Geometries with topological character

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A la memoria de mi padre

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Introduction

According to de F. Klein's Erlangen program [**34**], we know that a geometry $\mathcal{G}$ in a smooth manifold is determined by the choice of a subgroup $\mathrm{Mor}(\mathcal{G})$ of the group of diffeomorphisms of the manifold. Its study is that of those magnitudes, or more generally of those properties –the so called invariants of $\mathcal{G}$– that are preserved under the action of $\mathrm{Mor}(\mathcal{G})$.

It is necessary to ask ourselves what is the reason to select a geometry as the object of our study. In this sense the most important reason is the physical origin of some of them, being the paradigm symplectic geometry, or more generally Poisson geometry. In this case, our manifold is the phase space of a system and the geometry $\mathcal{S}$ is determined by the so called canonical transformations. Of course, further study shows that $\mathrm{Mor}(\mathcal{S})$ can be characterized as the group of diffeomorphisms preserving certain $(2,0)$ tensor, the *symplectic form*. One also observes that all the mechanisms and constructions depend on properties of this tensor that can be expressed in the language of differential geometry. To be more precise these are skew symmetry, non-degeneracy and closedness. From them one defines the notion of a symplectic structure in any smooth manifold.

More generally, Poisson geometry $\mathcal{P}$ is the right frame for the theory hamiltonian systems. The smooth manifold M (in our case finite-dimensional) is the space of states of the corresponding system; the observables correspond to a subalgebra $\mathcal{O}$ of the algebra of functions $C^\infty(M)$, which we assume to coincide with the whole algebra (o more generally it is a subsheaf of the sheaf of smooth functions on M). The evolution of the system is dictated by a one parameter family of diffeomorphisms, or infinitesimally by a vector field. Finally, there is a map $E\colon C^\infty(M) \to \mathfrak{X}(M)$, $f \mapsto X_f$, such that f is preserved by X_f ($X_f(f) = 0$), and E is a Lie algebra morphism for the bracket $\{f, g\} := X_g(f)$ (and the usual Lie bracket in vector fields). Again, it can be checked that the corresponding Poisson structure is described by a skew symmetric $(0,2)$ tensor (a bi-vector) subject to a closedness condition, and the geometry is determined by the diffeomeophisms that preserve that tensor.

Another important example is that of semi-riemannian geometry $\mathcal{R}$. Here the origin is the study of immersions of curves and surfaces in $\mathbb{R}^3$. In other words, the manifold is $\mathbb{R}^3$, the group is $O(3)$ and the invariant under study are the equivalence classes of immersions of curves and surfaces. The first result that shows the importance of the study of surfaces with metric is the celebrated Egregium theorem of Gauss; once the theory was generalized by Riemann, its important role is shown by its connection with the Theory of Relativity.

These two examples, symplectic and Poisson geometry on the one hand, and semi-riemannian geometry on the other, have quite a different character. Indeed, the corresponding tensors must satisfy pointwise properties, skew symmetry/symmetry and non-degeneracy (the latter in the symplectic and riemannian case), and in the symplectic and Poisson case the fulfilment of a partial differential equation. This last closedness condition, in contrast with the semi-riemannian case, imposes strong restrictions for the existence of this kind of structures. Maybe the fundamental difference is reflected in the "different size" of the corresponding group of transformations of the structures, which is finite-dimensional for $\mathcal{R}$ (a Lie group), and infinite dimensional for $\mathcal{S}$ y $\mathcal{P}$. This implies the existence of comparatively "less" invariants for $\mathcal{S}$ y $\mathcal{P}$. Actually there are no local invariants for $\mathcal{S}$ (resp. certain local structure theorems for $\mathcal{P}$), something which in the semi-riemannian situation cannot happen due to the presence of curvature. As a consequence, the invariants for $\mathcal{S}$ must have global nature. Thus, it is not strange that the study of this global phenomenon inherent to symplectic geometry is called symplectic topology. Something similar happens in Poisson geometry, where we can speak of a "Poisson topology" studying global aspects of the Poisson structure.

Therefore, and for a geometry $\mathcal{G}$ that can be defined as the previous by a smooth object (normally a section of a fiber bundle) with certain properties, so that the study of $\mathcal{G}$ reflects global aspects of the manifold, the previous paragraph raises the following natural questions.

(i) Given M a smooth manifold, which are the obstructions to the existence of such a geometry in M? Conversely, it is natural trying to show whether certain global properties of M obstruct or not the existence of the corresponding structure.

For symplectic structures, there is a first homotopic obstruction which refers to the existence of a $(2,0)$ tensor with the required pointwise properties. Regarding the closedness condition, in the case of a compact manifold one can only conclude the non-vanishing of the cohomology class defined by the symplectic structure. For open manifolds of dimension greater of equal than 6, it is a consequence of the h-principle proved by M. Gromov [**27**] that the only obstruction is the homotopic one.

It was R. Gompf who achieved striking results adopting the opposite point of view. In his paper [**24**] he exploited in full generality the so called normal connected sum for symplectic manifolds to construct symplectic manifolds with prescribed topological properties. Among others, the most important result of his work was the proof of the existence of compact symplectic manifolds (of any dimension ≥ 4) whose fundamental group is any finitely presented group.

For Poisson structures, and due to its generality, one cannot expect to obtain results similar to those of Gromov and Gompf for any Poisson structure. In any case, it is reasonable to study the corresponding problems for the class of regular Poisson structures, which are those for which the Poisson tensor has constant rank. For existence of regular Poisson structures with a prescribed foliation there is already a homotopic obstruction. It turns out that if the manifold is open in the sense of foliated manifolds [**6**], it is a consequence of the h-principle for foliated manifolds proven by M. Bertelson that indeed this is the only obstruction (see [**6**]).

We will devote the second chapter of this thesis to extending the results of R. Gompf for symplectic manifolds to regular Poisson manifolds. We will define the *normal connected sum* for Poisson manifolds and will show that for any dimension, rank (both ≥ 4) and group G finitely presented, the existence of closed regular Poisson manifolds whose fundamental group is G, showing thus that the fundamental group does not obstruct the existence of such structures.

The second natural question that arises in the context of the study of one of these geometries $\mathcal{G}$ is the following:

(ii) Given a manifold M and once the existence of structures $\mathcal{G}$ have been shown to be unobstructed, it is reasonable to give a classification of them, where two such structures $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be equivalent if $\mathrm{Mor}(\mathcal{G}_1)$ can be conjugated to $\mathrm{Mor}(\mathcal{G}_2)$ by a diffeomorphism (we can for example ask also the diffeomorphism to be isotopic to the identity).

To illustrate this question we will mention three examples, which are related among themselves both for the geometries that they refer to, and for the way the proofs are obtained. Firstly, we have the case of volume forms $\mathcal{V}$ (in compact manifolds). The obstruction to the existence of volume forms is the orientability of the manifold M. It is a classic result of J. Moser [**45**] that the volume form is totally determined by its volume, which in principle can be computed. That is, the geometry $\mathcal{V}$ is fully described by the choice of a multiple of an invariant of the homotopy type of M.

Secondly, we have the case of the symplectic structures (M closed). For the equivalence relation given by conjugation by diffeomorphisms isotopic to the identity, again after a result of J. Moser [**45**], we conclude that two symplectic 2-forms are in the same class if they can be joined by a path of symplectic structures with constant cohomology class.

The third structure we wanted to speak about regarding the classification problem, is that of stable Poisson structures $\mathcal{P}_{st}$ (or generic) in closed oriented surfaces. In a recent work [**51**], and for the equivalence relation defined by conjugation by diffeomeorphisms isotopic to the identity, O. Radko gave a description of the corresponding moduli space. The novelty of this example is that the number of connected component was in one to one correspondence with certain isotopy classes of hypersurface arrangements in the surface; for each arrangement, the correspondent connected component turned out to be diffeomorphic to a vector space with dimension the number of hypersurfaces in the arrangement plus one. What is more, the identification was given through a cohomology group associated to the structure.

There is a fourth geometry $\mathcal{N}$ which is a generalization of both $\mathcal{P}$ and $\mathcal{V}$. It is the so called *Nambu geometry*.

Nambu mechanics is a generalization of hamiltonian mechanics. In contrast with the latter, the dynamics are governed by a system of O.D.E's associated not to a unique hamiltonian, but to r of them ($r \geq 1$); the number of hamiltonians plus one is called the *order*. From the differential point of view that means that the Nambu structure is defined by a (smooth) section of $\mathfrak{X}^{r+1}$ with certain closedness properties, which in this case turn out to be and algebraic equation and a P.D.E. (see [**53**] for the precise definition).

The analog to stable Poisson structures in oriented closed surfaces are stable Nambu structures of maximal order. That is, for any closed oriented n-manifold, the sections of $\mathfrak{X}^n$ which cut the zero section transversely.

In the third chapter of this thesis we will make an study of these structures analog to that of O. Radko for stable Poisson structures. The main result will be a classification theorem for oriented isotopy classes. We will show that the connected components of the moduli space will coincide with the isotopy classes of certain hypersurface arrangements. Also, each connected component will be isomorphic to a vector space of dimension the number of hypersurfaces in the arrangement plus one. Again, the isomorphism will be given through the identification with certain cohomology group associated to the Nambu structure.

We will see that such classification is the result that observing that Radko's work [**51**], though stated in the language of Poisson geometry, relies mainly in differential topology results, together with the classification of area forms in surfaces, with can be generalized to arbitrary dimensions.

The final question we want to answer regarding geometries with topological character –and from our point of view that for which we will obtain the most interesting results– is the following:

(iii) Up to which point the geometry $\mathcal{G}$ is rich? That is, we raise the question of the existence of differential topological constructions compatible $\mathcal{G}$.

Observe that by definition we are considering geometries without local invariants. That means that the problems we aim to solve will be have local solutions, and the difficulty will be finding ways to obtain global solutions.

To clarify this third point we consider again the example of symplectic geometry $\mathcal{S}$. A natural problem in this setting is that of the existence of topologically non-trivial symplectic submanifolds. That is, solutions to the topological problem of the existence of submanifolds which are compatible with $\mathcal{S}$. Amazingly enough, such a natural and easy to state question has only been solved by S. Donaldson [**12**] in 1996, via the introduction of absolutely new techniques (*approximately holomorphic techniques*) which have been a mayor breakthrough in the research in this field.

Analogous problems to the existence of symplectic submanifolds are the existence of submersions with symplectic fibers, o more generally, stratifications with symplectic strata with are "close" to be fibrations.

The underlying philosophy is trying to reduce the understanding of symplectic structures in a manifold M to that of symplectic structures of lower dimensions (the fibers) together with a piece of topological information.

In the first chapter of this thesis we will make the same kind of analysis, but for a new geometry $\mathcal{C}_{2,1}$ that we will call 2-calibrated. We can understand this geometry as an analog of symplectic geometry in odd dimensions.

An example of a 2-calibrated manifold is an exact contact manifold (M^{2n+1}, α), $\alpha \in \wedge^1(M)$ (recall that $\alpha \wedge (d\alpha)^n$ is required to be a volume form). Its characteristic distribution $\ker \alpha$ has codimension 1, and the closed 2-form $d\alpha$ induces a symplectic vector space structure on each hyperplane of the distribution (it dominates or calibrates the distribution in a sense that we will make precise).

A *2-calibrated structure* is the generalization of the previous situation: a codimension 1 distribution (that explains the subindex 1 in $\mathcal{C}_{2,1}$) and a closed 2-forma which makes the distribution symplectic. Notice that this structure can be understood as a one parameter family of infinitesimal symplectic structures (that is we consider not only foliations but distributions) for which there is a closedness condition analog to that of symplectic structures. For the sake of brevity, we will speak from now on about calibrated structures instead of 2-calibrated structures.

Its is interesting to observe the difference between $\mathcal{C}_{2,1}$ and the regular Poisson structures with codimension 1 leaves. For the latter, if we try to state the closedness condition in covariant terms, this turns out to be a condition for a foliated 2-form (the 2-forms have to be non degenerate and closed on each leaf). It is not true in general that every such Poisson structure admits a lift to a calibrated structure, i.e., there does not exist in general a *closed* 2-form restricting on each leaf to the leafwise 2-form; we will see that the absence of such a lift will prevent the existence of the constructions that will be shown to exist for calibrated structures.

We will develop an *approximate holomorphic geometry* for compact calibrated structures, solving thus problem like the existence of calibrated submanifolds, calibrated stratifications and extending other constructions from approximately holomorphic geometry in compact symplectic manifolds.

We will devote one chapter of this thesis to answer each of the three questions that we have just raised. Each chapter will include a description of the most relevant results obtained and of the underlying ideas.

CHAPTER I

The geometry of calibrated manifolds

1. Introduction and results

1.1. Motivation.

Definition 1.1. *A (2-)calibrated manifold is a triple (M, D, ω), where M is a smooth manifold, D is a codimension 1 distribution, and ω is a closed 2-form which is non-degenerate over D. We say that ω is positive over D or that is dominated the distribution or that it calibrates it.*

The calibrated manifold is said to be of integer type if ω is in the image of $H^2(M;\mathbb{Z})$ in $H^2(M;\mathbb{R})$.

The dimension of M is therefore odd. Notice that the concept of calibrated manifold is an odd dimensional analog of the notion of symplectic manifold.

Remark 1.2: The concept of calibrated foliation is by no means new. A calibrated foliation is a foliation of arbitrary codimension for which a closed p-form dominating the foliation exists, where p is the dimension of the leaves [**29**]; they are also called geometrically tight or homologically tight foliations. This is of course a weaker condition that the one we are imposing, because a (2-)calibrated foliation of dimension $2n$ in our sense is of codimension 1, and the calibrating $2n$-form has to be of the form ω^{2n}, $d\omega = 0$. As far as the author knows, and for dimensions different from 3, there is no existing literature specific for the class of foliations we will deal with in this chapter.

All the structures we will work with will be of class C^∞ (smooth).

As we just mentioned an important example of calibrated varieties is that of closed 3-manifolds with 2-dimensional *smooth* calibrated (or taut) foliations.

Definition 1.3. *Let $(M^3, \mathcal{F})$ be an oriented manifold foliated by orientable surfaces (and hence co-orientable). We say that $\mathcal{F}$ is a calibrated (or taut) foliation if $M \neq S^2 \times S^1$ and a closed 2-form ω restricting to a leafwise area form of $\mathcal{F}$ exists.*

We recall that any closed oriented 3-manifold admits foliations $\mathcal{F}$ by oriented surfaces [**36**], but not every foliation is interesting enough to give us information about the topology of the manifold. Those of our interest are essentially the ones which do not have Reeb components.

It is a classical theorem that if a foliation does not have generalized Reeb components (see [**52**]), then the foliation is calibrated.

Calibrated foliations in 3-manifolds can be characterized by the existence of transverse cycles through any point (this result is a corollary of the works of Novikov and Sullivan [**52**]). We would like to start with this characterization to motivate the introduction of approximately holomorphic techniques in the study of calibrated structures.

Given M^n any smooth manifold, a natural way to construct submanifolds is to define them as the zero set of (transverse) functions, or more generally, of sections of certain vector bundles. If M is compact and orientable, and we have constructed a submanifold W in such a way, where W turns out to be orientable and of codimension 2, then the theory of characteristic classes allows us to reconstruct the bundle L from the homogical information that W provides: indeed, it will be the line bundle whose Chern class is the Poincaré dual of $[W] \in H_{n-2}(M;\mathbb{Z})$. Any other subvariety W_1 constructed as the zero set of a section of L will be cohomologous to W; actually, results from differential topology [**33**] guarantee the existence of a smooth cobordism inside M connecting both (or in other words, a Seifert hypersurface for $W_1 \amalg -W$). Thus, once the homological information that determines the line bundle has been analyzed, the problem of defining submanifolds cohomologous to W is a differential topology one: finding a section τ of L transverse to the $\mathbf{0}$ section so that $W_1 = \tau^{-1}(0)$.

In our situation there is an extra difficulty because the subvariety W_1 we look for, in principle of M^3, has to be transverse to $\mathcal{F}$. Locally, there is no obstruction to the existence of such sections, but there is a global one. The classic example is the Reeb foliation of S^3. If a cycle W transverse to the separating torus of the Reeb components existed, since $H_2(S^3;\mathbb{Z}) = 0$ it would be given as the zero set of a function $f\colon S^3 \to \mathbb{C}$. Since a cycle W with that property does not exist, we can deduce the existence of complex valued functions in S^3 which are not globally transverse to $\mathbf{0}$ along the directions of $\mathcal{F}$, even though they might be transverse to $\mathbf{0}$ in the usual sense. In other words, the submanifold W that they defined will be tangent to $\mathcal{F}$ in a non-empty set.

Even for calibrated foliations, the existence of a theorem of foliated transversality is not true. There are already counter examples at semi-local level.

In $\mathbb{R}^3$ with coordinates x, y, s foliated by horizontal planes $s = c$, $c \in \mathbb{R}$, consider the function $f(x, y, s) = x^2 + is$. It is clear that arbitrarily small perturbations of f cannot define a cycle transverse to the foliation. If we had $\tilde{f}$ such a perturbation, since transversality to $\mathcal{F}$ is an open property, small enough perturbations h would still have that property. Obviously the zero set W_h will not be transverse to the foliation, because the restriction to W_h of the projection onto the third coordinate will have a global maximum. At that point the rank of the leafwise differential of h is one. So a way to avoid such a situation is to work with a class of functions whose differential either vanishes or s surjective, i.e., the obvious choice is to work with leafwise holomorphic functions. We recall that we can always introduce a leafwise almost complex structure which necessarily is integrable. The difficulty that we might encounter is the lack of leafwise holomorphic function for a certain choice (or even for any) of almost complex structure. A first observation is that since the class of functions we are interested in has to be open (leafwise transversality is clearly an open condition), it is not exactly the leafwise holomorphic functions the class of our interest, but also those functions close enough to the latter in a sense that we will make precise.

Notice that the mentioned point of view has already been adopted for the study of compact symplectic manifolds of any dimension. It is the so called *approximately holomorphic theory* introduced by S. Donaldson in [**12**]. Not being very precise, this theory proves –for almost complex structures compatible with the symplectic structure ω– the existence of a strong transversality result for *approximately holomorphic sections* of the line bundles $L^{\otimes k}$, where L is the complex line bundle dual to $[\omega]$ and k is a large enough integer (we assume ω to be of integer type).

Therefore, we have a first hint indicating that it is reasonable to study the corresponding approximately holomorphic theory, at least for calibrated foliations in 3-manifolds (because we do have transverse cycles through any point).

Contact geometry provides a second motivation. Recall that an exact contact structure in a compact manifold M^{2n+1} is given by a non-degenerate 1-form α verifying that $\alpha \wedge (d\alpha)^n$ is a volume form. In particular $(M, \ker \alpha, d\alpha)$ is a calibrated structure. A rich approximately holomorphic theory has already been proven to exist for the sections of the bundles $L^{\otimes k}$, where L is again the dual of $d\alpha$ (and hence trivial). In this case the almost complex structure J is defined along $\ker d\alpha$, that has to be understood as the distribution of "holomorphic" directions. What is more, results of different scope has been achieved using on the one hand an intrinsic theory [**32**],[**50**], and on the other hand a relative theory applied to the symplectization $(M \times \mathbb{R}, d(t\alpha))$ [**43, 23**]. Thus, it is reasonable trying to check that all the mechanisms that make both theories (intrinsic and relative) work in the contact case, do not use any property which is not shared by all calibrated structures.

As we already mentioned, the definition of calibrated structure is new but it contains a number of existing geometries. The most interesting examples are on the one hand contact structures, the distribution D being maximally non integrable, and on the other hand $(2-)$calibrated foliations; we should at this point distinguish between 3-dimensional foliations and calibrated foliations in higher dimensions. For the former there exist a wealth of results in the literature which are characterized by the use of 3-dimensional topology techniques ([**19, 20, 17**]). For the latter the author does not know any research concerning codimension 1 $(2-)$calibrated foliations; anyhow there are important works dealing with codimension 1 foliations in $p+1$ dimensional manifolds calibrated by p-forms (see for example [**29, 49**]).

It is worth recalling that every calibrated structure (of integer type) in which D is integrable endows the manifold M with a regular Poisson structure (M, Λ) with codimension 1 symplectic leaves, which turns out to be integrable in the sense of Lie algebroids (see R. L. Fernandes and M.

Crainic papers [**10, 11**]). Hence, we also have the corresponding results coming from Poisson geometry for integrable Poisson structures.

Inside the calibrated structures which are Poisson (D integrable) it is worth mentioning the class of *cosymplectic structures*. Recall that a cosymplectic structure in a manifold M is given by a pair (α, ω), where α is a non-degenerate closed 1-form and ω is the closed 2-form dominating $\ker \alpha$. Observe that a cosymplectic structure is a kind of Poisson structure in which the symplectic foliation is quite special because it is defined by a closed 1-form. Indeed, from the topological point of view if we assume M to be closed and connected, the cosymplectic structure can be perturbed so that its description becomes really elementary; by compactness, it is possible to select a closed 1-form with integral periods arbitrarily close to the original one so that ω also dominates the foliation it defines. By an elementary result of Tischler, there exists a submersion to S^1 whose fibers are a finite number of leaves of the foliation (in the homotopy class of classifying maps associated to $[\alpha']$). Since M is assumed to be connected, it is possible to compose it with a self map of the circle isotopic to the appropriate root and so that the resulting map to S^1 is a submersion with connected fiber P. Therefore, M admits a description as the mapping torus associated to a diffeomorphism of P. Finally, ω endows P with a symplectic structure so that the diffeomorphism is indeed a symplectomorphism of P with the induced symplectic structure.

Besides contact and Poisson structures, and according to the behavior of the distribution D, it is worth mentioning the existence of literature concerning another kind of calibrated structures. In the monograph of W. Thurston and Y. Eliashberg [**17**], the authors define a *confoliation* as a codimension 1-distribution for which a defining 1-form α verifying $\alpha \wedge (d\alpha)^n \geq 0$ exists (the sign is not important, the point is that it does not change). Moreover, if M is 3-dimensional they say that the confoliation is taut if a dominating 2-form exists and a homotopic condition is fulfilled. That is, a confoliation is a calibrated structure for which the integrability of D varies between the contact and the integrability condition. The main result of in [**17**] implies that for any calibrated foliation in M^3 (different from $S^2 \times S^1$) it is possible to find a contact structure arbitrarily close so that we can interpolate between both structures using confoliations. Actually, the interpolation occurs at the level of 1-forms and it is given by a smooth path α_t. It is important to mention the motivation for this result: as we already said, not every foliation in an oriented closed 3-manifold gives topological information about the manifold. Similarly, contact structures are also divided into overtwisted and tight ones, which are the analog of foliations with generalized Reeb components and calibrated foliations respectively; all the information given by the latter is contained in its homotopy class as plane distributions. This analogy, together with other results, indicated a strong relation between tight contact structures and taut foliation, finally elucidated in the mentioned interpolation result (recall that a contact structure close enough to a calibrated confoliation was already known to be tight).

Finally, it should be mentioned that the most natural way to find calibrated structures is starting form the pair (M, D) and then trying to find closed 2-forms calibrating D, i.e., we think of calibrated structures as even dimensional codimension 1 distributions with a very special additional property, and from which –as we shall see– one can derive plenty of geometrical consequences. The problem of finding such a dominating 2-form is quite a complicated one. One can use a dual formulation in terms of the existence of certain *structural cycles* [**52**]. In dimension 3 it is possible to take advantage of this equivalence to derive existence results (and in higher dimensions there exist similar results for the existence of calibrating $2n$-forms, $2n$ being the dimension of D [**29**]). It is interesting to observe that in this dual formulation one already has to assume the existence of almost complex structures J in D to look for 2-forms positive in the complex lines defined by J in D [**52**].

But it is also possible to adopt the converse point of view. That is, we can start from a non-degenerate closed 2-form w in M^{2n+1}, and then select a codimension 1 distribution transverse to $\ker\omega$ as an auxiliary tool to try to deduce topological information of M (see [**38**]). We just observe that in dimensions bigger or equal than 5, once we assume the existence of a reduction of the structural group of TM to $U(n)$, it is a consequence of the h-principle that we can find closed non-degenerate 2-forms in any cohomology class [**27**].

The first new result that we want to mention is the extension to calibrated structures in closed manifolds of the existence of transverse cycles through any point for calibrated foliations in closed orientable 3-manifolds. Our "transverse cycles" will also inherit a structure of calibrated subvariety.

Definition 1.4. *Let (M, D, ω) be a calibrated structure. A calibrated submanifold of M is a submanifold W such that $TW \cap D$ is a codimension 1 distribution of TW and $\omega_{|TW\cap D}$ is non-degenerate.*

In other words, W is transverse to D and $TW \cap D$ is a symplectic subbundle of $(D, \omega_{|D})$.

Notice that when (M, D, ω) is compact, we can select a calibrating 2-form of integer type $\tilde{\omega}$ as close as we want to ω.

Theorem 1.5. *Let (M^{2n+1}, D, ω) be a closed calibrated manifold of integer type. For k large enough and for any $x \in M$ it is possible to find calibrated submanifolds W_k of M of codimension $2m$ through x which verify:*

- *The Poincaré dual of $[W_k]$ is $[k\omega]$.*
- *The inclusion $i\colon W_k \hookrightarrow M$ induces maps $i_*\colon \pi_j(W_k) \to \pi_j(M)$ (resp. $i_*\colon H_j(W_k;\mathbb{Z}) \to H_j(M;\mathbb{Z})$) which are isomorphisms for $j = 0, ..., n-m-1$ and epimorphisms for $j = n-m$.*

We must point out that this theorem can be also obtained as a simple corollary of the work of J. P. Mohsen[**43**] (see also [**23**]).

All the theory that we are to develop is based in a careful analysis of local situations, together with a globalization result whose origin is in the foundational paper of S. Donaldson [**12**].

First of all, let us see that theorem 1.5 is more or less obvious at local level.

Let us assume for simplicity that our calibrated manifold (M, D, ω) is Poisson, i.e., D is an integrable distribution giving rise to a foliation $\mathcal{F}$ by codimension 1 symplectic leaves. For each point $x \in M$ we take charts adapted to the foliation. The result is a local identification of $(M, \mathcal{F})$ with $\mathbb{R}^{2n} \times \mathbb{R}$. If our aim was to construct in principle a transverse cycle through x, a reasonable strategy would be trying to prolong the vertical $\{0\} \times \mathbb{R}$ transversely to $\mathcal{F}$, so that the resulting path C hits back the leaf F_x containing the point x.

In dimension 3 one checks that if had not such return, we could take a tubular neighborhood made transverse paths parallel to C that could be prolonged indefinitely. That would imply that the temporal evolution of the area of each disc (slice of the tubular neighborhood) would tend to zero very fast, because the neighborhood should have finite volume (that is exactly want happens inside a Reeb component). Thus, a way to avoid such a situation would be to take a globally defined vertical coordinate (the last in $\mathbb{R}^{2n} \times \mathbb{R}$) such that the evolution of the area associated to a certain leafwise 2-form does not tend to zero. This is equivalent (recall that we are in dimension 3) to the existence of a closed global 2-form calibrating D. The vertical coordinate globally defined is given by $\ker \omega$, because any vector field spanning this line field verifies $L_X \omega = 0$.

From the previous discussion we deduce that the choice of local chart for (M, D, ω) has to be any adapted to the foliation whose vertical direction coincides with $\ker \omega$. As we shall see it will be possible to find *Darboux charts* with coordinates $x^1, y^1, ..., x^n, y^n, s$ in which ω matches $\omega_0 = \sum_{i=1}^n dx^i \wedge dy^i$.

That means that calibrated foliations are a very special kind of foliations. In general it is possible to characterize foliated manifolds M^{2n+1} (with leaves of codimension 1) as those whose associated pseudogroup admits a reduction from $\mathrm{Diff}(\mathbb{R}^{2n+1})$ to $\mathrm{Diff}(\mathbb{R}^{2n} \times \mathbb{R}) \times \mathrm{Diff}(\mathbb{R})$. The calibrated structure induces another reduction to $\mathrm{Diff}(\mathbb{R}^{2n}) \times \mathrm{Diff}(\mathbb{R})$ (global vertical coordinate) and further to $\mathrm{Symp}(\mathbb{R}^{2n}, \omega_0) \times \mathrm{Diff}(\mathbb{R})$.

If the distribution of our calibrated structure is not integrable, then we cannot speak of absence of local invariants, but the local "picture" is anyhow very similar to that of calibrated foliations, because if we work in tiny open sets the distribution D will be very close to the one given in appropriate charts by "horizontal hyperplanes". In other words, it is possible to find coordinates $x^1, y^1, ..., x^n, y^n, s$ so that the pullback of ω is w_0, and D coincides in the origin with D_h, the foliation defined by the level hypersurfaces of s.

One can think of a construction of calibrated submanifolds in arbitrary dimensions analogous to that of transverse cycles in dimension 3. Locally, the submanifold would correspond to a symplectic submanifold V of $(\mathbb{R}^{2n}, \omega_0)$ multiplied by the vertical coordinate. Even more, since we have Darboux charts we might think of V as a symplectic vector space of $(\mathbb{R}^{2n}, \omega_0)$. Of course, the situation is far more complicated than for cycles in dimension 3. Firstly, and for each leaf F_x, it is necessary to make sure that all the local symplectic subspaces (or more generally symplectic submanifolds of $(\mathbb{R}^{2n}, \omega_0)$ "glue" into a symplectic submanifold of the leaf; this is exactly the kind of problem that the approximately holomorphic techniques solve in compact symplectic manifolds, the construction of symplectic submanifolds gluing local solutions. It has to be pointed out that even though the leaves will not be closed in general, the manifold it is indeed closed.

Secondly, once we have a symplectic subvariety in a certain leaf, it is necessary that it propagates along a transverse direction into symplectic submanifolds of the corresponding leaves, and it must return to the starting leaf; besides, from that return one has to be able to "close" the subvariety. Again, the is something to be expected to be unobstructed, because if the leaf is closed, the approximately holomorphic theory for symplectic manifolds gives us an isotopy through symplectic submanifolds joining both subvarieties.

Hence, one should expect an analog of approximately holomorphic theory in symplectic manifolds to work in closed calibrated manifolds.

We will now recall the main ideas underlying the approximately holomorphic theory for compact symplectic manifolds (M, ω), to be generalized to calibrated varieties.

1.2. Some ideas underlying the approximately holomorphic geometry in compact symplectic manifolds. One first element of this theory is the observation, at linear level, that a way to choose linear symplectic subspaces V is by introducing an almost complex structure J compatible with ω; automatically, every J-complex subspace becomes symplectic. Since being symplectic is an open condition, if a subspace V is "close enough" to JV, it will also be symplectic. Thus, for any almost complex manifold $(N, \bar{J})$, the existence of an "approximately" $(J, \bar{J})$-complex map $f\colon M \to N$ transverse to a J_1-complex subvariety $N_1 \subset N$, will give rise to a symplectic subvariety of M of the same codimension as that of N_1 in N (the subvariety is just $f^{-1}(N_1)$).

Therefore, the first problem would be to be able to find almost complex manifolds $(N, \bar{J})$ which might the target space of "enough" maps $f\colon M \to N$ "close" to be $J - J_1$-complex ("close" in a sense that we will make precise and "enough" in the sense that given one of this maps it is possible to perturb it within the class so that it becomes transverse to certain J_1-complex submanifolds of $(N, \bar{J})$). Instead of working with general almost complex manifolds as target space, the reasonable thing to do is to start by the complex vector spaces $(\mathbb{C}^m, J_0)$, where we are also allowed to add maps (something that will make easier to define the sought for perturbations). More generally, we will look for sections of complex vector bundles over M.

The hint indicating which ones are the appropriate vector bundles we should work with comes form the integrable situation. In that setting, since the Cauchy-Riemann equations are not over determined we can look for usual holomorphic sections. Results from complex geometry imply that a class of bundles, in principle line bundles, admitting a lot holomorphic sections, is that of very ample line bundles. Besides, there is a clear sufficient criteria to find such line bundles. It is enough to have a hermitian line bundle L with compatible connection ∇, so that its curvature F_∇ is positive (ample line bundle). Then, large enough tensor powers of L do have a lot of holomorphic sections (very ample line bundles); indeed "many" of these holomorphic sections turn out to be transverse to the $\mathbf{0}$ section of $L^{\otimes k}$ (Bertini's theorem), which is a complex submanifold of the total space of the line bundle, which is itself a complex manifold.

In the previous situation (ample bundle), iF_∇ defines a closed 2-form which endows M with a symplectic structure (a Kahler structure, because J is clearly compatible with the 2-form). Obviously, iF is an integer class, so what we have in M is a Hodge structure. When we start with a Hodge structure (M, J, ω), the previous process can be reversed to construct an ample line bundle (L, ∇), whose curvature is exactly $-i\omega$.

If we start from a pair (M, ω), with M compact and ω of integer type, and fix J-compatible almost complex structure, the construction of the hermitian line bundle (L, ∇) with curvature $-i\omega$ is still possible. Being J in general not integrable, the question is whether there will be enough sections of the powers $L^{\otimes k}$ ("very ample") close enough to be holomorphic. Locally, the existence of Darboux charts and the fact that the curvature of the line bundle coincides with ω_0, allow us to find also a model for the connection form of the line bundle, using a suitable trivialization. Assuming that in Darboux charts J coincides with J_0 (we can always find Darboux charts centered at any point and so at the origin $J = J_0$), it is possible to write down explicit solutions to the Cauchy Riemann equations which have very special properties. These (local) sections have *gaussian decay* and play the role of partition functions (sections) of the theory, because using them the global transversality problem can be localized and turned from a transversally problem for sections into a transversality problem for functions.

In the general case when J is not integrable we can still get Darboux charts with J matching J_0 at the origin. That will imply, provided the domain of the charts is small enough, that the previously mentioned solutions will almost be solutions to the corresponding Cauchy-Riemann equations. Of course, it is not enough to restrict the domain of the chart and keep the restriction of the previous solution, because this would be almost flat (constant) and hence would not have the appropriate decay; thus it would not be useful to localize the transversality problem. A way to overcome this difficulty is, instead of restricting to a smaller domain of the chart, contracting the whole picture. In particular, the contraction factor used is $k^{-1/2}$. As a consequence, the model solution once assumed to belong to a "contracted chart", turns out to come from a section of $L^{\otimes k}$. In other words, the symplectic form transforms into $k\omega$ (and the metric is also multiplied by k), so that the "contracted" chart is a Darboux chart for $k\omega$. Thus, by increasing the curvature of the line bundle one has access to smaller and smaller regions where the almost complex structure J necessarily looks like an integrable one (as much as we want by increasing the value of k).

As we mentioned, the existence of the so called *reference sections* (global sections constructed out of the model solutions) allows us to transform the global transversality problem into a lot of local transversality problems for functions (this number increases with k). An important observation is that if we want to add up local solutions (local perturbations), since the reference sections have support in a domain much bigger than the one where they are used to localize the transversality problem, there will be interference due to the overlap between the supports corresponding to different reference sections. The difficulty is that transversality does not behave well under addition of sections.

It is necessary to use the concept of *estimated transversality.* The idea behind it is quite simple: the sum of two sections transverse to the $\mathbf{0}$ section may fail to remain transverse; for example, because both vanish in a point with opposite differential (o more geometrically, because their graphs intersect with opposite angle). Estimated transversality requires that the graph of a section not only cuts the $\mathbf{0}$ section transversely, but also asks for the same property with respect to the parallel copies of the $\mathbf{0}$ section in a tubular neighborhood. Besides, the angle in all these points in the tubular neighborhood must be bounded by below in a sense to be made precise later.

The advantage is that this strongest notion of transversality is C^1-open in the sense that if a section has enough estimated transversality to the $\mathbf{0}$ section, if we add to it another section whose C^1-norm is small compared with the "amount of transversality", the result is a section which is still transverse to $\mathbf{0}$), and whose amount of transversality can be estimated in terms of the amount of transversality of the original section and the C^1-size of the added perturbation.

The most delicate element of the approximately holomorphic theory is showing that, indeed, every function close enough to be holomorphic admits arbitrarily small perturbations so that it becomes transverse to $\mathbf{0}$ (local transversality lemma), and with an amount of transversality such that it is possible to develop a process to solve each of the local transversality problems, and when we add up all the perturbations the result is a section which is transverse to $\mathbf{0}$ (globalization process).

The previous globalization process is valid, not only to achieve transversality to the $\mathbf{0}$ section, but to other (sequences of) holomorphic submanifolds of $L^{\otimes k}$, or of other appropriate bundles and, more generally, to certain stratifications by holomorphic submanifolds of the total spaces of the bundles (actually to submanifolds and stratifications "close enough" to be holomorphic). This should not come as a surprise because using the globalization process the construction relies on purely local results, and locally it is possible to find holomorphic coordinates and sections so that the corresponding holomorphic subvarieties of the total spaces are represented as the $\mathbf{0}$ section of a local trivial holomorphic bundle (again, instead of being holomorphic everything is actually close to be holomorphic in a certain sense).

Once the existence of (sequences of) *approximately holomorphic sections* with good transversality properties has been established, at least for some of them (the sequence of functions to projective spaces arising from projectivizations) it is natural to study its degenerations spaces, or in other words, to define the corresponding r-jet bundles and the subspaces or, more generally, the stratification given by the degeneration loci. The final goal is to proof a strong transversality theorem which would imply the existence of r-*generic* approximately holomorphic sections (i.e., sections whose r-jet is transverse to the corresponding stratification). One checks that the mentioned strong transversality result holds, the reason being that the corresponding bundles of r-jets and stratifications are suited to get transversality to them (though the technical complications are by no means trivial).

Once the existence of r-generic sections has been proven, the final step is trying to obtain normal forms for them. For example, analogs to complex Morse functions or functions to $\mathbb{CP}^2$ with the canonical singularity types. Those sections will give rise to a number of interesting structures in the symplectic manifold (M, ω) by objects (submanifolds, fibers) close to be J-complex and hence symplectic.

1.3. Description of the contents and results. In this dissertation we aim to make a study for calibrated structures analog to the previously described for symplectic manifolds. We will adopt a more general point of view similar to that introduced by D. Auroux [**4**], and will develop the whole theory for almost complex manifolds, whose definition is introduced in section 2, together with an analysis of their linear algebra. Next, we introduce the sequences of line bundles that are candidates to posses plenty of approximately holomorphic sections (definition 2.2). Also, an elementary symplectization procedure –which will be the key to obtain all the results through a refinement of the existing approximately holomorphic relative theory– is described.

The local model for almost complex manifolds (always of odd dimension unless otherwise stated) is introduced in section 3 (definition 3.1), and it turns out to be the one existing for even dimensional almost complex manifolds multiplied by a distinguished vertical coordinate. Given (sequences of sections), it will be necessary to make different estimates regarding its size and the size of other sections associated to them (covariant derivatives, certain projections of them, holomorphic and antiholomorphic components). We want to use the models so that the measures taken using the geometric elements of the model (euclidean metric and its Levi-Civita connection and distance, canonical almost complex structure J_0,...) are the same, up to a *uniform* constant , i.e., it must not depend neither in the point where the chart is centered nor in the bundle of the sequence in question (that is, there is no dependence in k either), as the measures computed using the original global geometric elements. The reason is that those of the model are much easier to handle. It is important that the final bounds that we get for the global sections are given using *uniform* constants, because some of them will be multiplied by a factor of the form $k^{-1/2}$ (for example the one governing the antiholomorphy of the sections) and thus by increasing k will be as close as we want to zero implying the fulfilment of the required properties. All this will summarized in the notion of approximate equality and local approximate property or equality (definition 3.6).

Depending on the estimates we want to obtain we will use different kinds of charts in the base manifold (the *approximate holomorphic coordinates* of definition 3.34) and even r-*comparable charts* (definition 3.4) in the total space of the bundles.

In the local model –whose domain is $\mathbb{C}^n \times \mathbb{R}$– the distribution is the one given by the complex hyperplanes (we also refer to it as the *horizontal distribution* and is denoted by D_h). We will need to have a notion of proximity between distributions (possibly of different dimensions), for example between the distribution D_h of the model and D. For that analysis it will be necessary to recall the concepts of maximal and minimal angle (definitions 3.10, 3.11).

The possibility of finding coordinate charts with suitable properties will imply that the situation for (odd dimensional) compact almost complex manifolds with a very ample sequence of line bundles (L_k, ∇_k) is essentially the same one has for (compact) calibrated manifolds (M, D, ω); the sequence of closed 2-forms iF_{∇_k} dominating D behaves as the sequence $k\omega$ would do, and similarly for the pairs (iF_{∇_k}, J) (lemma 3.21).

We will define what a sequence of approximately holomorphic (A.H.) sections is (also called approximately holomorphic sequence), together with the notion of gaussian decay with respect to a point of M (definition 3.25). The existence of *reference sections* will be proven using some previous lemmas which relate estimates using the elements of the integrable model (metric, connection, distribution, almost-complex structure,...) and the global ones.

All the mentioned material introduced in section 2 will be necessary to develop the intrinsic A.H. theory.

The content of the last subsection of section 2 is a discussion on the relation among the different intrinsic A.H. theories, where two notions of equivalence are introduced. The existence of different intrinsic theories follows from the fact that there is no natural retraction to the canonical projection $T^*M \to D^*$, and each one defines a different intrinsic theory.

We end up section 3 comparing the constructions of the intrinsic theory with those coming from the even dimensional theory, trivially available through the symplectization procedure, and describing the correspondent theories (intrinsic and relative) for very ample sequences of vector bundles of arbitrary rank.

In section 4 and for an appropriate sequence of hermitian vector bundles E_k over (M, D, J), the sequence of vector bundles of pseudo-holomorphic r-jets $\mathcal{J}^r_D E_k$ is introduced (definition 4.1). Similarly to what happens in the even dimensional case, the definition is such that for the local model what we consider is the bundle of foliated coupled holomorphic jets (there is a connection form coming from a suitable trivialization of the bundle). Observe that since the r-jets are defined recursively using the holomorphic part of the corresponding covariant derivative, the connection of E_k plays a very important role. The important consequence is that due to presence of curvature, for the obvious connection in $\mathcal{J}^r_D E_k$, the r-jet of an A.H. sequence of sections of E_k is not an A.H. sequence of sections of $\mathcal{J}^r_D E_k$ anymore. We fix this problem by introducing a new almost complex structure in $\mathcal{J}^r_D E_k$ (connection) for which this property is fulfilled. The fundamental result is that we start from a sequence of vector bundles E_k for which the transversality result for A.H. sequences holds, the same happens for the sequences of r-jets of sections of E_k (proposition 4.6).

The analogous constructions for the relative theory will be also introduced. In this case, the starting point is a J-complex distribution G (a *polarization*) of an even dimensional almost-complex manifold (M, J, g); there is a natural definition for the subbundle $\mathcal{J}^r_G E_k \subset \mathcal{J}^r E_k$, the subbundle of r-jets along G; similarly, given an A.H. sequence of sections of E_k, the sequence of r-jets along G is introduced. It is important to observe that we do not pretend to develop an approximately holomorphic theory for the sequence $\mathcal{J}^r_G E_k$, but similarly to what happens in the theory of foliated jets, we pretend to transfer transversality problems from $\mathcal{J}^r_G E_k$ to $\mathcal{J}^r E_k$ using the canonical submersion; of course, it will be necessary to study when this is possible, and that the solution of the induced problem in $\mathcal{J}^r E_k$ is indeed a solution of the original problem in the subbundle.

Our goal is not only to obtain transversality for sequences of A.H. sections of E_k (and $\mathcal{J}^r E_k$) to the $\mathbf{0}$ section of the corresponding bundles, but also to (sequences of) subvarieties and even stratifications of these sequences of bundles. The first part of section 5 is devoted to the introduction of the notion of (sequences of) *approximately holomorphic stratifications* (definition 5.2) –a natural extension of the corresponding concept in the case of even dimensional manifolds– together with a local description of them.

Next, we recall the concept of uniform transversality to the $\mathbf{0}$ section of a sequence of sections of the hermitian bundles E_k, an extend it in the obvious way to approximately holomorphic stratifications. The technical part that follows (subsection 5.2) takes advantage of the local description to conclude that uniform local transversality to this kind of stratifications is equivalent to estimated transversality of a function $h\colon \mathbb{C}^n \times \mathbb{R} \to \mathbb{C}^m$ to $\mathbf{0}$ along D_h (the complex hyperplanes) (lemma 5.9). A more detailed analysis of the concept of minimal angle and its variations is needed at this point. It is quite interesting that the previous discussion, in principle oriented to the intrinsic theory, works almost word by word in the relative case. This is another example of how the guiding principles of both theories coincide (they are just different foliated versions of the original theory for even dimensional almost-complex manifolds).

It is possible to weaken the notion of an A.H. stratification to the more general one of A.H. quasi-stratification (definition 5.23), so that transversality to the latter can be obtained using the same mechanisms as for the former. This new concept is introduced to work with the *Thom-Boardman-Auroux quasi-stratification* (subsection 5.3). In Kähler geometry the vector spaces of sections of $L^{\otimes k}$ (complete linear systems) are used to define maps to the projective spaces. The analog to linear systems of rank m in the A.H. setting are the A.H. sequences τ_k of the bundles $\mathbb{C}^{m+1} \otimes L_k$, L_k a very ample sequence. Out of the points that go to the $\mathbf{0}$ section (base points A_k), there is an induced A.H. sequence of maps $\phi_k\colon M - A_k \to \mathbb{CP}^m$. It is natural trying to perturb them to make them r-generic. A necessary step is to define $\mathcal{J}_D(M, \mathbb{CP}^m)$, the non-linear bundles of pseudo-holomorphic r-jets of maps to $\mathbb{CP}^m$, and in the total spaces the stratifications analogous to the Thom-Boardmann ones. Actually, the strategy is to use the submersion $\mathbb{C}^{m+1} - \{0\} \to \mathbb{CP}^m$ to define a quasi-stratification in $\mathcal{J}^r_D E_k$, the Thom-Boardman-Auroux quasi-stratification, so that transversality to if of the r-jets of the sections τ_k will imply estimated transversality to the Thom-Boardmann stratification of the r-jets of the projectivizations ϕ_k. The definition of the bundles $\mathcal{J}_D(M, \mathbb{CP}^m)$, the Thom-Boardmann-Auroux quasi-stratifications and the properties that allow to conclude the approximate holomorphicity of the strata, and the corresponding analysis in the relative case, are quite delicate (propositions 5.24 and 5.25 and lemma 5.27). The main difficulty is that since $\mathcal{J}^r_D E_k$ carries a modified almost-complex structure, it is quite hard to check that the strata are suited to obtain transversality to them (essentially that they are locally defined by A.H. functions w.r.t. the modified almost-complex structure of the total space), and ad hod constructions have to be used.

The content of section 6 is the proof of the strong transversality theorem to sequences of quasi-stratifications of the bundles $\mathcal{J}^r_D E_k$, and the relative version for polarized even dimensional almost-complex manifolds (theorem 6.1, corollary 6.3). We notice that we perturb the sections of E_k so that the r-jets become transverse (strong transversality). All the results from the previous sections, together with an appropriate globalization procedure, reduce the proof to an estimated transversality problem for A.H. functions $F\colon \mathbb{C}^n \times \mathbb{R} \to \mathbb{C}^m$. The required solution to this problem is a refinement of the one given in [**32**] or [**50**] for contact manifolds, extending S. Donaldson's result [**12**] (and D. Auroux' refinement [**2**]).

We would like to make a comment about the local transversality theorem. This result is based in the fact that the corresponding estimated transversality theorem for A.H. functions $h\colon \mathbb{C}^n \to \mathbb{C}^m$ holds also for one parameter families. Actually, it works because the parameter is real. It is not valid anymore for a complex parameter, because similarly to what happens in the real case, foliated transversality does not behave well with respect to the foliation itself. This is the reason to work with codimension 1 foliations and not dealing with higher codimension ones.

There is a second complication due to the non-integrability of D. To obtain a strong transversality theorem to quasi-stratifications of $\mathcal{J}^r_D E_k$, $r \geq 1$, it will be necessary to consider A.H. sections of E_k all whose derivatives are controlled ($C^{\geq r+h}$-A.H. sections).

The fundamental local result for relative constructions is J.-P. Mohsen's local relative transversality theorem Mohsen [**43**] for A.H. functions $h\colon \mathbb{C}^n \to \mathbb{C}^m$ respect to a fixed submanifold Q, that after a suitable choice of charts reduces to the local transversality theorem for A.H. sequences of functions $\bar{h}_k\colon \mathbb{C}^n \to \mathbb{C}^m$.

We will see how the different "quality" of the perturbations obtained form the local theorems (better in the relative theory than in the intrinsic), makes it possible to work in the relative setting with A.H. sequences of sections only requiring control on a finite number of derivatives.

In section 7 we will state a number of results obtained from the main theorems proved in section 6.

The applications for closed calibrated manifolds derived from transversality to 0-jets are, first of all, the already mentioned theorem 1.5, which follows from transversality to the **0** section of certain vector bundles E_k, and secondly the following result about the existence of determinantal subvarieties:

Proposition 1.6. *Let (M, D, ω) be a closed calibrated manifold of integer type and $L^{\otimes k}$ the sequence of powers of the pre-quantum line bundle. Let E, F hermitian vector bundles with connection and consider the sequence of bundles $I_k = E^* \otimes F \otimes L_k$. Then for k large enough there exist τ_k A.H. sequences of sections of I_k for which the determinantal loci $\Sigma_k^i(\tau_k) = \{x \in M |\ rk(\tau_k) = i\}$ are calibrated submanifolds (of integer type) stratifying M.*

The fundamental result of the approximately holomorphic theory for (closed) calibrated manifolds is a consequence of transversality to the Thom-Boardman-Auroux quasi-stratification.

Theorem 1.7. *Let (M, D, ω) be a closed calibrated manifold of integer type. fix J a compatible almost complex structure. Then it is possible to find A.H. sequences of r-generic maps to any $\mathbb{CP}^m$.*

As a corollary, we obtain an analog to the embedding theorem in projective spaces for symplectic manifolds (already proven for contact manifolds in [**47**]).

Corollary 1.8. *Let (M^{2n+1}, D, ω) a compact calibrated manifold of integer type. Let J be a compatible almost complex structure. Then it is possible to find sequences of maps $\phi_k \colon M \to \mathbb{CP}^{2n+2}$ so that for k large enough one has:*

- *ϕ_k is an A.H. immersion along the directions of D.*
- *$[\phi_k^* \omega_{FS}] = [k\omega]$, where ω_{FS} is the Fubini-Study 2-form of $\mathbb{CP}^{2n+2}$.*

Observe that these result, without making any reference to the almost-complex structure, is easily obtained applying the theory of characteristic classes together with the density of embeddings when the target space has large enough dimension. Thee interesting point is that if for example we have a (singular) holomorphic foliation of $\mathbb{CP}^{2n+1}$, it is possible to find embeddings transverse to the foliation along D, and thus inducing (singular) foliations in M with calibrated leaves.

Section 8 is devoted to the study of normal forms for A.H. maps to $\mathbb{CP}^1$, and the corresponding geometric corollaries for closed calibrated manifolds.

Observe that even in the even dimensional case, an r-generic A.H. map does not necessarily has the same behavior as a holomorphic one. If we go to odd dimensions, the behavior of a foliated holomorphic map is not that easy to describe due to the presence of a direction that we do not have control over. Anyhow, when the target space has (complex) dimension 1, we will see that it is possible to give a reasonable description of the map (and we only need to work with 1-jets to obtain genericity). Now going back to the more general setting of odd dimensional almost-complex manifolds, a 1-generic A.H. sequence of maps $\phi_k \colon M - Z_k \to \mathbb{CP}^1$ can be perturbed so that it conforms the required local models, where the perturbation takes place in a tubular neighborhood of the base locus and the degeneration locus of the 1-jet. Near this degeneration locus, which is a 1-dimensional manifold (a link), the small size of the antiholomorphic part of the derivative does not give much information, since the holomorphic part itself is equally very small around the vanishing locus. The sough for perturbation will be one making the antiholomorphic part of the derivative vanish in a small neighborhood of the degeneration locus. That will imply the existence of charts in which the map matches the required models, which are essentially a 1-parameter version of the holomorphic ones

An application of the existence of normal forms for appropriate A.H. maps to $\mathbb{CP}^1$ is the existence of Lefschetz pencil structures for closed calibrated manifolds.

