S + H^\nu f(ts))\phi_{ts}^*\frac{\omega^m}{m!} - f_S\frac{\omega^m}{m!}\right). \tag{13}$$

Pulling the time parameters out of the Hessian and taking limits we get

$$\lim_{t\to 0}\frac{1}{t^2}\left(\int_S \phi_{ts}^*(f\frac{\omega^m}{m!}) - f_S\frac{\omega^m}{m!}\right) = \int_S H^\nu f(s)\frac{\omega^m}{m!} + \left(\int_S f_S \lim_{t\to 0}\frac{\phi_{ts}^*\omega^m - \omega^m}{m!t^2}\right). \tag{14}$$

To compute the quadratic expansion in $t$ of $\phi_{ts}^*\omega^m - \omega^m$ we will need the three results:

- Given $X \in \Gamma(TS)$, if we let $\tilde{X}$ be its horizontal lift w.r.t. the connection $\nabla$, then
$$D\phi_s(X) = \tilde{X} + \nabla_X s \tag{15}$$
is a decomposition into horizontal and vertical part.
- The vertical and horizontal distributions are symplectically orthogonal.
- Recall that the curvature of the induced connection in the principal unitary bundle associated to $\nu(S)$ is a $\mathfrak{u}(n)$-valued 2-form, that we denote by $F_\nabla$. Any pair of vector fields $X_1, X_2$ defines a curvature vector field $F_\nabla(X_1,X_2) \in \Gamma(T\nu(S))$. This is of course a vertical vector field, and it is Hamiltonian (the fibers of $\nu(S)$ are identified with $(\mathbb{C}^m, h_{\mathrm{std}})$) with Hamiltonian function
$$\omega_c(\tilde{X}_1, \tilde{X}_2) \tag{16}$$
This Hamiltonian function vanishes on the zero section, so by 4 we have
$$\omega_c(s)(\tilde{X}_1,\tilde{X}_2) = \frac{i}{2}\mathrm{tr}(F_\nabla(X_1,X_2)ss^*). \tag{17}$$

We use 15 to write

$$\begin{aligned}\phi_{ts}^*\omega^m - \omega^m(0)(X_1,\dots,X_{2m}) =& \omega^m(ts)(\tilde{X}_1 + t\nabla_{X_1}s,\dots,\tilde{X}_2 + t\nabla_{X_{2m}}s) - \\ &- \omega^m(0)(X_1,\dots,X_{2m}).\end{aligned} \tag{18}$$

We want to understand the quadratic expansion in $t$ of 18. Let $\mathrm{Sh}(2,\overset{(m)}{\dots},2)$ be all possible ways of arranging m pairs of numbers in $1,\dots,2m$. By definition

$$\omega^m(ts)(X_1,\dots,X_{2m}) = \sum_{\alpha\in\mathrm{Sh}(2,\overset{(m)}{\dots},2)}\prod_{j=1}^m \omega(ts)(X_{\alpha(2j-1)},X_{\alpha(2j)}),$$

and we may write each summand

$$\prod_{j=1}^m \omega(ts)(\tilde{X}_{\alpha(2j-1)} + t\nabla_{X_{\alpha(2j-1)}}s, \tilde{X}_{\alpha(2j)} + t\nabla_{X_{\alpha(2j)}}s).$$

Because the horizontal and vertical distributions are symplectically orthogonal, on each factor no mixed terms appear. Since we are interested in the quadratic expansion each $\alpha \in \mathrm{Sh}(2, \overset{(m)}{\dots}, 2)$ just contributes with m+1 summands, which we can regroup into $m$ containing a pair of vertical vector fields

$$\sum_{i=1}^{m} t^2 \omega(ts)(\nabla_{X_{\alpha(2i-1)}} s, \nabla_{X_{\alpha(2i)}} s) \prod_{j \neq i} \omega(ts)(\tilde{X}_{\alpha(2j-1)}, \tilde{X}_{\alpha(2j)}), \tag{19}$$

and a final summand containing just horizontal vector fields

$$\prod_{j=1}^{m} \omega(ts)(\tilde{X}_{\alpha(2j-1)}, \tilde{X}_{\alpha(2j)}). \tag{20}$$

Because $\omega$ on vertical vector fields coincides with the coupling 2-form, we have

$$\omega(ts)(\nabla_{X_{\alpha(2i-1)}} s, \nabla_{X_{\alpha(2i)}} s) = -\mathrm{Im} h(\nabla_{X_{\alpha(2i-1)}} s, \nabla_{X_{\alpha(2i)}} s).$$

Since we can use the Hermitian metric to define a conjugate bundle and and induced connection, we can express 20 as

$$-\frac{i}{2} \nabla s \wedge \nabla \bar{s}(X_{\alpha(2j-1)}, X_{\alpha(2j)}). \tag{21}$$