Definition 1.9. *Let (M, D, ω) be a closed calibrated manifold. A Lefschetz pencil structure in M is given by a triple (A, f, B) so that:*

(1) *A is a closed calibrated subvariety of M real codimension* 4.
(2) *$f : M - A \to S^2$ is a smooth map.*
(3) *B, defined as the set of points where f fails to be a submersion along the directions of D, is a smooth calibrated subvariety of dimension 1. The image (by f) of each connected component of B is immersed and the intersections are transverse double points (i.e., $f_{|B}$ is generic).*

Besides f verifies the following properties:

- *For any point $a \in A$ there exists coordinates $z^1, ..., z^n, s$ centered at a and compatible with ω and a holomorphic chart of $\mathbb{CP}^1$, so that in a euclidean ball in the domain of the chart A is defined by the equations $z^1 = z^2 = 0$, and out of A, $f(z, s) = \frac{z^2}{z^1}$.*
- *For any point $b \in B$ there exists coordinates $z^1, ..., z^n, s$ centered at b and compatible with ω and a holomorphic chart of $\mathbb{CP}^1$, so that $f(z, s) = g(s) + (z^1)^2 + \cdots (z^n)^2$, where $g(0) = f(b)$ y $g'(0) \neq 0$.*

Out of the singular values $f(B)$, the inverse image of each regular point c defines an open calibrated submanifold of $M - A$. Using the local model in the points of A, it is straightforward that the closure of $f^{-1}(c)$ is the closed calibrated submanifold $f^{-1}(c) \cup A$. We refer to this compactification as a fiber of f.

Without entering into any detail about what the compatible charts with ω of the previous definition are (see definition 8.4), we state the following result.

Theorem 1.10. *Every closed calibrated manifold admits a Lefschetz pencil structure.*

In the last section we will consider the special case of calibrated foliations and the particular features of the theory in that particular case.

Finally, we will go back to the 3-dimensional calibrated foliations that served as motivation, and reinterpret some of the obtained results.

2. Almost-complex manifolds: linear algebra, very ample sequences of line bundles and symplectizations

Definition 2.1. *Let (M, D) be a manifold with a smooth distribution. An almost-complex structure adapted to D is a 4-tuple (M, D, J, g), where g is a riemannian metric , J an almost-complex structure on D and J is $g_{|D}$-antisymmetric. Therefore, the dimension of D has to be even.*

From now on and whenever there is no risk of confusion we will omit any reference to D as given data, and we will call the quartet (M, D, J, g) an almost-complex manifold.

The g-antisymmetry of J is used to define a hermitian metric on D by the formula $h(\cdot,\cdot) = g(\cdot,\cdot) + ig(\cdot, J)$.

In general D is allowed to have any (even) dimension, but for us, and from now on, D will either have codimension 1 or will be the whole tangent bundle (and in this last case we will mention it explicitly). The definitions to be given extend the existing ones [**4**] for almost-complex manifolds in which $D = TM$ (or almost-complex manifolds of even dimension, from now on).

For us, if M is odd dimensional it will be assumed to be compact (sometimes closed). Also, and unless otherwise stated, an almost-complex manifold of odd dimension (in which D has codimension 1) will be referred simply as an almost-complex manifold.

2.1. Linear algebra of almost-complex vector spaces.

Let V be a vector space and D a codimension 1 vector subspace endowed with an almost-complex structure J. We are mainly interested in studying the spaces of r-forms in V which are non-trivial when restricted to D.

Let us denote by $p\colon V^* \to D^*$ to the canonical projection. The kernel of p, $\mathrm{Ann}(D)$, is a line inside V^*. Let us consider $V^*_{D\neq 0}$ defined as the union of the complementary of $\mathrm{Ann}(D)$ inside V^* and the zero vector ($V^* = \mathrm{Ann}(D) \cup V^*_{D\neq 0}$, $\mathrm{Ann}(D) \cap V^*_{D\neq 0} = \{0\}$).

Similarly, we can consider the subspace $V^*_{\mathbb{C},D\neq 0}$ inside the complexification $V_{\mathbb{C}}$, which contains the subsets $V^{*1,0}_{D\neq 0}, V^{*0,1}_{D\neq 0}$ corresponding to 1-forms whose restriction to D is complex linear (resp. anticomplex linear), together with the zero vector. In other words, these subsets are the intersection of $V^*_{\mathbb{C},D\neq 0}$ with the inverse images of $D^{*1,0}$ and $D^{*0,1}$ respectively.

We adopt the notation $V^{*\otimes r} := V^* \otimes \overset{(r)}{\cdots} \otimes V^*$, $V^{*\odot r} := V^* \odot \overset{(r)}{\cdots} \odot V^*$ for the symmetric product and $\wedge^r V^*$ for the antisymmetric product. Again, we are only interested in forms whose restriction to D is non vanishing. That is, let us consider the obvious projection $p^r\colon V^{*\otimes r} \to D^{\otimes r}$, denote its kernel by $\mathrm{Ann}^r(D)$ and by $V^{*\otimes r}_{D\neq 0}$ to the union of its complementary and the zero vector (form). This last subset contains the subsets $V^{*\odot r}_{D\neq 0}$ and $\wedge^r V^*_{D\neq 0}$ of symmetric and anti-symmetric r-forms, defined as the inverse image in $V^{*\otimes r}_{D\neq 0}$ of $D^{*\odot r}$ and $\wedge^r D^*$. After complexifying, each subset $V^{*\otimes r}_{\mathbb{C},D\neq 0}$ (resp. $\wedge^r V^*_{\mathbb{C},D\neq 0}$) contains other subsets (with pairwise intersection the zero vector) according to the types determined by the almost-complex structure.

Back to calibrated manifolds (or more generally to almost-complex manifolds), we will work with certain sections σ, for example of the bundles $D^*_{\mathbb{C}}$, and we will be interested in defining and measure its covariant derivatives. A natural way to define a connection is as follows: first, choose a lift of σ to a section of TM^*, which will be necessarily contained in $TM^*_{\mathbb{C},D\neq 0}$, and then we compute the derivative using the Levi-Civita connection associated to the metric.

The reasonable way to define such lifts is by fixing a retraction $i\colon D^* \to V^*$ for p, which canonically induces retractions for p^r. If V carries a metric (inner product), it defines a retraction $\bar{\imath}$ whose image we denote by $\bar{D}^*$ (the forms vanishing in the orthogonal to D, denoted by $\langle \frac{\partial}{\partial s} \rangle$).

Let $\tilde{\imath}$ be any other retraction, and let us denote its image by $\tilde{D}^*$. We have induced retractions $\bar{\imath}^r, \tilde{\imath}^r$ for p^r.

If we decompose $V^* = \bar{D}^* \oplus \mathrm{Ann}(D)$, $\tilde{D}^*$ can be represented as the graph of a linear map $l\colon \bar{D}^* \to \mathrm{Ann}(D)$, ($\tilde{D}^* = (I + l)(\bar{D}^*)$), and a bound for the norm of l will follow from a bound by below for the angle $\angle(\tilde{D}^*, \mathrm{Ann}(D))$. There is also an induced map $l^r\colon \bar{D}^{*\otimes r} \to \mathrm{Ann}^r(D)$ so that $(I + l^r)(\bar{D}^{*\otimes r}) = \tilde{D}^{*\otimes r}$.

Inside the vector subspaces $\bar{D}^{*\otimes r}$ y $\tilde{D}^{*\otimes r}$ and their complexifications we have the subspaces of symmetric and anti-symmetric forms defined as the intersection of $V^{*\odot r}_{D\neq 0}$ and $\wedge^r V^*_{D\neq 0}$ with the usual subspaces of symmetric and anti-symmetric forms respectively; equivalently, they can be defined as the forms inside the cited subspaces whose restriction to D is symmetric (resp. anti-symmetric). We can proceed similarly in the complexifications to define the subspaces associated to the types defined by the complex structure in D. It is worth mentioning that the restriction of p^r to $\bar{D}^{*\otimes r}$ and $\tilde{D}^{*\otimes r}$ (and its complexifications), the inclusions $\bar{i}^r, \tilde{i}^r$ and the map $I+l^r \colon \bar{D}^{*\otimes r} \to \tilde{D}^{*\otimes r}$ preserve all this subspaces.

We stress again that the components with respect the two decompositions (for any value of r) are related by the linear map $I + l^r$. For example if $\alpha \in V^*$, denote its projection on D^* by $\alpha_{|D}$ (evaluation in D), its component in $\bar{D}$ by α_D, and its component in $\tilde{D}$ by $\alpha_{\tilde{D}}$. To compute $\alpha_{\tilde{D}}$ explicitly from α_D, we consider a 1-form ds vanishing in D and taking the value 1 over a vector $\frac{\partial}{\partial s}$ orthogonal to D. Next, we take a vector $v_k \in D$ so that $\tilde{D} = \mathrm{Ann}(\frac{\partial}{\partial s} + v_k)$. It is easy to check that $\alpha_{\tilde{D}} = \alpha_D - \alpha(v_k)ds$ (resp. $\alpha^{1,0}_{\tilde{D}} = \alpha^{1,0}_D - \alpha^{1,0}_D(v_k)ds$, $\alpha^{0,1}_{\tilde{D}} = \alpha^{0,1}_D - \alpha^{0,1}_D(v_k)ds$).

Thus, a bound for the norm of l implies the existence of a positive constant κ such that:

- $|\alpha_D| \leq |\alpha_{\tilde{D}}|$, $|\alpha^{*1,0}_D| \leq |\alpha^{1,0}_{\tilde{D}}|$, $|\alpha^{*0,1}_D| \leq |\alpha^{0,1}_{\tilde{D}}|$.
- $|\alpha_{\tilde{D}}| \leq \kappa|\alpha_D|$, $|\alpha^{1,0}_{\tilde{D}}| \leq \kappa|\alpha^{*1,0}_D|$, $|\alpha^{0,1}_{\tilde{D}}| \leq \kappa|\alpha^{*0,1}_D|$.

We also notice that for $\beta \in V^{*\otimes r}_{\mathbb{C}}$ $r > 1$, we can generalize the previous construction to compute explicitly $\beta_{\tilde{D}}$ form β_D.

2.2. Very ample bundles. In order to control the geometric properties of D, in principle locally, but as we shall see globally, we will ask for the existence of a closed 2-form calibrating D^{2n} (its n-th power). Being more precise, the 2-form will arise from the curvature of a hermitian line bundle. When $D = TM$, this is the already known concept of ample bundle that we generalize to the odd dimensional setting in the natural way.

Definition 2.2. [**4**] *Given c, δ positive real numbers, a hermitian line bundle with compatible connection $(L, \nabla) \to (M, D, J, g)$ is (c, δ)-D-ample (o simply ample) if its curvature F verifies $iF(v, Jv) \geq cg(v, v)$,*
$\forall v \in D$ (and hence it is non-degenerate and belongs to $\wedge^2 TM^*_{\mathbb{C},D\neq 0}$), and $|F_{|D} - F^{1,1}_{|D}|_g \leq \delta$, where we use the supremum norm.

A sequence of hermitian line bundles with compatible connection (L_k, ∇_k) is asymptotically very ample (or just very ample) if fixed constants δ, $(C_j)_{j\geq 0}$ and a sequence $c_k \to \infty$ exist, so that from some $K \in \mathbb{N}$ on the curvatures F_k verify:

(1) $iF_k(v, Jv) \geq c_k g(v, v), \forall v \in D$
(2) $|F_{k|D} - F^{1,1}_{k|D}|_g \leq \delta c_k^{1/2}$
(3) $|\nabla^j F_k|_g \leq C_j c_k$,

Remark 2.3: Since $iF_k \in \wedge^2 TM^*_{\mathbb{C},D\neq 0}$, this 2-form is almost completely determined by its restriction to D. Being D a codimension 1 distribution, we do have a line field $\ker F_k$ transverse to D and for which if $R_k \in \ker F_k$, $F_k(R_k, \cdot) = 0$.

From now on $\tilde{i}$ will denote the retraction associated to $\ker F_k$, and we will refer to it as the retraction associated to the curvatures or to $\omega_k := iF_k$. The image of D^* by this retraction will be denoted by $\tilde{D}^*$ (we will not use the subindex k, even though the retraction $\tilde{i}$ varies with k). Recall also that $\bar{i}$ denotes the retraction associated to the metric. It is clear that $iF_k = iF_{k,\tilde{D}}$. Therefore, the condition $|F_{k|D} - F_{k|D}^{1,1}|_g \leq \delta c_k^{1/2}$ is equivalent to having in the decomposition $iF_{k,\tilde{D}} = iF_{k,\tilde{D}}^{2,0} + iF_{k,\tilde{D}}^{1,1} + iF_{k,\tilde{D}}^{0,2}$ the norms $|iF_{k,\tilde{D}}^{2,0}|_g, |iF_{k,\tilde{D}}^{0,2}|_g$ bounded by $O(c_k^{1/2})$. As we saw in the subsection 2.1, this is also equivalent to having bounds of the same order for the components $(0,2)$ y $(2,0)$ of the projection of the curvature on $i(D_{\mathbb{C}}^*)$ parallel to de $\mathrm{Ann}(D)$, where i is any retraction. In particular for the retraction associated to the metric.

If we fix a smooth family of charts the previous computations can be performed using the euclidean metric in the domain of the charts; the result is the same kind of bounds but with constants $c_k' = Cc_k$, $C > 0$.

The meaning of the previous bounds is better understood once the metric g is rescaled to define the sequence of metrics $g_k := c_k g$ (or equivalently contracting the charts by the factor $c_k^{-1/2}$). The new bounds are of the form $iF_k(v, Jv) \geq g_k(v, v)$, $\forall v \in D$, $|F_{k|D} - F_{k|D}^{1,1}|_{g_k} \leq \delta c_k^{-1/2}$, $|\nabla^j F_k|_{g_k} \leq C_j c_k^{-1/2}$, where the constants transform into $C_j C {c_k'}^{-1/2}$ if in the rescaled charts we use the euclidean metric instead of g_k (actually the bounds in (3) are better because $\nabla^r(iF_k)$ is an $(r+2)$-form, but the exponent $-\frac{1}{2}$ will be enough for our purposes).

In the applications of the theory the starting point is a calibrated manifold of integer type with compatible almost-complex structure J. The closed 2-form ω calibrating D is used to define the pre-quantum line bundle (L, ∇) whose curvature is $-\omega$. The bundle (L, ∇) is $(1,0)$-ample and its tensor powers $L^{\otimes k}$ define a very ample sequence of line bundles.

In a calibrated manifold, the metric is defined using the 2-from ω: a compatible almost complex structure J is used to define $g_{|D}$, and then one extends it to a metric g for which the kernel of ω is orthogonal to D. In this situation, both the metric and the curvatures retraction are the same. Actually, one might choose any metric extending $g_{|D}$ and the theory would work equally.

Even though our choice of retraction is the metric retraction, it is not as good as the one associated to the curvatures to work out the local theory (at least at first sight). The curvatures retraction should be viewed as an auxiliary tool, because the notions it gives rise to are not as natural as the corresponding ones for the metric retraction. We should point out again that the notion of A.H. sequence will depend on the retraction. It might very well happen the certain retractions give rise to rich A.H. theories (with a lot of A.H. sections), whereas others do define A.H. theories for which we cannot conclude the existence of interesting A.H. sections. Anyhow, we will see that in the case of the metric and the curvatures retraction, the corresponding A.H. theories are *strongly equivalent* (lemma 3.30).

2.3. The relative theory. As well as an intrinsic theory for calibrated manifolds (and more generally almost-complex manifolds), it is possible to develop a similar theory using a relative construction for symplectic manifolds (more generally, almost-complex manifolds of even dimension). It will be necessary to generalize certain elements of the symplectic (even dimensional) theory that allows to give a special treatment to part of the holomorphic directions, so that in the end everything will be reduced to the local relative transversality result of J.-P. Mohsen [**43**]. Such generalization will be part of the contents of the current section, as well as of sections 4, 5 and 6.

The geometric results for calibrated manifolds deduced from both theories are exactly the same, but the intrinsic one presents much more technical complications. A second and more important advantage of the relative theory –that we do not exploit in this thesis– is that it makes it possible to derive relative constructions for example for pairs (M, N, ω), where (M, ω) is a compact symplectic manifold (maybe closed) and N is either a symplectic submanifold or a calibrated submanifold (calibrated by ω).

Definition 2.4. *A polarization of an almost-complex manifold* (M, D, J, g) *is a* J*-complex distribution* $G \subset D$*. In such a situation we speak of a polarized almost-complex manifold.*

Even though the relative theory to be developed works in more general situations, we will also consider polarizations in even almost-complex manifolds ($D = TM$). Thus, a polarized almost complex manifold will be given by the data (M, J, G, g).

Calibrated manifolds (M, D, ω, J) (in principle oriented and co-oriented) can be symplectized (in a canonical way once a metric has been fixed). It enough to choose β a 1-form defining D with norm 1 in each subspace. In $M \times [-1, 1]$ we define the closed 2-form $\Omega = \omega + d(t\beta)$, where ω (resp. β) is the pullback of the 2-form (resp. 1-form) in M, and t is the coordinate in $(-1, 1)$. It can be checked that Ω is non-degenerate in $M \times [-\epsilon, \epsilon]$. We extend J in $M \times \{0\}$ sending the unit vector orthogonal to D with positive orientation to $\frac{\partial}{\partial t}$ and; next we extend it to $M \times [-1, 1]$ independently of the coordinate. We also extend D independently of t. We finally extend g to the product metric, and keep the same notations for the three extensions. Actually, the metric g in $M \times [-1, 1]$ is only adapted to ω, J in $M \times \{0\}$, but the inequality $\Omega(v, Jv) \geq ag(v, v)$ holds, for certain positive constant a, and that is enough for our purposes. The tangent space splits as direct sum of the J-complex subbundles D and $D^{\perp}$, where the orthogonality is with respect to either the extended metric g or w.r.t. to the induced hermitian metric h (the extension J is g antisymmetric).

More generally, if we start from a very ample sequence $(L_k, \nabla_k) \to (M, D, J, g)$, we can consider the above described extensions of J and g to $M \times [-1, 1]$. Then we can pullback the sequence L_k to $M \times [-1, 1]$ and define the connections $\hat{\nabla}_k := \nabla_k + it\beta_k$, with $\beta_k := c_k\beta$ (being more precise one has to take tensor product of the pullback of (L_k, ∇_k) with the trivial hermitian line bundle with compatible connection form $it\beta_k$). The result is not quite a very ample sequence of hermitian line bundles over $(M \times [-1,1], J, g)$. Its curvature has three summands: F_k, $idt \wedge \beta_k$ $itd\beta_k$. One checks that the unique condition of definition 2.2 which is not fulfilled is (2), and due to the presence of the third summand above. Anyhow, the condition is indeed fulfilled in the strips $M \times [-\epsilon_k, \epsilon_k]$, with $\epsilon_k = c_k \epsilon$, and as we shall see, we only need to work for all k in a fixed strip of the form $M \times [-\epsilon, \epsilon]$ in the initial construction of reference sections. The final observation is that in the construction of reference sections condition 2 in definition 2.2 is only required to hold in the point where the chart is centered, and we are only interested in constructing reference sections centered at the points in $M \times \{0\}$ (where the third summand vanishes).

Therefore, to any given very ample sequence of line bundles over (M, D, J, g) we can canonically associate a "very ample" sequence of line bundles over the polarized almost complex manifold $(M \times [-\epsilon, \epsilon], J, D, g)$ (which is indeed very ample over the sequence of polarized almost-complex manifolds $M \times [-\epsilon_k, \epsilon_k], J, g, D)$).

3. Local theory: Local models, adapted charts and reference sections

3.1. The local model. In even dimensions (and without polarization), the reference model is that of positive line bundles over Kähler manifolds. The philosophy is that any deviation from the *local Kähler model* whose size –once the charts have been rescaled using the factor $c_k^{-1/2}$ and using the euclidean metric in the euclidean ball of radius $O(1)$– is smaller that $O(c_k^{-1/2})$, still makes it possible the construction of "many" approximately holomorphic sections. The first and most essential deviation is that of the almost-complex structure.

Without being very precise for the moment a sequence of sections of a certain bundle or, more generally, other objects like distributions, is said to enjoy a property in the approximate sense, if for any k bigger than some K the deviation from fulfilling the property (normally given in terms of an equality) is, measured in the metric g_k, at most of size $O(c_k^{-1/2})$. It is equally possible to speak about local properties in the approximate sense. In particular, for an appropriate choice of coordinate charts centered on each point, we can choose for a generic chart "local models" to compare with. One example is that of an arbitrary almost-complex structure J (thought as a constant sequence of sections of the bundle $D^* \otimes D$). If we fix charts suitably and compare it with an integrable one J_0, it will be at distance smaller than $O(c_k^{-1/2})$ of the latter (measured appropriately), and thus we will say that J is *approximately integrable.*

Another allowed deviation is considering connections whose curvature is approximately of size $(1,1)$ (the even dimensional analog of condition 2 in definition 2.2, already present in [**3**]). Actually, since in the integrable situation giving a unitary connection of type $(1,1)$ is equivalent to giving an integrable almost complex structure in the total space of the bundle (with the usual compatibility conditions), once approximate integrability in the base space has been allowed, it seems reasonable to weaken equally the requirements in the total space.

In the odd dimensional setting the global model would be a manifold with a codimension 1 foliation by holomorphic leaves. Locally, there should be charts with domain $\mathbb{C}^n \times \mathbb{R}$ adapted to the foliation in which the leafwise complex structure is of course integrable and independent of the "real" or "vertical" coordinate. Regarding the line bundle, the curvature of the connection has to be (already in the charts) leafwise positive and should define (using an appropriate trivialization) an integrable almost-complex structure in the total space independent of the vertical coordinate; in other words, the curvature has to be of type $(1,1)$ (actually in certain situations vanishing of the $(0,2)$ component will suffice) and independent of the vertical coordinate.

Similarly to what happens for even dimensional almost complex manifolds, we can allow almost-complex structures which locally (and for suitable charts) are integrable and independent of the vertical coordinate in the approximate sense, and also connections with are approximately of type $(1,1)$. But it turns out that any distribution $D \subset TM$ (thought as a constant sequence) is approximately integrable, i.e., for an appropriate choice of charts and appropriate model foliation D_h, the distance between D and D_h is of order $O(c_k^{-1/2})$. Hence, it makes sense to consider distributions D instead of only foliations. Actually, there is delicate point that is worth clarifying. If our goal is to develop a foliated analog of the even dimensional theory (at least the model should be a foliated version of the even dimensional model), in the integrable local model it should suffice with asking the leafwise curvature (the restriction of the curvature to each leaf) to be of type$(1,1)$ and independent of the vertical coordinate s. The results of subsection 2.1 imply that both conditions are equivalent to asking, for any retraction independent of the vertical coordinate, the corresponding projection of the curvature to be of type $(1,1)$ and independent of the vertical coordinate. In other words, in the integrable model we do not need to ask the line field spanned by $\frac{\partial}{\partial s}$ to coincide with the kernel of the curvature, i.e., it is not necessary that the vertical component of the curvatures vanishes (those summands including the factor ds). Anyhow, the vanishing of these vertical component turns out to be very useful for some of the local constructions.

A final way of summarizing the previous ideas is that the local integrable odd dimensional theory is a foliated theory for which there is no canonical choice of line field transverse to D.

For polarized almost-complex manifolds (M, J, G, g), the corresponding local model would be $\mathbb{C}^n$ decomposed as $\mathbb{C}^g \times \mathbb{C}^{n-g}$, corresponding to the distributions G and $G^{\perp_h}$, and so that the curvature and almost-complex structure restricted to the leaves $\mathbb{C}^g \times \{\cdot\}$ are independent of the coordinates $z^{g+1}, ..., z^n$.

Again, the theory is expected to work not only for J-complex foliations $\mathcal{G}$, but for any J-complex distribution G.

We should point out that the polarizations in the relative theory are used to construct certain sequences of vector bundles. To be able to describe applications of the even dimensional A.H. theory to these associated sequences of vector bundles, it will be necessary to give certain local descriptions that take into account the polarization.

Definition 3.1. *The Kähler model, or flat, or integrable is the domain $\mathbb{C}^n \times \mathbb{R}$ with coordinates $\{z^1, ..., z^n, s\}$, leafwise canonical complex structure J_0 and euclidean metric g_0. Regarding the line bundle, it has to admit a unitary trivialization so that the connection form has the expression $A_0 = \frac{1}{4}(\sum_{j=1}^n z_j d\bar{z}_j - \bar{z}_j dz_j)$. Hence, the curvature is $\omega_0 = \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\bar{z}_j$ (not its restriction to the leaves!) and its kernel coincides with the vertical direction. The euclidean metric is leafwise determined by J_0 and ω_0 (and the orthogonal to the leafs is $ker\omega_0$). The foliation D_h with holomorphic leaves $\mathbb{C}^n \times \{s\}$ is called horizontal foliation.*

The horizontal model in the even dimensional case is just a leaf of the previous one. If we have a polarization, the corresponding model will be the even dimensional one but with a further decomposition $\mathbb{C}^n = \mathbb{C}^g \times \mathbb{C}^{n-g}$.

For a sequence $c_k \to \infty$ the rescaled is the same as the flat one, but with metric $g_k = c_k g_0$ and connection $A_k = c_k A_0$. It is straightforward that the contraction with factor $c_k^{-1/2}$ sends the flat model into the rescaled one.

We notice that in the odd dimensional flat model we have asked for a condition on the full 2-form ω. We might have weaken the definition using the foliated 2-form, or even try to develop a theory for foliated non-degenerate closed 2-forms. Locally, there is no trouble in doing that, but the global obstruction comes from the fact that in general there is no pre-quantum line bundle associated to such a foliated 2-form (actually, there is a kind of Poisson structures for which such a pre-quantum line bundle exists, but the difficulty is that is that the leafwise curvature does not match the leafwise 2-from).

It would make sense to turn the condition on the curvature to a leafwise condition if we were able to obtain any other property in exchange; for example, being able to make the vertical direction with the orthogonal to D. But we will not we able to get something like that whereas we will indeed construct charts approximately matching the integrable model.

3.2. Adapted charts, r-comparable charts and approximate equalities. One of the first steps to be taken is to show the existence of charts and trivializations of a very ample sequence L_k that approximately coincide with the local model. Also, a more precise definition of the notion of approximate equality is in order.

Let us focus ourselves in the base space (M, D, g), with no reference to a very ample sequence of line bundles. Our aim is to bound certain sequences of sections and the sequences of their covariant derivatives; we want to compute such bounds –up to a constant– working in appropriate charts and using there the euclidean metric and distance, the flat covariant derivative d (just partial derivatives) and the splitting in horizontal and vertical coordinate.

Consider the sequence of riemannian manifolds (M, D, g_k), with $g_k := c_k g$.

Definition 3.2. *Given (M, D, g_k), a family of adapted charts is a set of smooth maps $\psi_{k,x}\colon (\mathbb{C}^n \times \mathbb{R}, 0) \to (U_{k,x}, x)$, where $(k, x) \in \mathbb{N} \times M$, so that:*

1. $\psi^*_{k,x} D_x = D_h(0)$
2. *For every k bigger than some K, there exist constants $\gamma, \rho_0 > 0$, with $\frac{1}{\gamma} g_0 \leq g_k \leq \gamma g_0$, and such that $|\nabla^j \psi^{-1}_{k,x}|_{g_k} \leq O(1)$ in $B_g(x, \rho_0 c_k^{1/2})$. In particular $\exists \rho > 0$ such that $\frac{1}{\rho} d_k(x, y) \leq |z^i_k(y)| \leq \rho d_k(x, y)$, $\frac{1}{\rho} d_k(x, y) \leq |s_k(y)| \leq \rho d_k(x, y)$ in the same ball, where $z^1_k, ..., z^n_k, s_k$ denote the coordinate functions for the charts $\psi_{k,x}$. It follows that $B_{g_k}(x, \frac{r}{\rho}) \subset \psi_{k,x}(B_{g_0}(0, r)) \subset B_{g_k}(x, \rho r)$, $r \in (0, c_k^{1/2})$.*

The derivatives of $\psi^{-1}_{k,x}$ in condition 2 of definition 3.2 are computed using the Levi-Civita connection associated to g, and d_k is the distance for g_k.

The required relation between the g_k-distance and the norm (g_0-distance) implies that for a given section defined in a ball $B_{g_k}(x, 2\rho)$ a bound for its pullback over the euclidean ball using the euclidean norm also bounds –up to a constant C_1 *independent of* k *and* x– the g_k-norm of the original section over $B_{g_k}(x, \rho)$.

Something similar holds for the covariant derivatives. Bounds of the covariant derivatives of $\psi^{-1}_{k,x}$ (which are given by bound in the derivative, the Christoffel symbols and the derivatives of these in the chart) imply that bounds in C^r-norm for the pullback of the section of order $O(c_k^{-1/2})$ using the flat connection and metric g_0 are equivalent –up to a constant independent of k, x– to the same C^r-bounds but using ∇_g, the Levi-Civita connection associated to g, and the metric g_k. What is more, for polynomials P_r, C^r-bounds of order $P_r(d_k(x, y))O(c_k^{-1/2})$ in a ball *fixed g-radius centered at x* measured with ∇_g and g_k are equivalent to bounds $Q_r(|(z_k, s_k)|)O(c_k^{-1/2})$, where Q_r is another polynomial (not depending on k and x), for the pullback of the section in a *a euclidean ball of radius* $O(c_k^{1/2})$ using d and g_0 (that is, all the metric elements associated to the flat model).

Adapted charts are always available. It is enough to make it for $\psi_{1,x}$ (we artificially set $c_1 = 1$ so that $g_1 = g$), and that is something elementary: One constructs charts depending smoothly on x and so that D equals D_h at the origin. Finally, we rescale them obtaining coordinates $z_k = c_k^{1/2} z$, $s_k = c_k^{1/2} s$.

It is worth noticing that to be able to compare the distances associated to g and g_0 it is not necessary that both metric match at the origin. We will make this comment more explicit in the next subsection, where we introduce class of charts with weaker properties than the adapted ones.

r-comparable charts. Adapted charts are a useful tool to estimate bounds using the euclidean metric, distance and connection. Depending on the kind of estimates we want to compute, we might be interested in working with a weaker kind of charts.

Let A be an invertible endomorphism of $\mathbb{R}^n$. Let us denote by $A(S^{n-1})$ the ellipsoid image of the unit sphere. A norm for A is defined as the maximum of the norms of the vectors in $A(S^{n-1})$, that we denote by $||A||$. The norm of the inverse is $1/d(A(S^{n-1}),0)$.

We will use the notation $\frac{1}{\gamma} \leq ||A|| \leq \gamma$ if the positive constant ρ bounds both the norm of A and of its inverse. To get bounds by below and above for the norm of A it is enough to obtain one of the bounds and the other for the determinant $\det(A)$, because the determinant computes the volume of the ellipsoid (it is possible to go from a bound for the determinant to a bound for the norm and viceversa using functions that only depend on the dimensions).

If we have a metric g (bilinear form) we can consider the corresponding matrix A_g associated to an euclidean orthonormal basis $\{e_1, ..., e_n\}$ ($A_{g,ij} = g(e_i, e_j)$).

Definition 3.3. *We define $||g||$ to be the norm of A_g in a euclidean basis. One checks that it does not depend on the basis. We will use both the notations $\frac{1}{\gamma} \leq ||g|| \leq \gamma$ and $\frac{1}{\gamma} \leq ||A_g|| \leq \gamma$ (and also $\frac{1}{\gamma}g_0 \leq g \leq \gamma g_0$, as we did in condition* (1) *in definition 3.2).*

One can give a different notion of norm for g by considering the norm of a transformation sending a g-orthogonal basis into an euclidean orthogonal basis. Again, the definition is independent of the choice of basis.

Both norms for g are equivalent. Indeed, if A_g is the symmetric matrix representing the bilinear form g in an euclidean orthonormal basis, and Q is a transformation sending a g-orthogonal basis into an euclidean orthogonal one, it follows that $A_g = QQ^t$. Hence, it is enough to relate one of the bounds (either by below or by above), and then relate the other through the determinant using the relation $\det(A) = \det(Q)^2$.

If we look at metrics g from the point of view of bilinear forms, another way to bound g by below and above is finding a constant $\gamma > 0$ so that $\frac{1}{\gamma} \leq g(v,v) \leq \gamma$, where v has euclidean norm 1.

The three described ways of bounding a metric by below and above are equivalent.

Definition 3.4. *Let $(g_\alpha)_{\alpha\in\Lambda}$ be a family of metrics defined in a neighborhood U_α of the origin of $\mathbb{R}^n$. The family $(g_\alpha)_{\alpha\in\Lambda}$ is said to be comparable to the euclidean metric if positive numbers γ, ρ_0 exist so that $B_{g_0}(0,\rho_0) \subset U_\alpha$ and $\frac{1}{\gamma}g_0 \leq g_\alpha \leq \gamma g_0$ in every point of $B_{g_0}(0,\rho_0)$ and for every α.*

Given any $r \in \mathbb{N}_+$, we say that the family is comparable to the euclidean to order r if it is comparable and the norm of the Christoffel symbols – computed in the usual euclidean orthonormal parallel basis– and that of its partial derivatives up to order $r-1$ are bounded by γ in every point of $B_{g_0}(0,\rho_0)$ and for every α.

Remark 3.5: In general and for a given open manifold M, the existence of charts centered at every point $x \in M$ so that the induced metrics give rise to a family of metrics g_x comparable to the euclidean is not straightforward. A sufficient condition is the existence of global bound for the curvature tensor.

The reason is that if for example we want to construct the charts using normal coordinates, we need information on the curvature to understand the behavior of the differential of the exponential map.

The kind of family that we have in mind is $\Lambda = \{x \in \coprod_{k\in\mathbb{N}} M_k\}$, where (M_k, g_k) is a family of riemannian manifolds. In particular, if $(M_k, g_k) = (M, c_k g)$, where M is compact, it is very easy to construct r-comparable charts (in this situation the requirements should hold not for all (k, x) but for all (k, x) with k bigger than some K).

The first property of our interest is that for a family of maps f_α defined in r-comparable charts, bounds (by above) of order $O(1)$ for $\nabla^{r-1}_{g_\alpha} f$ in $B_{g_0}(0, \rho_0)$ are equivalent to bounds of the same order for the partial derivatives up to order $r-1$ of f_α. Also, in comparable charts and as we anticipated in the end of the previous subsection, we can use either g or g_0 and the same for the associated distances.

The second property is the following: in the domain of r-comparable charts, $r \geq 1$, if we have a linear subspace $V \subset B_{g_0}(0, \rho_0)$ we can compare tubular neighborhoods of V for g_α and g_0. And what is more, inside these neighborhoods (in one of them) we can also compare the parallel transport of V (thought as tangent space to V itself) using both metrics. Indeed, that parallel transport is controlled by the Christoffel symbols; a bound for them allows us to study how the transverse geodesics to V differ (for both metrics), and how much V is modified when parallel translated by ∇_{g_α} along any curve whose first and second derivatives are controlled (in principle these curves will be geodesics for g_α normal to V).

We go back to our almost complex manifold (M, D, J, g) and the adapted charts.

Adapted charts are useful to characterize what an approximate equality/property is. They can be used as well to give a precise definition of an approximate equality when we compare with local models. Observe that the definition of approximate equality given at the beginning of subsection 3.1 works perfectly when we compare with global objects, but it is far from being precise when we have to compare, say, a distribution in the whole manifold with other integrable distributions only locally defined using charts.

Our objects will be sections of certain bundles. More precisely, they will be sequences of sections of fiber bundles (either hermitian or orthogonal), being the prototype the sequence $E \otimes (F \otimes L_k)$, where (L_k, ∇_k) is a very ample sequence of line bundles and (F, ∇) is an arbitrary hermitian vector bundle and E is a vector bundle associated to TM (built by complexification, taking duals, direct sums, and symmetric and antisymmetric products of the dual). We can use instead of E the vector bundle built through the same operations but out of D instead of TM. Since we will be interested in computing norms of the derivatives of sections of these sequence, we need to see the latter as a subbundle of E; we will use the notation E_D (resp. $E_{\tilde{D}}$) if we use the metric retraction (resp. the curvatures). The subbundle E_D (resp. $E_{\tilde{D}}$) inherits a connection from the Levi-Civita one and the corresponding retraction (recall that both subbundles have canonical complementaries inside E defined out of the corresponding complementaries of D in TM and $\mathrm{Ann}(D)$ in T^*M).

One first example is the tensor $J \in \Gamma(D \otimes D^*)$, where the sequence of bundles is constant . Once extended by zeroes, it gives rise to the sequence of tensors $\bar{J}_k \in \Gamma(D \otimes \bar{D}^*) \subset TM \otimes T^*M$ (constant sequence k) and $\tilde{J}_k \in \Gamma(D \otimes \tilde{D}^*) \subset TM \otimes T^*M$ (this sequence does vary with k). When taking covariant derivatives, we have two choices in both cases. The first one is just use repeatedly the connection on E; the second is using the derivatives in either E_D or $E_{\tilde{D}}$) (using the corresponding complementaries to project). The four options (i.e., extending by zeros using either of the splittings and then using either the full derivative or the one in the subbundle) give tensors with different norms, but under certain conditions –that our tensor will always fulfil– the bounds we look after will be equivalent for the four of them. Unless otherwise stated, in principle and for sections of the subbundles E_D and $E_{\tilde{D}}$ we will use the derivative inn E (the full derivative).

As it is the case of J, we will work with other sequences of sections of a fixed bundle E (resp. E_D or $E_{\tilde{D}}$) which will come from data from the base (M, D, J, g), without making any reference to very ample sequences of bundles.

The second kind of example is given by a sequence of sections τ_k of L_k. After taking the covariant derivative, we obtain $\nabla\tau_k \in \Gamma(T^*M \otimes L_k)$. Its restriction to D will have an antiholomorphic component $\bar{\partial}\tau_k$ that can be seen as a section of $D^{*0,1} \otimes L_k$. To take further derivatives (and hence being able to estimate the C^r-size of the antiholomorphic component), it is convenient to use the metric or the curvatures restriction and see $\bar{\partial}\tau_k$ as a section of either $\bar{D}^{*0,1}$ or $\tilde{D}^{*0,1}$.

Definition 3.6. *Let E_k be a sequence of either hermitian or unitary vector bundles with compatible connection ∇, and let T_k be a sequence of sections of them.*

T_k is said to vanish in the approximate sense (or that approximately vanishes) to order r and it is denoted by $T_k \approxeq_r 0$, if for all k bigger than some K the following inequalities hold:

$$|\nabla^j T_k|_{g_k} \leq O(c_k^{-1/2}),\ j = 0, ..., r,$$

where the higher order derivatives use the Levi-Civita connection on T^*M.

When the previous property holds for all r we speak of an approximate equality and denote it by $T_k \approxeq 0$.

Remark 3.7: We observe that if E_k is a constant sequence E (resp. E_D or $E_{\tilde{D}}$) as the previously described, once a family of adapted charts has been chosen, the previous statement is equivalent to

$$|\frac{\partial^{|p|}}{\partial x_k^p}(\psi_{k,x}^* T_k)|_{g_0} \leq O(c_k^{-1/2}),\ p = (p_1, ..., p_{2n+1}),\ |p| = p_1 + \cdots + p_{2n+1}, |p| \leq r,$$

in the points of $B_{g_0}(0, O(1))$ independently of k y x,where $x_k^1, ..., x_k^{2n+1}$ are the coordinates. Moreover, if the tensors are sections T_k of the subbundle E_D (resp. $E_{\tilde{D}}$), one can give an equivalent definition only using the covariant derivative in E_D (resp. in $E_{\tilde{D}}$). As we mentioned before, we will see that under certain conditions the both definitions are equivalent.

We will work with families of adapted charts having additional properties.

Definition 3.8. *A family of charts is adapted to g (resp. ω_k or the curvatures) if it is adapted and for k large enough the vector field $\frac{\partial}{\partial s_k}$ generates the g-orthogonal to D (resp. the kernel of ω_k).*

It is clear that there is no difficulty in finding charts adapted to g, because the metric splitting does not depend on k; one builds them for $k = 1$ and then rescale.

To show the existence of charts adapted to ω_k it is necessary to study the relation between both splittings. Such relation is coded in the bounds for ω_k and its derivatives of definition 2.2 and its relation with g over D, which control how the kernel of ω_k behaves. Anyhow, we point out that for calibrated manifolds both splittings coincide.

Let us call R_k to a g_k-unitary vector in the kernel of ω_k. Such vector (up to sign) will be in principle local if the manifold is not cooriented. The domain $\mathbb{C}^n \times \mathbb{R}$ of a chart $\psi_{k,x}$, say adapted to g, has a natural orientation defined as the one on $\mathbb{C}^n$ plus the vector $\frac{\partial}{\partial s_k}$. If (M, D, J, g) is oriented, the charts can be chosen such that both orientations agree. In that situation, R_k will have positive vertical component. In the non-oriented one, we choose r_k also to have positive vertical component.

Lemma 3.9. *The bound $|\omega_k|_{g_k} \leq O(1)$ implies $\angle(\ker w_k, D) \geq \varepsilon > 0$, for k bigger than some K.*

The inequalities $|\nabla^j \omega_k|_{g_k} \leq O(c_k^{-1/2})$, $j \geq 1$ imply $|\nabla^j R_k|_{g_k} \leq O(c_k^{-1/2})$, $j \geq 1$ (also for k bigger than some K).

Proof. Assume that the first statement is not true. That would imply, for any K and $\delta > 0$ the existence of a point $x_{k,\delta} \in (M, g_k)$, $k \geq K$, for which, if we denote by v_k the orthogonal projection of R_k over $D_{x_{k,\delta}}$, we would have $|v_k|_{g_k} > 1 - \delta$ and $|v_k - R_k|_{g_k} < \delta$. Therefore $\omega_k(v_k, Jv_k) \geq (1-\delta)^2$ and hence $\omega_k(v_k, Jv_k) = \omega_k(v_k - R_k, Jv_k)$, and $|\omega_k|_{g_k} > \frac{(1-\delta)^2}{(\delta||J||)}$.

Regarding the second statement, let us fix charts adapted to g and $\{e_1, ..., e_{2n+1}\}$ and orthonormal trivialization of TM for the metric g_k (e_{2n+1} orthogonal to D), and such that $|\nabla^r e_i|_{g_k} \leq O(c_k^{-1/2})$. We can restrict ourselves to a ball of g_k-radius of order $O(1)$.

We recall that since ∇ is the Levi-Civita connection, $\nabla_X R_k$ is orthogonal to R_k.

$$\nabla\omega_k(R_k, e_i, \cdot) = d(\omega_k(R_k, e_i)) - \omega_k(\nabla R_k, e_i) - \omega_k(R_k, \nabla e_i) \Rightarrow$$
$$\Rightarrow |\omega_k(\nabla R_k, e_i)|_{g_k} \leq O(c_k^{-1/2}).$$

Since $\nabla_X R_k$ is far from the kernel of ω_k the bound for $|\nabla R_k|_{g_k}$ is deduced from the bounds for ω_k together with the bound by below over D by a fixed multiple of g_k. Also, the bounds for $|\nabla^r \omega_k|_{g_k}$, for $|\nabla^j R_k|_{g_k}, 0 \leq j \leq r-1$ and the comparison over D of ω_k with g_k imply the required bound for the component of $\nabla^r R_k$ orthogonal to R_k. Regarding the component parallel to R_k, from $0 = \nabla^r \langle R_k, R_k\rangle$ we conclude that its size is measured in terms of the inner products $\langle \nabla^s R_k, \nabla^t R_k\rangle$, $0 < s, t \quad s+t = r$. □

From this results it follows that we can use either the metric g_k, or the restriction to D of this metric together with the local vector field R_k, and we obtain bounds of the same order (that is, we can use either the previous local basis $\{e_1, \cdots, e_{2n+1}\}$ or $\{e_1, \cdots, e_{2n}, R_k\}$, and for both all the local vector fields have derivatives of order $O(c_k^{-1/2})$).

The important consequence of the previous lemma is that we are now able to construct charts adapted to ω_k.

Before, it is necessary to give a notion of proximity between distributions. When we are given two of them, a natural thing to do it writing one of them as the graph of a map from the other to a transverse coordinate and them estimate the norm of the map. In our case we are also interested in comparing distributions of different dimensions. We recall the notions of maximal and minimal angle. [**46**].

Definition 3.10. *Let W be a vector space with non-degenerate inner product so that for any $u, v \in W$ we can compute the angle $\angle(u,v)$. Given $U \in Gr(p,W)$ y $V \in Gr(q,W)$ $p,q > 0$, we define $\angle_M(U,V)$, the maximal angle of U and V, as follows:*

$$\angle_M(U,V) \quad := \quad \max_{u\in U\setminus 0}\big(\min_{v\in V\setminus 0}\angle(u,v)\big)$$

In general, the maximal angle is not symmetric, but when $p = q$ it has symmetry and defines a distance in the corresponding grassmannian (see [**46**]).

The minimum angle between transverse complementary subspaces is defined as the minimum angle between two non-zero vectors, one on each subspace. An extension of this notion for transverse subspaces with non-trivial intersection is:

Definition 3.11. *(see [**46**]) Using the notation of definition 3.10, $\angle_m(U,V)$, the minimum angle between non-void subspaces U and V is defined as follows:*

- *If $\dim U + \dim V < \dim W$, then $\angle_m(U,V) := 0$.*
- *If the intersection is non-transverse, then $\angle_m(U,V) := 0$.*

- *If the intersection is transverse, we consider the orthogonal t the intersection and its intersections U_c and V_c with U and V respectively. We define* $\angle_m(U,V) := \underset{u\in U_c\smallsetminus 0}{min}\,(\underset{v\in V_c\smallsetminus 0}{min}\,\angle(u,v))$.

The minimum angle is symmetric.

The most important property relating maximal and minimal angle is:

Proposition 3.12. *(Proposition 3.5 in* [**46**]*) For non-void subspaces U, V, W of $\mathbb{R}^n$ one has the following inequality:*

$$\angle_m(U,V) \le \angle_M(U,W) + \angle_m(W,V).$$

The subspaces should be considered oriented so that the maximum angle of two smooth distributions of the same dimension, with the appropriate sign, is a smooth function.

Lemma 3.13. *It is possible to construct adapted charts $\psi_{k,x}\colon (\mathbb{C}^n\times\mathbb{R},0)\to (U_{k,x},x)$ sending the field of vertical directions spanned by $\frac{\partial}{\partial s_k}$ to $\ker w_k$.*

Proof. After lemma 3.9, there exist $\varepsilon > 0$, $K\in\mathbb{N}_+$, so that $\angle_m(\ker\omega_k, D)\ge \varepsilon$ for $k\ge K$. We can fix initial charts $\phi_{x,1}\colon(\mathbb{C}^n\times\mathbb{R},0)\to(U_{k,x},x)$ such that $\angle_M(D_h,D)\le\frac{\varepsilon}{2}$. Applying proposition 3.12 we deduce that $\angle_m(\phi^*_{1,x}\ker\omega_k, D_h)\ge\frac{\varepsilon}{2}$, for $k\ge K$. We define the family of charts $\phi_{k,x}$ by rescaling $\phi_{x,1}$.

Let us denote the flow of $\phi^*_{k,x}R_k$ by Φ^t_{x,R_k}. We rectify it using the map

$$\begin{array}{rrl}\chi_k\colon & \mathbb{C}^n\times\mathbb{R}\to & \mathbb{C}^n\times\mathbb{R}\\ & (z_k,s_k)\mapsto & \Phi^s_{R_k}(z_k,0)\end{array}$$

Since $|R_k|_{g_k} = O(1)$ and $\angle_m(R_k,D_h) > \frac{\varepsilon}{2}$ it follows that the map is defined for each x and k in a euclidean ball of radius en $r_1c_k^{1/2}$ (or of g-radius of order $O(1)$). What is more, it is possible to find a constant γ so that $\frac{1}{\gamma}\le|\chi_{k*}(z_k,s_k)|\le\gamma$. Thus, we can compare the euclidean metric in the target space of χ_k with the pushforward of the euclidean metric (and similarly we have the required bounds for the derivatives). Actually, we can undo the rescaling and consider, for each k, the map $\Phi^t_{x,c_k^{1/2}R_k}$ induced in the domain of the chart $\phi_{1,x}$. These maps fix the origin and $D_h(0)$, and for all their covariant derivatives bounds of order $O(1)$ are available. Therefore, bounds of order $O(c_k^{-1/2})$ are obtained for the compositions $\psi_{k,x} = \phi_{k,x}\circ\Phi^t_{x,c_k^{1/2}R_k}$. □

Once we have charts adapted to g (resp. ω_k) in which the splitting $D_h\oplus\frac{\partial}{\partial s}$ almost corresponds to the one given by D and the metric (resp. ω_k), we can introduce the notion of approximate equality for local tensor. Actually, the concept will be defined for any family of adapted charts, but since our local tensors will be related to D and both splittings, we restrict our attention to these two kinds of families of adapted charts.

We will also give the definition in principle for constant sequences of bundles E (resp. E_D or $E_{\tilde{D}}$), because the "local models" we want to compare with are related to the geometric data (M, D, J, g).

Definition 3.14. *Let us fix $\psi_{k,x}$ a family of charts adapted to the metric (resp. curvatures). Let T_k be a sequences of sections of the vector bundle E or the subbundles E_D (resp. $E_{\tilde{D}}$), and T another* local section *of the corresponding bundle; that means that we use the local distribution D_h and local splitting $D_h \oplus \frac{\partial}{\partial s}$ instead of D and $D^{\perp}$ (resp. $\ker \omega_k$) in the local definition of E_D (resp. $E_{\tilde{D}}$), which the local bundle where T belongs. The charts give canonical trivializations of E and of the local subbundles, and we ask T –defined in the domain of $\psi_{k,x}$– to be given by a fixed expression independent of k, x. Thus, T is the local model we want to compare with.*

T_k is said to be equal to T in the approximate sense (or approximately equal) to order r, and we denote it by $T_k \cong_r T$, if for all k bigger than some K the following bound hold:

$$|\frac{\partial^{|p|}}{\partial x_k^p}(T - \psi_{k,x}^* T_k)|_{g_0} \leq P_r((z_k, s_k))O(c_k^{-1/2}), \tag{3.1}$$

$p = (p_1, ..., p_{2n+1})$, $|p| = p_1 + \cdots + p_{2n+1}, |p| \leq r$, in $B_{g_0}(0, O(c_k^{1/2}))$ independently of k and x, where we have used the notation $z_k^i = x_k^{2i} + x_k^{2i+1}$, $s_k = x_k^{2n+1}$. In case we are only working $B_{g_0}(0, O(1))$, we ask for the same kind of inequalities but with $P_r = 1$ (the first inequality over $B_{g_0}(0, O(1))$ implies this last one).

When the previous property is fulfilled for all r we speak of an approximate equality and denote it by $T_k \cong T$.

We will speak about flatness in the C^r-approximate sense when $T = 0$ (i.e., $T_k \cong 0$), o more generally when T, being a local tensor, is constant (parallel w.r.t. the Levi-Civita connection associated to the euclidean metric).

Remark 3.15: Notice that we can use instead of d (usual partial derivatives) the covariant derivative ∇ and g_k in the definition. When we work with sections of the subbundles E_D (resp. $E_{\tilde{D}}$) we can give an equivalent definition using the restriction of d to D_h (i.e., the partial derivatives w.r.t. the horizontal coordinates).

Remark 3.16: It is possible to extend the notion of approximate equality for sequences of sections of $E \otimes (L_k \otimes F)$. Basically, we need to give (unitary) trivializations of $L_k \otimes F$ for each chart, so that the model tensor T is defined by a fixed formula (for example $T = 0$). We also need to couple d with the corresponding matrix of 1-forms in 3.1. Since the use of local models is oriented to simplify certain calculations, the reasonable thing to do is to choose trivializations so that the matrix of connection forms is independent of k and x and easy to handle.

Once adapted charts –which are nothing but an auxiliary tool– have been fixed (and not necessarily adapted to the metric or curvatures), the distance between D and D_h is given by a smooth map $\angle_M(D, D_h)$ vanishing at the origin. Thus, D is approximately integrable (or flat).

It is important to point out that the notions of approximate flatness do depend on the family of charts. Observe also that $T_k \approxeq_r T$ for a family of adapted charts is equivalent to $\phi_{k,x}^* T_k \approxeq_r T$, as long as $\phi_{k,x} \colon (\mathbb{C}^n \times \mathbb{R}, 0) \to (\mathbb{C}^n \times \mathbb{R}, 0)$ verifies $\phi_{k,x} \approxeq_r I$ and $\phi_{k,x}^* D_h(0) = D_h(0)$. Hence, we will not only use charts adapted to the metric (resp. curvatures) but also perturbations of the previous order; in other words, charts approximately adapted to the metric (resp. curvatures) for which $D \approxeq D_h$ and $D^\perp \approxeq \langle \frac{\partial}{\partial s} \rangle$ (resp. $\ker \omega_k \approxeq \langle \frac{\partial}{\partial s} \rangle$).

We have already mentioned that the theory for almost-complex manifolds generalizes the setting for calibrated manifolds with a compatible almost complex structure J and metric g. An important consequence in the latter situation is that sections close enough to be J-holomorphic (and transverse to submanifolds close to be J-complex) define submanifolds cutting D symplectically (at linear level). Recall that for any retraction i, by definition a 1-form $\Gamma \in i(D)^*_{\mathbb{C}}$ is J-holomorphic if its restriction to D has this property. Being close enough to be J-holomorphic means that the $(1,0)$ part of the restriction to D is big enough compared to the $(0,1)$ component. But we have seen that this is equivalent to the same assertion for the components $\Gamma^{1,0} \in i(D)^{*1,0}_{\mathbb{C}}$ and $\Gamma^{0,1} \in i(D)^{*0,1}_{\mathbb{C}}$.

Indeed, for any subspace $N_x \subset D_x$, $J(N_x) \perp_g N_x^{\omega_{|D}}$ so if $J(N_x)$ is close enough to N_x, the latter will be symplectic. The proximity is measured by $|\Gamma^{0,1}|/|\Gamma^{1,0}|$.

In the general situation of almost complex manifolds, and similarly to what happens in the even dimensional setting, this relation can also be recovered; being close to be J-holomorphic will imply symplecticity (of the corresponding submanifolds) w.r.t. ω_k. To show it, we construct almost-complex structures J_k compatible with ω_k and approximately matching J. For these new tensors, C^0-proximity to J is enough to deduce the desired result. It is a special feature of the approximately holomorphic theory that all the relations that hold in the models, are fulfilled in the approximate sense in (M, D, J, g). For the sake of completeness, we will prove that this indeed the case for J_k and J (they approximately coincide to any order).

Again, we face the problem of how to take derivatives of the tensors $J_k \in \Gamma(D^* \otimes D)$ (to be defined). On the one hand we can consider either the metric extensions $\bar{J}_k$ or the curvatures extensions $\tilde{J}_k$. On the other, we can either take full derivatives or the derivatives in the corresponding subbundles. In the next lemma we introduce conditions that eliminate this ambiguity.

To introduce the necessary notation, we recall that if T_k is a sequence of sections of a vector bundle $E_{|D}$ associated to D as in definition 3.14, for the extension $\bar{T}_k \in \Gamma(E_D)$ (resp. $\tilde{T}_k \in \Gamma(E_{\tilde{D}})$) we have a total derivative $\nabla^r \bar{T}_k \in \Gamma(T^*M^{\otimes r} \otimes E)$ (resp. $\nabla^r \tilde{T}_k \in \Gamma(T^*M^{\otimes r} \otimes E)$), and a derivative in E_D, $\bar{\nabla}^j \bar{T}_k \in \Gamma(T^*M^{\otimes r} \otimes E_D)$ (resp. $\tilde{\nabla}^j \tilde{T}_k \in \Gamma(T^*M^{\otimes r} \otimes E_{\tilde{D}})$) in which we compose with the projections

$$\pi_r^D : T^*M^{\otimes r} \otimes E \to T^*M^{\otimes r} \otimes E_D$$

$$\pi_r^{\tilde{D}} : T^*M^{\otimes r} \otimes E \to T^*M^{\otimes r} \otimes E_{\tilde{D}}$$

If we work locally in adapted charts, the projection is instead $\pi_r^{D_h} : T^*M^{\otimes r} \otimes E \to T^*M^{\otimes r} \otimes E_{D_h}$ (the identity in $T^*M^{\otimes r}$ tensorized with the projection parallel to the complementary subbundle).

We will also use the bundle isomorphism $q_r^{D,\tilde{D}} : T^*M^{\otimes r} \otimes E \to T^*M^{\otimes r} \otimes E$ defined as the identity in $T^*M^{\otimes r}$and in the E factor, it is defined as the isomorphism that sends E_D to $E_{\tilde{D}}$ induced by the automorphism of TM fixing D and sending $D^\perp$ into $\ker \omega_k$ parallel to D; equivalently it is induced also by the automorphism of T^*M which is the identity on $\mathrm{Ann}(D)$ and sends $\bar{D}$ into $\tilde{D}$ parallel to $\mathrm{Ann}(D)$.

In the domain of adapted charts (either to the metric or curvatures), the local map $q_r^{D,D_h} : T^*M^{\otimes r} \otimes E \to T^*M^{\otimes r} \otimes E$ is defined as the identity in $T^*M^{\otimes r}$, and sends E_{D_h} into E_D parallel to the common complement (again, induced by the automorphisms of TM fixing $\frac{\partial}{\partial s_k}$ and projecting D_h into D along the vertical direction).

The maps that have to do with the curvatures splitting do depend on k, but we omitted the dependence in the notation.

It is clear that the maps π_r^D have g_k-norm $O(1)$, and their derivatives are of size $O(c_k^{-1/2})$, because they are constant sequences. From the bounds for R_k and its derivatives obtained in lemma 3.9, we conclude the same kind of bounds for $\pi_r^{\tilde{D}}$ and $q^{D,\tilde{D}}$. We also have $q_r^{D,D_h} \approxeq I$ (I the identity). In the same way, $\pi_r^D \approxeq \pi_r^{D_h}$, because we go from one projection to the other by composing with q_r^{D,D_h}.

Notice that in the integrable situation, since D_h is parallel, the derivative in E (full derivative) and the derivative in E_{D_h} coincide. In the non-integrable situation a bound for the derivative in E_D does not imply the same kind of bound for the full derivative. In any case, we have the following result:

Lemma 3.17. *Let T_k be a sequence of tensors of $E_{|D}$ and let $\bar{T}_k$ and $\tilde{T}_k$ be the images of T_k by the immersions of $E_{|D}$ in E given by the metric and curvatures (we will refer to them as extension by zeros).*

First, $|\nabla^j \bar{T}_k|_{g_k} \leq O(1)$, $\forall j \in \mathbb{N}$ if and only if $|\nabla^j \tilde{T}_k|_{g_k} \leq O(1)$, $\forall j \in \mathbb{N}$.

If (any of) the previous bounds hold, then the following conditions are equivalent:

(1) $|\nabla^j \bar{T}_k|_{g_k} \leq O(c_k^{-1/2})$.
(2) $|\bar{\nabla}^j \bar{T}_k|_{g_k} \leq O(c_k^{-1/2})$.
(3) $|\nabla^j \tilde{T}_k|_{g_k} \leq O(c_k^{-1/2})$.
(4) $|\tilde{\nabla}^j \tilde{T}_k|_{g_k} \leq O(c_k^{-1/2})$.

The previous equivalence also holds for bounds of order $O(1)$ instead of $O(c_k^{-1/2})$.

PROOF. We will just proof the second equivalence, because the first is just the equivalence between (1) and (3) for bounds of order $O(1)$.

Let us first consider the case of sections of E_D (the equivalence between (1) and (2)). By definition $\bar{\nabla}\bar{T}_k = \pi_1^D(\nabla\bar{T}_k)$. Hence, we wish to show that if $|\nabla\bar{T}_k|_{g_k} \leq O(c_k^{-1/2})$ and $|\pi_1^D(\nabla\bar{T}_k)|_{g_k} \leq O(c_k^{-1/2})$, then $\pi_1^D(\nabla\bar{T}_k) - \nabla\bar{T}_k$ has norm bounded by $O(c_k^{-1/2})$. We can work in balls of g_k-radius $O(1)$ in the domain of charts adapted to the metric.

One implication is straightforward: from bounds of order $O(1)$ for the norm of both $\bar{T}_k$ and $\nabla\bar{T}_k$ we deduce that $(\pi_1^D\nabla\bar{T}_k - \nabla\bar{T}_k) - (\pi_1^D d\bar{T}_k - d\bar{T}_k)$ has norm bounded by $O(c_k^{-1/2})$, because the same estimate holds for the difference $d\bar{T}_k - \nabla\bar{T}_k$. In fact, by using the bounds $|\nabla^j\bar{T}_k|_{g_k} \leq O(1), \forall j \in \mathbb{N}$, we obtain bounds of order $O(c_k^{-1/2})$ for all the derivatives of the previous difference (here we only need control of order $O(1)$ for π_1^D and its derivatives; also, we use that for tensors $F^{a,b}$, $G^{b,c}$, the relation $\nabla G\circ F = \nabla G\circ F + G\circ\nabla F$ holds).

Similarly, $(\pi_1^D d\bar{T}_k - d\bar{T}_k) - (\pi_1^{D_h} d\bar{T}_k - d\bar{T}_k)$ and all its derivatives have norm bounded by $O(c_k^{-1/2})$, because $\pi_1^D \approxeq \pi_1^{D_h}$ and $d^j\bar{T}_k,\ \ \forall j \geq 1$, are at most of size $O(1)$.

Hence, the assertion for the first derivative is equivalent to the same assertion for the difference $\pi_1^{D_h} d\bar{T}_k - d\bar{T}_k$.

It is important to observe that all the higher order derivatives of the previous difference only require a bound of order $O(1)$ for $\nabla^j\bar{T}_k$ and the derivatives of the projections. After adding and subtracting the expression $q_0^{D,D_h}(\bar{T}_k)$ a $\bar{T}_k$, the problem reduces to find the same bounds for $\pi_1^{D_h} dq_0^{D,D_h}(\bar{T}_k) - dq_0^{D,D_h}(\bar{T}_k)$ (again, the difference approximately vanishes only requiring bounds $O(1)$ for $\nabla^j\bar{T}_k$ and the projections and their derivatives).

Define $B_{k,1} := \nabla\bar{T}_k - \bar{\nabla}\bar{T}_k$. From the previous considerations $B_{k,1} \approxeq 0$. Therefore $\nabla^2\bar{T}_k - \nabla\bar{\nabla}\bar{T}_k \approxeq 0$. so the proof of the case $r = 2$ reduces to find a similar bound for $\nabla\bar{\nabla}\bar{T}_k - \bar{\nabla}^2\bar{T}_k$, and the proof of this fact is just what we did, but using $\pi_2^D, \pi_2^{D_h}$ and q_1^{D,D_h}.

To bound $\nabla^r T_k$ we use the same ideas with the projections $\pi_r^D, \pi_r^{D_h}$ and q_{r-1}^{D,D_h}, together with the fact that $\nabla\nabla^{r-1}T_k - \nabla d^{r-1}T_k$ is of order $O(c_k^{-1/2})$, and that $B_{k,r-1} := \nabla^{r-1}\bar{T}_k - \bar{\nabla}^{r-1}\bar{T}_k$ approximately vanishes by induction.

The equivalence between (3) and (4) follows the same pattern.

Also the equivalence between (1) and (3) is obvious because $\tilde{T}_k = q_0^{D,\tilde{D}}(\bar{T}_k)$.

The equivalence for bound of order $O(1)$ is also straightforward.

□

Remark 3.18: We can use the previous idea to prove the same kind of result but equating the sequence not to zero, but to other tensor (just considering the difference), or to a tensor locally defined in adapted charts (a model). Observe also that to prove a global approximate equality $T_k \approxeq_r T$, we can use a certain family of adapted charts (working in g_k-balls of radius $O(1)$) and then switch to another family for other purposes.

Even though we still have not introduced charts adapted to polarizations G, is will be evident that under appropriate conditions, we will have similar results for tensor lying in the subbundles E_G.

Remark 3.19: Notice that to prove the equivalence between (1) and (3) we used the bounds of the map $q_0^{D,\tilde{D}}$ and its derivatives. Only bounds of order $O(1)$ for the derivatives are required in the proof (though the derivatives are actually bounded by $O(c_k^{-1/2})$).

Remark 3.20: The previous lemma holds because in the integrable situation we have strict equalities. We will see this kind of phenomenon more times.

We any of the four equivalent conditions hold, for the sake of simplicity we just denote it by $T_k \approxeq 0$.

Lemma 3.21. *Given a very ample sequence of hermitian line bundles over an almost-complex manifold, there is a canonical sequence of almost-complex structures J_k compatible with $\omega_k = iF_k$ and such that $J_k \approxeq J$.*

Proof. We will define instead $\tilde{J}_k$, the extension by zeros of J_k using the curvatures retraction. This should not be strange because the compatibility condition for J_k is stated in terms of ω_k. Throughout the proof, all the extensions by zeros will be done using the curvatures retraction.

We denote by $\tilde{J}$ and $\tilde{I}$ the extensions of J and I by zeros (the extensions do depend on k). Also denote by $\tilde{\omega}_k\colon TM \longrightarrow T^*M$ the bundle maps induces by the 2-forms ω_k. The composition $\tilde{\omega}_k \circ (-\tilde{J})$ restricts to D to a positive morphism from D to D^*. Hence, its symmetrization

$$\begin{aligned} S_k(u,v) &= \tfrac{1}{2}\omega_k(u,Jv) + \tfrac{1}{2}\omega_k(v,Ju) = \\ &= \omega_k(u,Jv) + \tfrac{1}{2}(\omega_k(v,Ju) - \omega_k(Jv,J^2u)) = \\ &= \omega_k(u,v) + \tfrac{1}{2}Re(\omega_k^{0,2}(v,Ju)), \forall u,v \in D, \end{aligned}$$

defines a metric on D, whose extension by zeros can be seen as a bundle map $\tilde{S}_k\colon TM \to T^*M$. Let us consider $A_k = \tilde{S}_k^{-1} \circ \tilde{\omega}_k$ and define $\tilde{J}_k = Q_k^{-1} \circ A_k$, where $Q_k^2 = -A_k^2$ and Q_k is self-adjoint; the inverses are taken along D and vanish in $\ker \omega_k$.

To show that $\tilde{J} \approxeq_0 \tilde{J}_k$, we notice that the bundle maps $\tilde{\omega}_k \circ (-\tilde{J})$, $\tilde{S}_k\colon TM \to T^*M$ are by definition at distance $O(c_k^{-1/2})$. The same holds for the inverses –due to the fact that $\omega_k(v,Jv) \geq g_k(v,v)$, $\forall v \in D$– and thus also for $\tilde{J} = (\tilde{\omega}_k \circ (-\tilde{J}))^{-1} \circ \tilde{\omega}_k$ and A_k. Besides,

$$|-A_k^2 - \tilde{I}|_{g_k} = |-A_k^2 - (-\tilde{J}^2)|_{g_k} \leq O(c_k^{-1/2}).$$

Therefore, $|Q_k - \tilde{I}|_{g_k} \leq O(c_k^{-1/2})$. If we use charts adapted to the curvatures and diagonalize $-A_k^2$ (for example using the Gauss method), it follows that $-A_k^2 = G_k \circ \tilde{D}(\lambda_{1,k}, ...\lambda_{2n,k}) \circ G_k^{-1}$, where $\tilde{D}(\lambda_{1,k}, ...\lambda_{2n,k})$ is the diagonal matrix $D(\lambda_{1,k},...\lambda_{2n,k},0)$, and both $|G_k-\tilde{I}|_{g_k}$ and $|\tilde{D}(\lambda_{1,k},...\lambda_{2n,k})-\tilde{I}|_{g_k}$ are of order $O(c_k^{-1/2})$. As a consequence,

$$|\tilde{D}\left(\frac{1}{\sqrt{\lambda_{1,k}}},...\frac{1}{\sqrt{\lambda_{n,k}}}\right)-\tilde{I}|_{g_k}\leq O(c_k^{-1/2}),$$

which gives the same bound for

$$Q_k^{-1}=G_k^{-1}\circ\tilde{D}\left(\frac{1}{\sqrt{\lambda_{1,k}}},...\frac{1}{\sqrt{\lambda_{n,k}}}\right)\circ G_k.$$

Finally, $|\tilde{J}-\tilde{J}_k|_{g_k}\leq|\tilde{J}-A_k|_{g_k}+|A_k-\tilde{J}_k|_{g_k}\leq O(c_k^{-1/2})$.

The bounds for the derivatives of the difference $\tilde{J}-\tilde{J}_k$ are much easier to obtain. One checks that $|\nabla^r(\tilde{\omega}_k\circ(-\tilde{J})-\tilde{S}_k)|_{g_k}\leq O(c_k^{-1/2})$ holds trivially. The reason is that the tensor is defined in terms of $\tilde{\omega}_k$ and $\tilde{J}$, and in the previous expression we have to estimate the size of a bounded number of summands each of which contains at least a derivative of either $\tilde{\omega}_k$ or $\tilde{J}$, together with terms of size at most $O(1)$. The control for derivatives of $\tilde{J}$ is obvious, because 3.17 implies that this follows from the corresponding control for $\bar{J}$ (which holds trivially because what we have is a constant sequence of tensors).

As we already mentioned –and for tensors T invertible along D (resp. $\tilde{D}^*$) and vanishing along $\ker\omega_k$ (resp. $\mathrm{Ann}(D)$)– the inverses that we have used are computed in the directions of D (resp $\tilde{D}^*$) and defined to be trivial along $\ker\omega_k$ (resp $\mathrm{Ann}(D)$). If T is one such tensor we have $T\circ T^{-1}=I\oplus 0=\tilde{I}$, $T^{-1}\circ T=I\oplus 0=\tilde{I}$. Since $|\nabla^r\tilde{I}|_{g_k}\leq O(c_k^{-1/2})$ (for the same reason as $\tilde{J}$), we have the usual formulas for the covariant derivatives of T^{-1}, up to terms of order $O(c_k^{-1/2})$. In particular $\nabla(\tilde{\omega}_k(-\tilde{J})\circ(\tilde{\omega}_k(-\tilde{J}))^{-1})=\nabla(I\oplus 0)$ implies that

$$\nabla(\tilde{\omega}_k(-\tilde{J}))^{-1}=-(\tilde{\omega}_k(-\tilde{J}))^{-1}\circ\nabla\tilde{\omega}_k(-\tilde{J})\circ(\tilde{\omega}_k(-\tilde{J}))^{-1}+(\tilde{\omega}_k(-\tilde{J}))^{-1}\circ\nabla(I\oplus 0)$$

We know use condition (2) in definition 2.2 to bound the g_k-norm of $(\tilde{\omega}_k(-\tilde{J}))^{-1}$ by $O(1)$. The consequence is:

$$\begin{aligned}|\nabla(\tilde{\omega}_k(-\tilde{J}))^{-1}|_{g_k} &\leq |-(\tilde{\omega}_k(-\tilde{J}))^{-1}\circ\nabla\tilde{\omega}_k(-\tilde{J})\circ(\tilde{\omega}_k(-\tilde{J}))^{-1}|_{g_k}+\\ &+ O(c_k^{-1/2})\leq O(c_k^{-1/2}).\end{aligned}$$

The last step is to check the appropriate bound for $|\nabla^r(Q_k-\tilde{I})|_{g_k}$. This can be done using charts adapted to the curvatures; in their domain (in euclidean balls of radius $O(1)$) we have to apply the Gauss method to $-A_k^2\in M_{2n\times 2n}(\mathbb{R})\subset M_{2n+1\times 2n+1}(\mathbb{R})$. The partial derivatives of any order of the functions $A^2_{k i,j}+\delta_{i,j}$ are of size $O(c_k^{-1/2})$. One notices that the difference between the matrix of change of basis G_k and the identity has its entries as linear combinations of products of the previous functions; that gives the desired bound for G_k, and similarly for its inverse. The final bounds for the square root are obtained using the same kind of considerations. □

When the starting point is a calibrated manifold (M, D, ω) and we apply the symplectization procedure described in subsection 2.3, the described extension of J turns out to be compatible only in the leaf $M \times \{0\}$. It is obvious that if we apply the corresponding simplified version for even dimensional almost-complex manifolds of lemma 3.21 to the symplectization, we obtain $\tilde{J}_k = \bar{J}_k$ (the matric and curvatures retractions do coincide) a sequence of compatible almost complex structures whose restriction to $M \times \{0\}$ is J.

3.3. Darboux charts and reference sections. Now we are in position to show the existence of charts approximately matching the flat model of definition 3.1. For polarized almost complex manifolds, we will also introduce Darboux charts adapted to the polarization (they are indeed a small modification of the Darboux charts defined in [**4**]).

Lemma 3.22. *For every point $x \in M$ and $k \in \mathbb{N}_+$, there exist adapted Darboux charts $\varphi_{k,x} \colon (\mathbb{C}^n \times \mathbb{R}, 0) \to (U_{k,x}, x)$ with coordinates $z_k^1, ..., z_k^n, s_k$. That is, charts adapted to the curvatures for which $\varphi_{k,x}^* \omega_k = \omega_0$. Regarding the relation between J and J_0 (more precisely between $\tilde{J}$ and $\bar{J}_0$, the extension by zeros given by the euclidean metric in the chart), one has $\varphi_{k,x}^* D \approxeq D_h$, $\varphi_{k,x}^*(\ker \omega_k) = \frac{\partial}{\partial s}$ y $\varphi_{k,x}^* \tilde{J} \approxeq \tilde{J}_0$.*

The precise bounds are:

$$|\varphi_{k,x}^* D - D_h|_{g_k} \leq O(|(z_k, s_k)| c_k^{-1/2}),\ |\nabla^j (\varphi_{k,x}^* D - D_h)|_{g_k} \leq O(c_k^{-1/2}),\ \forall j \geq 1,$$

where the inequalities hold in a ball of fixed g-radius *uniformly in k and x.*

For the antiholomorphic components,

$$\begin{aligned} |\bar{\partial} \varphi_{k,x}^{-1}(z_k, s_k)|_{g_k} &= O(c_k^{-1/2} + |(z_k, s_k)| c_k^{-1/2}), \\ |\nabla^j \bar{\partial} \varphi_{k,x}^{-1}(z_k, s_k)|_{g_k} &= O(c_k^{-1/2}), \end{aligned}$$

$\forall j \geq 1$, uniformly in k and x in a ball of fixed g-radius, *where $\bar{\partial} \varphi_{k,x}^{-1}$ is the antiholomorphic component of $\nabla_{\tilde{D}} \pi_{D_h}(\varphi_{k,x}^{-1})$, with $\pi_{D_h}(\varphi_{k,x}^{-1}) \colon U_{k,x} \to \mathbb{C}^n$.*

Proof. We want to prove the existence of adapted charts for with the distributions $\mathrm{Ann}(D)$, $D^{*1,0}$ and $D^{*0,1}$ approximately coincide with $ds, T^{*1,0}\mathbb{C}^n$ and $T^{*0,1}\mathbb{C}^n$ respectively, and such that $\omega_k \approxeq \omega_0$ (being able to obtain $\omega_k = \omega_0$ as well). The proof follows the lines of that of D. Auroux in [**4**] for even dimensional almost complex manifolds.

We start with an initial family of adapted charts $\phi_{1,x}$. For them we can assume $\phi_{1,x}^* J_x = J_0(0)$, by composing with a linear transformation if necessary. Then we rescale to obtain a family of charts $\phi_{k,x} \colon \mathbb{C}^n \times \mathbb{R} \to (U_x, x)$. If $\phi_{1,x}$ have been chosen depending smoothly on the center x, the bounds for $\phi_{k,x}^{-1}$ and their covariant derivatives are obvious. The result for the antiholomorphic components follows the same pattern. We just observe that to compute this components we have to project $\nabla \phi_{k,x}^{-1} \colon TB_g(x, c) \to T(\mathbb{C}^n \times \mathbb{R})$ over $T\mathbb{C}^n$, and $\bar{\partial} \phi_{k,x}^{-1}$ turns out to be the antiholomorphic component with respect to $\tilde{J}$ and $\bar{J}_0$ in $\mathbb{C}^n$. The desired bounds, once we know those of $\phi_{k,x}^{-1}$ and its derivatives, are equivalent to bounds of the same order for the maximal angle and its derivatives between the subbundles of antiholomorphic 1-forms for both almost complex structures in D_h. For the latter, the bounds follow from the smooth dependence of the charts on the center and the fact that both structures coincide at the origin. Actually, being absolutely precise, since the charts are not necessarily adapted to ω_k, the almost complex structure $\bar{J}_0$ for the charts it is not quite the one of the statement of the lemma, because the vertical component is not the image of the kernel of ω_k. In any case and from the comments of subsection 2.1 about the relation between holomorphic and antiholomorphic components for different retractions, since we can modify the charts to obtain charts adapted to the curvatures by composing with the maps Φ^t_{x,R_k} (lemma 3.13) –which fix $D_h(0)$ (not only at infinitesimal level) and have bound $O(1)$ for the first derivative and of order $O(c_k^{-1/2})$ for the derivatives– we obtain the bounds $O(|(z_k, s_k)|)$ for the antiholomorphic component of the inverse of this new charts and of order $O(c_k^{-1/2})$ for the subsequent derivatives.

The rest of the proof uses the arguments of the one for even dimensional almost complex manifolds. Since the vertical direction coincides with the kernel of ω_k, we can work in one of the symplectic leaves, say $\mathbb{C}^n \times \{0\}$, and apply the obtained transformation to all the leaves (the restriction of the 2-forms ω_k to each leaf satisfy the necessary requirements to apply the even dimensional theory). □

Remark 3.23: In a Darboux chart it is possible to obtain a suitable trivialization of L_k so that the connection form has a fixed formula. The price to pay is that J may not coincide with J_0 at the origin (though the difference will be bounded by $O(c_k^{-1/2})$). In certain circumstances, it may be convenient to have the equality; we only need to undo the last perturbation in the lemma, which is of size $O(c_k^{-1/2})$.

We speak of *approximately holomorphic coordinates* whenever we have adapted charts in which the splitting $T^*_{\mathbb{C}}M = \mathrm{Ann}(D)_{\mathbb{C}} \oplus \tilde{D}^{*1,0} \oplus \tilde{D}^{*0,1}$ approximately coincides with $T^*_{\mathbb{C}}(\mathbb{C}^n \times \mathbb{R}) = ds \oplus T^{*1,0}\mathbb{C}^n \oplus T^{*0,1}\mathbb{C}^n$, with bounds as in the statement of lemma 3.22. We also use the same terminology when the global splitting is $\mathrm{Ann}(D) \oplus \bar{D}^{*1,0} \oplus \bar{D}^{*0,1}$. Actually, we will allows more general kinds of approximately holomorphic coordinates, but we will postpone this discussion until the next section.

Remark 3.24: If we work with an even dimensional almost complex manifold with polarization G, we can compose the Darboux charts with an h_0-unitary (h_0 the canonical hermitian metric) such that G approximately coincides with $\mathbb{C}^g \times \{\cdot\}$ and $G^\perp$, represented for example as a $\mathbb{C}^g$-valued function, will be uniformly bounded by below (its minimal angle with $\mathbb{C}^g \times \{\cdot\}$ will be bounded by below), and all the derivatives of the function will be of order $O(c_k^{-1/2})$ (that is the distribution will be approximately constant). Also, we will have $G^{*1,0} \approxeq T^{*1,0}\mathbb{C}^g$, $G^{*0,1} \approxeq T^{*0,1}\mathbb{C}^g$. Since the transformation is h_0-unitary, ω_0 is preserved.

We will speak of *approximately holomorphic coordinates adapted to G* when the previous bounds measuring the difference between $G, G^{*1,0}, G^{*0,1}$ and $G^{\perp}$ with the corresponding models hold.

Going back to the odd dimensional situation, we will see that we will not be able to construct approximately holomorphic sequences of sections (without using symplectizations) with interesting transversality properties and so that we keep the same kind of control for the directions of D as for the whole tangent bundle.

To generalize the notion of approximately holomorphic section we need to choose a retraction i.

Definition 3.25. *Let i be a retraction to the canonical projection $T^*M \to D^*$. A sequence of sections τ_k of L_k is i-approximately J-holomorphic (or i-A.H.) if positive constants $(C_j^D, C_j)_{j\geq 0}$ exist such that,*

$$|\tau_k|_{g_k} \leq C_j^D, \ \ |\nabla^j_{i(D^*)}\tau_k|_{g_k} \leq C_r^D, \ \ |\nabla^j \tau_k|_{g_k} \leq C_r$$
$$|\nabla^{j-1}\bar{\partial}_{i(D^*)}\tau_k|_{g_k} \leq C_r c_k^{-1/2}.$$

A sequence of sections has i-gaussian decay w.r.t. x if positive constants $\lambda > 0$, $(C_j)_{j\geq 0}$ and polynomials exist $(P_j)_{j\geq 0}$ so that $\forall y \in M$ y $\forall j \geq 0$,

$$|\nabla^j_{i(D^*)}\tau_k(y)|_{g_k} \leq P_j(d_k(x,y))e^{-\lambda d_k(x,y)^2},$$
$$|\nabla^j \tau_k(y)|_{g_k} \leq C_r P_j(d_k(x,y))e^{-\lambda d_k(x,y)^2}$$

If we are only interested in controlling the first r-covariant derivatives, we will speak of sequences of i-C^r-A.H. sections (resp. with i-C^r-gaussian decay).

Notice that $\nabla^j_{i(D^*)}\tau_k$ is a section of $i(D^*)^{\otimes j} \otimes L_k$ constructed recursively using the confections induced by ∇_k, and the induced connection on $i(D^*)$ via the Levi-Civita connection and the curvatures splitting.

Remark 3.26: We observe that the previous notions *do depend on the chosen retraction.* If we use the metric retraction, that for us is the most natural, we will simply speak of A.H. sequences of sections and of sequences of sections with gaussian decay. The other important retraction, especially in local construction is $\tilde{i}$, the curvatures retraction.

Using the intrinsic theory we will be able to find sequences of $\tilde{i}$-A.H. with $\tilde{i}$-gaussian decay. These will be a fundamental tool to given any A.H. sequence, construct A.H. perturbations with arbitrarily small constants $C_0^D, ..., C_r^D$ so that when we add the perturbation the resulting sequence has interesting transversality properties (actually for both retractions $\tilde{i}$ and i).

We want to be able to verify the bounds of definition 3.25 for the retraction $\tilde{i}$ in Darboux charts (or in A.H.) using D_h, J_0, g_0 and d (recall that the Darboux charts are adapted to the curvatures). To prove this kind of statements we use ideas similar to those used in lemma 3.17. The new ingredient is the presence of the line bundle L_k. On each chart we fix a unitary trivialization so that the confection form is $A_0\frac{1}{4}(\sum_{i=1}^n z_k^i d\bar{z}_k^i - \bar{z}_k^i dz_k^i)$, the same for all k and x. Recall that d denotes the usual derivative or flat connection (partial derivatives) and d_{D_h} its projection over D_h parallel to ds (the partial derivatives w.r.t. the horizontal coordinates). Likewise, we denote by d^j (resp. $d^j_{D_h}$) the j-th iterate of the correspondent operator; d_{A_0} denotes the flat connection coupled with A_0, d_{A_0,D_h} and $d_{A_0,D}$ its projections over D_h and D parallel to ds, and $d^j_{A_0}, d^j_{A_0,D_h}$ and $d^j_{A_0,D}$ the j-th iterates of the correspondent operators.

Lemma 3.27. *Let E be one of the vector bundles of definition 3.6 and let τ_k be a sequence of sections of $E \otimes L_k$ such that $|\nabla^j \tau_k|_{g_k} \le O(1)$, $j = 0, ..., r$. For a family of Darboux charts and trivializations of L_k with associated connection forms A_0, the following equivalences hold:*

1. $|\nabla^j \tau_k|_{g_k} \le P_j(d_k(\psi_{k,x}(z_k, s_k))O(1)$, $j = 0, ..., r$ *is equivalent to* $|d^j \tau_k|_{g_0} \le Q_j(|(z_k, s_k)|)O(1)$, $j = 0, ..., r$ *in the points of* $B_{g_0}(0, O(c_k^{1/2}))$ *(or with polynomial equal to* 1 *over* $B_{g_0}(0, O(1)))$.
2. $|\nabla^j \tau_k|_{g_k} \le P_j(d_k(\psi_{k,x}(z_k, s_k))O(c_k^{-1/2})$, $j = 0, ..., r$ *is equivalent to* $|d^j \tau_k|_{g_0} \le Q_j(|(z_k, s_k)|)O(c_k^{-1/2})$, $j = 0, ..., r$ *in the points of* $B_{g_0}(0, O(c_k^{1/2}))$ *(or with polynomial equal to* 1 *over* $B_{g_0}(0, O(1)))$.
3. $|\nabla^j_{\tilde{D}} \tau_k|_{g_k} \le P_j(d_k(\psi_{k,x}(z_k, s_k))O(1)$, $j = 0, ..., r$ *is equivalent to* $|d^j_{A_0,D_h} \tau_k|_{g_0} \le Q_j(|(z_k, s_k)|)O(1)$, $j = 0, ..., r$ *or to* $|d^j_{D_h} \tau_k|_{g_0} \le S_j(|(z_k, s_k)|)O(1)$, $j = 0, ..., r$ *in the points of* $B_{g_0}(0, O(c_k^{1/2}))$ *(or with polynomial equal to* 1 *over* $B_{g_0}(0, O(1)))$.
4. $|\nabla^j_{\tilde{D}} \tau_k|_{g_k} \le P_r(d_k(\psi_{k,x}(z_k, s_k))O(c_k^{-1/2})$, $j = 0, ..., r$ *is equivalent to* $|d^j_{A_0,D_h} \tau_k|_{g_0} \le Q_r(|(z_k, s_k)|)O(c_k^{-1/2})$, $j = 0, ..., r$ *or to* $|d^j_{D_h} \tau_k|_{g_0} \le S_j(|(z_k, s_k)|)O(c_k^{-1/2})$, $j = 0, ..., r$ *in the points of* $B_{g_0}(0, O(c_k^{1/2}))$*(or with polynomial equal to* 1 *over* $B_{g_0}(0, O(1)))$.
5. *The bounds of* (1) *or* (2) *imply the same kind of bounds for* $|\nabla^{r-j} \nabla^j_{\tilde{D}} \tau_k|_{g_k}$, $j = 0, ..., r$. *In particular* (3) *follows from* (1) *and* (4) *follows from* (2).

The polynomials P_j, Q_j, S_j are obtained the ones from the others using formulas independent of k and x.

Proof. Regarding the full derivatives the difference $\nabla^r \tau_k - d^r \tau_k$ is the sum of *homogeneous terms*, each of which is a product containing some derivative $d^j \tau_k$ (with weight j), and other factors which are derivatives of A_0 and of the Christoffel symbols (with weight the order of the derivative plus one). The Christoffel symbols have size $O(c_k^{-1/2})$, A_0 is bounded by $|(z_k, s_k)|O(1)$, its first derivative is ω_0 –of size $O(1)$–, and its higher order derivatives are by hypothesis bounded by $O(c_k^{-1/2})$. The result follows from the bounds on τ_k and its derivatives, that are assumed to be of order $O(1)$.

Notice also that since each summands contains a derivative of τ_k, if for

$$\tau_k = \hat{\tau}_k \otimes e^{-\lambda|(z_k,s_k)|^2},\ \lambda > 0, \tag{3.2}$$

with all the derivatives of $\hat{\tau}$ bounded by $O(1)$, then τ_k, once multiplied by a suitable bump function β_k with support on $B(0, c_k^{1/6})$ verifies the second requirement to be sequence of sections with $\tilde{i}$-gaussian decay (see [**12**]).

The statement concerning the derivatives along the distributions is essentially that of lemma 3.17. We define

$$\pi_{k,j}^{\tilde{D}} \colon T^*M^{\otimes j} \otimes E \otimes L_k \to \tilde{D}^{*\otimes j} \otimes E \otimes L_k,$$

(resp. $\pi_{k,j}^{D_h} \colon T^*M^{\otimes j} \otimes E \otimes L_k \to \bar{D}_h^{*\otimes j} \otimes E \otimes L_k$) as the projection associated to the curvatures (notice that the bundles vary with k).

The morphisms $\pi_{k,j}^{\tilde{D}}$ (resp. $\pi_{k,j}^{D_h}$) are sections (resp. local sections) of $\mathrm{End}(T^*M^{\otimes j} \otimes E \otimes L_k)$, with are bundles with a metric induced by g_k (resp. g_0) and h_k, and a connection induced by ∇_g (resp. d) y ∇_k. Hence, for a sequence of sections of these bundles we have the notion of approximate equality of order r. In particular, the previous sequence of maps can be written in the form $p_{k,j}^{\tilde{D}} \otimes I$ (resp. $p_{k,j}^{D_h} \otimes I$), where the projections are sections of $T^*M^{\otimes j} \otimes E$ and the identity is a section of $\mathrm{End}(L_k) = L_k^* \otimes L_k = \mathbb{C}$ (with induced trivial connection). The identity has vanishing derivatives. Thus we deduce –in a ball of radius $O(c_k^{1/2})$– bounds of order $O(1)$ for the maps and of order $O(c_k^{-1/2})$ for the derivatives (using both ∇ and d_{A_0}). The difference of the projections is bounded by $|(z_k, s_k)|O(c_k^{-1/2})$ and its derivatives by $O(c_k^{-1/2})$.

Notice that in the whole discussion the choice of unitary trivializations of L_k does not play any role (we do not ask any requirement about their dependence on k, x).

Let us write $\nabla_{\tilde{D}} - d_{A_0,D_h} = (\nabla_{\tilde{D}} - d_{A_0,D}) + (d_{A_0,D} - d_{A_0,D_h})$. Using the previous ideas the difference $\nabla\tau_k - d_{A_0}\tau_k$ is a sum of products of components of τ_k multiplied by Christoffel symbols. Therefore $\nabla_{\tilde{D}}\tau_k - d_{A_0,D}\tau_k = \pi_{k,1}^{\tilde{D}}(\nabla\tau_k - d_{A_0}\tau_k) \approxeq 0$, where the approximate equality is for sections of $E \otimes L_k$ (see observation 3.16 after definition 3.14).

Similarly, $d_{A_0,D}\tau_k - d_{A_0,D_h}\tau_k = (\pi_{k,1}^{\tilde{D}} - \pi_{k,1}^{D_h})d_{A_0}\tau_k$, and the results are deduce from $|d_{A_0}\tau_k|_{g_k} \leq O(1)$, and $\pi_{k,1}^{\tilde{D}} \approxeq \pi_{k,1}^{D_h}$. Finally $(d_{D_h,A_0} - d_{D_h})\tau_k = A_0\tau_k$, and using the same ideas that proved the equivalence (1) we can check that the required bounds hold. Observe again that is the sequence can be written as in equation (3.2) (for some point x), then after multiplied by the mentioned bump function β_k, the projection of the first derivative over $\tilde{D}^*$ fulfils the required inequality to have $\tilde{i}$-gaussian decay w.r.t. x.

The differences for higher order derivatives are computed in the same way. For example $\nabla_{\tilde{D}}\nabla_{\tilde{D}} - d_{A_0,D_h}d_{A_0,D_h} = \nabla_{\tilde{D}}(\nabla_{\tilde{D}} - d_{A_0,D_h}) + (\nabla_{\tilde{D}} - d_{A_0,D_h})d_{A_0,D_h}$. Since already satisfies the bounds $\nabla(\nabla_{\tilde{D}} - d_{A_0,D_h})\tau_k$, the same happens for the projection $\nabla_{\tilde{D}}(\nabla_{\tilde{D}} - d_{A_0,D_h})\tau_k$ (and all its derivatives). For the second summand we apply what we did for the first derivative but to the sequence $d_{A_0,D_h}\tau_k$, which satisfies the required bounds. We do the same to bound the difference $d^2_{A_0,D_h} - d^2_{D_h}$. For the r-th derivative we simply write $\nabla^r_{\tilde{D}} - d^r_{A_0,D_h} = \nabla_{\tilde{D}}(\nabla^{r-1}_{\tilde{D}} - d^{r-1}_{A_0,D_h}) + (\nabla_{\tilde{D}} - d_{A_0,D_h})d^{r-1}_{A_0,D_h}$, being the size of the first summand bounded by $O(c_k^{-1/2})$ by induction, and also the second applying the construction for $r = 1$ to the appropriate sequence. The bounds $d^r_{A_0,D_h} - d^r_{D_h}$ are obtained similarly.

If the sequence, for some $x \in M$ can be written as in equation (3.2), after multiplying by β_k the second conditions required to have $\tilde{\imath}$-gaussian decay w.r.t. x holds.

Point five is straightforward for the decomposition $D_h \oplus \frac{\partial}{\partial s}$ and d_{A_0,D_h} (or d_{D_h}).

Thus, if τ_k is as in (3.2) for some x, the corresponding sequence has $\tilde{\imath}$-gaussian decay w.r.t. x.

□

Remark 3.28: In particular, associated to charts adapted to the curvatures and using trivializations of L_k as described, from the equivalence (2) we deduce that can easily define the notion of (local) approximate equality of order r for sequences of sections of $E \otimes L_k$, by requiring the corresponding bounds for the partial derivatives of order equal or smaller than r (as in equation 3.1 in definition 3.14). Besides, from the equivalence (4) we deduce that if the sections belong to $E_{\tilde{D}} \otimes L_k$ it is enough to consider partial derivatives w.r.t. horizontal coordinates.

The content of the following lemma is the existence of A.H. sequences of sections of L_k that play the role of partitions of the unity for the theory.

Lemma 3.29. *There exist $\kappa > 0$ and $K \in \mathbb{N}_+$, such that for all $x \in M$ $\tilde{\imath}$-A.H. sequences of sections $\tau^{ref}_{k,x}$ of L_k with $\tilde{\imath}$-gaussian decay w.r.t. x (and with constants in the bounds independent of x) can be constructed, $|\tau^{ref}_{k,x}| \geq \kappa$ in a ball of fixed g_k-radius centered at x (the bounds fulfilled from K on).*

Proof. We fix Darboux charts (adapted to the curvatures) and unitary trivializations for which the connection form is $A_0 = \frac{1}{4}(\sum_{i=1}^n z^i_k d\bar{z}^i_k - \bar{z}^i_k dz^i_k)$.

Following the ideas of S. Donaldson [**12**], we consider the local section (for each k) defined by the function $f(z_k, s_k) = e^{-|(z_k,s_k)|^2/4}$. The local section is multiplied by an appropriate cut-off function β_k with support in the ball of radius $c_k^{1/6}$. The gaussian decay for g_0, d, D_h y $|\cdot|^2$ is straightforward, and lemma 3.27 gives the result.

The approximate holomorphicity is trivial for J_0 (notice that for any polynomial Q_r a constant C_{Q_r} exists so that for any $(z_k, s_k) \in \mathbb{C}^n \times \mathbb{R}$ and any $\lambda > 0$, $|Q_r(|z_k, s_k|)e^{-\lambda|(z_k,s_k)|^2)|}| \leq C_{Q_r})$.

From the estimates for $|\bar{\partial}\varphi_k(z_k, s_k)|_{g_k}$ and $|\nabla^r\bar{\partial}\varphi_{k,x}(z_k, s_k)|_{g_k}$, we deduce the required bounds for $\tilde{J}$. □

3.4. Relation between the (intrinsic) A.H. theories and the relative theory. We mentioned the possibility of developing and A.H. theory which uses the metric retraction, instead of that given by the curvatures. We have focused ourselves in the second one because the existence of reference sections for it is more or less obvious.

Let i be any retraction for $T^*M \to D^*$. We denote by $q^{\tilde{i},i}\colon T^*M \to T^*M$ the bundle isomorphism sending $\tilde{D}^*$ to $i(D^*)$ parallel to $\mathrm{Ann}(D)$, and being the identity in the latter. Let $q^{\tilde{i},i}_{r,j}\colon T^*M^{\otimes r} \otimes L_k \to T^*M^{\otimes r} \otimes L_k$ be the bundle map defined to be identity in all the factors except for the j-th one of $T^*M^{\otimes r}$, in which we require it to coincide with $q^{\tilde{i},i}$.

Lemma 3.30. *Let τ_k be a sequence of sections of L_k. Suppose that bounds of order $O(1)$ exist for $q^{\tilde{i},i}$ and its derivatives (for k large enough). Then τ_k is a $\tilde{i}$-A.H. sequence of sections if and only if it is a i-A.H. sequence of sections. Suppose further that for the derivatives of $q^{\tilde{i},i}$ bounds of order $O(c_k^{-1/2})$ hold. Then it is possible to find a constant C such that if $|\nabla_{\tilde{D}}\tau_j|_{g_k} \le C^D$, $j = 0, ..., r$, then for $k \ge K$ $|\nabla_{i(D^*)}\tau_k|_{g_k} \le CC^D$. In this situation the gaussian decay is equivalent for both theories, and it is possible to estimate the constant C relating the bounds along D for both theories.*

If we have the first kind of bounds we speak of equivalent A.H. theories (for i and $\tilde{i}$). For the second and stronger bounds, we speak of strongly equivalent theories.

Proof. The ideas are the ones used in lemmas 3.17, 3.27. Let us check that $\tilde{i}$-approximate holomorphicity implies i-approximate holomorphicity.

Let us denote by $\bar{\partial}_{i(D^*)}$ the antiholomorphic component defined by i. By definition $\bar{\partial}_{i(D^*)}\tau_k = q^{\tilde{i},i}_{1,1}(\bar{\partial}_{\tilde{D}}\tau_k)$. Its r-th derivative is the sum of 2^r terms which are the composition of some derivative of $q^{\tilde{i},i}_{1,1}$ acting on some derivative of $\bar{\partial}_{\tilde{D}}\tau_k$, and thus the result follows easily. It is important to observe that there is only one summand in which there is no derivative of $q^{\tilde{i},i}_{1,1}$, and this is $q^{\tilde{i},i}_{1,1}(\nabla^r\bar{\partial}_{\tilde{D}}\tau_k)$ (being more precise the extension $q^{\tilde{i},i}_{1,1}$ acting on the corresponding covariant derivative is $q^{\tilde{i},i}_{r,1}(\nabla^r\bar{\partial}_{\tilde{D}}\tau_k)$). Hence, it we have strong equivalence the bound that we obtains is approximately the one of the mentioned summand.

The computation of the derivatives along D the situation is similar. We first notice that the bounds for $q^{\tilde{i},i}$ can be computed as follows: we select charts adapted to the metric and a (local) vector field of the form $\frac{\partial}{\partial s} + v_k$ spanning the complementary to D associated to i. The bound for the map is equivalent to one of the same order for the minimal angle between this complementary and D, which is itself equivalent to a bound by above for the euclidean norm of v_k. bounds for the covariant derivatives of the bundle map are equivalent to bounds of the corresponding order for $d^r v_k$. From this considerations it follows that the bounds on the derivatives imply bounds of the same order for $\nabla^{r-j}\nabla^j_{i(D^*)}q^{\tilde{r},r}$.

By definition, $\nabla_{i(D^*)}\tau_k$ is the projection along $\mathrm{Ann}(D)$ of $\nabla\tau_k$. Being this line field common for both splittings, we have $\nabla_{i(D^*)}\tau_k = q^{\tilde{i},i}_{1,1,}\nabla_{\tilde{D}}\tau_k$. Similarly to what happens with full derivatives $\nabla^r_{i(D^*)}\tau_k$ is the sum of terms of two kinds: firstly, we have those which include derivatives of $q^{\tilde{i},i}$ or derivatives restricted to $i(D^*)$. By hypothesis, we obtain bounds of the order of the derivatives of $q^{\tilde{i},i}_{1,1,}$. The remaining term is $q^{\tilde{i},i}_{r,r}\circ\cdots\circ q^{\tilde{i},i}_{r,1}(\nabla_{\tilde{D}}\tau_k)$. Hence, for k large enough and strongly equivalent theories, the total size is that of this summand up to a term of order $O(c_k^{-1/2})$ (and for equivalent theories we have a bound of order $O(1)$).

The assertion about the gaussian decay is obvious.

The proof in the opposite direction is the same. □

Since the map relating the metric and curvatures excisions has derivatives bounded by $O(c_k^{-1/2})$ (3.9), we deduce:

Lemma 3.31. *There exist $\kappa > 0$ and $K \in \mathbb{N}_+$, such that for all $x \in M$ A.H. sequences of sections $\tau^{ref}_{k,x}$ of L_k with gaussian decay w.r.t. x (and with constants in the bounds independent of x) can be constructed, $|\tau^{ref}_{k,x}| \geq \kappa$ in a ball of fixed g_k-radius centered at x (the bounds fulfilled from K on).*

From now on the A.H. theory of our choice will be the one associated to the metric retraction (and its strongly equivalent ones). Also, for the sake of brevity in the notation, we will denote the component of the derivative in $\bar{D}$ by ∇_D instead of $\nabla_{\bar{D}}$ or $\nabla_{\bar{i}(D^*)}$.

Remark 3.32: We insist on the fact that bounds of order $O(1)$ are enough to assure that the notion of sequence of A.H. section is the same for both retractions. This will be something useful to give normal forms for Lefschetz pencils.

The advantage of the strongly equivalent theories is the following: a fundamental problem that we will deal with (sections 4, 5) will be the study of the transversality properties (to certain stratifications) of the so called pseudo-holomorphic r-jets associated to an A.H. sequence τ_k. These new sections of sequences of vector bundles constructed out of $\bar{D}^{*1,0}$, to be introduced in the next section, approximately coincide with $\nabla^r_D\tau_k$ (we recall that the notation ∇^r_D substitutes $\nabla^r_{\bar{i}(D^*)}$). For a retraction i strongly equivalent ti $\bar{i}$ we can study the same transversality problem for the corresponding pseudo-holomorphic r-jets, which approximately coincide with $\nabla^r_{i(D^*)}\tau_k$. In the main applications we will see that the corresponding bundle morphism induced by $q^{\bar{i},i}$ preserves the stratifications in question (they will be invariant under the action of $Gl(n,\mathbb{C})$). The strong equivalence between the theories the r-jet associated to i will be approximately the image by the mentioned map of the r-jet associated to $\bar{i}$. The consequence is that transversality for one r-jet will be equivalent to transversality for the other. For equivalent theories this is in principle only equivalent for 1-jets (tough we will see that in certain regions the situation is better).

Remark 3.33: Actually, a consequence of this discussion is that for a given (compact) calibrated manifold of integer type, once we have chosen J c.a.c.s and defined $g_{|D} := \omega(\cdot, J)$, different extensions to a metric g in M give strongly equivalent A.H. theories.

Now we can give a precise definition of what approximately holomorphic coordinates are.

Definition 3.34. *We call approximately holomorphic coordinates to those associated to adapted families of charts $\psi_{k,x}\colon \mathbb{C}^n \times \mathbb{R} \to U_x$ for which:*

1. $\psi_{k,x}^* D \approxeq D_h$.
2. *The coordinate functions are $z_k^j\colon U_x \to \mathbb{C}$ are A.H.*
3. *The g_k-unitary local vector field v_k so that $\frac{\partial}{\partial s}_k + v_k \in D^\perp$ has minimum angle with D_h bounded by below and all its derivatives are bounded by $O(c_k^{-1/2})$.*

In other words, *approximate holomorphic coordinates* are those for which the corresponding local A.H. theory is strongly equivalent the (global) A.H. theory that we have fixed (the associated to the metric).