In particular if we divide 19 by $t^2$ and take the limit when $t$ goes to zero we obtain

$$\sum_{i=1}^{m} -\frac{i}{2} \nabla s \wedge \nabla \bar{s}(X_{\alpha(2j-1)} s, \nabla_{X_{\alpha(2j)}}) \cdot \prod_{j \neq i} \omega(0)(\tilde{X}_{\alpha(2j-1)}, \tilde{X}_{\alpha(2j)}),$$

and summing over all $\alpha \in \mathrm{Sh}(2, \overset{(m)}{\dots}, 2)$ and dividing by $m!$ we obtain

$$-\frac{i}{2} \nabla s \wedge \nabla \bar{s} \wedge \frac{\omega^{m-1}}{(m-1)!}(0). \tag{22}$$

It remains to compute

$$\lim_{t \to 0} \frac{\sum_{\alpha \in \mathrm{Sh}(2, \overset{(m)}{\dots}, 2)} \prod_{j=1}^{m} \omega(ts)(\tilde{X}_{\alpha(2j-1)}, \tilde{X}_{\alpha(2j)}) - \omega^m(0)(X_{\alpha(1)}, \dots, X_{\alpha(2m)})}{m! t^2}. \tag{23}$$

We have

$$\omega(ts)(\tilde{X}_{\alpha(2j-1)}, \tilde{X}_{\alpha(2j)}) = \omega_c(ts)(\tilde{X}_{\alpha(2j-1)}, \tilde{X}_{\alpha(2j)}) + \pi^* \omega_S(ts)(\tilde{X}_{\alpha(2j-1)}, \tilde{X}_{\alpha(2j)}).$$

By equation 17

$$\begin{aligned} \omega_c(ts)(\tilde{X}_{\alpha(2j-1)}, \tilde{X}_{\alpha(2j)}) &= \frac{i}{2} \mathrm{tr}(F_\nabla(X_{\alpha(2j-1)}, X_{\alpha(2j)}) t^2 s s^*) = \\ &= \frac{i}{2} t^2 \mathrm{tr}(F_\nabla(X_{\alpha(2j-1)}, X_{\alpha(2j)}) s s^*). \end{aligned}$$

Because

$$\pi^* \omega_S(ts)(\tilde{X}_{\alpha(2j-1)}, \tilde{X}_{\alpha(2j)}) = \omega(ts)(X_{\alpha(2j-1)}, X_{\alpha(2j)}),$$

when we take the limit in 23 we obtain

$$\frac{i}{2} \mathrm{tr}(F_\nabla s s^*) \wedge \frac{\omega^{m-1}}{(m-1)!}(0). \tag{24}$$

So the formula for the momentum map -which is the result of adding equations 11, 22, the first summand in 14 and 24- is

$$\int_S f^{(2)} \frac{\omega^m}{m!} - \frac{i}{2} f((\nabla s \wedge \nabla \bar{s} - \mathrm{tr}(F_\nabla s s^*) \wedge \frac{\omega^{m-1}}{(m-1)!}),$$

and this proves the theorem. □

4.4. **The proof of theorem 2.** More generally, we start with $\pi\colon (E,\nabla,h) \to (X,\sigma)$ a unitary vector bundle over a compact symplectic manifold. By the universal phase space construction we can find a coupling form $\sigma_{c(E)}$ so that for small $\epsilon > 0$ the 2-form

$$\pi^*\sigma + \epsilon\sigma_{c(E)} \tag{25}$$

is symplectic.

Because we want a compact ambient manifold, we may rather apply the symplectic coupling construction to the projectivization $\mathbb{P}(E)$ as in [16], section 6.4, and obtain $\omega = \pi^*\sigma + \epsilon\sigma_{c(\mathbb{E})}$ a symplectic structure which equals 25 near the zero section.

We would like to find a map

$$\tau\colon \mathrm{ham}(X,\sigma) \to \mathrm{ham}(\mathbb{P}(E),\omega)_X$$

such that the composition

$$\rho := \theta \circ \tau\colon \mathrm{ham}(X,\sigma) \to \mathrm{sp}(T_X\mathcal{M},\Omega_X)$$

is a Lie algebra homomorphism. In that way -and leaving aside the equivariance property- for the infinitesimal linear action $\rho$ we would have the momentum map

$$\mu_f(z) := \mu_{\tau(f)}. \tag{26}$$

The Hamiltonian bundle $(\mathbb{P}(E),\omega)$ carries a fiberwise Hamiltonian function $\zeta$. Its restriction to the fiber of $E$ is exactly 4 times $\epsilon$. We will assume now that $E$ is $L$ the k-th power of the prequantum line bundle. Then the fiberwise Hamiltonian is real valued, and its associated Hamiltonian vector field is the fundamental vector field of the $S^1$-action