It is possible to weaken the previous notion using families of charts centered at points of certain sequences of submanifolds of (M, D, J, g_k) for which the bounds of third condition in definition 3.34 are asked to be of order $O(1)$. That is, we use charts for which the corresponding local A.H. theory is only equivalent to ours; certain local constructions are not equivalent for both (though some of them indeed are equivalent, and we will take advantage of that). Unless otherwise stated, A.H. coordinates form now on are the ones defined in 3.34.

It is possible to obtain other families of reference sections with similar bounds using the symplectization. It is enough to consider reference sections centered at the points of $M \times \{0\} = M$, and check that when we restrict to M the bounds are preserved. To do that, we go to the more general setting of a polarized almost complex manifold.

Let χ_k be a sequence of C^r-A.H. of L_k. Then $\bar\partial_G \chi_k$ is clearly of size $O(c_k^{-1/2})$. We can Darboux charts adapted to G. In the local model it is clear that $\bar\partial_{\mathbb{C}^g} \chi_k \approxeq 0$. To compute $\bar\partial_G \chi_k$ and its derivatives we can write it as the projection of $\bar\partial_{\mathbb{C}^g} \chi_k$ by the bundle morphism that related both splittings. The bound for this map is of order $O(1)$, and of order $O(c_k^{-1/2})$ for its derivatives. Thus, if $|\nabla^r \bar\partial \chi_k|_{g_k} \le O(c_k^{-1/2})$, then $|\nabla^r \bar\partial_G \chi_k|_{g_k} \le O(c_k^{-1/2})$.

In our symplectization, $G = D$, and we still really need to compute bound for the restriction $\chi_{k|M}$. Since the metric makes the copies of M orthogonal to $\frac{\partial}{\partial t}$, we conclude that $|\nabla_M^r \bar\partial(\chi_{k|M})|_{g_k} \le O(c_k^{-1/2})$. We also get the required bounds for the full derivatives along TM and for the projections over $\bar D^*$ in the points of M; the same happens for the gaussian decay. Hence, if χ_k is a C^r-A.H.(C_r) sequence of sections defined $M \times [-\epsilon, \epsilon]$, then $\chi_{k|M}$ will be a C^r-A.H.(CC_r, CC_r), where C is a constant which does not depend on the initial sequence χ_k. In particular, from the reference sections centered in the points of $M \times \{0\}$ we obtain a family of reference sections for (M, D, J, g).

Observe that actually the metric has been extended to the symplectization in such a way that the induced holomorphic theory from the one in the symplectization is the one associated to the metric retraction. Actually, we could have chosen any other extension of the metric (and hence of J) in the symplectization, and from L_k a very ample sequence over (M, D, J, g) the bundle with connection over the symplectization defined in 2.3 is also a very ample sequence; its A.H. sequences of sections restrict to A.H. sections (for the metric retraction), and also reference sections centered at the points of M restrict to reference sections for the intrinsic theory.

It is important to observe that we can use the symplectization not only to induce the local intrinsic theory but also to solve transversality problems in M along D for a sequence $\chi_{k|M}$, where χ_k is an A.H. sequence in the symplectization. Indeed, transversality will be equivalent to a condition on $\nabla_D(\chi_{k|M})$. By definition $\nabla_D(\chi_{k|M})$ is the projection over $\bar{D}^*$ D of $\nabla_{TM}(\chi_{k|M})$. One checks that $\nabla_D \chi_k \in \Gamma(\bar{D}^* \otimes L_k)$ defined in the symplectization, extends $\nabla_D(\chi_{k|M})$. Thus we can try to turn the initial transversality problem into one for $\nabla_D \chi_k$ (which has been shown to approximately coincide with $\partial \chi_k$) in the points of M.

In fact the previous discussion applies in the relative situation (M, ω, G, N), where (M, ω) is a compact symplectic manifold, N (resp. (Q, D)) is a symplectic submanifold (resp. calibrated) and G is an almost complex distribution in a neighborhood U of N extending TN (resp. D). Observe that it is elementary to find a J making an arbitrary local extension of TN (resp. D) J complex.

When the submanifold is N (symplectic) we can find in the points of N approximate holomorphic coordinates (for M) adapted to G (rectifying N as well if we want and so that $G \oplus G^{\perp}$ approximately coincides with $\mathbb{C}^g \times \mathbb{C}^{n-g}$). We deduce that if χ_k is an A.H. sequence of sections of $L_k := L^{\otimes k}$, the restriction to N is also an A.H. sequence, and that $\nabla_G \chi_k$ extends $\nabla(\chi_{k|N})$.

If the submanifold is (W, D) calibrated (by the restriction of ω), one checks easily that the restriction of $L^{\otimes k}$ to (W, D, J) defines a very ample sequence of line bundles. By taking approximately holomorphic coordinates for M centered in the points of W, so that W is rectified, the restriction to W are A.H. coordinates for W and $G \oplus G^{\perp}$ approximately coincides with $\mathbb{C}^g \times \mathbb{C}^{n-g}$, if χ_k is an A.H. sequence defined in M, then it follows that its restriction is an A.H. sequence (for the metric retraction) and that $\nabla_G \chi_k$ extends $\nabla_D(\chi_{k|W})$.

3.5. Higher rank bundles. There are similar results for higher rank bundles which are locally of the form $\mathbb{C}^m \otimes L_k$ in the approximate sense.

Definition 3.35 (see [**3**]). *A sequence $E_k \to M$ of rank m hermitian bundles with unitary connection is asymptotically very ample (or just very ample) is positive constantsc$_k \to \infty$, $(C_j)_{j \geq 0}$ exist, so that the curvature verifies:*

(1) $\langle iF_k(v, Jv)u, u \rangle \geq g(v, v)|u|^2, \forall v \in D,$

(2) $|F_{k|D} - F_{k|D}^{1,1}|_{g_k} \leq C_r c_k^{-1/2}$,
(3) $|\nabla^j F_k|_{g_k} \leq C_j c_k^{-1/2} \forall j \geq 0$.

A sequence E_k of very ample hermitian bundles is approximately locally splittable (or just locally splittable), if for each $x \in M$ over a ball of fixed g-radius unitary sections $\tau_{k,1}, ..., \tau_{k,m}$ can be found so that $\tau_{k,1} \wedge ... \wedge \tau_{k,m}$ is bounded by below (they are comparable to a unitary frame), and for the induced local splitting $E_k = L_{k,1} \oplus \cdots \oplus L_{k,m}$, the matrix of 1-forms $\alpha_{k,x}$ representing the difference between the connection and the induced diagonal connection verifies $|\nabla^r \alpha_{k,x}|_{g_k} \leq O(c_k^{-1/2})$ for all $r \geq 0$ in the ball of fixed g-radius (which actually is a bit stronger than requiring a vanishing in the approximate sense, because there is no polynomial in the right hand side of the inequality).

In any case the main applications of the theory will be for bundles of the form $E_k = E \otimes L_k$, where E is a hermitian bundle with connection. For these bundles, as well as having the metric retraction we also have the one associated to the curvatures of L_k. The notion of A.H. sequence and sequence with gaussian decay w.r.t. a point is obvious; one easily checks that the result of tensoring reference sections for L_k with local unitary frames of E is a family of A.H. sections with gaussian decay $\tau_{k,x,1}^{\text{ref}}, ..., \tau_{k,x,m}^{\text{ref}}$ which trivialize the bundle in a ball of fixed g_k-radius.

For a general sequence of locally splittable bundles the only natural retraction is the metric one. Anyhow, if we want to construct reference sections without using the relative theory, we can consider the very ample sequence of local line bundles $L_{k,j}$ and obtain reference sections for them. In this local construction the local retraction given by the curvatures (different for each of the m sequences) can be used as an auxiliary tool to obtain A.H. reference sections, and apply lemma 3.30 to conclude that the sections are indeed reference sections for the metric retraction.

We have seen that there are as many (intrinsic) A.H. theories as retractions for the projection $T^*M \to D^*$. Some of them are equivalent and even strongly equivalent. In particular, the curvatures retraction and the metric retraction are strongly equivalent. We have shown the existence of reference sections for the curvatures retraction. From what we have just said in the previous paragraph, it implies the existence of reference sections for the metric retraction.

Finally, we have noticed that it is possible to obtain families of references sections using the relative theory. In light of these results, it might seem that having developed a local intrinsic theory is a useless task, but this is a wrong impression. In fact both theories, the intrinsic (which are as many as retractions) and the relative one are two version of an A.H. theory along distributions (foliations in the local models), and the kind of local constructions needed for both are essentially the same: firstly, Darboux charts adapted to the correspondent distribution are needed; the precise condition is that the distribution must coincide in the approximate sense with the foliation, and the complementary distribution –which in the intrinsic case is given by the retraction and in the relative is the orthogonal distribution–

has to be approximately constant (i.e., its minimum angle with the model foliation must be bounded by below and all the derivatives must vanish in the approximate sense). It turns out that in the intrinsic case and for the curvatures retraction the Darboux condition implies that the orthogonal line field actually coincides with the vertical line field.

Secondly, one uses this charts to write down explicit A.H. sequences coming from solution to the Cauchy Riemann equations in the model (analogs to S. Donaldson's reference section). At this point, the difference is that in the intrinsic theory we need a 1-parameter version of S. Donaldson's construction (with parameter suitably chosen to have Gaussian decay) and in the relative construction we just take Donaldson's solution and make sure –by using a family of charts adapted to the submanifold– that the restrictions to the calibrated submanifold have the desired properties.

4. Pseudo-holomorphic jets

The main goal of the theory is showing –for a very ample sequence of locally splittable bundles– the existence of A.H. sections with nice transversality properties w.r.t. suitably chosen stratifications. The conditions to be imposed on these stratifications will imply that *estimated transversality* will be reduced to a local estimated transversality result for 1-parameter families of holomorphic functions. Notice that this theory has already been developed for even dimensional almost complex manifolds, being the main result a strong transversality theorem for certain stratifications in the bundles of *pseudo-holomorphic* jets. We will develop a similar theory in the odd dimensional setting (both and intrinsic and a relative one).

In the relative setting we will show the existence of interesting stratifications associated to a polarization, and that under certain conditions (essentially those stated in [**4**]) uniform transversality reduces to local uniform transversality relative to submanifold, a problem already solved by J. P. Mohsen [**43**].

We introduce the (hermitian vector) bundles of pseudo-holomorphic jets.

4.1. Pseudo-holomorphic jets.

The integrable case. Let $E \to M$ be a hermitian bundle over a *complex manifold*. To define the holomorphic jets it is necessary to introduce a linear connection ∇ with $F_{\nabla}^{0,2} = 0$, that is, a structure of complex manifold in the total space of the bundle E (theorem 2.1.53 in [**15**]; we recall that this result works for any linear connection, and if it is unitary is equivalent to the curvature being of type $(1,1)$). Such structure gives rise to the notion of holomorphic section and hence of holomorphic r-jet. The space of r-jets has natural charts obtained out of holomorphic coordinates in the base, a holomorphic trivialization of the bundle, and using ∂_0 (defined using the canonical structure J_0 in the base and the trivial connection d) for holomorphic maps from $\mathbb{C}^n$ to $\mathbb{C}^m$; thus we obtain a local identification with $\mathcal{J}^r_{n,m}$, the usual r-jets for holomorphic maps from $\mathbb{C}^n$ to $\mathbb{C}^m$ (certain equivalence classes of germs of holomorphic maps from $\mathbb{C}^n$ to $\mathbb{C}^m$).

The connection on the fiber bundle can be used to give a different notion of local holomorphic r-jet (in principle chart dependent) by just considering the operator ∂_∇ (that is if the connection matrix in the trivialization is $A = A^{1,0}$, then the *coupled* 1*-jet* of a holomorphic section τ is defined to be $(\tau, \partial_0\tau + A\tau)))$. It is important to observe that the kind of information associated to the coupled r-jets does not coincide with that of the usual jets. At this point it is necessary to explain why this is a useful notion. First, we assume that the bundle E is of the form $\mathbb{C}^m \otimes L$, where (L, ∇) is a hermitian line bundle (usually very ample) for which $F_\nabla = F_\nabla^{1,1}$ (we point out again that is the connection is hermitian this is equivalent to having a holomorphic line bundle, and it this is not the case it is a stronger condition). A holomorphic section τ of E defines a map ϕ to $\mathbb{CP}^{m-1}$ out of the points where it does not vanish (its projectivization). We aim to study the genericity of ϕ through the genericity of τ. That amounts to transfer the corresponding transversality problem for the r-jet of ϕ in the non-linear bundle $\mathcal{J}^r(M, \mathbb{CP}^{m-1})$, to a transversality problem in a vector bundle of coupled r-jets for the coupled r-jet of τ. We start by describing the bundle of coupled r-jets.

We start by giving a local definition of the coupled r-jets: once we have chosen holomorphic coordinates, we can use the flat connection d acting on sections of $T^{*1,0}M \cong T^{*1,0}\mathbb{C}^n$; the r-jet of a section τ in the point of the chart is defined to be $(\tau, \partial_n\tau, ..., \partial_n^r\tau)$, where the iterates of ∂_n are constructed using d in the factors $T^{*1,0}M$.

As we said, in the domain of a holomorphic chart L can be trivialized with a holomorphic section; thus $A^{0,1}$ –the antiholomorphic component of the connection form– vanishes. Over each point the set of coupled r-jets can actually be identified with the set of usual ones: indeed, for each point we can trivialize the bundle with sections whose graph is tangent to the horizontal distribution of the connection $\mathcal{H}_\nabla$. Thus, the connection form vanishes. This, together with the vanishing of $F_\nabla^{2,0}$ imply the assertion. As a consequence, the local coupled r-jets fill the bundle $\sum_{j=0}^r ((T^{*1,0}\mathbb{C}^n)^{\odot j}) \otimes \mathbb{C}^m = \mathcal{J}^r_{m,n}$ (they are symmetric).

The coupled local r-jets do share an important property with the usual ones: given any coupled $(r+1)$-jet $\sigma = (\sigma_0, ..., \sigma_{r+1})$ over, say the origin, a local section α of $\mathcal{J}^r_{m,n}$ exists with the properties $\alpha(0) = (\sigma_0, ..., \sigma_r)$ and $\sigma = \partial_\nabla\alpha(0) = \nabla\alpha(0)$; it is enough to use the same construction as the one of the flat case ($A = 0$). One perturbs linearly $\pi_r^{r+1}(\sigma) = (\sigma_0, ..., \sigma_r)$ with appropriate complex homogeneous polynomials, and uses the vanishing of the connection form at the origin.

At this point, having a local model for coupled r-jets is enough for our purposes (actually we have described the even dimensional model, and the odd dimensional one will be the corresponding foliated version). Anyhow, and for the sake of completeness, we point out that it is possible to give a global definition of coupled r-jets with do keep the essential properties of the local ones.

We only need to introduce a connection on $T^{*1,0}M$. In particular, we can take the one induced by the Levi-Civita connection. When the metric is Kähler the resulting coupled r-jets are still symmetric. We can fix *holomorphic normal coordinates* for which the connection matrix vanishes at the origin (or equivalently the torsion on the connection vanishes at the origin [**25**]) and for which the curvature is of type $(1,1)$, and that implies symmetry of the global r-jet. Again, using normal coordinates it is easy to write an $(r+1)$-jet as the 1-jet of a local section of the bundle of coupled r-jets.

We must notice that the introduction of the metric changes the meaning of the coupled r-jets, because after projectivizing they will not correspond anymore to the usual r-jets. Anyhow, we point out that the definition of pseudo-holomorphic r-jets that we are about to give, will have the property that when k grows big they will approximately coincide with the local coupled r-jets (in other words, for large values of k the weight of the metric is approximately vanishing, so up to terms of order $O(c_k^{-1/2})$, we can use g_0). Thus, the pseudo-holomorphic r-jets will be important tools to study the genericity properties of the projectivizations.

Anyway it is interesting to observe that for Kähler metrics the (global) coupled jets do fill the right *vector bundle* and do have local representations (which are important tools to deal with transversality problems).

Another important advantage of the coupled r-jets is that they define a bundle with a simpler structural group. For the usual r-jets, the corresponding non-linear bundle has group ${}^0\mathcal{H}_n^r \times Gl(m,\mathbb{C})$, where ${}^0\mathcal{H}_n^r$ is the group of r-jets of germs of biholomorphic transformations of $\mathbb{C}^n$ fixing the origin.

Notice that if we wanted the global coupled are jets to coincide (locally) with the usual ones (imagine for the moment that the bundle L is trivial and we can use d as connection), then the metric should be flat and the transition functions of the coordinate charts, being isometries, would be linear. The result would be a reduction of the structural group from $Gl(n,\mathbb{C}) \times Gl(m,\mathbb{C}) \subset Gl(N,\mathbb{C})$, where N is the dimension of the fiber of $\mathcal{J}_{n,m}^r$. That reduction is exactly the one we get by introducing a connection (on $T^{*1,0}M$).

In the odd situation the model is essentially a foliated version of the latter. Assuming integrability for both D and J, we use local identifications adapted to the foliation integrating D, so that we have an identification with $\mathbb{C}^n \times \mathbb{R}$ in which we assume that J matches the leafwise canonical complex structure J_0. Also, we need a trivialization of L that gives rise to a connection form independent of the vertical coordinate and whose curvature restricts to each leaf to a $(1,1)$ form. The local definition of the coupled (and foliated) holomorphic r-jets is similar to the one for complex manifolds. One just consider the restriction of the function (section) to each leaf and applies the previous construction. Thus, the fiber over each point is that for complex manifolds (in particular the coupled (foliated) holomorphic r-jets are also symmetric (here we use that the complex structure is independent of the vertical coordinate and that the restriction of the connection form shares this property. In this trivialization any submanifold of $\mathcal{J}^r_{n,m}$ extends naturally to a submanifold in $\mathcal{J}^r_{D_h,n,m} = \mathcal{J}^r_{n,m} \times \mathbb{R}$ independently of the vertical coordinate). If we further assume that the bundle is $\mathbb{C}^m \otimes L$ and the curvature of L (maybe foliated) is $-i\omega_0$, the we also have holomorphic trivializations defined multiplying a unitary section whose connection form (maybe foliated) is A_0, times $f(z,s) = e^{-|(z,s)|^2/4}$ or $\check{f}(z,s) = e^{-z\bar{z}/4}$ (independent of s), that we know are solutions of the corresponding Cauchy-Riemann equations.

If we wanted to give a global model –even though the local one is enough for our purposes– it would be necessary to use a connection on $T^{*1,0}(\mathbb{C}^n \times \mathbb{R}) \cong T^{*1,0}\mathbb{C}^n$ (fiberwise). For example the one induced by the restriction of the Levi-Civita one to each leaf (the symmetry issue here would be more complicated).

As we have already mentioned, a local model will be enough for our purposes. If we wanted to give a global definition the requirement for the metric so it gives rise to the connection on $T^{*1,0}M$ with the right properties would be more delicate that in the complex case.

There is a final local model we wish to introduce. When we have polarized even dimensional almost complex manifolds, the local model is $\mathbb{C}^g \times \mathbb{C}^{n-g}$, and we work with foliated coupled jets along the leaves of $\mathbb{C}^g \times \{w\}$. We denote the corresponding bundle of coupled r-jets by $\mathcal{J}^r_{\mathbb{C}^g,n,m}$ (it actually coincides with $\mathcal{J}^r_{g,m} \times \mathbb{C}^{n-g}$). As we pointed out, the transversality problems for this bundle will be transferred to transversality problems in $\mathcal{J}^r_{n,m}$, so we need no further analysis of its properties, though we will be interested at some point in studying the natural map $\mathcal{J}^r_{n,m} \to \mathcal{J}^r_{\mathbb{C}^g,n,m}$.

Pseudo-holomorphic jets. Let E_k be a very ample sequence of (locally splittable) hermitian bundles over the almost complex manifold (M, D, J, g). We define the bundles $\mathcal{J}^r_D E_k := \sum_{j=0}^r ((\bar{D}^{*1,0})^{\odot j}) \otimes E_k$.

The hermitian metric h induced on D gives rise to a hermitian metric on the $(0,1)$ component of $D^*_{\mathbb{C}}$. The same happens with $(1,0)$, by just considering the same construction for $-J$. Using the metric retraction we have then hermitian metric on $\bar{D}^{*1,0}$ (just transferred by $\bar{i}$), which induces a hermitian metric on $(\bar{D}^{*1,0})^{\odot r}$; for $\mathcal{J}^r_D E_k$ the appropriate metric is $h_k := c_k h$, because in adapted charts is comparable to $h_0 = g_0$. The Levi-Civita connection induces a connection in $\bar{D}^{*1,0}$, which together with the symmetrization map $\mathrm{sym}_j \colon (\bar{D}^{*1,0})^{\otimes j} \to (\bar{D}^{*1,0})^{\odot j}$, induces a connection on $(\bar{D}^{*1,0})^{\odot j}$. This connection, together with the one on E_k define a connection $\nabla_{k,r}$ on $\mathcal{J}^r_D E_k$.

The definition of pseudo-holomorphic r-jets along D (or just pseudo-holomorphic r-jets) for a sequence E_k of hermitian vector bundles is the following (see [**4**]):

Definition 4.1. *Let τ_k be a section of (E_k, ∇_k). The pseudo-holomorphic r-jet $j^r_D \tau_k$ is a section of the bundle $\mathcal{J}^r_D E_k = \sum_r^{j=0} ((\bar{D}^{*1,0})^{\odot j}) \otimes E_k$ defined by induction taking the holomorphic 1-jet associated to $\nabla_{k,j}$ to obtain an element of $\bar{D}^{*1,0} \otimes (\sum_j^{i=0} (\bar{D}^{*1,0})^{\odot j}) \otimes E_k)$, and then composing with the symmetrizing map $(sym_{j+1} \otimes I, \cdots, sym_2 \otimes I, I \otimes I, I)$ to obtain a section of $\mathcal{J}^{j+1}_D E_k$.*

Remark 4.2: The previous definition incorporates the fact that the $(r+1)$-jets are defined as the symmetrization of the pseudo-holomorphic 1-jet of certain section (holonomic) of $\mathcal{J}^r_D E_k$ (actually in the definition we have considered the degree 1 component of the 1-jet, that after symmetrizing gives the components of degree $1, ..., r+1$ of the r-jet, that we complete adding the section itself coming from the degree 0-component to obtain a section of $\mathcal{J}^r_D E_k$). We obtain the same result if we consider the whole 1-jet and symmetrize; the only difference is that the components of degrees $1, ..., r$ appear twice.

Remark 4.3: The r-jet τ_k is the sum of $r+1$ homogeneous components. To define $j^{r+1}_D \tau_k$ we just need the degree r component.

Remark 4.4: The pseudo-holomorphic r-jets are useless for low values of k. When k is large enough, since the metric is approximately flat they approximately coincide with the local coupled r-jets (in approximately holomorphic coordinates). In particular is also easy to check that the non-symmetric part that is annihilated in the definition of the r-jet is approximately vanishing. Thus, the pseudo-holomorphic r-jets approximately coincide with those defined in $\mathbb{C}^n \times \mathbb{R}$ using J_0 and the flat metric fill the bundle $\mathcal{J}^r_D E_k$.

Actually, in the previous paragraph we might be interest in using a holomorphic trivialization in the model. For example, if $E_k = \mathbb{C}^{m+1} \otimes L_k$, choosing Darboux charts and a trivialization whose connection form is A_0, the ones associated to the solutions of the Cauchy-Riemann equations $f(z,s) = e^{-|(z,s)|^2/4}$ and $\check{f}(z,s) = e^{-z\bar{z}/4}$ (independent of s).

Remark 4.5: Though the connection on $\bar{D}^{*1,0}$ might not be compatible with the metric h_k it has this property in the approximate sense. Anyhow, this is not really important in the holomorphic case, because our goal will be to introduce certain connections –compatible or not with the metric– with curvature of type $(1,1)$ (not just having vanishing $(2,0)$ component).

Essentially all the properties and local constructions can be transferred from E_k to $\mathcal{J}^r_D E_k$. for each point x in M there is a local basis $\tau^{\mathrm{ref}}_{k,x,1}, ..., \tau^{\mathrm{ref}}_{k,x,m}$ made of reference sections. Once A.H. coordinates has been fixed (for example adapted to the metric), we have an identification of $\bar{D}^{*1,0}$ with $T^{*1,0}\mathbb{C}^n$ by considering each $dz^i_k \in T^{*1,0}\mathbb{C}^n$ and identifying it with its component along $\bar{D}^{*1,0}$. It should be stressed that this identification only makes sense in ball of g_k radius $O(1)$, which the region where our computation have to be more precise. The gaussian decay of the reference sections will take care of what happens out of this balls. We also notice that by writing dz^i_k we will mean its component along $\bar{D}^{*1,0}$ by the obvious local bundle map. The important observation is that this bundle map has norm bounded by $O(1)$ and derivatives bounded by $O(c_k^{-1/2})$ (multiplied by a suitable polynomial if we work in the ball of g_k-radius $O(c_k^{1/2})$), because the same bounds hold for the image of the local sections dz^i_k in $\bar{D}^{*1,0}$.

Once trivializations (by reference sections) and A.H. coordinates have been fixed, we have a local identification of $\mathcal{J}^r_D E_k$ with $\mathcal{J}^r_{n,m}$ associated to the basis $\mu_{k,x,I}$ defined as follows: for each $(n+1)$-tuple $I=(i_0,i_1,...,i_n)$, with $1<i_0<m$, $0\le i_1+\cdots+i_n\le r$, we set $\mu_{k,x,I} := dz_k^{1\odot i_1}\odot\cdots\odot dz_k^{n\odot i_n}\otimes\tau^{\text{ref}}_{k,x,i_0}$. It is elementary to check that this basis –comparable to a unitary one in balls of g_k-radius of order $O(1)$– is made of A.H. sections with gaussian decay w.r.t. x.

Hence, the sequence of locally splittable hermitian bundles $\mathcal{J}^rE_k$ is very ample (except for the fact that in the definition of ampleness the connection is required to be unitary, though this has here no real effect because this property is used to construct reference section, which are already at our disposal by other means).

When we have a polarized almost complex manifold (M,J,G,g), the bundle of pseudo-holomorphic r-jets along G will be defined to be $\mathcal{J}^r_GE_k := \sum_{j=0}^r((\bar{G}^{*1,0})^{\odot j})\otimes E_k$. Using the splitting $D=G\oplus G^\perp$ we can see $\mathcal{J}^r_GE_k$ as a subbundle of $\mathcal{J}^rE_k$. Extending by zeros, every section of the subbundle is a section of $\mathcal{J}^rE_k$. We use the same induction procedure as in the definition of pseudo-holomorphic r-jets along D, but either before or after symmetrizing, we project orthogonally $TM^{*1,0}$ over $\bar{G}^{*1,0}$ (or even before taking the $(1,0)$ component we project from $T^*M_{\mathbb{C}}$ to $\bar{G}^*_{\mathbb{C}}$).

In A.H. coordinates adapted to G and using $(g+1)$-tuples I_g as the previous ones, i.e., only for the coordinates $z_k^1,...,z_k^g$, one checks that μ_{k,x,I_g}, where dz_k^i, $1\le i\le g$, is identified with its projection first over $T^{*1,0}M$ and then over $\bar{G}^{*1,0}$, is a local basis of the subbundle $\mathcal{J}^r_GE_k$ made of A.H. sections with gaussian decay w.r.t. x, and A.H. as sections of the bundle $\mathcal{J}^rE_k$.

In this situation there is still a weak point. The main goal is to construct sections whose r-jets are transverse to certain stratifications. For that we need the r-jets to be A.H. sections of the bundles $\mathcal{J}^r_DE_k$ (or $\mathcal{J}^rE_k$ for even dimensional a.c. manifolds), so that we can apply the transversality results from A.H. theory (to be proved). We intend to use holonomic local basis defined as follows: if I is one of the $(n+1)$-tuples introduced before, we set $\nu_{k,x,I} := j^r_D\tau^{\text{ref}}_{k,x,I}$, where $\tau^{\text{ref}}_{k,x,I} := (z_k^1)^{i_1}\cdots(z_k^n)^{i_n}\tau^{\text{ref}}_{k,x,i_0}\in\Gamma(E_k)$. It is elementary to check that they are a basis comparable to a unitary one in a ball of fixed g_k-radius and they have gaussian decay; the r-jet is essentially a component of the r-th covariant derivative along D, and we can use the ideas of lemma 3.27 to check that C^{r+h}-bounds for τ_k transform into C^h-bounds for $j^r_D\tau_k$, having good control in how the new constants depend on those of τ_k.

Similarly we have local basis $\nu_{k,x,I_g} := j^r_G\tau^{\text{ref}}_{k,x,I_g}$, where the definition of $\tau^{\text{ref}}_{k,x,I_g}$ is the obvious one. Again, they are sections with gaussian decay and do form a basis in the appropriate ball.

It is an observation of D. Auroux that in the Kähler case (see [**5**]) the coupled jets are not anymore holomorphic sections of $\mathcal{J}^r_{n,m}$, w.r.t. the complex structure induced by the connection (due to the curvature).

This difficulty is overcome by introducing a new almost complex structure (a new connection) in $\mathcal{J}^r_DE_k$ (resp. in $\mathcal{J}^rE_k$ in the even dimensional case).

Proposition 4.6. *Let $E_k \to (M, D, J, g)$ be a very ample sequence of locally splittable hermitian bundles. The sequence $\mathcal{J}^r_D E_k$ –which is very ample for the connections $\nabla_{k,r}$ previously described– admits new connections ∇_{k,H_r} such that:*

1. *$\nabla_{k,r} - \nabla_{k,H_r} \in \bar{D}^{*0,1} \otimes End(\mathcal{J}^r_D E_k)$, and therefore both connections define the same pseudo-holomorphic jets (and similarly for polarizations when we modify the connection in $\mathcal{J}^r E_k$).*
2. *Let us denote by F_{k,H_r} and $F_{k,r}$ the curvatures of ∇_{k,H_r} and $\nabla_{k,r}$. Then $F_{k,H_r} \approxeq F_{k,r}$ and hence $(\mathcal{J}^r_D E_k, \nabla_{k,H_r})$ is a very ample sequence. Besides, a local basis of for the bundle is defined by $j^r_D \tau^{ref}_{k,x,I}$, where $\tau^{ref}_{k,x,j}$, $j = 1, ..., m$ is a sequence of reference sections.*
3. *If $\tau_k \colon M \to E_k$ is a C^{r+h}-A.H. sequence of sections, $j^r_D \tau_k \colon M \to \mathcal{J}^r_D E_k$ is a C^h-A.H. sequence of sections for the connections ∇_{k,H_r}. We also have that $j^r_G \tau_k \colon M \to \mathcal{J}^r_G E_k \subset \mathcal{J}^r_D E_k$ is a C^h-A.H. sequence of sections of $\mathcal{J}^r E_k$.*

For local coupled holomorphic jets if the curvature F_i of each line bundle L_i, $i = 1, ..., m$, restricted to the leaves has constant components (w.r.t. all the coordinates), the previous modification gives an equality for the restriction of the curvature to each leaf. As a consequence, the new almost complex structure in the total space of $\mathcal{J}^r_{D_h,n,m}$ is also integrable (its restriction to each leaf of the pullback of D_h). Also, if τ is a holomorphic section ($\mathbb{C}^m$-valued function), then $j^r_{D_h,n,m}\tau$ is holomorphic for the complex structure.

Proof. We omit the subindices k and r for the connections whenever there is no risk of confusion.

Let $\sigma_k = (\sigma_{k,0}, \sigma_{k,1})$ be a section (maybe local) of $\mathcal{J}^1_D E_k$. We define $\nabla_{H_1}(\sigma_{k,0}, \sigma_{k,1}) = (\nabla\sigma_{k,0}, \nabla\sigma_{k,1}) + (0, -F^{1,1}_D \sigma_{k,0})$, where $-F^{1,1}_D \sigma_{k,0} \in \bar{D}^{*0,1} \otimes \bar{D}^{*1,0} \otimes E_k$ (see [**5**]).

The previous formula defines a connection. All the identities can be checked locally in balls of fixed g_k-radius; what is more, since they are approximate identities we can use the local splitting $E_k = L_{k,1} \oplus ... \oplus L_{k,m}$, given by a local basis of C^{r+h}-A.H. sections $\tau_{k,1}, ..., \tau_{k,m}$, together with the induced diagonal connection –that we still call ∇– and its curvature F. Thus, it is enough to prove the theorem for line bundles.

Let L_k be a very ample sequence of line bundles. Using the metric splitting $TM = D \oplus D^\perp$ we write the connection $\nabla = \partial + \bar{\partial} + \nabla_{D^\perp}$. Since for the curvatures splitting $F_{\tilde{D}}$ is approximately of type $(1,1)$, due to the results of subsection 2.1 of section 2, we also have $F^{1,1}_D \approxeq F_D$, and thus $F_D \approxeq \bar{\partial}\partial + \partial\bar{\partial}$.

The additional term added to define the modified connection is better understood when it acts over A.H. sections τ_k. Recall that in coordinates, to compute the curvature the connection has to be composed with the operator $\nabla^1 \colon T^*M \otimes T^*M \otimes L_k \to T^*M \otimes L_k$ defined as follows: in a chart where T^*M is trivialized using the derivatives of the coordinates, we have the corresponding flat connection on T^*M; the operator ∇^1 is $d \otimes I - I \otimes \nabla$, that composed with the antisymmetrization map $\mathrm{asym}_2 \colon T^*M \otimes T^*M \to \wedge^2 T^*M$

gives rise to the curvature. Using A.H. coordinates one checks that F_D is approximately the composition of ∇_D with $\nabla^1_D := d_D \otimes I_D - I_D \otimes \nabla_D$ and then with the antisymmetrization map.

The term $\partial\bar{\partial}\tau_k \approxeq d_D\bar{\partial}\tau_k \approxeq d\bar{\partial}\tau_k$ approximately vanishes: we write $\bar{\partial}\tau_k = \sum_{i=1}^n d\bar{z}^i_k \otimes g_i\tau_k$, where $d\bar{z}^i_k \in \bar{D}^{*0,1}$. Thus, $(d_D \otimes I_D - I_D \otimes \nabla_D) \circ \bar{\partial}\tau_k \approxeq -(I_D \otimes \nabla_D) \circ \bar{\partial}\tau_k$. Therefore for τ_k an A.H. sequence $F_D\tau_k \approxeq \mathrm{asym}_2(-\bar{\partial}\partial\tau_k)$ (the symmetrization and antisymmetrization maps have norm bounded by $O(1)$ and have derivatives of size $O(c_k^{-1/2})$).

Hence $\bar{\partial}\partial$ can be written as the composition of $-\bar{\partial} \otimes \partial$ with the antisymmetrizing map. The conclusion is that its action $j^1_D\tau_k$ is approximately the action on τ_k of the non-antisymmetrized component of the curvature in D:

$$\nabla_H(\tau_k, \partial\tau_k) \approxeq (\nabla\tau_k, \nabla\partial\tau_k - \bar{\partial} \otimes \partial\tau_k).$$

Hence,

$$\nabla_{H,D}(\tau_k, \partial\tau_k) \approxeq (\nabla_D\tau_k, \partial\bar{\partial}\tau_k + \bar{\partial}\partial\tau_k - \bar{\partial} \otimes \partial\tau_k) \approxeq (\nabla_D\tau_k, \partial\bar{\partial}\tau_k),$$

and therefore,

$$\bar{\partial}_H j^1_D\tau_k \approxeq (\bar{\partial}\tau_k, 0) \approxeq 0.$$

In the integrable case (for a very ample line bundle) what we add is exactly $-\bar{\partial} \otimes \partial$ and the 1-jet of a holomorphic section is easily seen to be holomorphic for the new connection.

To check the identities of the curvature (point (2) in the definition) we fix A.H. coordinates and the mentioned basis of $\bar{D}^{*1,0}$ and $\bar{D}^{*0,1}$, completed with ds_k to a local basis of $T^*M \otimes \mathbb{C}$.

Consider the local basis of $\mathcal{J}^1_D L_k$ given by $(0, dz^1_k) \otimes \tau_k, ..., (0, dz^n_k) \otimes \tau_k, (1, 0) \otimes \tau_k$. The covariant derivative in this basis is:

$$\begin{aligned} \nabla_H(0, dz^i_k) &= (0, \nabla dz^i_k),\ i = 1, ..., n, \\ \nabla_H(1, 0) &= (\nabla 1, 0) = (A^i_k dz^i_k + B^i_k d\bar{z}^i_k + C_k ds_k, -F^{1,1}). \end{aligned}$$

The component of the curvature of ∇ in D is:

$$F_D \approxeq \sum_{i,j=1}^n \Omega_{ij} dz^i_k \wedge d\bar{z}^j_k, \quad \Omega_{ij} = \frac{\partial B^j_k}{\partial z^i_k} - \frac{\partial A^i_k}{\partial \bar{z}^j_k}.$$

For the subbundle spanned by $(0, dz^1_k), ..., (0, dz^n_k)$, the curvature of ∇_H is that of ∇.

Regarding the section $(1, 0)$,

$$\begin{aligned}
\nabla^2_H(1,0) &\approxeq \nabla_H(A^i_k dz^i_k + B^i_k d\bar{z}^i_k + C_k ds_k, \sum_{i,j=1}^n \Omega_{ij} d\bar{z}^i_k \otimes dz^j_k) \approxeq \\
&\approxeq (F_\nabla 1, \sum_{i,j,l=1}^n (\Omega_{ij} A^l_k d\bar{z}^i_k \wedge dz^l_k \otimes dz^j_k + \Omega_{ij} B^l_k d\bar{z}^i_k \wedge d\bar{z}^l_k \otimes dz^j_k) + \\
&+ \sum_{i,j=1}^n (C_k d\bar{z}^i_k \wedge ds_k \otimes dz^j_k) + \\
&+ \sum_{l,i,j=1}^n (A^l_k \Omega_{ij} dz^l_k \wedge d\bar{z}^i_k \otimes dz^j_k + B^l_k \Omega_{ij} d\bar{z}^l_k \wedge d\bar{z}^i_k \otimes dz^j_k) + \\
&+ \sum_{i,j=1}^n (C_k ds_k \wedge d\bar{z}^i_k \otimes dz^j_k)) = (F_\nabla 1, 0).
\end{aligned}$$

Observe that when the curvature along D is of type $(1,1)$ and $d\Omega_{ij} = 0$, something that happens in Darboux charts, the previous approximate equalities are indeed exact equalities. Actually this is the reason why we get approximate equalities in the non-integrable setting.

There is another way to prove the approximate equality of point (2) which amounts to choosing a special basis of holonomic sections. Given $\tau^{\mathrm{ref}}_{k,x}$ A.H., the component in $\bar{D}^*$ of the corresponding connection form approximately belongs to the subbundle $\bar{D}^{*1,0}$. We consider the local basis $j^1_D(z^l_k \tau_k)$, where z^l_k is a monomial of degree ≤ 1 (we do not really need to take reference sections, it is enough to take τ_k A.H. so that $j^1_D(z^l_k\tau_k)$ is a basis comparable to a unitary one in a ball of fixed g_k-radius). For the next computations τ_k can be any A.H. sequence.

$$\nabla_H(z^l_k\tau_k, \partial(z^l_k\tau_k)) \approxeq (\nabla z^l_k\tau_k, \nabla\partial(z^l_k\tau_k) - \bar{\partial}\otimes\partial z^l_k\tau_k),$$

and the curvature can be written:

$$F_H(z^l_k\tau_k, \partial(z^l_k\tau_k)) \approxeq F(z^l_k\tau_k, \partial(z^l_k\tau_k)) + (0, -\nabla\wedge\bar{\partial}\otimes\partial(z^l_k\tau_k) - F^{1,1}_D \wedge \nabla(z^l_k\tau_k)) \tag{4.3}$$

Being more precise, the second summand can be computed as follows:

$$\begin{aligned}
&-\nabla\wedge\bar{\partial}\otimes\partial(z^l_k\tau_k) - F^{1,1}_D \wedge \nabla(z^l_k\tau_k) \approxeq \\
&-\mathrm{asym}_2(d\otimes I - I\otimes\nabla(-\bar{\partial}\otimes\partial(z^l_k\tau_k))) + \mathrm{asym}_2(I\otimes\bar{\partial}\otimes\partial(\nabla z^l_k\tau_k).) \quad (4.4)
\end{aligned}$$

The operator $-\bar{\partial}\otimes\partial$ is approximately tensorial (because $F^{1,1}_D$ has this property). Therefore, writing $\nabla(z^l_k\tau_k) \approxeq \partial(z^l_k\tau_k) + C ds_k\otimes\tau_k = \alpha\otimes\tau_k + C ds_k \otimes \tau_k$, $\alpha\in\bar{D}^{*1,0}$, the second component in the r.h.s of 4.4 is:

$$\begin{aligned}
\mathrm{asym}_2(I\otimes\bar{\partial}\otimes\partial(\nabla z^l_k\tau_k)) &\approxeq \mathrm{asym}_2(I\otimes\bar{\partial}\otimes\partial(\alpha\otimes\tau_k + C ds_k\otimes\tau_k)) \\
&\approxeq \sum_{i,j} -\Omega_{ij}((\alpha + C ds_k)\wedge d\bar{z}^j_k \otimes dz^i_k \otimes \tau_k
\end{aligned}$$

The partial derivatives of the coefficients of $F_D^{1,1}$ are of order $O(c_k^{-1/2})$ (condition (3) in definition 2.2), so that the first component is:

$$\begin{aligned}
-\mathrm{asym}_2(d\otimes I - I\otimes\nabla(-\bar\partial\otimes\partial(z_k^l\tau_k))) &\approxeq \sum_{i,j} dz_k^l\otimes\Omega_{ij}\otimes d\bar z_k^j\otimes dz_k^i\otimes\tau_k - \\
-z_k^l\Omega_{ij}d\bar z_k^j\otimes dz_k^i\otimes(\partial\tau_k + Cds_k\otimes\tau_k) &\approxeq \sum_{i,j}\Omega_{ij}(\alpha + Cds_k)\wedge d\bar z_k^j\otimes dz_k^i\otimes\tau_k
\end{aligned}$$

Thus, $F_H \approxeq F$.

To show that the 1-jets of C^r-A.H. sections are C^{r-1}-A.H., we just notice that what we added to the connection is a component of the curvature, whose coefficients are of order $O(1)$ and has its derivatives bounded by $O(c_k^{-1/2})$.

The degree 0 component of the covariant derivative of $j_D^1\tau_k = (\tau_k, \partial\tau_k)$ is $\nabla^r\tau_k$. The degree 1 term is $\nabla^r\partial\tau_k$ plus r *homogeneous summands of order* $r+1$, which are products of derivatives $\nabla^j\tau_k$ (of order j) and derivatives $\nabla^{r-j-2}F_D^{1,1}$ (of order $r-j$). The bounds for the full derivatives are obvious. Those for the derivatives along D follow from the fact that for $k >> 0$ the mixture of types in the derivatives (according to the splitting of $T^*M^{\otimes r}\otimes\mathcal{J}_D^1E_k$ induced by the metric) is of size $O(c_k^{-1/2})$. In particular, the constant C_r^D for $\nabla_D^r\tau_k$ transforms into $C'C_r^D$ for $\nabla_D^{r-1}j_D^1\tau_k$ (here we apply the ideas and results of lemma 3.27).

The bounds for the antiholomorphic components and its derivatives follow from similar considerations together with $\bar\partial_H(\tau_k,\partial\tau_k) \approxeq (\bar\partial\tau_k, \partial\bar\partial\tau_k)$, when τ_k is an A.H. sequence of sections.

Being precise, we must notice that all the approximate equalities for sections of $\mathcal{J}_D^1L_k$ have been computed using the connection $\nabla_{k,1}$. But from the previous ideas we easily deduce that the approximate equalities for the connections $\nabla_{k,1}$ imply approximate equalities for ∇_{k,H_1}.

The gaussian decay of $j_D^1(\tau_{k,x,I}^{\mathrm{ref}})$ w.r.t. x is easily checked using A.H. coordinates. They also form a basis comparable to a unitary one.

To apply induction to the bundles of higher order jets we need the vector bundles $\mathcal{J}_D^rE_k$, $\mathcal{J}_D^1\mathcal{J}_D^rE_k$ to admit a modified connection with the previous properties. We cannot quite apply what we have done, because it was for line bundles. In any case, it can be checked that the previous proof also works because since property (2) holds, the curvature of the modified connection on $\mathcal{J}_D^rE_k$ is approximately tensorial in the sense that for ξ_k a section of $\mathcal{J}_D^rE_k$, $F_{H,D}^{1,1}\xi_k$ is approximately proportional to ξ_k (with equality in the integrable case). This property implies that if ξ_k is an A.H. sequence, the new term in the modification approximately coincides with $-\bar\partial\otimes\partial\xi_k$ (which is then approximately tensorial). At this point it can be checked that due to the expression of the new term, the proof for line bundles which develops equation (4.3) (and in which a couple of terms are cancelled) works also for the bundles $\mathcal{J}_D^rE_k$. Besides, it should be noticed that in the local model this modification induces (in balls of g_0-radius $O(1)$) in each leaf of $\hat D_h$ –the pull-back to the total space of the bundle of D_h– an integrable almost complex structure which is constant in the vertical coordinate, meaning this that for an appropriate holomorphic trivialization the connection form is independent of s (for example in Darboux charts with the usual trivialization, we can use the new trivialization associated to the function $\check{f}(z,s) = e^{-z\bar{z}/4}$).

Let us assume that the initial almost complex structure in $\mathcal{J}^r_D E_k$ has been modified so that $(\mathcal{J}^r_D E_k, \nabla_{H_r})$ is a very ample sequence of bundles for which the r-jets of certain local basis of A.H. sections give rise to a local basis of A.H. sections of $\mathcal{J}^r_D E_k$. To be able to apply induction we make the usual identification of $\mathcal{J}^{r+1}_D E_k$ with the subbundle of $\mathcal{J}^1_D \mathcal{J}^r_D E_k$ spanned by holonomic sections. The bundle $\mathcal{J}^1_D \mathcal{J}^r_D E_k$ is endowed with a connection $\tilde{\nabla}_{H_r}$ (using ∇_g in $\bar{D}^{*1,0}$ and ∇_{H_r} in $\mathcal{J}^r_D E_k$) which by induction can be modified to $\nabla_{H_{r+1}}$. We want to prove that the subbundle o $\mathcal{J}^{r+1}_D E_k$ inherits a connection with the desired properties.

Let us check first the situation in the integrable case: we consider the basis $\tau^{\mathrm{ref}}_{k,x,I}$, where $\tau^{\mathrm{ref}}_{k,x,j}$ is holomorphic and independent of the vertical coordinate of the chart (for certain trivialization). By definition, $j^{r+1}_{D_h} \tau^{\mathrm{ref}}_{k,x,I} = j^1_{D_h}(j^r_{D_h} \tau^{\mathrm{ref}}_{k,x,I})$. The connections induced by ∇ in $\mathcal{J}^1_{D_h,n} \mathcal{J}^r_{D_h,n,m}$ and $\mathcal{J}^{r+1}_{D_h,n,m}$ are the same. For the first bundle the holomorphic and vertical components of this induced connection coincide by induction with those of the modified connection $\tilde{\nabla}_{H_r}$ and also with those of the modification $\nabla_{H_{r+1}}$. Finally

$$\begin{aligned}
\nabla_{H_{r+1}} j^{r+1}_{D_h} \tau^{\mathrm{ref}}_{k,x,I} &= \left(\nabla_{H_{r+1}}\right) j^1_{D_h}(j^r_{D_h} \tau^{\mathrm{ref}}_{k,x,I}) = \\
&= \left(\partial_{H_{r+1}} + \nabla_{H_{r+1}, \frac{\partial}{\partial s_k}}\right) j^1_{D_h}(j^r_{D_h} \tau^{\mathrm{ref}}_{k,x,I}) = \\
&= \left(\partial + \nabla_{\frac{\partial}{\partial s_k}}\right) j^1_{D_h}(j^r_{D_h} \tau^{\mathrm{ref}}_{k,x,I}) = \left(\partial + \nabla_{\frac{\partial}{\partial s_k}}\right) j^{r+1}_{D_h} \tau^{\mathrm{ref}}_{k,x,I},
\end{aligned}$$

which by definition is a 1-form with coefficients in $\mathcal{J}^{r+1}_{D_h,n,m}$ (because in the end we get the holomorphic and vertical components of the connection induced by ∇). Since the connection preserves the subbundle for a local basis, ∇_H defines a connection on $\mathcal{J}^{r+1}_{D_h,n,m}$. Once more we are endowing the bundle $\mathcal{J}^{r+1}_{D_h,n,m}$ with an integrable almost complex structure in each leaf of the total space.

In the non-integrable case we use the symmetrization map

$$\mathrm{sym}^{r+1} := (\mathrm{sym}_{r+1} \otimes I, \cdots, \mathrm{sym}_2 \otimes I, I \otimes I, I) \colon \mathcal{J}^1_D \mathcal{J}^r_D E_k \to \mathcal{J}^{r+1}_D E_k,$$

composed with $\tilde{\nabla}_{H_r}$ to define a connection $\nabla_{H_{r+1}}$ on $\mathcal{J}^{r+1}_D E_k$. Notice that the holomorphic component $\partial_{H_{r+1}}$ and the vertical one $\nabla_{H_{r+1},D^\perp}$ of this connection coincide with the corresponding ones of ∇, the original connection of $\mathcal{J}^{r+1}_D E_k$. for the local basis $j^{r+1}_D \tau^{\mathrm{ref}}_{k,x,I}$,

$$\nabla_{H_{r+1}} j^{r+1}_D \tau^{\mathrm{ref}}_{k,x,I} = \bar{\partial}_{H_{r+1}} j^{r+1}_D \tau^{\mathrm{ref}}_{k,x,I} + \partial_{H_{r+1}} j^{r+1}_D \tau^{\mathrm{ref}}_{k,x,I} + \nabla_{H_{r+1},D^\perp} j^{r+1}_D \tau^{\mathrm{ref}}_{k,x,I}.$$

By the previous observation, the second and third summands belong to $\mathcal{J}^{r+1}_D E_k$. The first one is –by induction– of size $O(c_k^{-1/2})$. Since $j^{r+1}_D \tau^{\mathrm{ref}}_{k,x,I}$ is a local basis, the size of the non-symmetric component that we have to subtract from $\tilde{\nabla}_{H_r} j^{r+1}_D \tau^{\mathrm{ref}}_{k,x,I}$ to define $\nabla_{H_{r+1}} j^{r+1}_D \tau^{\mathrm{ref}}_{k,x,I}$ is bounded by $O(c_k^{-1/2})$;

in fact all its derivatives (taken with the connection $\tilde{\nabla}_{H_r}$) are also bounded by the same quantity (one just needs to use the bounds for the antiholomorphic component and the bounds of order $O(1)$ for sym^{r+1} and its derivatives).

The geometrical meaning is that the horizontal distribution of the connection ∇_{H_r} on $\mathcal{J}^1_D\mathcal{J}^r_D E_k$ is –in the points of $\mathcal{J}^{r+1}_D E_k$– at distance $O(c_k^{-1/2})$ of the tangent space of the subbundle (and also all the derivatives of the difference between $\nabla_{H_{r+1}}$ and ∇_{H_r}). Thus, the same approximate equalities will hold for both connections.

Regarding the curvature, using the local basis $j^{r+1}_D\tau^{\mathrm{ref}}_{k,x,I}$ one has:

$$F_{H_{r+1}} = \mathrm{sym}^{r+1}\circ\tilde{\nabla}_{H_r}\wedge\mathrm{sym}^{r+1}\circ\tilde{\nabla}_{H_r}\approxeq F_{\tilde{\nabla}_{H_r}}\approxeq F^{1,1},$$

where $F^{1,1}$ is the $(1,1)$ component of the original connection on $\mathcal{J}^{r+1}_D E_k$. The previous approximate equality is valid both for $\tilde{\nabla}_{H_r}$ and for $\nabla_{H_{r+1}}$.

Using similar considerations it can be deduced that the $(r+1)$-jet of a C^{r+1+h}-A.H. sequence of sections of E_k is a C^h-A.H. sequence of sections of $(\mathcal{J}^{r+1}_D E_k, \nabla_{H_{r+1}})$. The only difference is the composition with the symmetrization map. One checks that the different excision commute with the symmetrization. This, together with the bounds for the symmetrization map reduces the assertion on the bounds to the corresponding assertion for the sequence $j^1_D j^1_D\tau_k$ and the connection $\tilde{\nabla}_{H_r}$, which is fulfilled by induction.

The gaussian decay of $j^{r+1}_D\tau^{\mathrm{ref}}_{k,x,I}$ is checked similarly.

Regarding the relative theory, for any C^{r+h}-A.H. sequence τ_k of sections of E_k, $j^r_G\tau_k$ is a C^h-A.H. sequence of sections of $\mathcal{J}^r E_k$. The computations can be made in A.H. coordinates adapted to G. In that situation, the result is approximately that of the flat model. In the latter $j^r_G\tau_k$ is a component of the vector $j^r\tau_k$, and the same happens with the derivatives $\nabla^p_{H_r} j^r_G\tau_k$ and $\nabla^{p-1}_{H_r}\bar{\partial}_{H_r} j^r_G\tau_k$, $p=0,...,h$.

Again, gaussian decay of the sections $j^r\tau^{\mathrm{ref}}_{k,x,I_g}$ of the subbundle $\mathcal{J}^r_G E_k$ follows from the same ideas, as well as the fact that they define a local basis of the subbundle.

□

Remark 4.7: The approximate equality $F_{H_1,k}\approxeq F_k$ has important consequences. In A.H. coordinates and after the local identification of $\bar{D}^{*1,0}$ with $T^{*1,0}\mathbb{C}$, the connection form of ∇_H is approximately the sum of the connection forms for $A_{k,j}$ in $L_{k,j}$ (for suitable trivializations), plus the sum of the curvatures $\omega_{k,j}$. We have bounds of order $O(1)$ for the norm of the curvatures and of order $O(c_k^{-1/2})$ for the partial derivatives of its components. The consequence is that we will have the same kind of control on the metric on the total space of $\mathcal{J}^1_D E_k$ induced by the modified connection that we had for the initial connection. By induction we obtain the same kind of result for the sequences $\mathcal{J}^r_D E_k$. For sequences of the form $\mathbb{C}^m\otimes L^{\otimes k}$ over a calibrated manifold, in Darboux charts (the metric splitting matches that of the curvatures) and taking the appropriate trivialization the connection form and curvature coincide approximately with m copies of A_0 and ω_0, so we have (approximately) and explicit formula for the modified connection.

The above property will imply that if we have some kind of structure in the total space of the jet bundles, for example a sequence of stratifications, such that in the above mentioned stratifications are independent of k and x, then the different bounds associated to elements of those stratifications (basically those of functions defining locally the strata) will not depend on k and x (again we can compute them for the corresponding model in charts and with the euclidean metric elements).

We will also take advantage that in the holomorphic case what we have are equalities (and hence an induced holomorphic structure on $\mathcal{J}^r_{D_h,n,m}$ for which the r-jet of a holomorphic section of $\mathbb{C}^m \otimes L$ is holomorphic for the new structure).

5. Approximately holomorphic stratifications and transversality

Our main goal is to state a transversality principle for A.H. sequences to certain sequences of stratifications of the total space of a very ample sequence of bundles (a strong transversality principle if the sequence of bundles is $\mathcal{J}^r_D E_k$). It is precisely the fact of being A.H. plus certain conditions on the stratifications what will allow us to define the perturbation so that the desired transversality properties hold. Though so far we have only considered sequences of vector bundles, we can apply the theory to more general sequences of fiber bundles F_k with fiber an almost complex manifold (of even dimension), and a connection in the bundle compatible with the metric and almost complex structure on the fiber. For these *almost complex fiber bundles* we can readily generalize some of the previous concepts.

Definition 5.1. *Given positive constants c, C^D, C, a section τ of an almost complex bundle is C^r-A.H. with bounds c, C^D, C (C^r-A.H.(C^D, C, c)), is the following inequalities hold:*

$$\begin{aligned} |\tau| + |\nabla_D \tau| + ... + |\nabla^r_D \tau| &\leq C^D \\ |\nabla \tau| + ... + |\nabla^r \tau| &\leq C \\ |\bar{\partial}\tau| + ... + |\nabla^{r-1}\bar{\partial}\tau| &\leq Cc^{-1/2} \end{aligned}$$

When $c_k \to \infty$, a sequence of sections of a sequence of almost complex bundles is C^r-A.H. if positive constants C^D, C exist so that the sections τ_k are C^r- A.H.(C^D, C, c_k).

We speak of A.H. sequences when we have (C^D_j, C_j) a sequence of bounds controlling the norms of ∇^j and $\nabla^j_D, \nabla^{j-1}\bar{\partial}$ (the latter multiplied by $c_k^{-1/2}$) for all $j \in \mathbb{N}$.

5.1. Approximately holomorphic stratifications. The total spaces of a sequence of almost complex bundles inherit a metric $\hat{g}_k$, a distribution $\hat{D}$ of the same codimension as D and an almost-complex structure $\hat{J}_k$. We will consider stratifications $\mathcal{S} = (S^a_k), a \in A_k$, whose strata S^a_k intersect each fiber (transversely) with minimum angle uniformly bounded by below. The strata will verify certain constraints that under certain circumstances will be equivalent to the A.H. (w.r.t. g_k y $\hat{J}_k$ and $\hat{D}$) of functions locally defining them. The stratification will be required to be finite in the sense that $\#(A_k)$ must be bounded independently of k and the boundary of each strata $\partial S_k^a = \bar{S}_k^b - S_k^a$ will be the union of the strata of smaller dimension:

$$\partial S_k^a = \bigcup_{b<a} S_k^b$$

Finally we will work with uniform Whitney stratifications.

Definition 5.2. *(see [**3**]) Let F_k be a sequence of almost complex bundles over (M, D, J, g) and $(S_k^a)_{a\in A_k}$ finite Whitney stratifications of F_k whose strata are transverse to the fibers. Let $r \in \mathbb{N}$, $r \geq 2$. The sequence of strata is C^r-approximately holomorphic (C^r-A.H.) if for any bounded open set U_k of the total space of F_k and any $\epsilon > 0$ positive constants $C_\epsilon^D, C_\epsilon, \rho_\epsilon$ only depending on ϵ and on the size of U_k –but not on k– can be found, so that for any point $y \in U_k$ in a strata S_k^a for which $d_{\hat{g}_k}(y, \partial S_k^a) > \epsilon$, there exist complex valued functions $f_1, ..., f_p$ such that $B_{\hat{g}_k}(x, \rho_\epsilon) \cap S_k^a$ is given $f_1 = ... = f_p = 0$, and the following properties holds:*

1. *(Uniform transversality w.r.t. fibers + transverse comparability) The restriction of $df_1 \wedge ... \wedge df_n$ to $T^v F_k$ –the tangent space to the fibers– is bounded by below by ρ_ϵ.*
2. *(Approximate holomorphicity along the fibers) The restriction of the function $f = (f_1, ..., f_p)$ to each fiber is C^r-A.H.(C_ϵ^D, c_k).*
3. *(Horizontal approximate holomorphicity + holomorphic variation of the restriction to the fiber + estimated variation of the restriction to the fiber) For any λ^D, λ, c_k, and τ C^r-A.H.$(\lambda^D, \lambda, c_k)$ local section of F_k with image cutting $B_{\hat{g}_k}(y, \rho_\epsilon)$, $f_j \circ \tau$ is C^r-A.H.$(\lambda^D C_\epsilon^D, \lambda C_\epsilon, c_k)$. Moreover, if θ is a local C^r-A.H.$(\lambda^D, \lambda, c_k)$ section of $\tau^* T^v F_k$, $df_\tau(\theta)$ is C^r-A.H.$(\lambda^D C_\epsilon^D, \lambda C_\epsilon, c_k)$.*
4. *(Estimated Whitney condition) $\forall y \in S_k^b$ at distance smaller than some ϵ_0 of its boundary ∂S_k^b, with $S_k^a \subset \partial S_k^b$, the maximal angle between the distribution tangent to the level sets of $f = (f_1, ..., f_p)$ and the distribution tangent to the stratum S_k^b is bounded by $C_\epsilon d_{\hat{g}_k}(y, S_k^a)$.*

Remark 5.3: For the main applications of our theory (actually only if we use the intrinsic theory) we will need stratifications all whose derivatives are controlled (A.H. stratifications).

Remark 5.4: When D is the whole tangent bundle we recover the definition given by D. Auroux in [**4**].

Local description of the C^r-A.H stratifications. It is possible to give a local geometric description of a C^r-A.H. sequence of stratifications of almost complex bundles $p_k \colon (F_k, \hat{g}_k) \to (M, g_k)$, provided we have 1-comparable charts (subsection 3.2) for each $y \in \coprod_{k\in\mathbb{N}} F_k$ (the constants uniform in the family).

A first example of bundles with that property are the trivial bundles with trivial connection and fiber $(Q_k, \bar{g})$, Q_k compact. A family of r-comparable charts in the total space is obtained by multiplying a fixed family of r-comparable charts in the fiber times approximately holomorphic coordinates in the base.

The second class is that of very ample sequences of hermitian bundles with (linear) connection. In the domain of A.H. coordinates a basis comparable to a unitary one is fixed. If in this basis the connection form is uniformly bounded, then r-comparable charts are obtained multiplying A.H. coordinates times balls of some fixed radius in the fiber (the fiber is some $\mathbb{C}^N$ with euclidean metric). A bound of order $O(1)$ for the curvature gives a bound of the same order for the Christoffel symbols. In general bounds of order $O(1)$ in the partial derivatives of order $r-1$ of the curvature give bounds of the same order for the partial derivatives of the Christoffel symbols of order up to $r-1$.

In our applications the sequence of bundles will be of one of the two previously introduced classes. Besides, the bundles will always have "enough" A.H. sections in the sense that for any $y \in F_k$, it will be possible to find τ a C^r-A.H. section (with constants C^D, C only depending on the norm of y, if we are in a vector bundle which is the only example of almost complex fiber bundle with non-compact fiber we will deal with) such that $\tau(p_k(y)) = y$. Similarly, for any $u \in \tau^*T^vF_k$, a C^r-A.H. section $\theta \in \Gamma(\tau^*T^vF_k)$ exists with $\theta(p(y)) = u$ (the constants only depending in the norm of u, and in the norm of y also in a vector bundle).

Going to a family of 1-comparable charts of F_k, and for C^2-A.H. stratifications, The bounds that do not have to do with the anti-holomorphic required to hold in points (2) and (3) of definition 5.2 follow from (uniform) bounds of order $O(1)$ for $|f|$, for $|df|$ –the norm of the derivative– and for $|d^2f|$ –the norm of the partial derivatives of order 2– measured with the euclidean metric or the induced one (equivalently one can consider instead of the second partial derivatives only defined locally, ∇df). If we fixed a family of r-comparable charts, similar results are obtained for C^r-A.H. stratifications and bounds of order $O(1)$ for $|d^jf|$, $j = 0, ..., r$ –the norms of the partial derivatives of order smaller or equal than r.

It can also be checked that the assertion relative to the antiholomorphic components in points (2) and (3) of 5.2 follows –in r-comparable charts– from the corresponding assertion for $d^j\bar{\partial}f$, $j = 0, ..., r-1$ (the partial derivatives of $\bar{\partial}f$ of order smaller than r), where the almost complex structure can be any approximately coinciding with the induced one in the chart. When the F_k are vector bundles, in A.H. coordinates we can consider an A.H. trivialization so that the fiber is identified with $\mathbb{C}^N$; then the above calculations can be made w.r.t. to the standard almost complex structure $J_0^{n+N} := J_0^n \times J_0^N$. Indeed, since the connection form in the chosen A.H. trivialization is approximately complex linear, from the bound for the J_0^{n+N}-antiholomorphic component we deduce the corresponding one for the $\hat{J}$-antiholomorphic component; for higher derivatives of the latter, we use the bounds on the higher derivatives of the former together with control of order $O(1)$ relating the derivatives of the product horizontal distribution with the horizontal distribution for ∇ (such control follows in the fiber directions from the linearity of the connection and in the "base directions" form the control in the derivatives of the connection form).

Once a family of 1-comparable charts for the total space of the bundles has been fixed, we can modify them so that $f\colon \mathbb{R}^{2n+1+2N} \to \mathbb{R}^{2p}$ –which is a submersion– becomes the canonical projection in $2p$-coordinates, and so that the new charts are still 1-comparable.

To do that we denote the foliation defined by $\ker df$ the foliation defined by f. We consider the tangent space to the leaf of $\ker df$ through the origin and we use a linear transformation so that it can be assumed to coincide with the set of zeros of $2n+1+2N-2p$ coordinates –that we denote by x– and such that the zero set of the remaining coordinates –that we denote by t– is a subspace of the tangent space to the fiber; the norm of this linear transformation is uniformly bounded due to the bound by below in the minimal angle between fibers and the foliation $\ker df$ (this bound comes from the bound by below for $|df_1 \wedge \cdots \wedge df_p|$ of condition (1) in 5.2). We set $\phi(x,t) = (x, f(x,t))$. The bound is the partial derivatives of order 2 of f implies that the tangent spaces to leaves do not vary much. In particular a constant $r_1 > 0$ exists so that $\phi\colon B_{g_0}(0, r_1) \to \mathbb{R}^{2n}$ is a diffeomorphism. There re also positive constants r_2, r_3 such that $B_{g_0}(f(0), r_2) \subset \phi(B_{g_0}(0, r_1)) \subset B_{g_0}(f(0), r_3)$; the existence of r_3 follows from the bound for $|df|$. The euclidean orthogonal to the leaf at the origin when parallel translated to any point of $B_{g_0}(0, r_1)$ is still transverse to the leaf through that point. The differential of the projection restricted to this orthogonal has norm bounded by above (this projection is the meaningful part of the change of coordinates acting of this subspace). Similarly the image of the unit sphere in this subspace by df is an ellipsoid whose distance at the origin is bounded by below (again form the bound by below for $|df_1 \wedge \cdots \wedge df_p|$ in the fiber and also because df is approximately complex when restricted to $\hat{D}$). From these considerations we deduce a bound by below for the determinant of the change of coordinates ϕ (in particular the image of the level set $t = 0$ contains an euclidean ball of uniform radius r_2).

The conclusion is that the induced metric $\phi_* \hat{g}_k$ is comparable to the euclidean. This, together with the bound by above for $|d^2 f|$ gives a bound by above for the Christoffel symbols, so we even get a family of 1-comparable charts for which the local foliation is rectified (or if we want the model chart for a codimension $2p$-foliation).

5.2. Estimated transversality. We want to construct A.H. sequences of sections which are *transverse along the directions of D* to A.H. stratifications. The strategy is to focus into the local transversality problem, which by the use of reference frames and the functions locally defining the strata will be seen to be equivalent to a transversality problem for functions. Then we add all the local perturbations. The weak point is that by using the reference sections, we will be able to solve the problem in balls of g_k-radius $O(1)$; but the reference sections have support of order $O(c_k^{1/6})$, something that creates interference between the local solutions and eventually destroys the transversality. This difficulty is overcome using the strongest concept of *estimated transversality* instead of usual transversality.

Definition 5.5. *Let $(E, \nabla) \to (M, D, g)$ be a hermitian bundle with connection. Given $\eta > 0$ and τ a section of E, we say that τ is (η, D)-transverse to* $\mathbf{0}$ *(or η-transverse along D to* $\mathbf{0}$ *or simply η-transverse to* $\mathbf{0}$*), if in each point x where $|\tau(x)| < \eta$, we have $|\nabla_D \tau(x)| > \eta$.*

In the previous definition we can think of ∇_D either as the restriction of ∇ to D or as its component along $\bar{D}^*$, because both have the same norm. Actually, we can use any retraction i so that $q^{\tilde{i},i}$ has norm of order $O(1)$, because the norm of the restriction to D and of the component along $i(D^*)$ are then comparable.

It is possible to give a more geometric definition for which it is more convenient to think of ∇_D as the restriction of ∇ to D: the total space E has a metric and a distribution $\hat{D}$. The distribution tangent to the $\mathbf{0}$ section of E can be extended by parallel transport to a distribution $T^{||}$ defined in a tubular neighborhood of radius η; next, we intersect it with $\hat{D}$ and denote the resulting distribution by $T^{||}_D$. Let us denote by $T\tau$ the distribution tangent to the graph of τ and by $T_D\tau$ its intersection with $\hat{D}$. The definition of estimated transversality is equivalent to the existence of a constant $\eta' > 0$ such that $\angle_m(T^{||}_D, T_D\tau) > \eta'$ in the points where $\tau(x)$ enters in the tubular neighborhood of radius η' of the $\mathbf{0}$ section. In the original definition the distribution we use is H^{∇}_D, the intersection of $\hat{D}$ with H^{∇} – the horizontal distribution of ∇–. Since the connection is linear and hence tangent to the $\mathbf{0}$ section, H^{∇}_D and $T^{||}_D$ will be as close as needed in a small enough tubular neighborhood of the $\mathbf{0}$ section. This argument proves also that estimated transversality using different connections gives comparable quantities (and the comparability constant is deduced form an upper bound for the connection matrix relating both connections).

The notion of estimated transversality can be easily extended to finite Whitney stratifications $\mathcal{S} = (S^a)_{a \in A}$ of E. For each stratum S^a, let us denote by $T^{||}S^a$ the parallel transport (Levi-Civita connection) of the tangent bundle of S_a to a small tubular neighborhood; define $T^{||}_D S^a$ to be its intersection with $\hat{D}$. If the parallel transport is transverse to the fibers then automatically $T^{||}_D S^a$ will have the expected dimension ($\dim S^a - 1$). For a section τ of E we still use the notation $T\tau$ for the tangent distribution to its graph and $T_D\tau$ to denote its intersection with $\hat{D}$. Given any point $x \in M$, $T_D\tau(x)$ will denote the vector subspace of the distribution $T_D\tau$ in the point $\tau(x)$. Once we work with a fixed section τ, $T^{||}_D S^a(x)$ will be the subspace of the corresponding distribution in the point $\tau(x)$ (if $\tau(x)$ belongs to the points where the distribution is defined).

Definition 5.6. *Let η be a positive number. The section τ is (η, D)-transverse to S (or simply η-transverse) if in each point x where τ is at distance smaller than η of a stratum S^a, $T^{||}_D S^a(x)$ has the expected dimension and $\angle_m(T_D\tau(x), T^{||}_D S^a(x)) > \eta$.*

A sequence of sections is uniformly transverse to $\mathbf{0}$ *(resp. to a sequence of stratifications) if a constants* $\eta > 0$ *exists so that for* k *bigger than some* K*, the sections* τ_x *are* η*-transverse to* $\mathbf{0}$ *(resp. to the sequence of stratifications)*

The definition of estimated transversality to a stratum makes sense only out of a tubular neighborhood of its boundary, because in the points at distance of the boundary smaller than some ϵ, we might have not enough control on the geometry of the stratum so that for example the tubular neighborhood where $T^{||}S^a$ is defined tend to zero with ϵ. Thus, it is more convenient to work in a compact region of the stratum (points at distance of the boundary bigger than some ϵ) and define estimated transversality in the complementary *only in the points of the strata*, i.e., if $\tau(x)$ hits the stratum in one of this points very close to the boundary, we demand that the intersection inside $\hat{D}$ happens with minimal angle bounded by below. We will see that to solve the uniform transversality problem in this region the uniform Whitney condition will be enough (condition (4) in definition 5.2).

The notion of uniform transversality to an A.H. sequence of stratifications admits a nice local formulation, as long as we have 1-comparable charts as described in the previous subsection (with the local foliation ker *df* rectified).

Before it is necessary to study a bit further the notion of minimal angle.

Variations of the maximal and minimal angle. Recall that to measure the minimum angle between transversal subspaces $U, V \subset \mathbb{R}^n$ we proceed as follows:

Let us first suppose that U, V are complementary subspaces: let V_1 be the intersection of the unit sphere with V. For each point $v \in V_1$, its distance to U coincides with the distance of $\pi_U(v)$ –its orthogonal projection onto U– to U. The corresponding angle is comparable to the norm of the orthogonal projection of v onto $U^\perp$. It follows that the minimal angle can be compared with the distance to the origin of the ellipsoid $\pi_{U^\perp}(V_1) \subset U^\perp$.

With the previous interpretation of the minimum angle it is clear that if instead of using the euclidean metric in $U^\perp$ we use a comparable one, the corresponding minimal angle is comparable to the original one. We can also use a comparable metric in V and obtain a comparable minimum angle. In particular we can represent V as the graph of a linear function $\tau_*\colon U \to U^\perp$. By definition, the amount of transversality of τ_* to $\mathbf{0}$ is the distance to the origin of $p_{U^\perp}(\mathrm{graf}(\tau_*(U_1)))$. Notice that $\mathrm{graf}(\tau_*(U_1))$ is the unit sphere in V for the pushforward by τ_* of the euclidean metric in U. This induced metric is comparable to the euclidean with comparison constant γ obtained out of an upper bound for τ_*. Therefore, provided we can control the norm of τ_*, a bound by below for the minimal angle is equivalent to a bound by below for the amount of transversality for the function τ_*.

More generally, we can change the metric in the whole space to a comparable one; it is easily checked that for any U, V complementary subspaces, the restriction of the new metric to V and $U^\perp$ is comparable to the euclidean one: the unit ellipsoid contains an euclidean ball of radius ρ_1 and it is contained in an euclidean ball of radius ρ_2. This property still holds when we intersect with any subspace (and obviously the comparison constant for the total space is valid for the restrictions of the metric to any subspace).

When U and V are transverse subspaces with nontrivial intersection, the situation is similar. If we change the euclidean metric to a comparable one g we obtain comparable notions of minimal angle: to check that let us denote by W the orthogonal to $U \cap V$ w.r.t. the new metric. Imagine for the moment that W is also the euclidean orthogonal. As we observed, the induced metric by g in W is comparable to the euclidean with the same comparison constant as in the whole space. In general W will not coincide with the euclidean orthogonal, but one checks that the map $\pi_{(U\cap V)^\perp} \colon W \to (U \cap V)^\perp$ sends the complementary subspaces in W whose angle has to be measured to the intersection of U and V with $(U \cap V)^\perp$. Hence, we have to measure in $(U \cap V)^\perp$, but with the pushforward by $\pi_{(U\cap V)^\perp}$ of the euclidean metric in W. This is equivalent to finding a bound by below for the norm of the map with follows form a bound by below for $\angle_m(W, U \cap V)$. This last bound is a consequence of the comparison between g and g_0 in the total space: indeed, such bound by below for the angle would not exist we might find a matrix relating to orthonormal basis (one for each metric) with determinant arbitrarily small. We proceed by contradiction: we take $u \in (U \cap V)^\perp$ of unit euclidean norm such that $u + z \in W$ is a vector with large norm and rescale it to $\lambda(u + v)$ so that its g-norm becomes 1; thus, λ will be very small. It is possible to complete it to a g-orthogonal basis in W, and then in the whole space by adding a g-orthogonal basis in $U \cap V$. The change of basis can be written w.r.t. an euclidean orthogonal basis $\{e_1, ..., e_n\}$, where $e_1, ..., e_s$ span $U \cap V$ and $e_{s+1}, ..., e_n$ span $(U \cap V)^\perp$. According to this splitting the matrix have four blocks. The upper left one represents the change of basis in $U \cap V$ and hence by hypothesis its determinant is bounded by above. Since the upper left block of the matrix in vanishing we only need to study the lower right one. By hypothesis, its components are bounded by above. Since one of the lines are the components in $(U \cap V)^\perp$ of λu, we conclude that the determinant can be made as small as desired by letting λ go to zero.

It is an easy exercise to check that the from biggest quantity for the minimal angle is achieved when we use as complementary subspace the orthogonal to the intersection.

In certain circumstances there will be a natural choice of complementary W to measure the angle which will make the calculations easier. But we must make sure that the minimum angle between W and the intersection $U \cap V$ is bounded by below, so that we obtain a comparable quantity when we measure the minimum angle in W.

A way of choosing a complementary is as follows: we consider $W_1 \subset U \cap V$ and W_1^c a complementary subspace (in the total space); next we select $W_2 \subset (U \cap V) \cap W_1^c$ and W_2^c a complementary in W_1^c. We iterate the construction. In some step, we will have taken W_t such that W_t is the whole intersection $(U \cap V) \cap W_{t-1}^c$. The complementary W is defined to be W_t^c. By construction, W is complementary to $U \cap V$.

Lemma 5.7. *If in the previous construction we have bounds $\angle_m(W_j, W_j^c) > \delta_j$ (the minimum angle as complementary subspaces of W_{j-1}^c), then a constant $\eta(\delta_1, ..., \delta_j) > 0$ exists so that $\angle_m(U \cap V, W) > \eta$.*

PROOF. Assume that the total space has dimension 3 and that $t = 2$ and $\dim W_1 = \dim W_2 = \dim W = 1$. It is easier to translate the problem into the corresponding one of spherical geometry: we can think of $(U \cap V) \cap S^2$ as the equator of the sphere S^2. W_1 is then a point in the equator (and its antipodal) and W a point in the sphere; we have to show that it is far enough form the equator. The hypothesis on W_1^c implies that the corresponding geodesic it is not very close to the point W_1 in the equator. Thus, it cuts the equator with angle bounded by below; otherwise it would be contained in an arbitrarily small tubular neighborhood of the equator and hence arbitrarily close to W_1. The mentioned condition on the angle, together with the fact that W is a point in the geodesic far enough form the intersection with the equator –which by definition is W_2– implies that the distance of W to the equator is bounded by below.

When the dimensions of W_1, W_2, W are arbitrary, the proof can be reduced to the previous one. If $w \in W$, $u = u_1 + u_2 \in W_1 + W_2 = U \cap V$, the angle $\angle(w, u)$ can be measured in the subspace $\mathbb{R}^3$ spanned by w, u_1, u_2, with the induced metric (comparable to the induced by the euclidean with the same comparison constant). It is obvious that the lower bounds for $\angle_m(W_1, W_1^c), \angle_m(W_2, W)$ do hold for $W_1 \cap \mathbb{R}^3 = \langle u_1 \rangle$, $W_1^c \cap \mathbb{R}^3 = \langle u_2 \rangle \oplus \langle w \rangle$, $W_2 \cap \mathbb{R}^3 = \langle u_2 \rangle$ and $W \cap \mathbb{R}^3 = \langle w \rangle$ (because the measured minimum angles are between complementary subspaces). Therefore, $\angle(w, u)$ is bounded by below.

If $t \geq 2$, we can apply induction. We just set $\tilde{W}_1 = W_1 + W_2$ and $\tilde{W}_1^c = W_2^c$. From what we just showed, $\angle_m(\tilde{W}_1, \tilde{W}_1^c)$ is bounded by below. Thus the hypothesis still hold but now for $t - 1$. □

A consequence of this lemma is the following result.

Corollary 5.8. *Let $U, V, \hat{V}$ subspaces of $\mathbb{R}^n$, with $V \subset \hat{V}$. If a bound $\angle_m(U, V) \geq \delta$ is available, then a constant $\eta(\delta)$ exists such that $\angle_m(U, \hat{V}) \geq \eta$.*

PROOF. We can choose an appropriate complementary. We choose $W_1 = U \cap V$ and $W_1^\perp$ as complementary. We know that for the intersections $U^\perp, V^\perp$ –which are complementary subspaces inside $W_1^\perp$– the bound $\angle_m(U^\perp, V^\perp) \geq \delta$ holds . We set $W_2 = \hat{V} \cap U^\perp$ and as a complementary –which will already be W– the span of $V^\perp$ and the orthogonal to W_2 in $U^\perp$. If a lower bound for $\angle_m(W_2, W)$ is assumed, then W can be safely used as complementary: by definition $W \cap V = V^\perp$ and $W \cap U \subset U^\perp$. Hence $\angle_m(W \cap U, W \cap V) \geq \delta$.

The bound on $\angle_m(W_2, W)$ that we assumed can be checked using lemma 5.7. We start with $V^\perp \subset W$; the complementary (inside $W_1^\perp$) is chosen to be $U^\perp$ and the bound for the minimal angle follows from hypothesis. Inside $U^\perp$ we select the orthogonal to $U^\perp \cap \hat{V}$ and $U^\perp \cap \hat{V}$ as complementary. Applying again lemma 5.7, we obtain a bound for the minimal angle between

$U^\perp \cap \hat{V} = W_2$ and the span of $V^\perp$ and the orthogonal to $U^\perp \cap \hat{V}$ which is W. $\square$

Now we can move into a local characterization of uniform transversality w.r.t. a sequence of stratifications as in definition 5.2.

Lemma 5.9. *Let $\mathcal{S}^a$ be a sequence of strata as those in the stratifications of definition 5.2. Let $y \in F_k$ be a point in the stratum at distance more than ϵ of the boundary, and let $f_1, ..., f_p$ the corresponding local functions defining the stratum in $B_{\hat{g}_k}(y, \rho_\epsilon)$. Then uniform transversality of τ_k to S_k^a (along D) in that ball is equivalent (comparable) to uniform transversality (along D) of the function $(f_1 \circ \tau_k, ..., f_p \circ \tau_k)$ to $\mathbf{0}$.*

Thus, for the points of the strata at distance more than ϵ of the boundary, uniform transversality of the sequence to these regions follows from a uniform lower bound for the amount of transversality to $\mathbf{0}$ of the corresponding local transversality problems for functions.

Proof. By simplicity we omit the subindices for the sections τ_k.

The proof consist of to parts. The first one amounts to proving that estimated transversality of $f \circ \tau$ (along D) is equivalent to estimated transversality (of $T_D\tau$) w.r.t. the distribution $T_D^{||} S_k^a$ and it can be proven as follows:

Let us forget for the moment about D and assume that we are trying to state the same result but for full transversality (considering TM instead of D).

To measure the minimum angle we take instead of $T\tau \cap \ker df$, W defined to be the span of the orthogonal to $T\tau \cap \ker df$ in $\ker df$, and the subspace $d\tau(L)$, where L is the orthogonal (in the base) to the kernel of $d(f \circ \tau)$. By construction $\angle_m(W, T\tau \cap \ker df)$ coincides with $\angle_m(d\tau(L), T\tau \cap \ker df)$ measured in $T\tau$. A lower bound for the latter is deduced from an upper bound for the norm of $d\tau$.

Next, we will do to changes of metric: the first one amounts to taking inside $d\tau(L)$ the pushforward by τ of the euclidean in L (again an upper bound for $|\tau_*|$ guarantees control for the distortion of the metric). The second one occurs in the orthogonal to $\ker df \cap W$ in W, where we use the pullback by df of the euclidean metric in $\mathbb{C}^p$; this subspace has minimum angle with $\ker df$ bounded by below. The consequence is that controlling the change of metric in it is equivalent to controlling it in $\ker df^\perp$ (the pullback of the euclidean metric in $\mathbb{C}^p$), or in any other complementary V to $\ker df$ such that $\angle_m(V, \ker df)$ is bounded by below. Our choice of V is the orthogonal to $\ker df \cap T^v F_k$ inside $T^v F_k$ (the tangent space to the fibers). Observe that if $\angle_m(T^v F_k, \ker df) \geq \eta$ holds, then $\angle_m(V, \ker df) \geq \eta'$ follows by lemma 5.7.

Let us call U to the intersection of $\ker df$ with the orthogonal to $T^v F_k \cap \ker df$. We want to show that U is not very close to $T^v F_k$. If U is written as the graph of a linear map from $T^v F_k^\perp$ to the orthogonal to $T^v F_k \cap \ker df$ inside $T^v F_k$, we look for and upper bound for the norm of that map. If such a bound did not exist we could find a vector u in the orthogonal to $T^v F_k \cap \ker df$ inside $T^v F_k$ such that $df(v)$ is arbitrarily small. But this would contradict the bound by below for the restriction of $df_1 \wedge ... \wedge df_n$ to the fiber.

It is clear that with the new choices of complementaries and metric what we are computing is exactly the amount of transversality of $d(f \circ \tau)$ to $\mathbf{0}$.

Thus, estimated transversality of $d(f \circ \tau)$ to $\mathbf{0}$ is equivalent to a bound by below $\angle_m(T\tau, \ker df)$, and the equivalence depends on the norms of τ_*, f_* and $\angle_m(\ker df, T^v F_k)$.

We we intersect everything with $\hat{D}$, the previous argument equally works. The only difference is that we only use the bound for the restriction of τ_* to D. It is important to notice that the previous changes of complementaries and metric do occur inside $\hat{D}$ (see that with the notation of the previous paragraphs, the complementary V is inside $T^v F_k$ which is itself contained in $\hat{D}$, so we can apply property (1) in 5.2 to conclude the equivalence).

The second part of the proof reduces to proving that for any $\varepsilon > 0$ $\angle_M(T_D^{||} S^a, \ker df \cap \hat{D}) \leq \varepsilon$ in a tubular neighborhood of radius $\varrho(\varepsilon)$ of the stratum. This is equivalent to proving the same result for $\angle_M(T^{||} S^a, \ker df)$ and then use the bound by below for $\angle_m(\hat{D}, \ker df)$. The equivalence follows from proposition 3.7 in [**46**], where in their notation $V = V' = \hat{D}$, $U = \ker df$, $U' = T^{||} S^a$. Anyway, this result can easily be proven using the ideas about the alternative definitions of the minimum angle. One just notices that U and U' of the same dimension and $\angle_M(U, U')$ small enough, U' is the graph of a linear map from U to $U^{\perp}$. The maximal angle is comparable to the distance of the ellipsoid $p_{U^\perp}(U'_1) \subset U^{\perp}$. One checks that intersecting with a transversal enough subspace of V corresponds to working in a subspace with comparable metric.

The bound for $\angle_M(T^{||} S^a, \ker df) \leq \epsilon$ is a consequence of the existence of 1-comparable charts for which f is the projection in $2p$-coordinates. As we mentioned in subsection 3.2 of this section, since the stratum becomes a vector space (of dimension $2p$) it is possible to compare tubular neighborhoods for the induced metric and the euclidean. In the corresponding g-tubular neighborhood of radius ϱ, each point q is the endpoint of a g-geodesic. Any vector $v \in T_q^{||} S^a$ is the result of parallel translating certain vector $u \in T_y S^a$. In the geodesic, parallel transport is controlled by the Christoffel symbols (also for the tangent field to the geodesic). Thus, in the point of the geodesic for time $t \in [0, r]$, the difference between the g-parallel transport of U and its g_0-parallel transport is bounded by $e^{t\Gamma} - 1$, where $\Gamma > 0$ is a constant depending on the bounds for the induced connection. In particular $|v - u| \leq e^{t\Gamma} - 1$. By definition u (over the point q) belongs to $\ker df$. Therefore $\angle_M(T^{||} S^a, \ker df) \leq e^{r\Gamma} - 1$. □

Remark 5.10: It is important to notice that for even dimensional a.c. manifolds and *for any distribution* $Q \subset TM$, there is an obvious definition of estimated transversality to a sequence of A.H. stratifications. The role of $\hat{D}$ is replaced by $\hat{Q}$, the pullback of Q to the total space of the bundle. It can be checked that the proof of lemma 5.9 works for any distribution. Thus, transversality to a stratum far from the points of its boundary is equivalent to estimated transversality to $\mathbf{0}$ of the function $f \circ \tau$ along the directions of Q. This is specially interesting when the distribution is integrable, because we can study the transversality along one single leaf instead of in the whole manifold, where it turn out to be equivalent to the transversality of the restriction of the function $f \circ \tau$ to the leaf. We can generalize this situation by considering just a submanifolds $Q \subset M$. Estimated transversality of τ to the strata along the directions of Q (in the points of Q) is equivalent to full estimated transversality to $\mathbf{0}$ of the function $(f \circ \tau)_{|Q}$.

For a stratum whose codimension $2p$ is less or equal than the dimension of Q (submanifold), it is clear that estimated transversality of $f \circ \tau$ to $\mathbf{0}$ along Q implies estimated transversality of the function along all the directions of TM. When the codimension is smaller then we conclude than in neighborhoods of uniform g_k-radius of Q, the sections do not touch the corresponding stratum (assuming control of order $O(1)$ in the derivatives). Observe that for odd dimensional a.c. manifolds and the distribution D, if $2p > \dim D$ then $2p > \dim D + 1 = \dim M$. From this observation we deduce that transversality along D –which has been seen to imply transversality along all the directions of TM– is unobstructed in the sense that the expect codimension for which transversality along D implies empty intersection is the same as for full transversality.

Lemma 5.11. *Let $\mathcal{S} = (S_k^a)_{a \in A}$ be a sequence of A.H. stratifications as in definition 5.2. Assume that the sequence τ_k is uniformly transverse to $\mathcal{S}$ along the directions of a distribution Q whose dimension is greater of equal than the codimension of the strata, and that the uniform bound $|\nabla \tau_k|_{g_k} \leq O(1)$ holds. Then for each $a \in A$, $\tau_k^{-1}(S_k^a)$ is a subvariety of M uniformly transverse to Q.*

Proof. We omit the subindices for sections and stratifications. The proof of the result is specially easy for those points of M that are sent by τ far from the boundary of the strata S_k^a. We need to find a subspace $Q^c \subset \ker d(f \circ \tau)$ complementary to Q whose minimum angle with Q is bounded by below (such a complementary always exist because of $\dim Q \geq 2p$). We take $u_1, ..., u_{n-q}$ a basis of $Q^{\perp}$. There exists a unique v_i in the orthogonal to $\ker d(f \circ \tau) \cap Q$ inside Q so that $u_i + v_i \in \ker d(f \circ \tau)$. The ideas used to define different notions of minimum angle show that the bound we look for is equivalent to a bound by above in the norm of v_i, which follows from the hypothesis on the norm of $\nabla \tau$ (the norm of df is bounded and the minimum angle between the horizontal distribution of ∇ and the fiber is bounded by below) and from the bound by below for the norm of v_i (deduced from the bound by below along the directions of Q).

We mention that this argument can be modified to work in the total space of F_k instead of with the function f. Thus, it is also valid for those points in the strata close to the boundary. $\square$

In particular, the following corollary is deduced.

Corollary 5.12. *Let $\mathcal{S} = (S_k^a)_{a \in A}$ be a sequence of A.H. stratifications over the a.c. manifold (M, D, J, g) as in definition 5.2. Assume that the A.H. sequence τ_k is uniformly transverse to $\mathcal{S}$ (along D) and that the uniform bound $|\nabla \tau_k|_{g_k} \leq O(1)$ holds. Then for each $a \in A$, $\tau_k^{-1}(S_k^a)$ is either empty*

–if the codimension of S_k^a is bigger than the dimension of D (or M)– or a subvariety uniformly transverse to D.

For a even dimensional a.c. manifold and transversality along the directions of a (compact) subvariety Q (assuming the mentioned bound on $|\nabla \tau_k|_{g_k}$), then either $\tau_k^{-1}(S_k^a)$ is at g_k-distance of Q bounded by below or is a subvariety (at least defined in a g_k-neighborhood of Q) uniformly transverse to Q.

We would like to finish this section giving sufficient conditions for a sequence of stratifications of a sequence of very ample bundles E_k to be A.H. (and finite and Whitney).

We introduce the following definitions:

Using A.H. trivializations of E_k, and A.H. coordinates each bundle is locally identified with $(\mathbb{C}^n \times \mathbb{R}) \times \mathbb{C}^m \to \mathbb{C}^n \times \mathbb{R}$.

Definition 5.13. *We say that a submanifold of the total space of $(\mathbb{C}^n \times \mathbb{R}) \times \mathbb{C}^m$ is constant if it is fiberwise constant. The (constant) submanifold is holomorphic it its intersection with a fiber is holomorphic submanifold of $\mathbb{C}^m$. We also say that a submanifold of the total space is $Gl(n,\mathbb{C}) \times Gl(m,\mathbb{C})$-invariant when its intersection with each fiber has this property.*

We say that a stratification of E_k is constant (resp. holomorphic) when we have local identifications for each point so that the corresponding strata are identified with a fixed constant (resp. holomorphic) submanifold (for $k >> 0$). A constant stratification of E_k is Whitney if the model has this property. We say that the stratification is $Gl(n,\mathbb{C}) \times Gl(m,\mathbb{C})$-invariant is the intersection of the stratification with each fiber of E_k has this property.

Lemma 5.14. *Let $(S_k^a)_{a\in A}$ be a holomorphic finite Whitney stratification of E_k invariant under the action of $Gl(n,\mathbb{C}) \times Gl(m,\mathbb{C})$ or $Gl(n,\mathbb{C}) \times \mathbb{C}^*$. Then the sequence $(S_k^a)_{a\in A_k}$ is as in definition 5.2.*

Conversely, from a stratification of $(\mathbb{C}^n \times \mathbb{R}) \times \mathbb{C}^m \to \mathbb{C}^n \times \mathbb{R}$ with the mentioned properties, using the local identifications it is possible to induce an A.H. sequence of finite Whitney stratifications of E_k.

Proof. Using the local identification conditions (1), (2), (3) y (4) in definition 5.2 hold trivially. Maybe we should point out that the strata (in the fiber over the origin for example) are submanifolds of $\mathbb{C}^m$ and the bounds we get might be not independent on the points (for example for the Whitney condition). But we are only interested in working on compact regions, we the bounds are uniform.

The previous comments proof the lemma for even dimensional a.c. manifolds. In the odd dimensional situation the total space has structure of a.c. manifold (we use the splitting given by the metric). That is the, the orthogonal to $\hat{D}$ is defined by $D^\perp$ and the connection over this line bundle. That means that when using A.H. coordinates and holomorphic sections to trivialize E_k we obtain coordinates $z_k^1, ..., z_k^n, u_k^1, ..., u_k^m, s_k$ in which the strata are defined by A.H. functions (usually independent of $z_k^1, ..., z_k^n, s_k$) *w.r.t. these* *coordinates*. Again, it might very well happen that these coordinates are not strictly A.H. for the total space, because $\frac{\partial}{\partial s_k}$ does not approximately coincide with $\hat{D}^\perp$ (due to the presence of the connection). Anyway, the control on the connection and its derivatives imply that we have A.H. coordinates in a weak sense that assures that if f is A.H. in these coordinates, then it is A.H. for the structure of A.H. manifold in the total space of the bundle (and the metric retraction). □

5.3. Quasi-stratifications of $\mathcal{J}^r_D E_k$. The main application we seek for is for a generalized stratification (*quasi-stratification*) of $\mathcal{J}^r_D E_k$, where E_k is $\mathbb{C}^{m+1} \otimes L_k$, and L_k is a very ample sequence of line bundles (the powers of the prequantizable line bundle if we start form a calibrated manifold). It is through this quasi-stratification that we want to study the genericity properties of the projectivization of a section of E_k away from its vanishing set.

In contrast to what happen for 0-jets, it is not easy to find non-trivial A.H. stratifications for higher order jets. There are even some which arise naturally but which do not have the required properties. The difficulty comes from the fact that the modification of the connection that makes the r-jets of A.H. sections also A.H. sections of $\mathcal{J}^r_D E_k$, makes it difficult to make sure that the strata are given by functions whose composition with an A.H. section is an A.H. function. A sufficient condition will be that the functions f are A.H., but this is precisely the condition hard to be checked.

Example 5.15: Let $L^{\otimes k}$ the sequence of powers of the prequantum line bundle of a *symplectic manifold*. Let us consider the following sequence of strata in $\mathcal{J}^1 L^{\otimes k}$:

$$\Sigma_{k,n} = \{(\sigma_0, \sigma_1) | \sigma_1 = 0\}$$

Using the base $\mu_{k,x,I}$, where $I = 1, ..., n$, and taking reference sections in Darboux charts, we get coordinates $z^1_k, ..., z^n_k, v^0_k, v^1_k, ..., v^n_k$ for the total space. $\Sigma_{k,n}$ is then defined by the zeros of the function $f = (v^1_k, ..., v^n_k) \colon \mathbb{C}^{2n+1} \to \mathbb{C}^n$, which is not holomorphic (or A.H.) w.r.t. the modified almost complex structure of the total space. Otherwise, the composition $f \circ j^1(z^1_k \tau^{\mathrm{ref}}_{k,x})$ would be A.H., but that composition is $(1 + z^1_k \bar{z}^1_k, z^1_k \bar{z}^2_k, ..., z^1_k \bar{z}^n_k)$. It is not only that for these functions the strata are not A.H., but for any choice. Such choice, when composed with the mentioned 1-jet would just be the composition of $(1 + z^1_k \bar{z}^1_k, z^1_k \bar{z}^2_k, ..., z^1_k \bar{z}^n_k)$ with a self diffeomorphism of $\mathbb{C}^p$ fixing. It is easy to see that one cannot obtain a holomorphic map in such a way, because $f \circ j^1(z^1_k \tau^{\mathrm{ref}}_{k,x})$ has image in $\mathbb{R}^n$ (it cannot be A.H. either).

For the main application we need to weaken the notion of stratification:

Definition 5.16. *(see* [**5**]*) Let S be a submanifold of $\mathcal{J}^r_{D_h,n,m}$. We define Θ_S to be the set of points $\sigma \in S$ for which it can be found an $(r+1)$-jet whose r-jet is σ (truncating it), and that seen as a 1-jet (along D_h or foliated) of a local section of $\mathcal{J}^r_{D_h,n,m}$, cuts S in σ transversely along D_h (its restriction to D_h cuts S transversely).*

We refer to Θ_S as the holonomic transversal subset of S.

We mention that when we represent an $(r+1)$-jet σ as a local section of $\mathcal{J}^r_{D_h,n,m}$, in order to check whether $\pi_r^{r+1}\sigma \in S$ belongs to Θ_S, the local representation is essentially unique. Regarding transversality, it is enough to consider the degree 1 part of the Taylor expansion in the coordinates $z_k^1, \bar{z}_k^1, ..., z_k^n, \bar{z}_k^n$ (we turn the section into a function using the basis μ_I). The degree 0 part is determined by the r-jet, the hypothesis imply that the antiholomorphic part is vanishing and the holomorphic part is determined by the $(r+1)$-jet. That means in particular that we can restrict our attention to holomorphic representations if necessary.

If we use the trivialization given by the basis μ_I, some of the notions introduced in the previous section can be extended to stratifications of $\mathcal{J}^r_{D_h,n,m}$.

A submanifold of $\mathcal{J}^r_{D_h,n,m}$ is constant if in the previous trivialization is fiberwise constant.

A stratification is of $\mathcal{J}^r_{D_h,n,m}$ is *constant* if all its strata are constat. Each strata is thus defined by its intersection with one fiber. It is clear that if S is constant, Θ_S is also constant. It can also be checked that if S is invariant under the fiberwise action of $Gl(n,\mathbb{C}) \times Gl(m,\mathbb{C})$, then Θ_S has the same invariance property.

Definition 5.16 can be translated to the bundles $\mathcal{J}^r_D E_k$ using local representations for ∇_H. The problem is that since the connection has been modified if the strata S_k –identified with S– is given by conditions that involve the components of degree higher than zero, once $S_k \subset \mathcal{J}^r_D E_k$ is identified with $S \subset \mathcal{J}^r_{D_h,n,m}$, Θ_{S_k} is not necessarily identified with Θ_S because we cannot compare the local representations for d_{D_h} and $\nabla_{H,D}$. Anyway, there will be examples where this identification occurs.

It is convenient to use local A.H. representations for the $(r+1)$-jets.

Definition 5.17. *Let σ_k be a pseudo-holomorphic $(r+1)$-jet over a point x. A section α_k of $\mathcal{J}^r_D E_k$ is c-local (C^D, C)-representation of σ_k if α_k is local C^1-A.H.(C^D, C) section defined in $B_{g_k}(x,c)$ and we have:*

(1) $\alpha_k(0) = \pi_r^{r+1}\sigma_k$
(2) $\nabla_{H,D}\alpha_k(0) = \sigma_k$,

where $\nabla_{H,D}$ is the component of ∇_H along D (using as usual the metric splitting).

The next step is showing that any $(r+1)$-jet admits (global) (C^D, C)-representations with constants not depending on k, x (though they depend on the norm of the $(r+1)$-jet). This fact is a consequence of certain features of the modified connection ∇_H.

Lemma 5.18. *Let E_k be a very ample sequence of locally splittable hermitian bundles over M and $(\mathcal{J}^{r+1}_D E_k, \nabla_H)$ the sequences of bundles of pseudo-holomorphic $(r+1)$-jets with the modified connection. For any $(r+1)$-jet σ_k over a point $x \in M$, there exists a natural K and constants C^D, C depending* *on the norm of σ_k and the geometry of M, but not on k and x, such that we have a (global) (C^D, C)-A.H representation α_k of σ_k.*

PROOF. Let us fix A.H. coordinates with $J = J_0$ in the origin. Using the local identification given by these coordinates and a trivialization (for example given by reference sections $\tau^{\text{ref}}_{k,x,j}$), we push the connection ∇_H to a connection in $\mathcal{J}^r_{D_h,n,m}$ still denoted by ∇_H. Since the local representations are tools to deal with the points of the stratum where an A.H. sections cuts in a non transversal way (approximately), we assume for the moment the existence of a sequence $\mathcal{S}_k$ of A.H. stratifications of $\mathcal{J}^r_D E_k$, which in the previous local identifications coincides with a $Gl(n,\mathbb{C}) \times Gl(m,\mathbb{C})$-invariant stratification of $\mathcal{S}$ of $\mathcal{J}^r_{D_h,n,m}$. For each stratum S^a_k it would be reasonable to define $\Theta_{S^a_k}$ as the subset of S^a_k of those r-jets σ which have at least an $(r+1)$-jet extending σ, with a local representation (not necessarily A.H.) cutting S^a_k transversally along D. We would like the local identification with $\mathcal{J}^r_{D_h,n,m}$, to send $\Theta_{S^a_k}$ to Θ_{S^a}, but this is not true in general. A sufficient condition would be to be able to relate the local representations w.r.t. d_{D_h} and $\nabla_{H,D}$, or being more precise, the value of the component of the connection form on $\bar{D}^*$ in the point in question. Since $\mathcal{S}$ has been assumed to be $Gl(n,\mathbb{C}) \times Gl(m,\mathbb{C})$-invariant, we can use this action to modify the connection form of $\nabla_{H,D}$ (while we keep d and $\mathcal{S}$). If we could get the vanishing of the connection form on the point with this action, then the subsets of transverse holonomy would coincide.

Let us see what happens in the models. In the Kähler case (even dimension) –and for the original connection– would be enough to trivialize E_k with holomorphic sections such that the connection form vanishes at the origin and use normal coordinates. It turns out that once we modify the connection we will always have an antiholomorphic component. Hence, it seems reasonable to use a holonomic trivialization (jets of holomorphic sections).

In the non-integrable case –and now back to odd dimension– it is elementary to kill the connection form along D at the origin. It already has this property in the approximate sense provided we select appropriate A.H. trivializations of E_k killing its connection form. Since the metric has weight $O(c_k^{-1/2})$, a small modification in the choice of $dz^i_k \in \bar{D}^{*1,0}$ makes the work. Notice that the modification does not come necessarily form a change of A.H. coordinates, but it only uses the action of $Gl(n,\mathbb{C})$.