$$X_\zeta = R. \tag{27}$$

For any $f \in \mathrm{ham}(X,\sigma)$ we define

$$\tau(f) := \pi^* f(1 + k\zeta). \tag{28}$$

By applying the Cavalieri's principle to the fibers, it is clear that $\tau(f) \in \mathrm{ham}(\mathbb{P}(E),\omega)$. Because $\tau(f)$ has vanishing linear part along the normal directions at the zero section, we conclude that $\tau \in \mathrm{ham}(\mathbb{P}(E),\omega)_X$.

We let $\mathcal{K}$ denote the group of unitary bundle automorphisms of $(L,h)$ which preserve the Hermitian connection. It is well known that we have the following Lie algebra monomorphism

$$\begin{array}{rcl} \mathrm{ham}(X,\sigma) & \longrightarrow & \mathrm{Lie}(\mathcal{K}) \\ f & \longmapsto & X_{f,K} := \tilde{X}_f + k\pi^* f R, \end{array}$$

and by taking the linearization about $X$ we have a Lie algebra homomorphism

$$\begin{array}{rcl} \mathrm{ham}(X,\sigma) & \longrightarrow & \mathrm{sp}(T_X\mathcal{M},\Omega_X) \\ f & \longmapsto & \mathcal{X}^{(1)}_{f,K}. \end{array}$$

Thus if we prove that

$$\rho(f) = \mathcal{X}^{(1)}_{f,K} \tag{29}$$

we can conclude that $\rho$ is a Lie algebra homomorphism.

To check that 29 holds we need to show that $X_{f,K}$ and $X_{\tau(f)}$ induce vector fields in $T_X\mathcal{M}$ with the same linearization. According to subsection 4.2 it suffices to check that their difference vanishes at $X$ linearly.

We have

$$d\tau(f) = (1 + k\zeta)\pi^* df + k\pi^* f d\zeta$$

Because $\zeta$ is constant along sphere bundles and the connection is Hermitian, $d\zeta$ vanishes on the horizontal distribution. Because $\pi^* f$ is constant along the fibers

$\pi^* df$ vanishes along the vertical distribution. Since these two distributions are symplectically orthogonal

$$X_{\tau(f)} = (1 + k\zeta)X_{\pi^* f} k\pi^* f + X_\zeta,$$

where the Hamiltonian vector fields can also be computed for the restriction of the symplectic form to vertical and horizontal distributions respectively. Using 27 we obtain

$$X_{\tau(f)} = (1 + \zeta)X_{\pi^* f} + k\pi^* f R.$$

Because we can neglect quadratic terms it suffices to show

$$X_{\pi^* f} = \tilde{X}_f + O(2). \tag{30}$$

The Hamiltonian vector field can be computed w.r.t. restriction of $\omega$ to the horizontal distribution, which we identify with the tangent bundle of $X$. By 25 the corresponding 2-form is

$$\sigma + \epsilon\omega_{c|H}.$$

The coupling form vanishes along the zero section, and closedness implies that this vanishing is necessarily linear, thus 30 holds and therefore $\rho$ is a Lie algebra homomorphism.

The momentum map formula is

$$\mu_f(s) := \mu_{\tau(f)}(s) = \int_S \tau(f)^{(2)} \frac{\sigma^m}{m!} - \frac{i}{2} f(\epsilon(\nabla s \wedge \nabla \bar{s} - \mathrm{tr}(F_\nabla s s^*) \wedge \frac{\sigma^{m-1}}{(m-1)!}).$$

Using that $f \in \mathrm{ham}(X, \sigma)$, $F_\nabla = -ki\sigma$ and

$$H^\nu \tau(f)(s) = \frac{1}{2}\epsilon f k s\bar{s}$$

we get

$$\mu_f(s) = \epsilon \int_S f \left( -\frac{i}{2} \nabla s \wedge \nabla \bar{s} \wedge \frac{\sigma^{m-1}}{(m-1)!} + \frac{1}{2} k s\bar{s} \frac{\sigma^m}{(m-1)!} + \frac{1}{2} k s\bar{s} \frac{\sigma^m}{m!} \right).$$

This is a momentum map for the infinitesimal action on the symplectic vector space $(\Gamma(L), \Omega_X)$, where

$$\Omega_X(s_1, s_2) = \int_X \epsilon\sigma(s_1, s_2) \frac{\sigma^m}{m!}.$$

Dividing the symplectic form $\Omega_X$ by $\epsilon$ we recover the formula in 2.

Regarding the equivariance issue, since $\mathrm{ham}(X, \sigma)$ injects in $\mathcal{K}$, one has the adjoint action of $\mathcal{K}$ in $\mathrm{ham}(X, \sigma)$. It is easy to verify that the momentum map in 2 is equivariant w.r.t. this action.

As mentioned at the beginning there is a slight discrepancy between formula 11 in [6] and ours. With our conventions, we believe formula 13 in [6]

$$\nabla_{X_f} s \wedge w^{m-1} = -df \wedge \nabla s \wedge \sigma^{m-1}$$

should rather be

$$\nabla_{X_f} s \wedge w^{m-1} = -m df \wedge \nabla s \wedge \sigma^{m-1}. \tag{31}$$

This would follow from

$$0 = i_{X_f}(\nabla s \wedge \omega^m) = \nabla_{X_f} s \wedge \omega^m + m df \wedge \nabla s \wedge \omega^{m-1}.$$

Using 31 instead of formula 13 in [6], then formula 11 in [6] becomes our formula 2.

## References


[1] Atiyah, M. F.; Bott, R. The Yang-Mills equations over Riemann surfaces. Philos. Trans. Roy. Soc. London Ser. A 308 (1983), no. 1505, 523615.

[2] Arnold, V. Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. (French) Ann. Inst. Fourier (Grenoble) 16 1966 fasc. 1, 319–361.

[3] Lee, Brian Geometric structures on spaces of weighted submanifolds. (English summary) SIGMA Symmetry Integrability Geom. Methods Appl. 5 (2009), Paper 099, 46 pp.

[4] Crainic, Marius; Fernandes, Rui Loja Integrability of Lie brackets. Ann. of Math. (2) 157 (2003), no. 2, 575620.

[5] Donaldson, S. K. Symmetric spaces, Kähler geometry and Hamiltonian dynamics. Northern California Symplectic Geometry Seminar, 13–33, Amer. Math. Soc. Transl. Ser. 2, 196, Amer. Math. Soc., Providence, RI, 1999.

[6] Donaldson, S. K. Scalar curvature and projective embeddings. I. J. Differential Geom. 59 (2001), no. 3, 479–522.

[7] Ebin, David G.; Marsden, Jerrold E. Groups of diffeomorphisms and the solution of the classical Euler equations for a perfect fluid. Bull. Amer. Math. Soc. 75 1969 962–967.

[8] Gay-Balmaz, François; Ratiu, Tudor S. The geometric structure of complex fluids. Adv. in Appl. Math. 42 (2009), no. 2, 176–275.

[9] Gay-Balmaz, François; Vizman, Cornelia Dual pairs in fluid dynamics. Preprint arXiv:1007.1347.

[10] Hamilton, Richard S. The inverse function theorem of Nash and Moser. Bull. Amer. Math. Soc. (N.S.) 7 (1982), no. 1, 65–222.

[11] Haller, Stefan; Vizman, Cornelia Non-linear Grassmannians as coadjoint orbits. Math. Ann. 329 (2004), no. 4, 771–785.

[12] Holm, Darryl D.; Marsden, Jerrold E. Momentum maps and measure-valued solutions (peakons, filaments, and sheets) for the EPDiff equation. The breadth of symplectic and Poisson geometry, 203–235, Progr. Math., 232, Birkhuser Boston, Boston, MA, 2005.

[13] Kriegl, Andreas; Michor, Peter W. The convenient setting of global analysis. Mathematical Surveys and Monographs, 53. American Mathematical Society, Providence, RI, 1997.

[14] Marsden, Jerrold E.; Raiu, Tudor; Weinstein, Alan Semidirect products and reduction in mechanics. Trans. Amer. Math. Soc. 281 (1984), no. 1, 147–177.

[15] Marsden, Jerrold; Weinstein, Alan Coadjoint orbits, vortices, and Clebsch variables for incompressible fluids. Order in chaos (Los Alamos, N.M., 1982). Phys. D 7 (1983), no. 1-3, 305–323.

[16] McDuff, Dusa; Salamon, Dietmar Introduction to symplectic topology. Second edition. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1998. x+486 pp.

[17] Ono, K. Floer-Novikov cohomology and the flux conjecture. Geom. Funct. Anal. 16 (2006), no. 5, 981–1020.

[18] Weinstein, A. A universal phase space for particles in Yang-Mills fields. Lett. Math. Phys. 2 (1977/78), no. 5, 417–420.

[19] Weinstein, Alan Connections of Berry and Hannay type for moving Lagrangian submanifolds. Adv. Math. 82 (1990), no. 2, 133–159.



Centro de Análise Matemática, Geometria e Sistemas Dinâmicos, Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisboa, Portugal

*E-mail address*: `martinez@math.ist.utl.pt`