For the modified connection form it is not possible to get such property starting form the basis μ_I (which homogeneous in the sense that each section only has non-vanishing components of a certain order) and using the action of $Gl(n,\mathbb{C}) \times Gl(m,\mathbb{C})$; such action preserves the homogeneous subbundles of $\mathcal{J}^r_D E_k$, whereas the connection does not preserve its sections. In particular for $\mathcal{J}^1_D L_k$ and any section τ of E_k with $\tau(0) \neq 0$, $\nabla_{H_1}(\tau, 0) = (\nabla\tau, -F^{1,1}_D \tau)$, where $-F^{1,1}_D \tau(0) \neq 0$ (somehow this reflects what happens in example 5.15).

It is evident what we have to do to obtain local A.H. representations. We start with the basis $\nu_{k,x,I}$ of $\mathcal{J}^{r+1}_D E_k$. Over the origin, any $(r+1)$-jet σ is a linear combination $\sum_I \beta_I \nu_{k,x,I}$. By linearity $\sigma = j^{r+1}_D(\beta_I \tau^{\text{ref}}_{k,x,I})$, and therefore $\pi^{r+1}_r \sigma = j^r(\beta_I \tau^{\text{ref}}_{k,x,I})(0)$. Since is an A.H. section, $\sigma - \nabla_{H,D} j^r(\beta_I \tau^{\text{ref}}_{k,x,I})(0)$

has size $O(c_k^{-1/2})$. Thus, a linear perturbation of size $O(c_k^{-1/2}))$ in the coordinates $x_k^1, y_k^1, ..., x_k^n, y_k^n$ gives the desired representation, whose bounds are given in terms of those of $\tau_{k,x,j}^{\text{ref}}$ and of the norm of the $(r+1)$-jet; they do not depend neither on k nor in x. □

Definition 5.19. *Let S_k be a submanifold of $\mathcal{J}_D^r E_k$ transverse to the fibers, and let $C^D, C, c > 0$. we define $\Theta_{S_k(C^D,C,c)} \subset S_k$ as the set of points σ_k in S_k that have a c-local (C^D, C)-A.H. representation α_k for a lift of σ_k cutting S_k transversely (in σ_k) along the directions of D.*

In certain circumstances we will forget about the constants and will speak of Θ_{S_k} as the set of points for which there are lifts with transverse local representations. This will only happens when in the local identifications this subsets can be identified with the corresponding Θ_S.

Remark 5.20: The definition of the sets $\Theta_{S_k(C^D,C,c)}$ is only useful when k is very large, being the reason that if a sequence of r-jets of C^{r+1}-A.H.(C^D, C) sections of E_k is uniformly transverse to a sequence of strata $S_k \subset \mathcal{J}_D^r E_k$, then the intersection will occur in points of $\Theta_{S_k^a(\bar{C}^D,\bar{C},c)}$, where the constants $\bar{C}^D, \bar{C}$ only depend on C^D,C and on the geometry of E_k and M, but not k (and c is any positive number smaller than the injectivity radius).

Remark 5.21: If an $(r+1)$-jet $\tilde{\sigma}_k$ is the derivative along D of certain C^{r+1}-A.H.(C^D, C) local section, then $||\sigma_k|| \leq C^D + O(c_k^{-1/2})$.

We recall the following important example (see [**5**]):

Example 5.22: Let us denote by Z_k the set of r-jets corresponding to sections intersecting the **0** section of E_k.

$$Z_k = \{\sigma = (\sigma_0, ..., \sigma_r) \, | \sigma_0 = 0\}$$

If we look at Z_k as a submanifold Z of $\mathcal{J}_{D_h,n,m}^r$ and work in the integrable setting (where there is no need of using constants that measure the lack of integrability) one checks that Θ_Z are those jets for which σ_1 is onto (and hence is empty if $m > n$). If now we think of Z a sequence Z_k of submanifolds of $\mathcal{J}_D^r E_k$, the subsets $\Theta_{Z_k(C^D,C,c)}$ will be empty if the dimension of the fiber is bigger than that of D. Otherwise, they will the points σ such that the $r+1$-jet $(\sigma, 0)$ has a c-local representation with the appropriate bounds and σ_1 is onto.

Observe that this is a rather special example because it is defined by conditions only involving 0-jets. That means that the modification of the connection does not affect, and thus in the local identifications Θ_{Z_k} goes to Θ_Z.

We want to perturb A.H. sequences of sections so that their r-jets become transverse to certain stratification (quasi-stratifications). Moreover, the perturbations will be arbitrarily small in the directions of D (say in C^h-norm, i.e., controlling the first h covariant derivatives along D). That means that the set of r-jets and $(r+1)$-jets we will work with will be uniformly bounded. Thus if we consider the subsets $\Theta_{S_k(C^D,C,c)}$, where the constants (C^D, C) are chosen to be bigger than those controlling the A.H. to order 1 of $j^r_D\tau_k$ (and such that there are c-local representations of a uniformly bounded set of $(r+1)$-jets containing $j^{r+1}_D\tau_k$ and its nearby perturbations), the lack of transversality to S_k of r-jets of A.H. sequences close enough to $j^r_D\tau_k$ can be stated in terms of its (approximate) belonging to the complementary to $\Theta_{S_k(C^D,C,c)}$ en S_k.

Definition 5.23. *(see also [**5**]) Let $(A, \prec)$ be a subset with a binary relation without cycles ($a_1 \prec \cdots \prec a_p \Rightarrow a_p \not\prec a_1$). A finite Whitney quasi-stratification of $\mathcal{J}^r_{D_h,n,m}$ indexed by A is a finite family of smooth submanifolds $(S^a)_{a\in A}$ not necessarily disjoint such that:*

1. $\partial S^a \subseteq \bigcup_{b\prec a} S^b$,
2. *for any point in the boundary $q \in \partial S^a$ there has to be $b \prec a$ such that either $q \notin \Theta_{S^b}$ or $S^b \subset \partial S^a$ and the Whitney condition holds for $S^b \subset \partial S^a$ (or at least q belongs to those points far from the boundary where that condition holds).*

The quasi-stratification will be said to be constant (resp. holomorphic, $Gl(n,\mathbb{C})\times Gl(m,\mathbb{C})$ invariant) if its strata have this property.

It is possible to give a similar definition of finite Whitney A.H. stratifications for the bundles $\mathcal{J}^r_D E_k$: the family of not necessarily disjoint smooth submanifolds $(S^a_k)_{a\in A_k}$ has to be transverse to the fibers, with local equations as in definition 5.2 for those points far from the boundary, defined in balls of uniform radius proportional to the distance to the boundary and verifying conditions (1), (2) y (3) in that definition. The difference occurs in the points close to the boundary: there we have $\partial S^a_k \subseteq \bigcup_{b\prec a} S^b_k$, and there has to exist a natural number K such that for any $k \geq K$, in any point of the boundary $q \in \partial S^a_k$ there is an index $b \prec a$ such that $q \in S^b_k$ and either of the following conditions hold:

- i $S^b_k \subset \partial S^a_k$ and the uniform Whitney condition (the fourth in definition 5.2) holds in all the points of S^b_k, or at least in those which are far from the boundary, being q in that subset (and not necessarily for all the precedent indices).
- ii *q approximately does not belong to $\Theta_{S^b_k}$*; that is, for each triple (C^D, C, c) another positive constant $\check{C}$ exists depending on the triple but not on k, such that for any $(r+1)$-jet σ with $\pi^{r+1}_r\sigma = q$, any local (C^D, C)-representation α of σ cuts S^b_k (in q) with minimum angle at most $\check{C}c_k^{-1/2}$.

It is obvious that for even dimensional a.c. manifolds, and for $\mathcal{S}_k$ a finite Whitney A.H. stratification of $\mathcal{J}^r E_k$ we can define the sets $\Theta_{S^a_k(C,c)}$, and give the corresponding definition of quasi-stratification.

5.4. The Thom-Boardman-Auroux quasi-stratification for maps to projective spaces. To study the genericity of maps to $\mathbb{CP}^m$ defined as projectivizations of sections of $E_k = \mathbb{C}^{m+1} \otimes L_k$, we introduce the non-linear bundle of pseudo-holomorphic r-jets of maps to $\mathbb{CP}^m$. Its main properties will be analyzed in the next to propositions that follow. Before we recall that Z_k denotes the sequence of strata of $\mathcal{J}^r_D E_k$ –already introduced in example 5.22– of r-jets whose degree 0-component vanishes. We define $\mathcal{J}^r_D E^*_k := \mathcal{J}^r_D E_k - Z_k$. When the almost complex manifold is even dimensional, we set $\mathcal{J}^r E^*_k := \mathcal{J}^r E_k - Z_k$, and $\mathcal{J}^r_G E^*_k := \mathcal{J}^r_G E_k - Z^G_k$ for the subbundle associated to a polarization G, where again Z_k denotes the set of r-jets of $\mathcal{J}^r E_k$ whose degree 0 component vanishes, and $Z^G_k = Z_k \cap \mathcal{J}^r_G E_k$.

Definition-Proposition 5.24. *A non-linear bundle of pseudo-holomorphic r-jets (along D) $\mathcal{J}^r_D(M, \mathbb{CP}^m)$ can be defined so that for any function $\phi\colon (M, D, J, g) \to \mathbb{CP}^m$ a notion of r-jet $j^r_D\phi \in \Gamma(\mathcal{J}^r_D(M, \mathbb{CP}^m))$ with the following properties can be given:*

1. *There exist a bundle map $j^r\pi\colon \mathcal{J}^r_D E^*_k \to \mathcal{J}^r_D(M, \mathbb{CP}^m)$ which is a submersion. For any section τ_k of E_k, in the points where it does not vanish it defines a projectivization ϕ_k and the following relation holds:*
$$j^r\pi(j^r_D\tau_k) = j^r_D\phi_k. \tag{5.5}$$
2. *The fibers of $\mathcal{J}^r_D(M, \mathbb{CP}^m)$ admit a canonical integrable almost complex structure so that the map $j^r\pi$ is fiberwise holomorphic and for any A.H. sequence τ_k of E_k, $j^r\pi(j^r_D\tau_k) \in \Gamma(\mathcal{J}^r_D(M - \tau_k^{-1}(0), \mathbb{CP}^m))$ is an A.H. sequence of sections of $\mathcal{J}^r_D(M, \mathbb{CP}^m)$.*

*There is an analogous definition of the bundle $\mathcal{J}^r(M, \mathbb{CP}^m)$ for even dimensional a.c. manifolds (see [**4**]). It also has a canonical integrable a.c. structure on the fibers so that the map $j^r\pi\colon \mathcal{J}^r E^*_k \to \mathcal{J}^r(M, \mathbb{CP}^m)$ is a fiberwise holomorphic submersion. Given $\phi\colon M \to \mathbb{CP}^m$, there is a corresponding notion of pseudo-holomorphic r-jet for which the following relation holds:*
$$j^r\pi(j^r\tau_k) = j^r(\pi \circ \tau_k), \tag{5.6}$$
For each A.H. sequence τ_k A.H. of E_k, $j^r\pi(j^r\tau_k) \in \Gamma(\mathcal{J}^r(M - \tau_k^{-1}(0), \mathbb{CP}^m))$ is an A.H. sequence.

Given a polarization G we can define $\mathcal{J}^r_G(M, \mathbb{CP}^m) \overset{i^r_G}{\hookrightarrow} \mathcal{J}^r(M, \mathbb{CP}^m)$ so that the following commutative square of submersions holds:

$$\begin{array}{ccc}
\mathcal{J}^r E^*_k & \xrightarrow{p^r_G} & \mathcal{J}^r_G E^*_k \\
\Big\downarrow{\scriptstyle j^r\pi} & & \Big\downarrow{\scriptstyle j^r\pi} \\
\mathcal{J}^r(M, \mathbb{CP}^m) & \xrightarrow{p^r_G} & \mathcal{J}^r_G(M, \mathbb{CP}^m)
\end{array}$$

The maps p^r_G are induced by the orthogonal projection $T^*M \to \bar{G}^*$ and if $j^r_G\tau_k$ is a section of $\mathcal{J}^r_G E^*_k$, for the restriction of $j^r\pi$ we have:
$$j^r\pi(j^r_G\tau_k) = j^r_G(\pi \circ \tau_k). \tag{5.7}$$

If τ_k is an A.H. sequence, $j^r_G\phi_k$ is also an A.H. sequence of sections of $\mathcal{J}^r(M - \tau_k^{-1}(0), \mathbb{CP}^m)) \subset \mathcal{J}^r_G(M - \tau_k^{-1}(0), \mathbb{CP}^m)$.

Definition-proof. We define the non-linear bundle $\mathcal{J}^r_D(M, \mathbb{CP}^m)$ as follows: we fix a system of holomorphic charts for $\mathbb{CP}^m$. For example in $\mathbb{C}^{m+1}$ with coordinates $z_0, ..., z_m$ we consider the canonical projection $\pi\colon \mathbb{C}^{m+1} - \{0\} \to \mathbb{CP}^m$, and take the charts $\varphi_i^{-1}\colon U_i \to \mathbb{C}^m$ sending $[z_0, ..., z_m]$ to $(\frac{z^1}{z^0}, ..., \frac{z^m}{z^0})$; we denote the change of coordinates $\varphi_j^{-1} \circ \varphi_i$ by Ψ_{ji}. For each chart φ_i we consider the bundle

$$\mathcal{J}^r_D(M, \mathbb{C}^m)_i := (\sum_{j=0}^{r} (\bar{D}^{*1,0})^{\odot j}) \otimes \mathbb{C}^m.$$

Over each point x, in the intersection $U_i \cap U_j$ the fiber over x of $\mathcal{J}^r_D(M, \mathbb{C}^m)_i$ and $\mathcal{J}^r_D(M, \mathbb{C}^m)_j$ are identified using the same transformation $j^r\Psi_{ji}$ in $\mathcal{J}^r_{n,m}$ induced by the holomorphic change of coordinates Ψ_{ji}. In other words, if we take A.H. coordinates containing x and make the corresponding local identification con $T^{*1,0}\mathbb{C}^n$, we get an induced identification of $\mathcal{J}^r_D(M, \mathbb{C}^m)_i$ with $\mathcal{J}^r_{D_h,n,m}$. Thus, an r-jet σ is represented as the r-jet of a holomorphic function $F\colon \mathbb{C}^n \to \mathbb{C}^m$. The identification we made is the one that identifies σ with the r-jet of $\Psi_{ji} \circ F$ as an element of $\mathcal{J}^r_{D_h,n,m} \cong \mathcal{J}^r_D(M, \mathbb{C}^m)_j$. This map does not depend on the local identification with $T^{*1,0}\mathbb{C}^n$ (because we can compose on F with the corresponding element of $Gl(n, \mathbb{C})$), and therefore it is globally defined (in all the base space because in the overlaps of the charts the definition has been seen to be compatible). This identifications $j^r\Psi_{ji}$ define an equivalence relation, that is, the cocycle condition holds – because it happens so in the integrable case– and thus give rise to a well defined locally trivial fiber bundle $\mathcal{J}^r_D(M, \mathbb{CP}^m)$.

Let $\phi\colon (M, J, D) \to \mathbb{CP}^m$. Its r-jet $j^r_D\phi$ is defined as follows: the charts of the projective space induce maps $\phi_i := \varphi_i^{-1} \circ \phi\colon M \to \mathbb{C}^m$. Using the trivial connection d in this trivial vector bundle, and using as usual the induced connection on $\bar{D}^{*1,0}$ we can define the corresponding symmetrized r-jet $j^r_D\phi_i$ (see definition 4.1). We must check $j^r_D\phi_j = j^r\Psi_{ji}(j^r_D\phi_i)$. More generally, instead of using a holomorphic diffeomorphism $\Psi_{ji}\colon \mathbb{C}^m \to \mathbb{C}^m$ we will consider any holomorphic map $H\colon \mathbb{C}^{m_1} \to \mathbb{C}^{m_2}$ that will give rise to a map $j^rH\colon \mathcal{J}^r_D(M, \mathbb{C}^{m_1}) \to \mathcal{J}^r_D(M, \mathbb{C}^{m_2})$, so that for a function $\phi\colon M \to \mathbb{C}^{m_1}$ the equation $j^r_D(H \circ \phi) = j^rH(j^r_D\phi)$ holds.

The proof uses induction on r. First we may assume $m_2 = 1$ and it is enough to check the equality for the component of order r of the r-jet. The computation of $j^1_D(H \circ \phi)$ is done by firstly taking in $\nabla(H \circ \phi)$ its projection over $\bar{D}^*$. Next the holomorphic component is singled out and then we keep its symmetrization. Thus,

$$\nabla(H \circ \phi) = \sum_{a=1}^{m_1} \frac{\partial H}{\partial z^a} \nabla\phi_a,$$

is the sum of partial derivatives of H multiplied by the components $\nabla\phi_a$ of $\nabla\phi$. The algebraic expression is exactly the same as the one for $j^1H(j^1_D\phi)$ except for the fact that in the latter the partial derivatives of H multiply the components $\nabla\phi_a$ of the 1-jet $j^1_D\phi$. In any case, taking $\nabla_D(H \circ \phi)$ amounts to substitute in the previous algebraic expression the factors $\nabla\phi_a$ by $\nabla_D\phi_a$.

Since H is holomorphic, $\partial(H\circ\phi)$ is equivalent to taking the component $\partial\phi_a$ of $\nabla_D\phi_a$ (in the algebraic expression we can consider partial derivatives of H in the variables $z, \bar{z}$, where the latter are vanishing). The symmetrization is not needed for 1-jets.

We proceed similarly to compute $j^2_D(H\circ\phi)$.

$$\nabla j^1_D(H\circ\phi) = \sum_{b,a=1}^{m_1} \frac{\partial^2 H}{\partial z^a \partial z^b}\nabla\phi_a \otimes \partial\phi_b + \sum_{c=1}^{m_1}\frac{\partial H}{\partial z^c}\nabla\partial\phi_c. \tag{5.8}$$

In this algebraic expression we have $(2,0)$ terms –those containing a second covariant derivative or equivalently a first partial derivative of H– and $(1,1)$ terms which contain a second partial derivative of H or the (tensor) product of two first covariant derivatives. Taking the component along D and then the holomorphic part does not alter the algebraic expression; we just write $\partial\phi_a$ instead of $\nabla\phi_a$ and $\partial^2\phi_c$ instead of $\nabla\partial\phi_c$. It is easily checked that the algebraic expression is the same as that of $j^2H(j^2_D\phi)$, being the only difference that in the latter we have $\partial^2_{\mathrm{sym}}\phi_c$, the symmetrization of the term $\partial^2\phi_c$ in the former. Thus, our problem reduces to showing that the symmetrization of 5.8 is the same algebraic expression but changing $\partial^2\phi_c$ by its symmetrization.

We observe that what we just said holds for any function H. Let x be the point in whose fiber we are working,. We define $H'\colon \mathbb{C}^{m_1}\to\mathbb{C}$ as the degree 2 homogeneous component of the Taylor expansion of H around x. It is clear that $j^2_D(H'\circ\phi)$ are the $(1,1)$ terms in 5.8. Thus, the algebraic expression of $j^2_D(H'\circ\phi)$ coincides with that of $j^2H'(j^2_D\phi)$. But in these case we have equality because the difference in the factors only occurs in $(2,0)$ terms. Since by hypothesis $j^2H'(j^2_D\phi)$ is symmetric, then $j^2_D(H'\circ\phi)$ is also symmetric. Thus, the $(1,1)$-terms in $j^2_D(H\circ\phi)$ are symmetric. Therefore, the symmetrization being a linear projector does not alter them. Now one checks that the symmetrization of each summand $\frac{\partial H}{\partial z^c}\nabla\partial\phi_c$ is exactly $\frac{\partial H}{\partial z^c}\partial^2_{\mathrm{sym}}\phi_c$.

By definition, to compute $j^rH(j^r_D\phi)$ after the local identification using A.H. coordinates we take F holomorphic whose usual r-jet coincides with $j^r_D\phi$, and define $j^rH(j^r_D\phi)$ to be $j^r_{D_h}(H\circ F)$. The summands are tensor products with factors $\partial^{r_i}F_{a_1}$, $\sum r_i = r$, multiplied by a partial derivative of F in the variables z_{a_i} of order the number of factors in the tensor products. By hypothesis we assume that when we substitute in these expression $\partial^{r_i}F_{a_1}$ by $\partial^{r_i}_{\mathrm{sym}}\phi_{a_i}$, we obtain $j^r_D(H\circ\phi)$. From that we deduce that the algebraic expression for $\nabla j^r_D(H\circ\phi)$ coincides with that of $j^{r+1}(H\circ F)\cong j^{r+1}H(j^{r+1}_D\phi)$. Taking the holomorphic part does not change it. In each summand of $\partial j^r_D(H\circ\phi)$ all the factors but at most one in the tensor product are of the form $\partial\partial^{r_i}_{\mathrm{sym}}\phi_{a_1}$, and hence already symmetric. We want to show that $\mathrm{sym}_r(\partial j^r_D(H\circ\phi)) = j^{r+1}H(j^{r+1}_D\phi)$. Since $\mathrm{sym}_r(j^{r+1}H(j^{r+1}_D\phi)) = j^{r+1}H(j^{r+1}_D\phi)$, proving that the symmetrization of each summand amounts to putting $\partial^{r_i}_{\mathrm{sym}}\phi_{a_i}$ instead of $\partial^{r_i}F_{a_1}$ and symmetrizing the resulting expression would be enough. But this is an elementary result concerning symmetric products.

Therefore, we conclude that the pseudo-holomorphic r-jet of a map to $\mathbb{CP}^m$ is well defined.

If M is an even dimensional a.c. manifold the definition of $\mathcal{J}^r(M, \mathbb{CP}^m)$ is the same (we just do not need to project the full derivative into the subspace $\bar{D}$). When we have a polarization G there is an analogous definition of the bundle of pseudo-holomorphic r-jets along G. Using the previous holomorphic charts of the projective space we consider the subbundles $\mathcal{J}^r_G(M, \mathbb{C}^m)_i := (\sum_{j=0}^r (\bar{G}^{*1,0})^{\odot j}) \otimes \mathbb{C}^m$, where $\mathcal{J}^r_G(M, \mathbb{C}^m)_i \subset \mathcal{J}^r(M, \mathbb{C}^m)_i$, using the splitting $G \oplus G^\perp = TM$. It is easily checked that the identifications $j^r\Psi_{ji} \colon \mathcal{J}^r(M, \mathbb{C}^m)_i \to \mathcal{J}^r(M, \mathbb{C}^m)_j$ preserve these subbundles, because the elements of these subbundles are characterized by being vanishing when they act over any vector of $G^\perp$; the algebraic expression that gives rise to the identification between $\mathcal{J}^r(M, \mathbb{C}^m)_i$ and $\mathcal{J}^r(M, \mathbb{C}^m)_j$ has this property.

The proof that shows that the $j^r\phi$ is well defined is exactly the same we gave for odd dimensional a.c. manifolds; a small modification shows that $j^r_G\phi$ is well defined (instead of keeping the component ∇_D of the odd dimensional case, we project over $\bar{G}^*$).

The next property to be checked is the existence of a submersion $j^r\pi \colon \mathcal{J}^r_D E^*_k \to \mathcal{J}^r_D(M, \mathbb{CP}^m)$ (resp. $j^r\pi \colon \mathcal{J}^r E^*_k \to \mathcal{J}^r(M, \mathbb{CP}^m)$ that restricts to a submersion $\mathcal{J}^r_G E^*_k \to \mathcal{J}^r_G(M, \mathbb{CP}^m)$), so that for any section τ_k of E_k, in the points where it does not vanish $j^r_D(\pi \circ \tau_k) = j^r\pi(j^r_D\tau_k)$ (resp. $j^r(\pi \circ \tau_k) = j^r\pi(j^r\tau_k)$ y $j^r_G(\pi \circ \tau_k) = j^r\pi(j^r_G\tau_k)$), i.e., the equations 5.5, 5.6 and 5.7 hold.

Going back to odd dimensional manifolds, we define $j^r\pi$ to have the same expression as in the integrable case. That means that we fix A.H. charts and a section of L_k to trivialize so that the r-jet σ in question is identified with the usual holomorphic r-jet in a point of a holomorphic function F. We compose with the appropriate chart φ_i^{-1} of the projective space and $j^r\pi(\sigma)$ to be the r-jet of $\varphi_i^{-1} \circ F$; the arguments that showed that the holomorphic r-jets are well defined –together with a small observation– also prove that $j^r\pi(\sigma)$ is well defined independently of the A.H. coordinates and of the chart of $\mathbb{CP}^m$ we used. The observation is that also that the map is independent of the trivialization of L_k, or in other words, of the connection ∇ we use in L_k, so we can work with d (and then we are exactly in the same situation that proved that the $\jmath^r_D\phi$is well defined). The reason is that $\varphi_i^{-1} \circ F$ is a section of $\mathbb{C}^m \otimes (L_k \otimes L_k^{-1})$ with connection $\nabla \otimes \nabla^{-1} = d$ (composing with a chart amounts to dividing by one of the coordinates). It is clear that the map is a submersion.

The equality $j^r_D(\pi \circ \tau_k) = j^r\pi(j^r_D\tau_k)$ holds because when we compose with a chart φ_i^{-1}, by definition $j^r_D(\pi \circ \tau_k)$ is $j^r_D(\varphi_i^{-1} \circ \pi \circ \tau_k)$. Also by definition $j^r\pi(j^r_D\tau_k)$ coincides with $j^r(\varphi_i^{-1} \circ \pi)(j^r_D\tau_k)$, and the equality between both expression has already been proved (recall that the definition of $j^r\pi$ does not depend on the connection in L_k, so we can trivialize it and use d).

To show that for even dimensional a.c. manifolds $j^r\pi \colon \mathcal{J}^r E^*_k \to \mathcal{J}^r(M, \mathbb{CP}^m)$ is a well defined submersion and that the equation 5.6 holds, the previous ideas work word by word. If we have a polarization G, the commutativity of the diagram 5.24 follows from the commutativity in the holomorphic case (because going through one of the sides we restrict the holomorphic function to the leaf $\mathbb{C}^g \times \{\cdot\}$ and then compose with π, and w.r.t. the other we compose and the restrict, and the result is the same). It is also clear that $j^r\pi\colon \mathcal{J}^r_G E^*_k \to \mathcal{J}^r_G(M, \mathbb{CP}^m)$ is a submersion.

Regarding point (2) of this definition-proposition, the fibers of $\mathcal{J}^r_D(M, \mathbb{CP}^m)$ (resp. $\mathcal{J}^r(M, \mathbb{CP}^m)$) admit a canonical holomorphic structure because using local identifications the fiber is some $\mathbb{C}^N$ and the change identifications $j^r\Psi_{ji}$ (the formulas are those of the holomorphic case).

$j^r\pi$ is fiberwise holomorphic for the same reason; in each fiber we have a map from some $\mathbb{C}^{N_1}$ to some $\mathbb{C}^N$ (after composing with a chart φ_i) whose formula is that of the integrable case which is trivially holomorphic.

To be able to say when a sequence of functions of $\mathcal{J}^r_D(M, \mathbb{CP}^m)$ is A.H. we need to introduce a metric and an a.c. structure in the total space of the r-jets. This can be done using a connection (for example out of the Levi-Civita connection associated to the Fubini-Study metric in the projective space and of the connection on T^*D). In our case we choose to do something different but equivalent. We just fix a system of *holomorphic charts* for $\mathbb{CP}^m$ so that on each charts we only work in a compact domain of $\mathbb{C}^m$, and such that for the change of coordinates for these regions we have uniform bounds in the family. For example, in $\mathbb{CP}^1$ we can take all the charts that are the result of removing a hyperplane (point), and work in the complement of a uniform ball of the point.

Each chart φ defines $\mathcal{J}^r_D(M, \mathbb{C}^m)_\varphi \subset \mathcal{J}^r_D(M, \mathbb{CP}^m)$ which is of the form $(\sum_{j=0}^r (\bar{D}^{*1,0})^{\odot j}) \otimes \mathbb{C}^m = \mathcal{J}^r_D\mathbb{C}^m$. In each of these vector bundles we introduce the metric and almost-complex structure induced by considering the sequence of trivial hermitian bundles $M \times \mathbb{C}^m \to M$ with trivial connection.

If we compose with one of these charts, say φ, it is straightforward that if τ_k is an A.H. sequence of sections of E_k, then where its projectivization ϕ_k is defined it defines an A.H. sequence of functions. It follows that $j^r_D\tau_k$ is an A.H. sequence of sections of $\mathcal{J}^r_D(M, \mathbb{C}^m)_\varphi = \mathcal{J}^r_D\mathbb{C}^m$. Notice that the notion is well defined because when we make a change of chart, the identification of $\mathcal{J}^r_D(M, \mathbb{C}^m)_\varphi$ with $\mathcal{J}^r_D(M, \mathbb{C}^m)_{\varphi'}$ is given by an A.H. map. Indeed, we just need to go to the models and realize that the almost-complex structure induced in $\mathcal{J}^r_{D_{h,n,m,}}$ by that of $\mathcal{J}^r_D(M, \mathbb{C}^m)_\varphi$ approximately coincides with the canonical one (and the metrics are comparable). Since the change of coordinates in $\mathbb{CP}^m$ induces a holomorphic map in $\mathcal{J}^r_{D_{h,n,m,}}$, we are done.

The situation is the same for even dimensional a.c. manifolds. For a given polarization, one checks using adapted charts to G that $j^r_G\phi_k$ is an A.H. sequence of sections of $\mathcal{J}^r(M, \mathbb{CP}^m)$ whose image lays in $\mathcal{J}^r_G(M, \mathbb{CP}^m)$. $\square$

Once we have defined the non-linear bundles of r-jets of maps to $\mathbb{CP}^m$ and indicated its relation with the bundles $\mathcal{J}^r_D E^*_k$ (resp. $\mathcal{J}^r E^*_k$, $\mathcal{J}^r_G E^*_k$), we want to pullback transversality problems for $j^r_D(\pi \circ \tau_k)$ to quasi-stratifications of $\mathcal{J}^r_D(M, \mathbb{CP}^m)$, to transversality problem for $j^r_D\tau_k$ to the pullback of the quasi-stratification to $\mathcal{J}^r_D E^*_k$. Being more precise, we need to define –at least for certain kinds of strata $\mathbb{P}S^a_k$ of $\mathcal{J}^r_D(M, \mathbb{CP}^m)$– the corresponding subsets of transversal holonomy $\Theta_{\mathbb{P}S^a_k}$. That will not be very difficult due to the fact that this non-linear bundles have been defined identifying bundles of the form $\mathcal{J}_D^r\mathbb{C}^m$. The purpose two define this subsets it twofold: on the one hand they are necessary to define the notion of quasi-stratification of $\mathcal{J}_D^r(M,\mathbb{CP}^m)$ (definition 5.23 and its extension to the non-integrable situation), and the important property is that if inside $S_k^a := j^r\pi^*\mathbb{P}S_k^a$ we define $\Theta_{S_k^a} := j^r\pi^*\Theta_{\mathbb{P}S_k^a}$, the latter does not belong to $\Theta_{S_k^a(C^D,C,c)}$ in the approximate sense, for certain constants. The consequence is that if we start from an appropriate Whitey stratification of $\mathbb{P}\mathcal{S}$, then if we consider its pullback $\mathcal{S}$ and add the strata Z_k, we obtain a Whitney quasi-stratification, up to checking that when the strata S_k^a approach Z_k they accumulate in the points of $Z_k - \Theta_{Z_k}$. The second purpose in the introduction of the subsets $\Theta_{\mathbb{P}S_k^a}$ is that they are used in the definition of the Thom-Boardman-Auroux quasi-stratification (though we will see that we will refine it to a genuine Whitney stratification of $\mathcal{J}_D^r(M,\mathbb{CP}^m)$ so that when it is pullbacked to $\mathcal{J}_D^rE_k$ the quasi-stratification condition in the boundary of each stratum (see definition 5.23 and its extension to the non-integrable situation) is only used for the closure of each strata in Z_k).

In the relative theory we start form an appropriate sequence of strata $\mathbb{P}S_k^G$ of $\mathcal{J}_G^r(M,\mathbb{CP}^m)$, and our purpose it to obtain transversality of $j_G^r(\pi\circ\tau_k)$ to them by getting transversality of $j_G^r\tau_k$ to $S_k^G := j^r\pi^*\mathbb{P}S_k^G \subset \mathcal{J}_G^rE_k^*$; again, this is done by making $j_G^r\tau_k$ uniformly transverse to $S_k := p_G^{r*}S_k^G \subset \mathcal{J}^rE_k^*$. Thus, we need to define also the subsets $\Theta_{\mathbb{P}S_k^G}$ and study its properties.

Definition-Proposition 5.25. *If $\mathbb{P}S_k$ is a sequence of strata of $\mathcal{J}_D^r(M,\mathbb{CP}^m)$ (resp. $\mathcal{J}^r(M,\mathbb{CP}^m)$), so that for a fixed choice of charts of $\mathbb{CP}^m$ and A.H. coordinates it is identified with a stratum $\mathbb{P}S$ of $\mathcal{J}_{D_h,n,m}^r$ (resp. $\mathcal{J}_{n,m}^r$) invariant under the action of $Gl(n,\mathbb{C})$, then we can give an obvious definition of the subsets $\Theta_{\mathbb{P}S_k}$ by asking them to coincide in the local identifications with $\Theta_{\mathbb{P}S}$ (defined in 5.16). Thus, for S_k –the pullback of $\mathbb{P}S_k$ to $\mathcal{J}_D^rE_k^*$ (resp. $\mathcal{J}^rE_k^*$)– we can define Θ_{S_k} to be the pullback of $\Theta_{\mathbb{P}S_k}$. Moreover, if we take A.H. trivializations of L_k and consider the submersion $\mathcal{J}_{D_h,n,m+1}^r - Z \to \mathcal{J}_{D_h,n,m}^r$ (resp. $\mathcal{J}_{n,m+1}^r - Z \to \mathcal{J}_{n,m}^r$) in which S_k is identified with a stratum S, then the image of Θ_{S_k} in the chart coincides with Θ_S (see definition 5.16). Finally, the points of $S_k - \Theta_{S_k}$ approximately do not belong to $\Theta_{S_k(C^D,C,c)}$ (resp. $\Theta_{S_k(C,c)}$), for certain constants.*

For the relative theory we assume that for a choice of A.H. charts adapted to G and holomorphic charts of the projective space, the sequence $\mathbb{P}S_k^G \subset \mathcal{J}_G^r(M,\mathbb{CP}^m)$ is identified with an stratum $\mathbb{P}S^G$ of $\mathcal{J}_{\mathbb{C}^g,n,m}^r = \mathcal{J}_{g,m}^r \times \mathbb{C}^{n-g}$, invariant under the action of $Gl(g,\mathbb{C})$. Then there is an obvious definition of $\Theta_{\mathbb{P}S_k^G}$ which in local identifications in nothing but $\Theta_{\mathbb{P}S^G}$. We then define the subset $\Theta_{S_k} \subset S_k \subset \mathcal{J}_D^rE_k^*$ by pulling back this subset to $\mathcal{J}_D^rE_k^*$ using either of the sides of the commutative diagram 5.24. Using the lower part of the diagram 5.24 we define $\mathbb{P}S_k := p_G^{r*}\mathbb{P}S_k^G$. It follows that $\Theta_{\mathbb{P}S_k}$ (as defined in the first paragraph of this definition-proposition) coincides with $p_G^{r*}\Theta_{\mathbb{P}S_k^G}$. Thus, we can apply the results of the preceding paragraph to conclude that the point of $S_k - \Theta_{S_k}$ approximately do not belong to $\Theta_{S_k(C,c)}$, for certain constants.

DEFINITION-PROOF. We want to define the subsets $\Theta_{\mathbb{P}S_k}$ y Θ_{S_k} and see th relation of the former with the subsets $\Theta_{S_k(C^D,C,c)}$ of S_k. Once we have fixed A.H. coordinates and holomorphic charts of the projective space, we have the bundles $\mathcal{J}^r_D(M,\mathbb{CP}^m)_i = \mathcal{J}^r_D\mathbb{C}^m$ for which the definition of $\Theta_{S_{k,i}}$ and its identification with Θ_{S_i} is obvious, because we are working with a sequence of trivial bundles with trivial connection and the strata are assumed to be $Gl(n,\mathbb{C})$-invariant. Thus, our only task reduces the showing that the subsets Θ_{S_i} glue well under the maps $j^r\Psi_{ji}$ (everything can be checked in the integrable case, due to the local identifications).

Let ψ be an r-jet in $\Theta_{\mathbb{P}S_i}$. Hence, we have a lift $\tilde{\psi}$ and a local representation α of the lift cutting S_i transversally in ψ. As we mentioned, regarding transversality the local representation is essentially unique. That means in particular that any other representation α' will also share the transversality property. By definition $\tilde{\psi}$ is the $(r+1)$-jet of a local holomorphic function F (independent of s_k if we want). Then $j^r_{D_h}F(0) = \psi$ and $d_{D_h}j^r_{D_h}F(0) = \partial j^r_{D_h}F(0) = j^{r+1}_{D_h}F(0) = \tilde{\psi}$. Therefore $j^{r+1}\Psi_{ji}\tilde{\psi}$ a lift of $j^r\Psi_{ji}\psi$ with local representation $j^r\Psi_{ji}j^r_{D_h}F = j^r_{D_h}(\Psi_{ij}\circ F)$, which is obviously transverse to $j^r\Psi_{ji}\mathbb{P}S_i = \mathbb{P}S_j$, because $j^r\Psi_{ji}$ is a diffeomorphism.

Since $\Theta_{\mathbb{P}S_k}$ is well defined, its pullback to $\mathcal{J}^r_D E^*_k$ is a well defined subset of S_k. Next, we want to show that the pullback of $\Theta_{\mathbb{P}S}$ by the submersion $j^r\pi\colon \mathcal{J}^r_{D_h,n,m+1} - Z \to \mathcal{J}^r_{D_h,n,m}$ is Θ_S, which follows easily form the previous arguments. Let σ be an r-jet projection over ϕ. Any lift $\tilde{\psi}$ is the projection of a lift $\tilde{\sigma}$ of σ. The latter admits a local representation $j^r_{D_h}H$, with H holomorphic, and it is clear that $j^r_{D_h}(\pi\circ H)$ is a local representation for $\tilde{\psi}$. Being $j^r\pi$ a submersion, transversality of $j^r_{D_h}H$ to S is equivalent to transversality of $j^r_{D_h}(\pi\circ H)$ to $\mathbb{P}S$.

We now prove that an r-jet $\sigma \in S_k - \Theta_{S_k}$ does not belong to $\Theta_{S_k(C^D,C,c)}$ in the approximate sense, for constants that essentially depend on the norm of the lifts $\tilde{\sigma}$ and on the constants associated to the basis $\nu_{k,x,I}$. σ does not belong to Θ_{S_k} if and only if $j^r\pi(\sigma) = \psi$ does not belong to $\Theta_{\mathbb{P}S_k}$. We know that for any lift $\tilde{\sigma}$ of σ, we can find local A.H. representations of the form $\alpha = j^r_D\xi_k + h_I j^r_D\tau^{\mathrm{ref}}_{k,x,I}$, where the h_I are degree 1 polynomials in the coordinates $x^1_k, y^1_k, ..., x^n_k, y^n_k$ of size at most $O(c_k^{-1/2})$ (lemma 5.18). Thus, $j^r\pi(\alpha) - j^r_D(\pi\circ\xi_k)$ and its derivative along D have size $O(c_k^{-1/2})$, which implies that $j^r\pi(\alpha)$ is at distance $O(c_k^{-1/2})$ of defining a lift for ϕ. It follows that the minimum angle of intersection between $j^r\pi(\alpha)$ and $\mathbb{P}S_k$ in ϕ is at most of order $O(c_k^{-1/2})$, and thus the same happens for α and S_k.

In the relative theory for each chart φ_i^{-1} we have a sequence of vector subbundles $\mathcal{J}^r_G(M,\mathbb{C}^m)_i \hookrightarrow \mathcal{J}^r(M,\mathbb{C}^m)_i$. The subbundle inherits a connection form that of $\mathcal{J}^r(M,\mathbb{C}^m)_i$ (it comes from the Levi-Civita connection and the metric retraction for $\bar{G}^*$). We can slightly modify the trivialization of $\bar{G}^{*1,0}$ coming from A.H. coordinates adapted to G (the perturbation of size $O(c_k^{-1/2})$), so that the connection form vanishes at the origin. If the stratification is $Gl(g,\mathbb{C})$-invariant, then there is an obvious definition of the subset $\Theta_{\mathbb{P}S^G_{k,i}}$ which under the local identification is nothing but $\Theta_{\mathbb{P}S^G_i}$ (because for each $\mathcal{J}^r_G(M,\mathbb{C}^m)_i$ the action of $Gl(g,\mathbb{C})$ kills the connection form at the origin). Using the identification with holomorphic foliated r-jets one proves that the holonomic transverse subsets are well defined. For $\mathbb{P}S_k := p^{r*}_G\mathbb{P}S^G_k$, there is already a definition of the subsets $\Theta_{\mathbb{P}S_k}$. Using the local identification given by charts adapted to G, the pseudo-holomorphic r-jets along G go to $\mathcal{J}^r_{\mathbb{C}^g,n,m} = \mathcal{J}^r_{g,m} \times \mathbb{C}^{n-g}$, and the projection $\mathcal{J}^r_{n,m}$ over the subbundle is the trivial projection suppressing some coordinates (those corresponding to elements of the base $\mu_{k,x,I}$, where I is not of the form I_g). It follows that $\Theta_{\mathbb{P}S}$ is the pullback of $\Theta_{\mathbb{P}S^G}$, because the projection $\mathcal{J}^r_{n,m} \to \mathcal{J}^r_{\mathbb{C}^g,n,m}$ is induced by the holomorphic submersion $\mathbb{C}^n \to \mathbb{C}^g$, and then the same ideas that proved that $\Theta_S \subset \mathcal{J}^r_{n,m+1}$ coincided with the pullback of $\Theta_{\mathbb{P}S} \subset \mathcal{J}^r_{n,m}$ apply. Going through the lower side of the commutative diagram 5.24 we obtain a local description of Θ_{S_k} (defined to be the pullback of $\mathbb{P}S^G_k$ using either of the sides) as $j^r\pi^*\Theta_{\mathbb{P}S_k}$. Hence, applying the first part of this proposition the points of $S_k - \Theta_{S_k}$ approximately do not belong to $\Theta_{S_k(C,c)}$, for certain constants.

□

The Thom-Boardman-Auroux is nothing but the pullback to $\mathcal{J}^r_D E^*_k$ by $j^r\pi$ of the analog of the Thom-Boardmann stratification of $\mathcal{J}^r_D(M,\mathbb{CP}^m)$ (see for example [**1, 7**]), together with the strata Z_k. The definition is the one given for even dimensional a.c. manifolds by D. Auroux in [**4**].

Given $\sigma \in \mathcal{J}^r_D E^*_k$, let us denote by $\phi = (\phi_0, ..., \phi_r)$ its image in $\mathcal{J}^r_D(M,\mathbb{CP}^m)$. Let us define

$$\Sigma_{k,i} = \{\sigma \in \mathcal{J}^r_D E^*_k | \dim_{\mathbb{C}} \ker \phi_1 = i\}$$

If $\max(0, n-m) < i \leq n$, the strata $\Sigma_{k,i}$ are smooth submanifolds whose boundary is the union of $\cup_{j>i}\Sigma_{k,j}$ and of a subset of $Z_k - \Theta_{Z_k}$.

It is clear that $\Sigma_{k,i}$ is defined by conditions on ϕ. Thus, the strata are pullback of constant and holomorphic strata $\mathbb{P}\Sigma_{k,i}$, and the given definition of their closure is easy to check (also having into account the description of Θ_{Z_k} described in example 5.22).

For $r \geq 2$, $\Theta_{\Sigma_{k,i}}$ is the subset of r-jets $\sigma = (\sigma_0, ..., \sigma_r) \in \Sigma_i$ so that

$$\Xi_{k,i;\sigma} = \{u \in D^{1,0}, (i_u\sigma, 0) \in T_\sigma \Sigma_{k,i}\}$$

has the expected codimension in $D^{1,0}$, which is the codimension of $\Sigma_{k,i}$ in $\mathcal{J}^r_D E_k$.

Indeed, we can work with the projection and observe that $\Theta_{\mathbb{P}\Sigma_{k,i}}$ are exactly those points of $\mathbb{P}\Sigma_{k,i}$ which have a lift with a transverse local representation. Since the term that we add to the r-jet to define the lift is of order $r+1 > 2$, the transversality of the local representation does not depend on the lift, that can be chosen to have vanishing $r+1$-order component. Let us define

$$\mathbb{P}\Xi_{k,i;\sigma} = \{u \in D^{1,0}, (i_u\phi, 0) \in T_\phi \mathbb{P}\Sigma_{k,i}\},$$

One checks that $\Theta_{\mathbb{P}\Sigma_{k,i}}$ are those ϕ for which $\mathbb{P}\Xi_{k,i;\sigma}$ has the codimension of $\mathbb{P}\Sigma_{k,i}$ in $\mathcal{J}^r_D(M,\mathbb{CP}^m)$. It is easy to see using the ideas of proposition 5.25 that $\Theta_{\Sigma_{k,i}}$ –the pullback of $\Theta_{\mathbb{P}\Sigma_{k,i}}$– is the subset previously described.

If $p+1 \le r$, we define inductively

$$\Sigma_{k,i_1,\dots,i_p,i_{p+1}} = \{\sigma \in \Theta_{k,i_1,\dots,i_p}, \dim\ker(\phi_1 \cap \Xi_{k,i_1,\dots,i_p;\sigma}) = i_{p+1}\},$$

with

$$\Xi_{k,I;\sigma} = \{u \in D^{1,0}, (i_u\sigma, 0) \in T_\sigma \Sigma_{k,I}\}.$$

As in the previous case, we define $\Theta_{\Sigma_{k,I}}$ as the points such that the codimension of $\Xi_{k,I;\sigma}$ in $D^{1,0}$ is the same as the codimension of $\Sigma_{k,I}$ in $\mathcal{J}^r_D E_k$.

If $i_1 \ge \cdots \ge i_{p+1} \ge 1$, $\Sigma_{k,i_1,\dots,i_{p+1}}$ is a smooth submanifold (constant and holomorphic) whose closure in $\Sigma_{k,i_1,\dots,i_p}$ is the union of the $\Sigma_{k,i_1,\dots,i_p,j}, j > i_{p+1}$, and a subset of $\Sigma_{k,i_1,\dots,i_p} - \Theta_{k,i_1,\dots,i_p}$ [**7**]. The problem is that for large values of r, n, m, the closure of the strata is hard to understand, and what we have defined, once Z_k has been added, might very well not be a Whitney quasi-stratification. For low values of r, n, m we have a Whitney quasi-stratification of $\mathcal{J}^r_D E_k$, because it comes from a Whitney stratification of $\mathcal{J}^r_D(M, \mathbb{CP}^m)$, and because the strata do not accumulate in points of Θ_{Z_k}.

Anyhow, we can use the results of Mather [**39**] to refine *the stratification of* $\mathcal{J}^r_D(M, \mathbb{CP}^m)$ (which is constant, holomorphic and $Gl(n,\mathbb{C}) \times \mathcal{H}^r_m$-invariant), so that locally (after the identifications we obtain a constant, holomorphic Whitney stratification invariant under the action of $Gl(n,\mathbb{C}) \times \mathcal{H}^r_m$ on each $\mathcal{J}^r_{D_h,n,m}$, and such that the submanifolds $\mathbb{P}\Sigma_{k,I}$ are unions of strata of the refinement. A consequence of the mentioned invariance is that the refinements –defined locally using the identifications provided by A.H. coordinates and holomorphic charts in $\mathbb{CP}^m$– glue well under the identifications $j^r\Psi_{ji}$ defining a global refinement, which is indeed independent of the choices. Thus, its pullback is a finite Whitney stratification of $\mathcal{J}^r_D E^*_k$ and such that the $\Sigma_{k,I}$ are union of strata. It is important to notice that since all the strata are contained in the closure of $\Sigma_{k,\max(0,n-m)+1}$, they accumulate near Z_k in points of $Z_k - \Theta_{Z_k}$. Therefore, by adding Z_k we obtain a quasi-stratification of $\mathcal{J}^r_D E_k$.

If we have a polarization, we use exactly the same definitions but in the subbundle $\mathcal{J}^r_G E_k$ instead of $\mathcal{J}^r_G(M, \mathbb{CP}^m)$; the result is a stratification $\mathbb{P}\mathcal{S}^G$ of $\mathcal{J}^r_G(M, \mathbb{CP}^m)$ that locally –given the identification of $\mathcal{J}^r_{\mathbb{C}^g,n,m}$ with $\mathcal{J}^r_{g,m} \times \mathbb{C}^{n-g}$– coincides with the corresponding Thom-Boardman stratification of $\mathcal{J}^r_{g,m}$ multiplied by $\mathbb{C}^{n-g}$ (this is exactly what happens in the odd dimensional situation, where we get a 1-parametric version of the even dimensional model). Using the lower part of the commutative diagram, the pullback of this stratification $\mathbb{P}\mathcal{S}$ is locally multiplying by the remaining coordinates of $\mathcal{J}^r_{n,m}$, because the submersion $\mathcal{J}^r_{n,m} \to \mathcal{J}^r_{\mathbb{C}^g,n,m}$ amounts to suppress certain coordinates. Thus, to refine $\mathcal{S}$, the pullback of $\mathcal{S}^G$ to $\mathcal{J}^r E^*_k$, we first locally refine $\mathbb{P}\mathcal{S}^G$ (this is made by refining one fiber of $\mathbb{P}\mathcal{S}^G$ en $\mathcal{J}^r_{g,m} \times \mathbb{C}^{n-g}$). The local refinements of $\mathbb{P}\mathcal{S}^G$ glue well, because they are $Gl(g,\mathbb{C}) \times \mathcal{H}^r_m$-invariant and thus define a sequence of Whitney stratifications $\mathcal{J}^r_G(M, \mathbb{CP}^m)$ which does not depend neither on the A.H. coordinates adapted to G nor in the chosen holomorphic charts of $\mathbb{CP}^m$. Its pullback to $\mathcal{J}^r_G E^*_k$ refines $\mathcal{S}^G$ to a sequence of Whitney stratifications. Finally, its pullback by $p^r_G \colon \mathcal{J}^r E^*_k \to \mathcal{J}^r_G E^*_k$ is another sequence of Whitney stratifications refining $\mathcal{S}$. Once we add Z_k (also the pullback of $Z_k^G \subset \mathcal{J}_G^r E_k$), we obtain a sequence of Whitney quasi-stratifications of $\mathcal{J}^r E_k$(the local descriptions imply that the strata only accumulate in points of $Z_k - \Theta_{Z_k} \subset Z_k$).

Definition 5.26. *(see [**5**]). Given (M, D, J, g) (resp. (M, J, G, g)) and a very ample sequence of hermitian line bundles L_k, let us denote $\mathbb{C}^{m+1} \otimes L_k$ by E_k. The Thom-Boardman-Auroux (T-B-A) quasi-stratification of $\mathcal{J}_D^r E_k$ (resp. $\mathcal{J}^r E_k$, $\mathcal{J}_G^r E_k$) is the quasi-stratification given by the submanifolds $Z_k \subset \mathcal{J}_D^r E_k$ (resp. $Z_k \subset \mathcal{J}^r E_k$, $Z_k^G \subset \mathcal{J}_G^r E_k$) and a refinement as described of the analog of the Thom-Boardmann stratification of $\mathcal{J}_D^r E_k^*$ (resp. of $\mathcal{J}^r E_k^*$, $\mathcal{J}_G^r E_k^*$).*

We say that an A.H. sequence τ_k A.H. of sections of $E_k \to (M, D, J, g)$ (resp. $E_k \to (M, J, g)$) is r-generic if it is uniformly transverse to the Thom-Boardman-Auroux quasi-stratification of $\mathcal{J}_D^r E_k$. In such situation we will speak about r-genericity of ϕ_k, the sequence of projectivizations of τ_k.

Lemma 5.27. *The Thom-Boardman-Auroux quasi-stratification of $\mathcal{J}_D^r E_k$ (resp. $\mathcal{J}^r E_k$) is approximately holomorphic (and also finite and Whitney).*

In the relative case the pullback to $\mathcal{J}^r E_k$ of the T-B-A quasi-stratification of $\mathcal{J}_G^r E_k$ is also an approximately holomorphic quasi-stratification.

Proof. The description of the closure of the strata inside Z_k implies that the quasi-stratification condition holds. We observe that for low values of r, n, m for which we do not need to refine to obtain Whitney stratification in $\mathcal{J}_D^r(M, \mathbb{CP}^m)$, its pullback is a quasi-stratification of $\mathcal{J}_D^r E_k$, once the Z_k have been added.

The delicate point is checking that the strata are A.H. (for the modified connection). First we study the sequence Z_k. Though for this sequence the approximate holomorphicity is obvious, we will give a proof that works for other sequences of strata.

Indeed, $Z_k \subset E_k$ is obviously an A.H. sequence of E_k. We will see that the natural projection $\pi_{r-h}^r \colon \mathcal{J}_D^r E_k \to \mathcal{J}_D^{r-h} E_k$ is A.H. Thus, the pullback of an A.H. sequence of strata of $\mathcal{J}_D^{r-h} E_k$ will define an a.H. sequence of strata of $\mathcal{J}_D^r E_k$. In particular, $Z_k \subset \mathcal{J}_D^r E_k$ will be an A.H. sequence because is the pullback of the sequence of $\mathbf{0}$ sections of E_k.

The projection π_{r-h}^r is clearly A.H. because locally the choice of A.H. coordinates and of a basis made of reference sections $j_D^r \tau_{k,x,I}^{\mathrm{ref}}$, gives rise to A.H. coordinates $z_k^1, ..., z_k^n, u_k^I, s_k$ (I is one of $(n+1)$-tuples introduced at the beginning of section 4) in the total space of the bundles. Since the image of $j_D^r \tau_{k,x,I}^{\mathrm{ref}}$ is $j_D^{r-h} \tau_{k,x,I}^{\mathrm{ref}}$ –which is A.H. (and hence will be given by A.H. functions $v_k^{I'}(z_k, s_k)$, for a basis of sections $j_D^r \tau_{k,x,I'}^{\mathrm{ref}}$, where I' is an appropriate $(n+1)$-tuple)– and the projection is fiberwise complex-linear, the projection turns out to be A.H. (again we also have the problem of the choice of s_k as "vertical" coordinate of the total space, but the comment of the proof of lemma 5.14 also applies).

We want to do something similar with the strata Σ_I and the projection $j^r \pi \colon \mathcal{J}_D^r E_k^* \to \mathcal{J}_D^r(M, \mathbb{CP}^m)$.

We have similar properties because the image of a trivialization $j^r_D\tau^{\mathrm{ref}}_{k,x,I}$ is $j^r_D(\pi\circ\tau^{\mathrm{ref}}_{k,x,I})$, also A.H. Equally, the map is fiberwise holomorphic (definition-proposition 5.24), but the difference is the non-linearly of the restriction to the fibers.

We adopt a different strategy. The approximate holomorphicity of the map to order 0 (the size of the norm of the $(1,0)$ component of the differential) is obvious. If both structures were integrable, checking the holomorphicity of this differential wold be enough. We can locally find (in the product of a ball of g_k-radius $O(1)$ in the ball times a ball in the fiber of radius $O(1)$) new distributions and almost-complex structures such that they approximately coincide with the initial ones, they are integrable and the map $j^r\pi$ is holomorphic for them.

We just take Darboux charts for L_k and substitute D, J by D_h, J_0. The result of proposition Los 4.6 in the integrable case (and for curvature with trivial derivative, as is the case in Darboux coordinates) imply that the perturbations of the connection define a new almost-complex structure $\bar{J}$ in the total space of $\mathcal{J}^r_{D_h,n,m}$ which is integrable. The integrability for $\bar{J}_0$ in $\mathcal{J}^r_{D_h}(\mathbb{C}^n\times\mathbb{R},\mathbb{CP}^m)$ is obvious. Recall that after the local identification of the jets with $\mathcal{J}^r_{D_h,n,m+1}$ and $\mathcal{J}^r_{D_h,n,m}$, it has been checked that the definition of $j^r\pi$ matches that of the holomorphic situation. For this map the holomorphicity is clear because it is fiberwise holomorphic and sends "enough" holomorphic sections of $\mathcal{J}^r_{D_h,n,m+1}$ to holomorphic sections of $\mathcal{J}^r_{D_h,n,m}$. To be more precise, for any point $\sigma\in\mathcal{J}^r_{D_h,n,m+1}$, and any vector v in its tangent space not tangent to the fiber, we can find a holomorphic section F whose r-jet in x is σ and such that the tangent space to its graph contains v. Since $j^r\pi(j^r_{D_h}F)=j^r_{D_h}(\pi\circ F)$ is also a holomorphic section, we deduce that $j^r\pi_*(\bar{J}v)=\bar{J}_0(j^r\pi_*(v))$.

It is easy to check that in the domain of these local models the new distributions and A.H. structures approximately coincide with the original ones.

Exactly the same proof shows that $j^r\pi\colon\mathcal{J}^rE^*_k\to\mathcal{J}^r(M,\mathbb{CP}^m)$ is A.H. In the relative case, and for a sequence of strata $\mathbb{P}S_k$ fulfilling the conditions of proposition 5.25, the approximate holomorphicity of $p^{r*}_G j^r\pi^*S_k$ follows from the commutativity of the diagram 5.24, and from the approximate holomorphicity of $j^r\pi\colon\mathcal{J}^rE^*_k\to\mathcal{J}^r(M,\mathbb{CP}^m)$ and of $p^r_G\colon\mathcal{J}^r(M,\mathbb{CP}^m)\to\mathcal{J}^r_G(M,\mathbb{CP}^m)$ (for the latter is follows from the J-complex linearity of the map $T^{*1,0}M\to\bar{G}^{*1,0}$).

□

Remark 5.28: Now we can be more precise about the way in which the relative theory is to be applied. If we work with the symplectization (M,D,J,g), or even with (M,ω) a symplectic manifold with N (resp. (Q,D)) a symplectic submanifold (resp. calibrated submanifold), G is a local J-complex distribution extending TN (resp. D), and we start from an A.H. sequences of sections χ_k of $E_k=\mathbb{C}^{m+!}\otimes L_k$, we know from subsection 2.3 that the restriction of χ_k to N (resp. Q) is an A.H. sequence of sections. Thus, we can projectivize it to obtain $\phi_{k|N}$ (resp. $\phi_{k|Q}$) an A.H. sequence of maps to $\mathbb{CP}^m$, whose r-genericity is a transversality problem for $j^r(\phi_{k|N})$ (resp. $j^r_D(\phi_{k|Q})$), which a section of $\mathcal{J}^r(N, \mathbb{CP}^m)$ (resp. $\mathcal{J}^r_D(Q, \mathbb{CP}^m)$). Once we take holomorphic charts of the projective space, we obtain a chart for the non-linear bundle of r-jets of the form $\mathcal{J}^r(N, \mathbb{C}^m)$ (resp. $\mathcal{J}^r_D(Q, \mathbb{C}^m)$). Since this linear bundle has no curvature (the connection is trivial), the r-jet approximately coincides with $\nabla^r(\phi_{k|N})$ (resp. $\nabla^r_D(\phi_{k|Q})$). Similarly, $j^r_G\phi_k$, the r-jet along G –defined in the points of M where the distribution G is defined– approximately coincides with $\nabla^r_G\phi_k$. Exactly in the same way as we did in subsection 2.3, and having into account that the connection is trivial– one checks that $(\nabla^r_G\phi_k)_{|N} \approxeq \nabla^r(\phi_{k|N})$ (resp. $(\nabla^r_G\phi_k)_{|Q} \approxeq \nabla^r_D(\phi_{k|Q})$). Therefore the r-jet along G $j^r_G\phi_k$ approximately extends the r-jet of the restriction ofϕ_k. The last observation in order is that the identification of $\mathcal{J}^r(N, \mathbb{CP}^m)$ (resp. $\mathcal{J}^r_D(Q, \mathbb{CP}^m)$) with $\mathcal{J}^r_G(M, \mathbb{CP}^m)$ in the points of the submanifold coming form the identification $T^*N \cong \bar{G}^*$ (resp. $\bar{D}^* \cong \bar{G}^*$) given by the metric, preserves the T-B-A quasi-stratifications. Thus, if we obtain uniform transversality of $j^r_G\phi_k$ to the T-B-A quasi-stratification of $\mathcal{J}^r_G(M, \mathbb{CP}^m)$ in the points of the subvariety, we also obtain uniform transversality for the r-jet of the restriction to the corresponding T-B-A quasi-stratification.

6. The main theorem

We aim to perturb A.H. sections of E_k so that its r-jets are transverse to an A.H. quasi-stratification of $\mathcal{J}^r_D E_k$. It will possible to control the size of the perturbation along D to any fixed finite order. We introduce a new notation: we say that a sequence τ_k is $C^{\geq h}$-A.H.(C^D) if the sequence is A.H., i.e., we have bounds for the derivatives of all the orders, and the constant C^D controls the norm of the sections and of the derivatives along D up to order h. We will also speak of $C^{\geq h}$-A.H. without explicitly mentioning the bound.

Theorem 6.1. *Let E_k be a sequence of locally splittable very ample hermitian bundles and $\mathcal{S} = (S^a_k)_{a \in A_k}$ a $C^{\geq h}$-A.H. sequence of finite Whitney quasi-stratifications of $\mathcal{J}^r_D E_k$ ($h \geq 2$). Let δ be a positive constant. Then a constant $\eta > 0$ exists such that for any $C^{\geq r+h}$-A.H.(C^D) sequence τ_k of E_k, it is possible to find an A.H. sequence σ_k of E_k so that for every k bigger than some K,*

(1) $|\nabla^j_D(\tau_k - \sigma_k)|_{g_k} < \delta, j = 0, ..., r + h.$
(2) *$j^r_D\sigma_k$ is η-transverse to $\mathcal{S}$.*

In the statement of the theorem the natural number K and the constants $\tilde{C}_j$ controlling the approximate holomorphicity of σ_k will depend on the whole sequence of bounds (C^D_j, C_j) of τ_k. The constant η will not be in general independent of a finite number of the (C^D_j, C_j)

It is important to make sure that all our constructions do not depend on the size of the constants C_j of the sequences to which they will be applied. The reason is that these bounds will change when we add a new local perturbation (the final perturbation will be the result of adding a huge number of local perturbations). The only dependence that we allow is on the choice of constant K from which the assertions start to hold: for example to go from bounds along D_h in a chart to bounds along D *we need bounds of order* $O(1)$ *in the full derivatives* no matter how big its value might be (we avoid the effect of its size by increasing k); the same happens when we want to make the holomorphic component arbitrarily small (it is of size $C_r c_k^{-1/2}$). We will also make some of the constructions depend on the bound C_{r+2} of the initial section τ_k to be perturbed.

We also have an analogous transversality result for even dimensional a.c. manifolds with polarization along compact subvarieties. The proof is a modification of that of D. Auroux in [**4**] together with the local transversality result to submanifolds of J. P. Mohsen [**43**]. Anyway, we state the corresponding theorem.

Theorem 6.2. *Let E_k be a very ample sequence of locally splittable hermitian bundles over (M, J, G, g), and let Q be a compact submanifold of M. Let us consider $\mathcal{S} = (S_k^a)_{a\in A_k}$ a C^h-A.H. sequence of finite Whitney quasi-stratifications of $\mathcal{J}^r E_k$ whose strata are pullback of strata of $\mathcal{J}_G^r E_k$ ($h \geq 2$). Let δ be a positive constant. Then a constant $\eta > 0$ and a natural number K exist such that for any C^{r+h}-A.H.(C) sequence τ_k of E_k, it is possible to find a C^{r+h}-A.H. sequence σ_k of E_k so that for any k bigger than K,*

1. $|\nabla^j(\tau_k - \sigma_k)|_{g_k} < \delta, j = 0, ..., r+h$ *(σ_k es C^{r+h}-A.H.(δ)).*
2. $j_G^r \sigma_k$ *is η-transverse to $\mathcal{S}$ along Q (along the directions of TQ in the points of Q).*

Observe that in the relative case we do not need to work with sequences all whose derivatives are controlled, and both K and η depend only on C. The odd dimensional situation is different because the quality of the perturbations is much worse due to the non-integrability of D; basically the derivatives along the directions of D (up to some finite order h) will be arbitrarily small only if we have control for the full derivative of all the orders and k is chosen to be very large. The exact reason will become clear along the proof.

Theorem 6.2 is not quite the result we look for, but almost; our goal is a uniform transversality theorem to sequences of "quasi-stratifications" $\mathcal{S}^G$ of $\mathcal{J}_G^r E_k$ (what is asked to be a quasi-stratification is its pullback $\mathcal{S} \subset \mathcal{J}^r E_k$). The reason to work in the bundle $\mathcal{J}^r E_k$ is avoiding the task of checking the ampleness of the bundle $\mathcal{J}_G^r E_k$.

If we have a stratification $\mathcal{S}_k^G$ in $\mathcal{J}_G^r E_k$ the way to define estimated transversality w.r.t. a distribution TQ of TM is the obvious one; we just note that the parallel transport is made using the metric in the total space of $\mathcal{J}_G^r E_k$ induced by that of $\mathcal{J}^r E_k$; using local trivializations of E_k, A.H. coordinates adapted to G and the correspondent euclidean metric, we obtain local identifications of $\mathcal{J}^r E_k \cong \mathcal{J}_{n,m+1}^r$ and $\mathcal{J}_G^r E_k \cong \mathcal{J}_{\mathbb{C}^g,n,m+1}^r$ such that $\mathcal{J}_{n,m+1}^r = \mathcal{J}_{g,m+1}^r \times \mathbb{C}^u$; what is more, we have seen that the stratification $\mathcal{S}$ of $\mathcal{J}_{n,m+1}^r$ also is the product of $\mathcal{S}^G$ times $\mathbb{C}^u$. The euclidean metric is comparable to $\hat{g}_k$ and its restriction (also euclidean) is also comparable to the induced one in $\mathcal{J}_G^r E_k$; for the euclidean metric estimated transversality of $j^r_G\tau_k$ to $\mathcal{S}$ is exactly the same as estimated transversality of $j^r_G\tau_k$ to $\mathcal{S}^G$. The consequence is the following result:

Theorem 6.3. *Let E_k be a very ample sequence of locally splittable hermitian bundles over (M, J, G, g), and let Q be a compact submanifold of M. Let us consider $\mathcal{S} = (S^a_k)_{a\in A_k}$ a C^h-A.H. sequence of finite Whitney quasi-stratifications of $\mathcal{J}^r E_k$ whose strata are pullback of strata of $\mathcal{J}^r_G E_k$ ($h \geq 2$). Let δ be a positive constant. Then a constant $\eta > 0$ and a natural number K exist such that for any C^{r+h}-A.H.(C) sequence τ_k of E_k, it is possible to find a C^{r+h}-A.H. sequence σ_k of E_k so that for any k bigger than K,*

1. *$|\nabla^j(\tau_k - \sigma_k)|_{g_k} < \delta, j = 0, ..., r+h$ (σ_k es C^{r+h}-A.H.(δ)).*
2. *$j^r_G\sigma_k$ is η-transverse to $\mathcal{S}^G$ along Q (along the directions of TQ in the points of Q).*

This corollary has an important consequence. Let us start from (M, w) symplectic with (Q, D) compact calibrated submanifold and G a J-complex polarization extending D (and exactly the same if what we have a symplectic submanifold), and suppose that $\mathcal{S}^G$ in the points of Q comes from a stratification $\mathcal{S}^D$ of $\mathcal{J}^r_D E_k \to (Q, D)$ through the identification $\mathcal{J}^r_D E_k \cong \mathcal{J}^r_G E_k$ in the points of Q (the metric identifies T^*Q with a subset of T^*M and $\bar{D}^*$ goes to $\bar{G}^*$). One checks that estimated transversality to $\mathcal{S}^D$ (that requires the use of the induced metric on Q) is comparable (over Q) to the already given definition of estimated transversality to $\mathcal{S}^G$ (using A.H. coordinates adapted to G that rectify Q and induce A.H. coordinates on it). Thus, we deduce uniform transversality of $(j^r_G\tau_k)_{|Q}$ to $\mathcal{S}^D$. When there is no curvature (for example when we work with the projectivizations) or when $r = 0$ –circumstances that will occur in all our applications– there is an approximate identification between $(j^r_G\tau_k)_{|Q}$ and $j^r_D(\tau_{k|Q})$, which implies uniform transversality to $\mathcal{S}^D$ of the latter, something which is the final objective of the relative theory: constructing A.H. sequences of sections whose restriction to the submanifold has good uniform transversality properties (and possible the sequence itself inside M).

6.1. Proof of the main theorem. The proof follows the same pattern as that of D. Auroux in [**4**] for the even dimensional case, but with certain technical complications.

Definition 6.4. *A family of properties $\mathcal{P}_k(\eta, x)_{x\in M, \eta>0, k>>0}$ of sections of E_k is called local C^{r+1}-open if given a sequence τ_k C^{r+h}-A.H. verifying property $\mathcal{P}(\eta, x)$ (h will depend on the property in question $\mathcal{P}$), for any C^{r+h}-A.H. sequence of sections χ_k such that $|\nabla^j_D(\tau_k - \chi_k)|_{g_k} \leq \epsilon$, $j = 0, ..., r+1$, χ_k verifies $\mathcal{P}(\eta - L\epsilon, x)$ for every k bigger than some $K(C^R)$, where the constant $L > 0$ is independent of η, ϵ, x and of the sequence χ_k.*

We point out that in the previous definition only the derivatives along D are taken into account.

Let us fix a sequence of strata S^b_k of $\mathcal{J}^r_D E_k$. A section τ_k verifies $\mathcal{P}_k(\eta, x)$ if either $j^r_D\tau_k(x)$ is at distance of ∂S^b_k smaller than $N_1\eta$ and if it cuts the stratum it does it with minimum angle $\angle_m(T_D S_k^b, T_D \tau_k) \geq \eta$, or it remains at distance of ∂S_k^b bigger than $N_2 \eta$ and it is η-transverse to S_k^b. The constants $N_1 > N_2 > 0$ –most likely very large– *depend on* η. The meaning of the constant N_1 is that in points whose distance at the boundary of the stratum is close to $N_1 \eta$, the radius of the ball where the local η. N_2 can chosen to be bigger than $N_1 - 1$ and it simply allows us to have an overlap region for both notions of estimated transversality. We also require η to be small enough compared with the constants (C^D, C) of τ_k.

Uniform transversality w.r.t. a sequence of strata S_k^b is defined to be the existence of some $\eta > 0$ such that $\mathcal{P}_k(x, \eta)$ holds for every x and $k >> 0$. If $\mathcal{P}_k(x, \eta)$ –as defined for a sequence of strata S_k^b of an stratification of $\mathcal{J}_D^r E_k$– is shown to be a local C^{r+1}-open (or just open) property, if we perform perturbations of size along D a fraction of η, we will still have for the new sequence, say, $\frac{\eta}{2}$-transversality w.r.t. S_k^b in all the points of M.

The strata of $\mathcal{S}_k$ (for each k) are reordered in such a way that for any index b, $\partial S_k^b \subset \cup_{a<b} S_k^a$. To show that $\mathcal{P}_k(x, \eta)$ w.r.t. S_k^b is an open property it is necessary to assume that $\mathcal{P}_k(x, \alpha)$ holds for all the preceding indices, where α is an appropriate multiple of η. We need to assume this property in order to prove the openness in points close to the boundary of S_k^b.

We note that once uniform transversality w.r.t. the preceding indices is assumed, the constructions (essentially the amount of transversality to be obtained) will depend on the constants C of τ_k (it actually depend on the size of the first covariant derivative of $j_D^r \tau_k$ which depends on C_{r+2}). As we mentioned, once we have achieved uniform transversality to a sequence of strata, we will not be able to determine the constants $\tilde{C}_j$ of the perturbed section. If the number of indices is greater or equal than 2 and having into account that the amount of transversality obtained depends on the constant $\tilde{C}_{r+2}$, we will not be able to determine it (we will see that the exception occurs for 0-jets).

Lemma 6.5. *Let τ_k be a $C^{\geq r+h}$-A.H.(C^D) sequence so that $\mathcal{P}(x, \alpha)$ holds w.r.t. all the indices preceding b and in all the points of M ($h \geq 2$). Then if $\eta > 0$ is small enough (depending on the size of C_{r+2}) and ϵ is again small enough when compared to η, then $\mathcal{P}_k(x, \eta)$ for τ_k w.r.t. S_k^b is a local C^{r+1}-open property.*

Proof. Notice that we need to assume that the property we want to prove holds w.r.t. all the strata of preceding indices in order to deal with the points close to the boundary. This is not a contradiction because at least the first strata has empty boundary so we start the induction.

Assume that $\mathcal{P}(\eta, x)$ holds for τ_k w.r.t S_k^b. If χ_k is another C^{r+1}-A.H.(ϵ, C_{r+1}) sequence, then for k large enough $|j_D^r \chi_k| < B_1 \epsilon$ and $|\nabla_D j_D^r \chi_k| < B_1 \epsilon$ (B_1 as close as we want to 1).

Recall that the distance along the fiber is comparable with the $\hat{g}_k$ distance. By choosing L large enough, it is possible to find new constants $N_1^\epsilon, N_2^\epsilon$ so that if $y = j_D^r(\tau_k + \chi_k)(x)$ is at distance of the boundary bigger than $N_2^\epsilon(\eta - L\epsilon)$, the distance to the boundary of $q = j_D^r \tau_k(x)$ is bigger than

$N_2\eta$ (the distance in the total space of $\mathcal{J}^r_D E_k$ between points of the same fiber is comparable to the hermitian distance along the fibers). Also if the distance for y is smaller than $N_1^\epsilon(\eta - L\epsilon)$, then the corresponding distance for q is smaller than $N_1\eta$.

Let us first try the case when y, q are far from the boundary of the stratum. we can assume than both y and q (and the corresponding vector subspaces $T_D j^r_D(\tau_k+\chi_k)(x)$ and $T_D j^r_D\tau_k(x)$) are in the domain of a 1-comparable chart, which is the product of A.H. coordinates times a ball in the fiber; we use the euclidean metric $\hat{g}_0$ and its parallel transport, because it is comparable to the parallel transport w.r.t. $\hat{g}_k$. If ϵ is a small enough fraction of η and k is large enough, then $\angle_M(T_D j^r_D(\tau_k+\chi_k)(x), T_D j^r_D\tau_k(x)) \leq B_2\epsilon$, where B_2 does not depend on k, x. The reason is that the tangent space to the graph of each section along D approximately coincides with the $(r+1)$-jet, and the difference using ∇_D is comparable to that obtained using d_D. Also the $\hat{g}_k$-parallel transport of $T^{||}S^b_k(y)$ along the segment in the fiber joining y and q differs from the $\hat{g}_0$-parallel transport is a quantity proportional to the distance. If this variation is small enough compared to $\angle_m(T^{||}S^b_k, \hat{D})$, exactly the same happens for $T^{||}_D S^b_k(y)$ and $T^{||}_D S^b_k(q)$. Therefore, for ϵ small enough compared to η, we deduce $\mathcal{P}(\eta - L\epsilon, x)$ for χ_k to w.r.t. S^b_k. We observe that for η small enough the constant κ_1 so that $\epsilon = \kappa_1\eta$, does not depend on η (and either on the bounds of the section because we can make the antiholomorphic part arbitrarily small by increasing k appropriately).

The second possibility is that q is at distance of the boundary less than $N_1\eta$. We will see that if this quantity is chosen appropriately, a point $p \in S^a_k$, $a < b$, at distance of q smaller than the mentioned quantity will never avoid $\Theta_{S^a_k}$ in the approximate sense. Thus, the Whitney condition can be applied to deduce transversality in a small neighborhood of the boundary of the strata.

If τ_k is a C^{h+r}-A.H.(C^D, C) sequence, for any $x \in M$ and $q = j^r_D\tau_k(x)$, positive constants ρ_1, ρ_2 exist such that $j^r_D\tau_k(B_{g_k}(x, \rho_1)) \subset B_{\hat{g}_k}(q, \rho_2)$ (we recall that $j^r_D\tau_k$ is C^h-A.H.$(\bar{C}^D, \bar{C})$); the choice of constants depends on $\bar{C}$.

Let $p \in B_{\hat{g}_k}(q, \rho_2)$ and let its projection over M be the point x'. Let us call $p - q' = j^r_D\tau_k(x')$. There exist unique coefficients β_I so that $q' = \beta_I\nu_{k,x,I}$. By the linearity of the r-jets $j^r_D(\tau_k + \beta_I\tau^{\mathrm{ref}}_{k,x,I})(x') = p$. We select ρ_2 in such a way that the size of these coefficients is a small fraction of the amount of transversality α of $j^r_D\tau_k$ in x. We want to show that α-transversality of $j^r_D\tau_k$ in q (the image of x) implies $\alpha - B_3 d_{\hat{g}_k}(p, q)$-transversality of $j^r_D(\tau_k + \beta_I\tau^{\mathrm{ref}}_{k,x,I})$ in p (the image of x'); that would contradict —for an appropriate choice of ρ_2– the fact that p does not belong to $\Theta_{S^a_k}$ in the approximate sense. We simply observe that we have to show a similar relation for the variation of the angle from $T^{||}_D S^a_k(q)$ to $T^{||}_D S^a_k(p)$ and from $T_D j^r_D\tau_k(x)$ to $T_D j^r_D(\tau_k + \beta_I\tau^{\mathrm{ref}}_{k,x,I})(x')$ (measured for example using $\hat{g}_0$). The first relation has already been proved; for the second, we use the triangular inequality comparing firstly $T_D j^r_D\tau_k(x)$ with $T_D j^r_D\tau_k(x')$, and secondly the latter with $T_D j^r_D(\tau_k + \beta_I\tau^{\mathrm{ref}}_{k,x,I})(x')$. The last comparison follows from the size of the coefficients β (it is the same situation that we have just proved for points far from the boundary of the stratum); for the first comparison we use the bound on $\nabla\nabla_D j^r_D \tau_k$ which controls the variation of $T_D j^r_D \tau_k$ (in the approximate sense, due to the non-integrability of D).

Thus, we impose $N_1\eta$ to be smaller than ρ_2. In particular the point p is far from the boundary of $S^a_k \subset \partial S^b_k$ and it belongs to those points in the stratum S^a_k where the Whitney condition applies. That means that a constant $\rho > 0$ exists so that for any $y \in B_{\hat{g}_k}(p,r) \cap S^b_k$, $\angle_M(T^{||}S^a_k(y), TS^b_k(y)) \leq B_4\rho$, where both ρ and B_4 do not depend neither on k nor in y. The quantities $\angle_m(\hat{D}, T^{||}S^a_k), \angle_m(\hat{D}, T^{||}S^b_k)$ are bounded by below, and we deduce that $\angle_M(T^{||}_D S^a_k(q), T_D S^b_k(q)) \leq B_5\rho$, and thus the existence of a small constant $\kappa_2 > 0$ such that for any $\alpha > 0$ small enough, α-transversality of $j^r_D\tau_k$ to S^a_k implies $\frac{\alpha}{2}$-transversality to S^b_k in the neighborhood of radius $\rho\kappa_2$.

Hence, we further impose $N_1\eta$ to be at most one half of $\min(\rho_2, \rho\kappa_2)$, and this concludes the proof. □

Remark 6.6: Notice that in the induction process the amount of transversality obtained in the points close to the boundary of the corresponding stratum is not used at all; is in that region where the constant C_{r+2} is important in order to choose the size of the tubular neighborhood of ∂S^b_k where $\frac{\alpha}{2}$-transversality is deduced form the Whitney condition.

To give a proof for theorem 6.2, we define property $\mathcal{P}(x,\eta)$ for the C^{r+h}-A.H.(C) sequence τ_k as η-transversality in x to S^b_k of $j^r_G\tau_k$ along Q. One proves similarly that this is a local C^{r+1}-open property (the definition analogous to 6.4 uses the whole derivative ∇ instead of ∇_D). One just simply has to make sure that $|\nabla^{r+1}\tau_k|_{g_k} \leq \epsilon$, implies $|\nabla_Q j^r_G\tau_k|_{g_k} \leq L\epsilon$, and that $|\nabla^{r+2}\tau_k|_{g_k} \leq C$ gives rise to a uniform bound for $|\nabla\nabla_Q\tau_k|$. It has already been shown that a C^{r+h}-bound C for τ_k a C^h-bound $\bar{C} = B'C$ for $j^r_G\tau_k$ is obtained; thus the first question is straightforward and the second can be checked using a family of charts adapted to $TQ \times TQ^\perp$ in the points of Q. We equally notice that if $x \in Q$ and $j^r_G\chi_k(x)$ cuts S^a_k along TQ with minimum angle bounded by below, then it belongs to the subset of holonomic transversality $\Theta_{S^a_k}$ of S^a_k. This comments are enough that the previous arguments works also for transversality in the points of Q of r-jets along G.

The proof of theorem 6.1 requires the choice of a constant c_1 so that for any points $x, x' \in M$ with $d_k(x,x') \leq c_1$, we have $d_{\hat{g}_k}(\tau_k(x), \tau_k(x')) \leq \frac{\eta}{2}$. The constant will depend on (C^D, C_{r+2}). The set $\mathcal{B}^\tau_k$ of "good points" for τ_k is defined as the set of points $x \in M$ such that $\tau_k(x)$ is at distance smaller than 2η of those points in S^b_k at distance of ∂S^b_k greater than $N_2\eta$. The interesting property is that if $x \in \mathcal{B}^\tau_k$, then $B_{g_k}(x, c_1)$ belongs to the region where the local functions of definition 5.2 are available. Also, if χ_k is a perturbation whose C^{r+1}-size along D is smaller than $\frac{\eta}{2}$, then the image of $B_{g_k}(x,c_1)$ by $j^r_D(\tau_k + \chi_k)$ stays in that region. If $x \notin \mathcal{B}^\tau_k$, then both $j^r_D\tau_k$ and $j^r_D(\tau_k+\chi_k)$ send the ball $B_{g_k}(x,c_1)$ out of the set of points at distance of $S^b_k \cap B_{\hat{g}_k}(\partial S^b_k, N_2\eta)$ greater η.

Summarizing what we said, if $x \notin \mathcal{B}^\tau_k$ and χ_k is of C^{r+1}-size along D ϵ smaller than $\frac{\eta}{2}$, then $\mathcal{P}_k(x', \eta - L\epsilon)$ holds for $\tau_k + \chi_k$ w.r.t. S^b_k in all the points $x' \in B_{g_k}(x, c_1)$. Therefore, we will only need to work in the points of $\mathcal{B}_k^\tau$ to obtain estimated transversality to the stratum.

In the relative case we can also find a constant c_1 with similar properties, and single out a set of good points $\mathcal{B}_k^\tau$ to work in.

Proposition 6.7. *Let $\mathcal{P}(\eta, x)_{x\in M, \eta>0, k>>0}$ be a family of local C^{r+1}-open properties of sections of E_k. Let us assume the existence of positive constants $c, c', c,'' e$, so that for any $\delta > 0$ and ξ_k a $C^{\geq r+h}$-A.H.(C^D) sequence of sections of E_k ($h \geq 2$), it is possible to find for any $x \in M$ $C^{\geq r+h}$-A.H. sections $\chi_{k,x}$ with the following properties for $k >> 0$:*

1. *$\chi_{k,x}$ is $C^{\geq r+h}$-A.H.($c''\delta$).*
2. *the sections $\frac{1}{\delta}\chi_{k,x}$ have gaussian decay w.r.t. x with bounds for all the derivatives, so that the bound controlling the derivatives along the directions of D to order $r+h$ does not depend on the sequence ξ_k).*
3. *$\mathcal{P}(\gamma, y)$ holds for $\xi_k + \chi_{k,x}$ for every $y \in B_{g_k}(x, c)$ with $\gamma = c' log(\delta^{-1})^{-e}$.*

Then given any $\alpha > 0$ and any $C^{\geq r+h}$-A.H. sequence of sections τ_k of E_k, there exist a $C^{\geq r+h}$-A.H. sequence σ_k of sections such that for k large enough $\tau_k - \sigma_k$ is $C^{\geq r+h}$-A.H.(α), and property $\mathcal{P}(\eta, x)$ holds for the sections σ_k for certain uniform η and for every point $x \in M$.

Proof. See for example [**50**]. □

In the previous proposition the constant c is uniform and can be chosen to be arbitrarily small; for our starting sequence τ_k we impose c to be smaller than ρ_1. For any point $x \notin \mathcal{B}_k^\tau$ we select as perturbation the zero section. Thus, the proof of theorem 6.1 (resp. 6.2) is reduced to prove the existence of the perturbations χ_k centered at the points in $\mathcal{B}_k^\tau$, and for any given $C^{\geq r+h}$-A.H. (resp. C^{r+h}-A.H.) sequence ξ_k.

The local perturbation. (Continuation of the proof of the main theorem). Let x be a point in $\mathcal{B}_k^\tau$ and ξ_k a C^{r+h}-A.H($\bar{C}^D$) sequence of sections such that $|\nabla_D^j j_D^r \tau_k - \nabla_D^j j_D^r \xi_k|_{g_k} \leq \delta$, $j = 0, ..., h$, where $\delta < \frac{\eta}{2}$. Thus, we deduce that $j_D^r \xi_k(B_{g_k}(x, c_1))$ lays in the region where the local description of the stratification of definition 5.2 holds. By condition 3 in that definition, the function $F = (f_1 \circ j_D^r \xi_k, ..., f_p \circ j_D^r \xi_k)$ is $C^{\geq h}$-A.H.($C_{1,\eta}\bar{C}^D$), $C_{1,\eta}$ being a uniform constant. It is a consequence of lemma 5.9 that for $\gamma > 0$ small enough, γ-transversality of F to $\mathbf{0}$ is equivalent to $A\gamma$-transversality of $j_D^r \xi_k$ to the stratum, where A is a uniform constant. If c_1 is chosen small enough the ball $B_{g_k}(x, c_1)$ will be in the domain of A.H. coordinates (we can also ask the coordinates to be adapted to the metric) where we have the local basis $j_D^r \tau_{k,x,I}^{\mathrm{ref}}$ of the bundle of r-jets; we will define the perturbation using elements of this basis spanning the complementary directions to the stratum.

Rescaling th sections, it can be assumed that $|\tau_{k,x,I}^{\mathrm{ref}}|_{C^{r+h}} \leq \frac{1}{\delta}$. Being more precise it is important that the function to be perturbed is F, and for that reason we will use an appropriate basis of A.H. sections (functions) of $\mathbb{C}^p$, which is the target space for F. For each I, the $\mathbb{C}^p$-valued function $\Theta_I = (df_1(j_D^r \xi_k) j_D^r \tau_{k,x,I}^{\mathrm{ref}}, ..., df_p(j_D^r \xi_k) j_D^r \tau_{k,x,I}^{\mathrm{ref}})$ is $C^{\geq h}$-A.H.($C_{2,\gamma}\bar{C}^D$), for a certain uniform constant $C_{2,\gamma}$. Using condition 1 in 5.2, and maybe making c_1 smaller, we conclude the existence of complex numbers $\lambda_{I,i}$, $i = 1, ..., p$, $\sum_I |\lambda_{I,i}| < 1$, such that the functions $\Theta_i = \Theta_I$ give rise to a basis of $\mathbb{C}^p$ comparable to a unitary one (the determinant $|\Theta_1(x) \wedge \cdots \wedge \Theta_p(x)|$ is uniformly bounded by below), basically because the r-jets we chose conform a basis for the orthogonal to $\ker df$ comparable to a unitary one). In this new basis $F = \mu_1\Theta_1 + \cdots \mu_p\Theta_p$, where the properties of F and of the Θ_i imply that $\mu = (\mu_1, ..., \mu_p)$ is $C^{\geq h}$-A.H. over the ball $B_{g_k}(x, c_1)$ (an $C^{\geq h}$-A.H. function in terms of an A.H. basis has $C^{\geq h}$-A.H. coordinates). We define the corresponding perturbation to be $\zeta_{k,x,i} = \sum_I \lambda_{I,i}\tau^{\mathrm{ref}}_{k,x,I}$; notice that these are sections of E_k.

If necessary, a neighborhood of x can be rescaled so that the image of $B_{g_k}(x, c_1)$ in the domain of the A.H. coordinates contains $B^+ \times [0, 1]$, where $B^+ \subset \mathbb{C}^n$ is the euclidean ball of radius 1. We fix also $c < c_1$ so that the image of $B_{g_k}(x, c)$ is contained in $B_{1/2} \subset \mathbb{C}^n \times \mathbb{R}$, the euclidean ball in $\mathbb{C}^n \times \mathbb{R}$ of radius $\frac{1}{2}$. We pullback μ to the chart to obtain a function $\tilde{\mu}\colon \mathbb{C}^n \times \mathbb{R} \to \mathbb{C}^p$ to which we can apply the local estimated transversality result extending the original one of S. Donaldson for A.H. functions from $\mathbb{C}^n$ to $\mathbb{C}^p$, whose proof we postpone until the end of this section.

Proposition 6.8. *Let $F\colon B^+ \times [0, 1] \to \mathbb{C}^p$, $0 < \delta < \frac{1}{2}$ a constant and $\sigma = \delta(log(\delta^{-1})^{-e}$, where e is a fixed integer depending only on the dimensions n, p. Assume that for any $s \in [0, 1]$, the following estimates hold for F_s in B^+:*

$$|F_s| \leq 1,\ |\bar{\partial} F_s| \leq \sigma,\ |\nabla\bar{\partial} F_s| \leq \sigma$$

Then, a smooth curve $w\colon [0, 1] \to \mathbb{C}^p$ exists such that $|w| < \delta$ and the function $F - w$ is σ-transverse to $\mathbf{0}$ over $B_{1/2}$ along the directions of $\mathbb{C}^n$ (or along D_h). Moreover, if $|\nabla^j \partial F/\partial s| < C_j$ for all $j \in \mathbb{N}$, then w can be chosen so that $|d^j w/ds^j| < \Phi_j(\delta)$, for all $j \in \mathbb{N}$ and $d^j w/ds^j(0) = 0$ y $d^j w/ds^j(1) = 0$ for all $j \in \mathbb{N}$, where Φ_j is a function only depending on the dimensions n, p.

In proposition 6.8 the norms are computed using the euclidean metric, the covariant derivative is the flat one and the almost-complex structure is J_0.

Once k is large enough, we can apply this proposition (and possibly after rescaling), because for example using the results of lemma 3.27 we know that approximate holomorphicity of the function w.r.t. D, g_k, J is equivalent to approximate holomorphicity w.r.t. D_h, J_0, g_0 (and the change of bounds can be controlled is k is large enough).

Therefore, we can find a smooth curve in $\mathbb{C}^p$, $(w_1, ..., w_p)$, so that $\tilde{\mu} - w$ is γ-transverse to $\mathbf{0}$ over $B_{1/2}$ along D. This implies $A_1\gamma$-transversality of $\mu - w$ to $\mathbf{0}$ over $B_{g_k}(x, c_1)$, for a uniform A_1 and thus $A_2\gamma$-transversality to $\mathbf{0}$ of $F - w_1\Theta_1 - \cdots - w_p\Theta_p$ over the same ball.

It is important to point out that what we have obtained is a solution for the transversality problem in the bundle of r-jets, but what we look for is a solution to the strong transversality problem, that is, a perturbation which is the r-jet of a sequence of sections of E_k. The natural candidate is

$$\chi_{k,x} = -(w_1\zeta_{k,x,1} + \cdots + w_p\zeta_{k,x,p}).$$

It is clear that $\chi_{k,x}$ is an A.H. sequence with gaussian decay w.r.t. x. In order to prove that the constants governing the derivatives and gaussian decay along D are comparable to δ, it is necessary to choose the curve w with vanishing derivatives along D or at least approximately vanishing. By construction –and in contrast to the even dimensional case where this functions are constants– the derivatives along D do not vanish. In the corresponding A.H. coordinates the w are constant along D_h. Since we have *uniform bounds for the derivatives along the vertical direction* we can conclude that the derivatives along D vanish in the approximate sense. The subtlety is that if we worked with C^{r+h}-A.H. sequences, we would be able to conclude uniform bounds for the derivatives of the local perturbations (or the derivatives of w) up to order h; but to keep on with the inductive process we need control to order $r+h$ (of order h for the r-jets). This is the precise reason that have forced us to work with $C^{\geq r+h}$-A.H. sequences.

Regarding estimated transversality, if we denote

$$\tilde{F} = (f_1 \circ j^r_D(\xi_k + \chi_k), ..., f_p \circ j^r_D(\xi_k + \chi_k)),$$

it is enough a bound for $|\nabla^j_D(\tilde{h} - (h - w_1\Theta_1 - \cdots - w_p\Theta_p)|_{g_k}$, $j = 0, 1$, of size $A_2\frac{\gamma}{2}$ to obtain $A_2\frac{\gamma}{2}$-transversality for $\xi_k + \chi_k$. Notice that the difference between both functions comes from the fact that when we compose with f and perturb linearly, the corresponding perturbation is not linear on the fibers of $\mathcal{J}^r_D E_k$. In other words, in the comparable 1-charts that rectify the foliation $\ker df$ the fibers $\mathcal{J}^r_D E_k$ of are not linear subspaces. In any case, the lack of linearity is controlled by the second derivative of f. Being more precise,

$$\tilde{F} = F - w_1\Theta_1 - \cdots - w_p\Theta_p + O(c_k^{-1/2}) + O((\delta + c_k^{-1/2})^2),$$

by the bounds on the second derivatives of the f_i and because $j^r_D\sigma_{k,x} - (w_1\Theta_1 + \cdots + w_p\Theta_p)$ is of size $O(c_k^{-1/2})$. The important observation is that we obtain not only C^0-control of size $O(c_k^{-1/2}) + O((\delta + c_k^{-1/2})^2)$, but also the same kind of control for the derivatives along D. Due to the formula of γ in terms of δ, if δ is small enough the latter is a quadratic term in δ much smaller than $A_2\gamma$. The outcome is $A_3\gamma$-transversality for $\tilde{F}$ over $B_{g_k}(x, c)$, which gives $A_4\gamma$- transversality of $j^r_D\tilde{\tau}_k$ to S^a_k over the ball. Therefore, $\mathcal{P}(A_4\delta\log(\delta^{-1})^{-e}), y)$ holds for $\xi_k + \chi_k$ in all the points $y \in B_{g_k}(x, c)$, and this finishes the proof of theorem 6.1. □

The proof of theorem 6.2 is the same but with to small modifications. The first one is that we can span the directions complementary to the distribution $\ker df$ using the r-jets along G, because by hypothesis the strata are pullback of submanifolds in $\mathcal{J}^r_G E_k$ by the orthogonal projection $\mathcal{J}^r E_k \to \mathcal{J}^r_G E_k$. The second modification is J. P. Mohsen's local transversality result to a subvariety Q (see section 5 in [**43**]), which essentially amounts to compose on the right with A.H. coordinates adapted to G an rectifying the submanifold. The advantage is that the corresponding perturbation is a linear combination of the sections $\tau^{\mathrm{ref}}_{k,x,I_g}$ and that means that we can use C^{r+h}-A.H. sections. Besides, the final amount of transversality only depends on the geometry of the manifold and stratification and on the constant that controls the derivatives of order smaller of equal than $r+h$ of the sequence to be perturbed.

Proof of the local estimated transversality theorem. (Proposition 6.8). The proof is a modification of proposition 5.1 in [**50**], which is an extension of the proof of lemma 11 in [**32**], that again refines preposition 3 in [**2**] and Donaldson's original local estimated transversality result in [**12**] (proposition 25).

For the function $F_s\colon \mathbb{C}^n \to \mathbb{C}^p$, the sets $U(F_s, w, \delta, \sigma)$ are defined as the image in $B(0,\delta)$ of the set of points in $B' = B(0, \frac{1}{2}) \subset \mathbb{C}^n$ for which $F_s - w_s$ is σ-transverse to $\mathbf{0}$ (here we mean full transversality, because all the directions are those of D_h).

The key point is being able to show that $U(F_s, w, \delta, \sigma)$ contains the complementary of a set W which is union of a certain number not greater than $N(\delta)$ of balls of radius σ. Assuming that result, for a couple of points x, y in the complementary, we consider a curve (in the complementary) defined as follows: the segment $[x, y]$ cuts ∂W in at most $2N$ points. The pieces of the segment in the interior of W are removed and we draw instead piecewise smooth curves which are the result of patching geodesics in the subsets of the spheres that conform the boundary of W. We the perturb this curve w_0 to a smooth curve w_1.

The proof that it is possible to find a curve w_1 with its derivatives of order i bounded in terms of a function $\Phi_i(\delta)$ is as follows: each smooth piece of w_0 is either a segment or a piece of geodesic in the corresponding sphere. Thus, all the derivatives of this pieces are bounded in terms of σ, and therefore in terms of δ. We actually need the length of each piece to be bounded by below by a fraction of δ, because to define w_1 we will just glue contiguous pieces using a cut-off function, which will be the result of rescaling a fixed one. If we are able to use in both sides a piece of curve of length bounded by below by a multiple of σ, then the desired result will hold.

The only thing that we need to do is choose the balls that define W with some care. Actually, what we do is taking a fixed covering of $B(0,\delta) \subset \mathbb{C}^p$ by balls of radius σ, and W will be the union of those balls intersecting the complementary of $U(F_s, w, \delta, \sigma)$. Since we look uniform bounds in δ and σ we use our usual strategy. We take a covering for $\sigma = 1$ and rescale it.

For these value $\sigma = 1$, we consider the lattice of integer points and actually cover the whole $\mathbb{C}^p$ with balls of some radius ρ centered at the points of the lattice. The radius $\rho = O(1)$ is chosen so that:

(1) $\rho > \frac{\sqrt{2p}}{2}$, so that we get a covering.
(2) The spheres are in general position.

We want to cover the whole $\mathbb{C}^p$, so that when we contract the picture multiplying by an arbitrary σ we are still able to cover the ball of radius δ.

Although the covering has an infinite number of balls, since it is translation invariant the condition on the intersection has to be checked only for a finite number of balls. That implies the existence of a radius $\rho = O(1)$ with the desired properties. The important point is that S^{n-1} –the boundary of each ball– splits into regions with the following properties: for each region R, its boundary ∂R is stratified with maximal strata ∂R_i. For each $r > 0$ we define $\partial R_{i,r}$ as the set of points in ∂R_i at distance of its boundary grater than r. It is possible to find $r_0 > 0$ so that for any $x \in \partial R_{i,r_0}$, $y \in \partial R_{j,r_0}$,$i \neq j$, there exists a piecewise geodesic joining them whose total length is bounded by above and the length of each piece bounded by below. Again this is a consequence of the fact that we only have to check things for a finite number of balls of the covering.

Given δ and σ arbitrary, we rescale the construction so that ρ becomes σ and we just consider the balls intersecting $B(0,\delta) \subset \mathbb{C}^p$. Recall that W is now defined as the union of balls intersecting the complementary of the image of $U(F_s, w, \delta, \sigma) \subset B(0,\delta)$, and it is a consequence of for example lemma 11 in [**32**] that the number of balls of W is bounded by $N(\delta)$.

One checks that two points in ∂W can be joined by a piecewise geodesic, so that the number of pieces is bounded by a multiple of the number of balls N; each piece has length bounded by below by $b\sigma$, with b independent of σ. If the starting of ending point of the curve is too close to the boundary of the corresponding region $\partial R_{i,r}$, we can go back along the piece of geodesic that of the sphere (so that the length of the piece would violate the bound by below), we can go back toward the interior again, and then start from that point; this change only amounts to increase the number $N(\delta)$ bounding the number of smooth pieces to $N + 2$. Regarding the segments, those which are not either the initial or the final lay in balls not belonging to W, if they connect points of balls in W with empty intersection, then its length is bounded by below appropriately; if the intersection is non-empty, we change the segment by one going from the initial point to the center of the ball not in W containing the segment, and another from the center to the endpoint. For the initial and final segment, if they are very short, we move back to the center of the ball (not in W) and then draw another segment towards the intersection point.

We simply recall that the final curve w is built taking slices $\mathbb{C}^n \times \{s\} \subset \mathbb{C}^n \times [0,1]$, constructing there curves w_q as described and connecting them with vertical segments (see lemma 11 [**32**]). Since the length of such segments has the appropriate bound by below, the perturbation using cut-off functions as described gives the desired bounds for w. $\square$

7. Applications

We now prove the results stated in the introduction.

Proof of theorem 1.5. We consider a more general situation than that of the statement of theorem 1.5. Let $E_k \to (M, D, J, g)$ be a very ample sequence of locally splittable hermitian vector bundles of rank m. Notice that for a calibrated manifold of integer type and example of such sequence is $E \otimes L^{\otimes k}$, E a rank m hermitian bundle.

Given any point x, the construction of a calibrated submanifold through x of real codimension $2m$ is as follows (see [**38**]). It is necessary to select reference sections "adapted" to x; in A.H. coordinates (for example adapted to the metric) we take the sections $z_k^j \tau_{k,x,j}^{\text{ref}}$, $j = 1, ..., m \leq n$ and consider its direct sum, a section of E_k This sequence of sections vanishes on x and it is uniformly transverse to the **0** section in a g_k-ball of radius $O(1)$ centered at x. In the globalization procedure we start by these balls and consider no local perturbations over them. Now the point is not altering the section in x when we add other local perturbations. We are specially concerned with those points at g_k-distance smaller than $O(c_k^{1/6})$, because that is the size of the support of the reference sections. If y is one of those points, we multiply the reference sections $\tau_{k,y,j}^{\text{ref}}$ by a function $h_{y,x}$ that is required to be J-complex at y, vanishing at x, bounded by below in a ball of g_k-radius $O(1)$ centered at y and with derivatives uniformly bounded by above. The result is still a sequence of A.H. sections with gaussian decay w.r.t. y, trivializing the bundle in a ball $B_{g_k}(y, O(1))$ and vanishing at x. Notice that as long as the g_k-distance between x and y is uniformly bounded by below, the choice of such $h_{y,x}$ is possible. But that is enough for our purposes, because we do not need to perturb in the balls $B_{g_k}(x, O(1))$.

Therefore, it is possible to find sequences of A.H. sections τ_k of E_k uniformly transverse to **0** and vanishing at x (since we deal with 0-jets, we can work with C^2-A.H. sequences). Let us call W_k to the submanifolds $\tau_k^{-1}(\mathbf{0})$ (for $k >> 0$), that are uniformly transverse to D (corollary 5.12) and approximately almost complex.

To study its topology we proceed very much as in the symplectic and contact case (see [**12, 2, 32**]).

The function $f_k = \log |\tau_k|^2 \colon M - W_k \to \mathbb{R}$, is a Morse function that will give a surgery construction of M from W_k.

We need to proof that the critical points of f_k are isolated (Morse condition) and that their index is bigger than $n - m$.

It is straightforward that f_k tends $-\infty$ when we approach W_k, and that is x is a critical point for f_k, then $|\tau_k(x)| \geq \eta$, where η is the amount of estimated transversality.

If the critical point is sent to the tubular neighborhood of the zero section where the estimated transversality condition holds, the surjectivity of $\nabla_D \tau_k$ implies that a $v \in D$ exists such that $\nabla_v \tau_k(x) = \overline{\tau_k(x)}$.

Since

$$df_k = \frac{1}{|\tau_k|^2}(\langle \nabla \tau_k, \tau_k \rangle + \langle \tau_k, \nabla \tau_k \rangle),$$

the derivative of $df_{kv}(x)$would not vanish, and that leads to a contradiction.

In particular we can find a perturbation of size $O(c_k^{-1/2})$ out of a tubular neighborhood of W_k of g_k-radius $O(1)$ (whose points are regular for f_k), so that the new f_k is Morse. Observe that the Morse condition in the directions of D holds for the original f_k but without perturbation the critical points might form 1-dimensional manifolds transverse to D.

From the previous comments we deduce

$$\partial f_k = \frac{1}{|\tau_k|^2}(\langle \partial\tau_k, \tau_k\rangle + \langle \tau_k, \bar\partial\tau_k\rangle). \tag{7.9}$$

Since on the critical points all the components of the derivative vanish, using 7.9 we have the following equality on the point x,

$$|\langle \partial\tau_k, \tau_k\rangle| = |\langle \tau_k, \bar\partial\tau_k\rangle|, \tag{7.10}$$

whose importance comes from the fact that we know the size of the r.h.s.

If we differentiate 7.9 and evaluate at x, we have:

$$\bar\partial\partial f_k = \frac{1}{|\tau_k|^2}(\langle \bar\partial\partial\tau_k, \tau_k\rangle - \langle \partial\tau_k, \partial\tau_k\rangle + \langle \bar\partial\tau_k, \bar\partial\tau_k\rangle + \langle \tau_k, \partial\bar\partial\tau_k\rangle). \tag{7.11}$$

Recall that we still use the metric g_k. Having into account the approximate equality $\partial\bar\partial + \bar\partial\partial \approxeq F_D^{1,1}$ and the estimates for the antiholomorphic components, we can transform 7.11 into

$$\bar\partial\partial f_k = \langle F_D^{1,1}\tau_k, \tau_k\rangle - \langle \partial\tau_k, \partial\tau_k\rangle + O(c_k^{-1/2}). \tag{7.12}$$

Since the norm of τ_k is bounded by below in the critical point and $F_D^{1,1}$ is the curvature of a locally splittable bundle, for any vector $u \in D$ of g_k-norm uniformly bounded by below, $\langle F_D^{1,1}(u, Ju)\tau_k, \tau_k\rangle = O(1)$.

Let us consider the subspace H_x of D_x of vectors which are sent by $\partial f_k(x)$ to the complex line of E_k spanned by τ_k. H_x is J-complex and its real dimension is at least $2n - 2m + 2$.

If $u \in H_x$,

$$|\tau_k||\partial_u\tau_k| = |\langle \partial_u\tau_k, \tau_k\rangle| = |\langle \tau_k, \bar\partial\tau_k\rangle|,$$

so $|\partial_{|H_x}\tau_k| = O(c_k^{-1/2})$, and thus over this subspace the dominant term of the r.h.s. of 7.12 is $\langle F_D^{1,1}\tau_k, \tau_k\rangle$.

Having into account that the restriction to D of the Hessian H_{f_k} verifies $H_{f_k}(u) + H_{f_k}(Ju) = -2i\bar\partial\partial f_k(u, Ju)$, over H_x will necessarily be negative.

Suppose for the moment that the index of the Hessian is smaller than $n-m-1$. That implies the existence of a subspace $V \subset D_x$ of real dimension at least $n+m$ non-negative for H_{f_k}. The dimension of $V \cap JV$ is at least $2m$, but this contradicts that the Hessian is negative over H_x, because $(V \cap JV) \cap H_x$ is non-trivial.

The results for the homology and homotopy groups follows from a classical argument from Morse theory.

$\square$

Remark 7.1: Notice that the perturbation we made of τ_k in order to get the Morse condition for f_k, did not affected the points of W_k. Hence the results we obtained are really for the relative topology of the pair (M, W_k).

The next theorem we want to prove is about the existence of determinantal submanifolds, that is still a transversality result for 0-jets (vector bundles E_k), but not anymore to the $\mathbf{0}$ section but to a non trivial sequence of (non-linear) A.H. stratifications.

PROOF OF PROPOSITION 1.6. Let $L_k \to (M, D, J, g)$ be a very ample sequence of line bundles over an a.c. manifold. For a calibrated manifolds of integer type, L_k will be for example the sequence of powers of the prequantum line bundle.

Let E, F be hermitian bundles with connection, and let us define the sequence of very ample vector bundles $I_k := E^* \otimes F \otimes L_k$. In the total space of I_k we consider the sequence of stratifications S_k whose strata are $S_k^i = \{A \in I_K |\ \mathrm{rk}(A) = i\}$, where $A \in \mathrm{End}(E, F \otimes L_k)$ and $\mathrm{rk}(A)$ is its rank.

If we apply 5.14, we deduce that S_k^i is an A.H. sequence of finite Whitney stratifications. Therefore we can apply theorem 6.1 to construct an A.H. sequence of sections τ_k of I_k uniformly transverse to S_k.

Hence, for k large enough M is stratified by the submanifolds $S_{\tau_k}^i = \{x \in M |\ \mathrm{rk}(\tau_k(x)) = i\}$, which are uniformly transverse to D and approximately almost complex.

If the original manifold was calibrated and of integer type, the previous stratification is by calibrated submanifolds.

□

Theorem 7.2. *Let L_k be a very ample sequence of line bundles over (M, D, J, g) and set $E_k = \mathbb{C}^m \otimes L_k$. Any A.H. sequence of sections of E_k admits an arbitrarily small perturbation (in the sense of $C^{\geq r+h}$ or C^{r+h}-A.H. sequences, depend on whether we use the intrinsic or the relative theory) such that the corresponding A.H. sequence of projectivizations $\phi_k \colon M - A_k \to \mathbb{CP}^m$ is r-generic (A_k submanifold of base points of real codimension $2m + 2$).*

PROOF. The proof is just the transversality theorem applied to the T-B-A quasi-stratification. □

We must point out that the situation is not as good as in the even dimensional situation. The description of the A.H. functions close to the points of the degeneration loci (the different strata induced by the T-B-A quasi-stratification) is more complicated.

Firstly, and similarly to what happens for even a.c. manifolds, to obtain normal forms it is necessary to add perturbations so that the function becomes holomorphic (at least in certain directions); otherwise the approximate holomorphicity is not significative due to the vanishing of the holomorphic part (or more generally degenerated). Secondly, we have an extra non-holomorphic direction that we do not control. At most, we can apply the usual genericity results to that direction (but perturbations of size $O(c_k^{-1/2})$ so as not to destroy the other properties). In certain circumstances that will be enough to obtain useful normal forms.

One instance of the preceding situation is when the target space has large dimension so that the generic map is an immersion without self intersections, as is the case of corollary 1.8, whose proof we give now.

PROOF OF COROLLARY 1.8. Let $L_k \to (M, D, J, g)$ be a very ample sequence of line bundles over an a.c. manifold and set $E_k = \mathbb{C}^m \otimes L_k$, where $m \geq n + 2$.

We apply theorem 6.1 or 6.2 to the T-B-A quasi-stratification of $\mathcal{J}^1_D E_k \to (M, D, J, g)$ (resp. $\mathcal{J}^1_G E_k \to (M \times [-\epsilon, \epsilon], J, G, g)$ with $G = D$ and along the submanifold $Q = M \times \{0\}$)), to obtain maps ϕ_k 1-generic. From the choice of m it follows that the set of base points and of points where $\partial\phi_k$ is not injective, are empty. It is clear that by construction $\phi_k^*[\omega_{FS}] = [w_k]$.

Also the choice of m allows us to perturb the section $\tau_k \in \Gamma(E_k)$ such that ϕ_k is an embedding. Moreover, if the perturbation is of size $O(c_k^{-1/2})$ none of the properties of ϕ_k is lost (we still have a 1-generic sequence of sections). □

We finish this section by mentioning that it is possible to obtain uniform transversality to a finite number of quasi-stratifications of the same sequences of bundles. For example we can obtain the genericity result that gives rise to embeddings in $\mathbb{CP}^m$ transverse to a finite number of complex submanifolds of $\mathbb{CP}^m$, and more generally analogs to complex codimension 1 foliations for symplectic manifolds [**9**]. In the fist case we just need to consider for each submanifold the sequence of stratifications of $\mathcal{J}^1_D(M, \mathbb{CP}^m)$ whose unique stratum (for each k) is defined to be the 1-jets whose degree 0-components is a point of the submanifold; next we pull it back to a stratification $\mathcal{S}'$ of $\mathcal{J}^1_D E_k^*$ (the structure near Z_k is not relevant because transversality to T-B-A means that the sections stay away from Z_k). Therefore, we have defined a stratification of $\mathcal{J}^1_D E_k$ which is trivially A.H. because is the pullback by A.H. maps of an initial A.H. stratification of $\mathcal{J}^0_D(M, \mathbb{CP}^m)$.

Any 1-generic sequence of A.H. sections of E_k uniformly transverse to $\mathcal{S}'$, once perturbed to give an embedding in $\mathbb{CP}^m$, gives rise to maps $\phi_k \colon M \hookrightarrow \mathbb{CP}^m$ uniformly transverse to the submanifold along the directions of D.

Recall that a codimension 1 holomorphic foliation of $\mathbb{CP}^m$ is given by a holomorphic section ϖ of $T^{*1,0}\mathbb{CP}^m \otimes L$, where L is a holomorphic line bundle. We consider in $\mathcal{J}^1_D(M, \mathbb{CP}^m)$ the set $\mathbb{P}\mathcal{S}^{\varpi}$ of points sent to the $\mathbf{0}$ section by $\varpi \colon \mathcal{J}^1_D(M, \mathbb{CP}^m) \to L$. It can be partitioned into strata corresponding to the inverse images of the vanishing set of ϖ and its complementary, of codimension n. It can be seen that transversality to the former implies that the 1-jet does not touch the latter in a tubular neighborhood [**9**] (a kind of Whitney condition).

In this way –and for $k >> 0$– $\phi_k^*\varpi$ defines after a suitable stratification a sequence of calibrated foliations. Basically this perturbation is needed to obtain normal forms in the zeros of $\varpi(j^1_D\phi_k)$ that do not come from the singular set (real dimension 1); in particular it guarantees that when the leaves approach this singular set, they are still calibrated submanifolds.

Let us point out that the immersions in projective spaces with extra transversality properties w.r.t. foliations is a non-trivial result, in contrast with the embeddings of corollary 1.8 that can be obtained using the theory of characteristic classes (though without any reference to the almost complex structure).

Another possible application is, as proposed by D. Auroux for even dimensional a.c. manifolds [**4**], to obtain r-generic applications to $\mathbb{CP}^m$ whose composition with certain projections $\mathbb{CP}^m \to \mathbb{CP}^{m-h}$ are still r-generic (the corresponding stratifications are A.H. because they are pullback of A.H. stratifications by the A.H. map; the structure near Z_k is also seen to be appropriate).

Ir is also possible to develop an analogous construction but for A.H. maps to grassmanians $Gr(r, m)$, starting from sections of $\mathbb{C}^{r+1} \otimes E_k$, E_k of rank m (see [**46, 5**]).

8. Normal forms for A.H. maps to $\mathbb{CP}^1$

In this section E_k will denote the sequence of bundles $\mathbb{C}^2 \otimes L_k$, where L_k is a very ample sequence of line bundles over (M, D, J, g).

In the bundle $\mathcal{J}^1_D E_k$ the Thom-Boardman-Auroux quasi-stratification has only to strata: Z_k and $\Sigma_{k,n}$.

Any $C^{\geq 1+h}$-A.H. (resp. C^{1+h}-A.H. using the relative theory) sequence, $h \geq 2$, can be perturbed to a sequence τ_k –with control on the derivatives to order $1 + h$ along the directions of D (resp. in the directions of the whole tangent space)– uniformly transverse to Z_k and $\Sigma_{k,n}$.

Therefore, we obtain $\phi_k : M - A_k \to \mathbb{CP}^1$ an A.H. sequence (by A.H. we mean form now on either $C^{\geq 1+h}$-A.H. or C^{1+h}-A.H. depending on whether we use the intrinsic or the relative theory) of functions with the following properties:

(1) The set of base points $A_k = \tau_k^{-1}(Z_k)$ is a compact submanifold of M of real codimension 4 cutting D transversely (in a uniform way), and such the subspace $D \cap TA_k \subset D$ is approximately J-complex.
(2) $\Sigma_n(\phi_k)$, defined as the set of points where $\partial\phi_k$ is singular (and where in principle $d_D\phi_k$ does not vanish due to the presence of the anti-holomorphic component), is a compact submanifold of codimension $2n$ (a link) uniformly transverse to D.

So far the kind of A.H. coordinates we have used –except for the total spaces of the bundles– have been those adapted to the metric and the strongly equivalent ones. In this section we will use more general A.H. coordinates (equivalent to the usual ones, i.e., with bounds of order $O(1)$ for the derivatives of $\angle_m(D_h, D^\perp)$) which will be centered only in the points of a sequence of submanifolds.

The content of the following propositions is that we can find A.H. coordinates $z_k^1, ..., z_k^n, s_k$, so that in the points of A_k the function ϕ_k is the quotient of two coordinates z_k^1, z_k^2 , and in those of $\Sigma_n(\phi_k)$ is the analog to a complex Morse function (quadratic in the z_k).

Proposition 8.1. *For every point $a \in A_k$ A.H. coordinates $z_k^1, ..., z_k^n, s_k$ centered at a can be found (and a holomorphic chart of $\mathbb{CP}^1$) so that in a* *ball of fixed g_k-radius in the domain of the coordinates A_k is described as $z_k^1 = z_k^2 = 0$, and the function out of the points of A_k has the expression $\phi_k(z_k, s_k) = \frac{z_k^2}{z_k^1}$.*

Proof. Let us start form usual A.H. coordinates centered in $a \in A_k$ with an A.H. trivialization of L_k. Thus the section is represented by a couple of A.H. functions $f_k^i \colon \mathbb{C}^n \times \mathbb{R} \to \mathbb{C}$, $i = 1, 2$, A.H.; both define a foliation with real codimension 4 leaves uniformly transverse to D_h, and intersecting D_h in A.H. submanifolds. We can assume, maybe composing with a complex linear transformation composed with another linear map whose size does not exceed $O(c_k^{-1/2})$, that in the origin the intersection with D_h has tangent space $z_k^1 = z_k^2 = 0$ (in the origin J approximately coincides with J_0). The coordinates we look for are obtained by rectifying the foliation, that is, using $f_k^1, f_k^2, z_k^3, ..., z_k^n, s_k$ as new coordinates. The properties of ϕ_k are clearly satisfied. Uniform transversality implies that the domain of the new charts contains a ball of uniform g_k-radius $O(1)$ (the image of the functions in the original A.H. coordinates fills a ball in $\mathbb{C}$ of radius uniformly bounded by below). The bounds of order $O(1)$ for the full derivatives of f_k^1, f_k^2 –together with those measuring the lack of antiholomorphicity– imply that what we have defined are generalized A.H. coordinates. □

Remark 8.2: In the new A.H. coordinates the curvature restricted to $D_h(0) = D(0)$ is approximately of type $(1,1)$. For certain results we might be interested in not making any reference to the almost complex structure J of the calibrated manifold. Then it does not make any sense to speak about A.H. coordinates. Instead we can use modify a bit the sections to obtain the previous proposition in coordinates with a compatibility condition w.r.t. ω.

Proposition 8.3. *It is possible to find a perturbation of τ_k of order at most $O(c_k^{-1/2})$ so that for the corresponding projectivization ϕ'_k we can find A.H. coordinates centered in the points b of $\Sigma_n(\phi'_k)$, and holomorphic charts of $\mathbb{CP}^1$, such that*

$$\phi'_k(z_k, s_k) = \phi'_k(0, s_k) + (z_k^1)^2 + \cdots (z_k^n)^2$$

Being more precise, we can find radius $\rho_2 > \rho_1 > 0$, new distributions, almost-complex structures J_k and functions ϕ'_k such that:

1. *$D_k \approxeq D$, $J_k \approxeq J$, with $D_k = D$, $J_k = J$ out of the tubular neighborhood of $\Sigma_n(\phi_k)$ of radius ρ_2, and both integrable in the tubular neighborhood of radius ρ_1.*
2. *$\phi_k \approxeq \phi'_k$, $\phi_k = \phi'_k$ out of the tubular neighborhood of $\Sigma_n(\phi_k)$ of radius ρ_2 and ϕ'_k holomorphic in the tubular neighborhood of radius ρ_1.*

Proof. For each component of $\Sigma_n(\phi_k)_\lambda$, $\lambda \in \Lambda_k$, we consider the model of the tubular neighborhood constructed using the exponential map along the directions of $D(b)$, b any point in the component. This is a vector bundle with almost complex fibers. Thus, is a complex bundle. Since its base space is S^1, it has to be trivial.

Since each component of $\Sigma_n(\phi_k)_\lambda$ –once reparametrized by the arc (for the metric g_k)– has bounds for its derivatives of order $O(1)$, it is possible to find a trivialization of the bundle whose sections have angle with D bounded by below (their graphs as functions from S^1 to the fiber $\mathbb{C}^n$) and all its derivatives bounded by $O(1)$. Using the mentioned trivialization we obtain a map

$$\varphi_{k,\lambda} \colon \mathcal{N}_\rho(\Sigma_n(\phi_k)_\lambda) \to \mathbb{C}^n \times S^1$$

It is important to point out that the radius of the tubular neighborhood ρ is independent of k, and fills a tubular neighborhood of $\{0\} \times S^1$ whose euclidean radius is also independent of k and λ. These maps are also complex in the points of $\Sigma_n(\phi_k)_\lambda$ (J is sent into J_0).

Generalized A.H. coordinates $z_k^1, ..., z_k^n, \theta_k$ are obtained by just by composing with the canonical projections to $\mathbb{C}^n$, where we can think of θ_k either taking values on an appropriate interval or in $[0, 2\pi]$ if we want to parametrize the whole tubular neighborhood.

The fibers and complex structure J_0 of the vector bundle induce a local foliation and integrable almost complex structure that are denoted by D_h and J_0. In the domain of $\varphi_{k,\lambda}$ it is evident that $D \approxeq D_h$ and $J \approxeq J_0$. The distribution and almost complex structure of the statement of the proposition are the result of interpolating from the integrable structures to the original ones (in the annulus of radii ρ_1, ρ_2). It is worth pointing out some details.

i. The interpolation is made using bump functions that only depend on the euclidean distance to $\{0\} \times S^1$. It the functions do not vary too fast, the bounds of order $O(c_k^{-1/2})$ measuring the difference between the initial structures D, J and the final ones will hold trivially. Since the radius ρ is independent of k, λ, we can make such choice of bump functions.
ii. D_k, the result of interpolating from D to D_h is obtained by writing D_h as the graph of a function $D \to D^\perp$; in other words, we perturb in the direction of $D^\perp$) (we could equally have used the angular coordinate θ_k).
iii. The case of the almost-complex structures is similar. We think of them as sections of $T^*M \otimes TM$ vanishing along $D^\perp$. Therefore, both restrict to almost-complex structures on D_k. We interpolate using the linear structure of the space of endomorphisms to define new tensors $\check{J}_k$, which approximately coincide with J_0 and J. The result of squaring these tensor is not necessarily $-I$, but it is approximately true. It is not difficult to perturb them to define J_k with $\check{J}_k \approxeq J_k$ and $J_k^2 = -I$: it is enough to choose a trivialization of the form $e_1, J_0 e_1, ..., e_n, J_0 e_n$, with $|\nabla^j e_i|_{g_k} \leq O(1)$, $\forall j \in \mathbb{N}$. J_k is defined acting on the previous basis by the formula $J_k(e_i) = \check{J}_k(e_i)$, $J_k^2(e_i) = -e_i$. The bounds for the derivatives of the basis imply that the new tensor approximately coincides with $\check{J}_k$. By definition J_k is an almost complex structure; in the points where $\check{J}_k$ coincides with J_0 and J respectively, J_k has this property also.

In the normal bundles parametrized by $\mathbb{C}^n \times S^1$ it is possible to consider a different A.H. theory, we will investigate further in the next section. Essentially, since D_h is integrable it is not necessary to use retractions to define covariant derivatives of sections of D_h^*. We can restrict the metric g_k to each leaf and use the corresponding Levi-Civita connection. In any case, since in the leaf the Christoffel symbols and its derivatives are of size $O(c_k^{-1/2})$, we can use as usual the trivial connection on each leaf. The sequence $\phi_k \colon \mathbb{C}^n \times S^1 \to \mathbb{CP}^1$ is A.H. (for D, J or D_k, J_k) if and only if is A.H. for the foliated theory in D_h with J_0 and g_0, where g_0 is the euclidean metric in $\mathbb{C}^n \times [\theta_0 - \epsilon, \theta_0 + \epsilon]$, for small intervals of g_k-length $O(1)$ covering S^1 (it is equivalent to consider the product metric g' in $\mathbb{C}^n \times S^1$ with factors the euclidian and the spherical). Notice that by definition the foliated A.H. theory coincides with the A.H. theory associated to the metric retraction for g_0 (or g'); it is clear that fro this retraction is not strongly equivalent to $\bar{i}$, but the bounds in the charts imply that they are equivalent, so we can apply lemma 3.30.

Moreover, $\partial\phi_k$ y $\partial_0\phi_k$ are related by a bundle map $q^{\bar{i},i_0}$ (the comparison of the euclidean metric with g) and the same happens with $\nabla_D\partial\phi_k \approxeq \partial^2_{\mathrm{sym}}\phi_k$ and $\partial_0^2\phi_k$, but here in the approximate sense and in a small enough tubular neighborhood.

Notice that in light of observation 3.32 this is something characteristic of the strongly equivalent theories and not of the equivalent ones. But we have:

$$\partial_0^2\phi_k \approxeq q_2^{\bar{i},i_0}(\partial^2_{\mathrm{sym}}\phi_k) + d_D q^{\bar{i},i_0}(\partial\phi_k).$$

The second term vanishes in $\{0\} \times S^1$ and the first one is bounded by below, and thus in a neighborhood of small enough radius ρ the mentioned result holds.

The first consequence is that in $\mathcal{N}_\rho(\{0\} \times S^1)$, the variation of $\partial\phi_k$ is equal to that of $\partial_0\phi_k$ en . Thus, if ρ has been chosen small enough the zeros of $\partial_0\phi_k$ are diffeomorphic to the zeros of $\partial\phi_k$ (that is, only an S^1 in the domain of $\varphi_{k,\lambda}$). Besides, since both holomorphic components are related by a bundle map which is the identity in the $\mathbf{0}$ section, the zeros of $\partial_0\phi_k$ are exactly $\{0\} \times S^1 = \Sigma_n(\phi_k)_\lambda$. For the same reason the zeros of $\partial_{J_k}\phi_k$ coincide with the zeros of $\partial\phi_k$.

Working with the foliated theory, we write ϕ_k in the coordinates z_k, θ_k. By simplicity we omit the subindex for the angular coordinate.

At this point it is reasonable to interpret our approximation problem for ϕ_k as an approximation problem for a smooth family of function with parameter S^1 of A.H. functions from $\mathbb{C}^n$ in $\mathbb{C}$. Let us point out that in principle the image of ϕ_k lays in $\mathbb{CP}^1$, and we want to find charts $\varphi_k \colon \mathbb{C} \hookrightarrow \mathbb{CP}^1$ so that $\varphi_k^{-1} \circ \phi_k(\mathcal{N}_\rho(\Sigma_n(\phi_k)_\lambda))$ has image uniformly bounded, so we obtain uniform bounds for the family of functions. In principle it is possible to find points in $\mathbb{CP}^1$ missed by the image of the solid torus $\mathcal{N}_\rho(\{0\} \times S^1)$. Finding balls of uniform radius missing the image is of course a uniform transversality problem.

Indeed, we fix say, $\infty = [0 : 1] \in \mathbb{CP}^1$, that defines an obvious A.H. sequence of strata $\mathbb{P}S_k^\infty$ in $\mathcal{J}_D^0(M, \mathbb{CP}^1)$. Its pullback to $\mathcal{J}_D^1(M, \mathbb{CP}^1)$ defines a sequence of strata of $\mathbb{P}S_k^\infty$ which is also A.H. One checks that the sequence is transverse to $\mathbb{P}\Sigma_{k,n}$, so the intersection defines $\mathbb{P}\Sigma_{k,n}^\infty$ an A.H. sequence of strata. We can decompose $\mathbb{P}\Sigma_{k,n}$ into this intersection and its complementary subset $\mathbb{P}\Sigma_{k,n}^{\mathbb{C}}$. After pulling it back to $\mathcal{J}_D^1 E_k$, we obtain $\Sigma_{k,n}^\infty, \Sigma_{k,n}^{\mathbb{C}}, Z_k$ an finite Whitney A.H. quasi-stratification. We can make τ_k uniformly transverse to it. Therefore $\phi_k(\mathcal{N}_\rho(\Sigma_n(\phi_k)_\lambda))$ will be at least at distance η of $\infty \in \mathbb{CP}^1$, for $k >> 0$.

With this observation we can suppose that $\phi_k(z_k, \theta)$ is an A.H. sequence of maps in the coordinates z_k, θ and with bounded image in $\mathbb{C}$).

In this point and following the ideas of [**12**] and [**50**], we can make a first choice of function ϕ_k' satisfying all the requirements of the statement of this proposition, but its closeness to ϕ_k.

Let us call $H_\theta(z_k)$ to the quadratic form associated to $\frac{1}{2}\partial_0\partial_0\phi_k(0, \theta)$, the foliated hessian in the points of $\{0\} \times S^1$. ϕ_k' is defined interpolating between $H(z_k) + \phi_k(0, \theta)$ and ϕ_k in a suitable annulus (as we did with the distribution and almost-complex structure).

ϕ_k' is J_k-holomorphic in the corresponding tubular neighborhood.

Regarding the difference between ϕ_k and $H(z_k) + \phi_k(0, \theta)$, we simply observe that on each leaf $\mathbb{C}^n \times \{\theta\}$ the second is –in the approximate sense– the Taylor expansion to order 2 of the first. Thus

$$\phi_k(z_k, \theta) - (H(z_k) + \phi_k(0, \theta)) = O(c_k^{-1/2}(|z_k| + |z_k|^2)) + O(|z_k|^3).$$

In the points where ϕ_k' coincides with $H(z_k) + \phi_k(0, \theta)$ one has $|\partial\phi_k'| \geq |\bar\partial\phi_k'|$, where the equality only holds in the points of $\Sigma_n(\phi_k)$: for ∂_0 and $\bar\partial_0$ the assertion is evident; the desired result is obtained by observing that the derivative of ϕ_k along $\frac{\partial}{\partial\theta}$ is bounded by above and the quadratic form H bounded by below. By choosing appropriately the size of the annulus where the interpolation occurs, the mentioned inequality holds for ϕ_k that is still A.H. and with the required transversality problem.

It is possible to choose ϕ_k' with better properties; basically instead of taking the holomorphic part of the Taylor expansion to order 2, we take the whole series. Being more precise and following th ideas of S. Donaldson y D. Auroux ([**12, 2**]), we observe that the restriction of ϕ_k to on each leaf $\mathbb{C}^n \times \{\theta\}$ is A.H. A function $H'(z_k, \theta)\colon B(0, \rho') \times S^1 \to \mathbb{C}$, $r' < r$, exists, so that:

(1) H' is smooth.
(2) $H_\theta'\colon B(0, \rho') \subset \mathbb{C}^n \to \mathbb{C}$ is holomorphic for every $\theta \in S^1$.
(3) For all $j \in \mathbb{N}$ a positive constant C_j exists such that for $k >> 0$ the partial derivatives up to order j of $\phi_k - H'$ and bounded by $C_j c_k^{-1/2}$, or in other words, $\phi_k \approxeq H'$ as functions from $B(0, \rho') \times S^1$ to $\mathbb{C}$.

This result is a consequence of a parameter version of the solution to the $\bar\partial$-problem in the ball of radius 1 in $\mathbb{C}^n$. For the sake of completeness, we include in appendix A a rather elementary proof of the parametric solution (corollary A.2) out of the usual solution to the $\bar{\partial}$-problem.

ϕ'_k is defined interpolating between ϕ_k and H'. Since ϕ'_k approximately coincides with ϕ_k, the points where $\partial_0\phi'_k$ vanishes approximately coincide with the vanishing set for $\partial\phi_k$. Moreover, we can add a perturbation of size $O(c_k^{-1/2})$ to make them match (it would be a translation on each leaf). Also the hessian $\frac{1}{2}\partial_0\partial_0\phi'_k$ is uniformly bounded by above and below.

The previous properties for ϕ'_k remain valid if we add perturbations of size $O(c_k^{-1/2})$ that are leafwise holomorphic and at least quadratic at the origin. In particular, with one of these we can make the Hessian H'_θ at the origin to have different eigenvalues (generic property).

The only remaining step is to apply Morse's lemma with parameter to find a map $\Psi\colon \mathbb{C}^n \times S^1 \to \mathbb{C}^n \times S^1$, $(z_k, \theta) \mapsto (w(z_k), \theta)$,so that $\phi'_k(z_k(w)) = \phi'_k(0,\theta) + (w^1)^2 + \cdots (w^n)^2$.

We simply point out that a proof of Morse's lemma is based in the diagonalization of symmetric matrices (see for example [**42**]). In each of the steps of the algorithm it is necessary to make a linear transformation so that the upper left entry of certain symmetric matrix M_θ is non-vanishing; it is at this point where we use that the eigenvalues are different (so that the eigenspace have complex dimension 1 and the linear transformation is reduced to choose certain eigenspace, being this choice smooth in θ). □

Combining the previous propositions we obtain the existence of "Lefschetz pencils" structures for calibrated manifolds.

Definition 8.4. *In a calibrated manifold a chart $\varphi\colon \mathbb{C}^n \times \mathbb{R} \to M$ is said to be compatible with ω (we speak about compatible coordinates) if in the origin D_h matches D and ω is positive and of type $(1,1)$ w.r.t. J_0.*

As we pointed out before, we use these charts when we do not want to make any reference to the compatible almost-complex structures J, so that the concept of A.H. coordinates does not make much sense. It is obvious that a chart in which J_0 matches J in the origin is compatible with ω.

Proof of theorem 1.10. Let J be a c.a.c.s. and τ_k an A.H. sequence of $\mathbb{C}^2 \otimes L_k$ uniformly transverse to the quasi-stratification of $\mathcal{J}^1_D E_k$ with strata $Z_k, \Sigma^\infty_{k,n}, \Sigma^{\mathbb{C}}_{k,n}$. Once propositions 8.1 and 8.3 have been applied, for large values of k the triple $(A_k, \phi'_k, \Sigma_n(\phi'_k))$ defines a Lefschetz pencil structure. We simply mention that the genericity of $\phi'_k(\Sigma_n(\phi'_k))$ follows from one of the perturbations of lemma 8.3 independent of the holomorphic coordinates. We also note that proposition 8.1 gives A.H. coordinates which are not necessarily compatible with ω (the problem is that $TA_k \cap D$ is not necessarily J-complex). In any case, following the ideas of S. Donaldson or F. Presas ([**14, 50**]) it is possible to modify the sequence of sections to obtain coordinates compatible with ω in the points of A_k with the needed description for A_k and ϕ'_k. □

Very much as for contact manifolds, the existence of pencils through A.H. sequences of sections of $\mathbb{C}^2 \otimes L_k$ allows to relate the topology of any two (sequences of) divisors of (M, D, ω) constructed as the zeros of A.H. sections of L_k (possibly for different almost complex structures).

Proposition 8.5. *Let (M^{2n+1}, D, ω) be a closed calibrated manifold of integer type and let J_1 and J_2 c.a.c.s Let τ_k^1 and τ_k^2 two sequences of A.H. sections of L_k w.r.t. J_1 and J_2 respectively, both uniformly transverse to $\mathbf{0}$. Then a natural number $K \in \mathbb{N}$ exists such that for $k \geq K$ the "divisor" W_k^1 is cobordant to W_k^2 by a cobordism that amounts to adding only n-handles.*

In particular we deduce that $H_i(W_k^1; \mathbb{Z}) \cong H_i(W_k^2; \mathbb{Z})$ and $\pi_i(W_k^1) \cong \pi_i(W_k^2)$, for $i = 0, ..., n-2$ (a weaker result than the hyperplane Lefschetz theorem for divisors of theorem 1.5).

Proof. The proof is almost word by word the one given for contact manifolds in [**50**]. For the sake of completeness, we give a rough idea of it goes.

Let us assume $J_1 = J_2$. With the sections τ_k^1 and τ_k^2 an A.H. sequence of sections (τ_k^1, τ_k^2) of $\mathbb{C}^2 \otimes L_k$ is constructed. We perturb it to obtain a Lefschetz pencil but without normal forms for $B_k := \Sigma_n(\phi_k)$. If the perturbation is small enough compared with the amount of transversality η for both sequences, the new sequences –to be denoted as the old ones– will give divisors isotopic to the initial ones. Thus we can assume 1-genericity for (τ_k^1, τ_k^2). In the corresponding Lefschetz pencil $\phi_k \colon A_k \to S^2$ we have to compare the fibers over 0 and ∞. The cobordism will be the inverse image of a segment joining both points. Being more precise, we need to blow up M in the points of A_k and along the *complex* directions of D; the tubular neighborhood of the base locus has fiber $\mathbb{C}^2$ and each point of the $\mathbf{0}$-section (base locus) is substituted by a $\mathbb{CP}^1$. We notice that this operation only occurs at differentiable level, and we do not put any calibrated structure in the $\tilde{M}$.

The immersed curve $\phi_k(B_k)$ splits S^2 into connected components isomorphic to disks. It is clear that fibers (now we really have fibers in we work in $\tilde{M}$) over points in the same connected component are isomorphic, because ϕ_k is a submersion there. Therefore, we must study what happens when the segment joining 0 and ∞ crosses $f(B_k)$.

Now we make the necessary perturbation to obtain normal forms for ϕ_k in the points of B_k. It does not affect the points of W_k^1 and W_k^2.

If the segment intersects B_k in a point b, we take a chart centered at b and modify it so that $\phi_k'(z, s) = s + iO(s^2) + (z^1)^2 + \cdots (z^n)^2$ (the cobordism occurs in an arbitrarily small neighborhood of the origin of the coordinate chart). We also alter the segment to make it coincide with the imaginary axis of $\mathbb{C}$ ($f(B_k)$ is tangent to the real axis at the origin)

With this expression for the function one can construct an appropriate Morse function for the cobordism and compute the index of the critical point, which is n (see [**50**]).

When J_1 and J_2 are different we just need to show that once a distance in the space of c.a.c.s. have been fixed, for each J a positive ϵ_J exists so that if J' is at distance of J smaller than ϵ_J, then we can find A.H sequences τ_k and τ'_k whose corresponding sequences of divisors are isotopic for $k >> 0$. We refer to the reader to [**50**] to check this assertion. □

9. Almost-complex foliated manifolds

Let (M, D, J, g) be an almost-complex manifold for which D integrates into a foliation $\mathcal{F}$. In this situation we do not need to use a retraction for $T^*M \to D^*$ to define a covariant derivative in the latter bundle. On each leaf we consider $g_{|\mathcal{F}}$ and its Levi-Civita connection. This gives rise to a new A.H. theory that is the most natural one in this situation. We use the subindex $\mathcal{F}$ to denote the operators associated to this theory.

We also have all the strong transversality and normal forms results that we had proven for the intrinsic theory associated to the metric retraction. We do not need to repeat all the constructions.

For any E_k very ample sequence of locally splittable hermitian bundles, we have the bundle maps

$$\bar{r}_j (D^{*1,0})^{\odot j} \otimes E_k \to (\bar{D}^{*1,0})^{\odot j} \otimes E_k,$$

induced by the retraction $\bar{r}$ associated to the metric. One checks in A.H. coordinates (adapted to the metric), that not only τ_k is A.H. if and only if is A.H. for the foliated theory, but $\bar{r}_j(\partial^j_{\mathrm{sym},\mathcal{F}} \tau_k)$ approximately coincides with $\partial^j_{\mathrm{sym}} \tau_k$ as well. Thus, the image of $j^r_{\mathcal{F}} \tau_k$ by the corresponding bundle map approximately coincides with $j^r_D \tau_k$. Also the corresponding T-B-A quasi-stratifications are preserved by the bundle map $\mathcal{J}^r_{\mathcal{F}} E_k \to \mathcal{J}^r_D E_k$ induced by $\bar{r}$. Therefore, if $j^r_D \tau_k$ is uniformly transverse to it, for $k >> 0$ large enough $j^r_{\mathcal{F}} \tau_k$ will also have this property.

If we decide to repeat all the constructions of the intrinsic theory, it is convenient to use charts adapted to the foliation in the whole domain and not only in the origin. The advantage that we get by identifying D_h with D is that the local perturbations $w\tau^{\mathrm{ref}}_{k,x,j}$ are constant along D_h and thus along D. That means that even for r-jets ($r \geq 0$) we can work with C^{r+h}-A.H. sequences, ($h \geq 2$), instead of considering $C^{\geq r+h}$-A.H. sequences that require control in all the derivatives.

Notice that for certain foliated a.c. manifolds some of the geometric results of sections 7,8 are a direct corollary of the 1-parametric results for symplectic manifolds. This is something we mentioned in the introduction that can now be stated in a more precise way.

If we want to avoid local considerations we have to restrict ourselves to 1-parameter families of symplectic manifolds. In other words, the foliated calibrated manifold $\mathcal{M}(M, \omega, \varphi)$ is the mapping torus of a symplectomorphism φ of (M, ω).

$$\mathcal{M}(M, \omega, \varphi) := \frac{M \times [-1, 1]}{\sim_\varphi}$$

To construct divisors W (or more generally Lefschetz pencils (A, f, B)), we just need to find a smooth family W_t, $t \in [-1, 1]$, with $\varphi(W_1) = W_{-1}$. We fix J_{-1} compatible and construct $W_{k,-1}$ as the zero set of an A.H. sequence τ_k uniformly transverse to $\mathbf{0}$. It is straightforward that $\varphi_*^{-1}(J_{-1})$ is compatible with ω, and that $\tau_k \circ \varphi$ is $\varphi_*^{-1}(J_{-1})$-A.H. and uniformly transverse to $\mathbf{0}$. The zeros are the symplectic submanifolds $W_{k,1} = \varphi^{-1}(W_{k,-1})$. The uniparametric results guarantees –for k large enough– the existence of smooth families $W_{k,t}$ interpolating between $W_{k,-1}$ and $W_{k,1}$.

For Lefschetz pencils –at least concerning genericity– we equally interpolate between the sequences τ_k and $\tau_k \circ \varphi$ of $\mathbb{C}^2 \otimes L_k$. Since φ is an $\varphi_*^{-1}(J) - J$-complex map, uniform transversality of $j^1_D \tau_k$ implies the same result for $j^1_D(\tau_k \circ \varphi)$ (where the amount of transversality obtained is related by the norm of φ). The isotopy result in [**14**] (that can be modified to get not only continuity, but also smoothness) gives an interpolation $(A_{k,t}, \phi_{k,t}, B_{k,t})$ between the triples $(A_{k,1}, \phi_{k,1}, B_{k,1})$ and $(A_{k,-1}, \phi_{k,-1}, B_{k,-1})$, where $A_{k,t}, B_{k,t}$ are symplectic and $\phi_{k,t}$ A.H. for certain a.c. structures J_t. The computation of normal forms is –have we have seen– a natural generalization of that for symplectic manifolds. Therefore, the constructions of Lefschetz pencil structures in this case can be seen as a corollary of the A.H. theory for symplectic manifolds.

9.1. Calibrated foliations in closed 3-manifolds. We want to write explicitly some of our applications for 3-dimensional calibrated foliations (smooth taut foliations).

When a foliation in a 3-dimensional manifold is defined, the changes of coordinates are usually asked to be leafwise C^r and at least continuous in the transverse direction. We speak of a foliation defined by a C^r cocycle.

Our theory needs smoothness. In other words, we need the foliation to be given by a smooth 1-form. In this point we can use a result that states that for any foliation given by a C^r cocycle (in particular C^∞) a new (conjugate) smooth structure can be chosen so that the foliation is defined by a C^r 1-form (see comment 1.1.2 in [**17**]).

By a recent result of D. Calegari [**8**], any foliation $\mathcal{F}$ defined by a C^r cocycle is isotopic to a foliation defined by a C^∞ cocycle and so that the leaves give smooth immersions with continuous variation in the C^∞ topology. We can apply the result of the precedent paragraph to this new foliation $\mathcal{F}'$ and thus obtain a smooth foliation in M^3. What is more, the new foliation is still taut if $\mathcal{F}$ had this property.

Therefore, our techniques can be applied too any topological taut foliation (we apply them to $\mathcal{F}'$ and undo the isotopy).

We write again some of the applications for 3-dimensional smooth taut foliations.

Theorem 1.5 gives us the existence of transverse cycles through any point and is of little importance because this is the characterization of a taut foliation ($M \neq S^2 \times S^1$).

Regarding Lefschetz pencils, the first observation is the absence of base points (the set has codimension 4).

Corollary 9.1. *Let* $(M^3, \mathcal{F})$ *be a taut foliation in a closed* 3*-dimensional manifold and* ω *a closed* 2*-form of integer type that dominates/calibrates the foliation.*

There exist pairs (f, B)*, where* B *is a transverse link and* $f\colon M^3 \to S^2$ *a smooth map such that:*

(1) f *is a submersion along* D *in* $M^3 - B$*.*
(2) $f(B)$ *is in general position (inducing in* S^2 *a particular CW-complex structure).*
(3) *On each point* $b \in B$ *there are coordinates* z, s *compatible with* ω *and holomorphic coordinates in* $S^2 \cong \mathbb{CP}^1$*, so that* $f(z,s) = g(s) + z^2$*, with* $g(0) = 0$*,* $g'(0) \neq 0$*.*

We now give an interpretation of the Lefschetz pencils as an uniparametric version of an existing construction for surfaces.

For any closed Riemann surfaceΣ we want to find the simplest kind of map to the simplest Riemann surface, that is, with $\mathbb{CP}^1 \cong S^2$. The map has to be a holomorphic branched covering $f\colon \Sigma \to S^2$, which local models of the form $z \mapsto z^2$ (index 2 points).

The uniparametric version, or else the foliation version of the previous result would be a map defining a branched cover on each leaf (with index 2 points). It is reasonable that the branching set is given by a 1-parametric family of divisors –i.e., a transverse link– and the ramification points –which will not be isolated– are asked to be a set of curves in S^2 in general position.

The Lefschetz pencil structure is thus a natural extension to smooth taut foliations of the mentioned result for Riemann surfaces.

CHAPTER II

A new construction of Poisson manifolds

1. Introduction and results

The use of almost complex methods in symplectic geometry together with new surgery techniques have increased notably our understanding of the topology of symplectic manifolds (see for instance the foundational papers [**26, 12, 24**]). These results constitute an extraordinary mixture of "soft" and "hard" mathematical ideas in the sense of Gromov [**27**]. In spite of all this success very little is known for nontrivial families of symplectic manifolds. Families of symplectic manifolds lead naturally to the notion of Poisson manifolds.

Definition 1.1. *A Poisson structure on a manifold M is a Poisson algebra structure on its sheaf of functions. That is to say, given two local functions f, g on M we define on its common domain of definition a bilinear bracket $\{f, g\}$ verifying the following properties:*

(1) *Skew-symmetry,* $\{f, g\} = -\{g, f\}$.
(2) *Leibnitz' rule,* $\{f, gh\} = g\{f, h\} + \{f, g\}h$.
(3) *Jacobi identity,* $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$.

Alternatively, a Poisson structure on a manifold M is given by a bivector Λ such that $[\Lambda, \Lambda] = 0$, where $[\cdot, \cdot]$ denotes the Schouten bracket (see for instance [**54**] and references therein). The Poisson bracket $\{f, g\}$ of two functions is given in terms of Λ by $\Lambda(df, dg)$. Moreover, the Poisson tensor Λ defines a natural bundle morphism $\#\colon T^*M \to TM$ whose range defines an involutive distribution $\mathcal{S}_\Lambda$ whose integral leaves are equipped with a canonical symplectic structure. Conversely, any foliation $\mathcal{S}$ by symplectic manifolds of a manifold M such that for any smooth function the hamiltonians of the restriction of the function to each leaf glue into a smooth vector field, induces a unique Poisson structure whose symplectic foliation is precisely $\mathcal{S}$ [**54**]. In this sense we make precise the idea above that Poisson structures on manifolds provide the geometrical setting to describe smooth families of symplectic structures.

Looking at the known examples of Poisson manifolds, we see that in most occasions the starting point for them is an algebraic structure (a Lie algebra, a cocycle, etc.) and then we construct the manifold whose Poisson structure is related with the initial algebraic one.

However it would be most interesting to explore the converse viewpoint. Given a manifold M determine the nontrivial Poisson structures that it supports. This is of course an extremely difficult task because of the intrinsic nonlinearity of Poisson structures. Some work in this direction has been already done by M. Bertelson who has studied in [**6**] the problem of characterizing regular foliations which arise from Poisson structures. In order to accomplish part of this task, another step which is worth taking is checking the possibility of extending the smooth topological constructions to the Poisson category. Some of them have already been carried to the symplectic setting; D. McDuff [**40**] has defined the blowing up of a symplectic submanifold. R. Gompf [**24**] has used the normal connected sum to construct symplectic manifolds with arbitrary fundamental group. In this sense symplectic geometry is "flexible" in sharp contrast with Kähler geometry. Of course, trivial families of symplectic manifolds can be constructed just by taking the product of a symplectic manifold M with an arbitrary compact manifold Q. Such trivial product Poisson structures will have the same fundamental group as M provided that $\pi_1(Q) = 0$. Hence unless we are looking for one-parameter families of symplectic manifolds, the existence of families with arbitrary fundamental group is trivial (not at all its classification).

Thus the problem of constructing Poisson manifolds with arbitrary fundamental group is reduced to the particular situation of codimension 1 symplectic foliations, and more concretely to the search of 5-dimensional compact Poisson manifolds of constant rank 4 with arbitrary fundamental group.

In the process of constructing them, we will introduce a surgery operation for Poisson manifolds that naturally extends Gompf's construction (theorem 4.4).

We will start recalling in section 2 how any orientable closed 3-manifold can be endowed with a regular rank 2 Poisson structure. This construction –based on classical results of foliation theory in 3-manifolds– contains the essential ideas that lead to a surgery construction for Poisson manifolds.

Roughly speaking, out of the normal connected sum for symplectic manifolds we will introduce a parametric version for Poisson manifolds (with compact parameter). In order to do that certain properties of *fibred Poisson structures* will be recalled (section 3), because the analog of the symplectic submanifold used in the symplectic case will be a *transverse fibred Poisson submanifold* (definition 3.1).

Section 4 is devoted to the proof of the main construction: any foliated manifold constructed as *the normal connected sum* admits a unique Poisson structure (unique in a sense to be made precise) extending the ones on the summands (theorem 4.4). To way to prove this result is clear. Without being very precise, we want to find certain models for the Poisson structures of the subsets to be identified. These models will be parametric versions of the symplectic ones, and we will be able to find them because in the symplectic case they are obtained with the aid of certain operators, that of course will also exist for compact families.

In section 5 we study the *modular class* (see [**57**]) of some of the constructed varieties via surgery. The results indicate the strong topological character of the construction because certain properties of the Poisson structures of the summands may very well change in the normal connected sum.

It is this flexibility what will allow us to prove in section 6 the main application of the surgery construction to be introduced.

Theorem 1.2. *Let G any group finitely presented. Then for any natural numbers $n, d \geq 4$, d even, a closed oriented Poisson manifold $(M^{n,d}, \Lambda)$ of dimension n and rank d exists such that $\pi_1(M^{n,d}) \cong G$. Moreover, this foliated manifolds have vanishing Godbillon-Vey class, but those with a codimension 1 symplectic foliation are not unimodular (and neither calibrated). They can also be chosen to admit spin structures.*

We will conclude this charter in section 7 with an application that we find of interest in light of the study of calibrated manifolds of the previous chapter. We will describe conditions under which the normal connected sum of two calibrated manifolds –which is a Poisson manifold with codimension 1 leaves– admits a lift to a calibrated structure extending the ones of the summands (theorem 7.1).

2. Poisson structures on 3-dimensional manifolds

A regular Poisson structure on a 3-manifold M^3 is just a foliation by surfaces with a leafwise smooth area form. In particular, if M^3 is orientable, finding such a structure turns out to be an easy problem of differential topology, whose non trivial part is to endow the manifold with the foliation. The problem of finding a codimension one foliation on an oriented 3-manifold is a classical one which is already solved. We now give an outline of a solution (in which we do not ask much to the foliation), because it essentially contains the ideas that give rise to a surgery construction for Poisson manifolds.

Recall that every orientable compact 3-manifold M can be obtained from S^3 by surgery on a link with components k_j. Moreover, the framings are of the form $(m_j \pm 1 l_j)$, where m_j, l_j are the meridian and longitude of the boundary tori. Notice that the components of the link can be chosen to be very close to the unknot and hence transversal to the Reeb foliation $\mathcal{R}$ of S^3, i.e., the knots are *submanifolds transversal to the leaves inheriting the trivial Poisson structure.* Once open tubular neighborhoods of k_j have been removed, in the solid tori $T_j = D^2 \times S^1$ to be glued, the boundary of the leaves of $\mathcal{R}$ will be non separating curves on ∂T_j (and cutting once the meridian). This curves are non-trivial in the homology of T_j, so we cannot hope to add a punctured surface to get closed leaves, but if we remove a small tubular neighborhood $\mathcal{N}_j$ of the longitude $\alpha_j = \{0\} \times S^1$ we can find a map $\phi_j : T_j - \mathcal{N}_j \to S^1 \times I \times S^1$ such that the image of the curve m_j on T_j goes to $S^1 \times \{0\} \times \{e\}$, the meridian of $S^1 \times I \times S^1 \subset D^2 \times S^1$. Hence, pulling back the Reeb foliation of $S^1 \times I \times S^1$ we get a foliation except inside a solid torus, where we again put a Reeb component. We have thus proved the following well known proposition.

Proposition 2.1. *Every oriented compact 3-manifold admits a regular rank 2 Poisson structure.*

Using the ideas above we see that any fibred knot of a 3-manifold gives a foliation with a Reeb component and a "modified" Reeb component, where instead of having disks approaching to the torus we have punctured oriented surfaces (the Seifert surfaces of the knot).

It is for dimensions bigger than 3 where surgery constructions are a powerful tool to construct manifolds with prescribed topology (for example arbitrary finitely presented fundamental group). Hence having one such construction compatible with Poisson structures would allow us to conclude the existence of Poisson manifolds with a wide range of different topological properties.

3. Fibred Poisson structures

We have seen in the previous section that to perform surgery in an oriented Poisson 3-manifold we do not need to worry about the Poisson tensor itself but only about extending the symplectic foliation we had. It is not difficult to propose a surgery technique for Poisson manifolds which is indeed an extension of the normal connected sum for symplectic manifolds. Roughly speaking, we will use a transversal submanifold that intersects symplectically with the leaves of the symplectic foliation. This will allow us to perform the normal connected sum along the symplectic submanifolds and we will show that the resulting manifold admits a Poisson structure determined (up to sum extent) by the ones we had initially. We will see that with the appropriate setting the proofs will be natural generalizations of those of Gompf [**24**].

3.1. Poisson structures compatible with fiber bundle structures. Let $\pi\colon P \to Q$ be a fiber bundle. We call a Poisson structure Λ_P on P *compatible with the fiber bundle structure* if the symplectic leaves of Λ_P are the fibers of π (hence the fibers are connected). We will also call the triple (P, π, Λ_P) a *fibers Poisson manifold.* If P is compact this is equivalent to saying that the space of leaves is a smooth manifold Q such that the projection $\pi\colon P \to Q$ is a submersion.

We begin by noticing that whenever one has a foliation, one can do the usual exterior calculus in the bundles associated to the distribution. In our case we will have a locally trivial fibration $\pi\colon P \to Q$ and the bundle we are interested in is the one of vertical vectors, i.e., the kernel of π. We will speak of vertical vector fields and k-forms, Lie derivatives in the direction of vertical vector fields and exterior derivative of vertical k-forms. We shall denote the set of vertical k-forms by $\Omega^k_{\text{fib}}(P \to Q)$, and by d_π the exterior vertical derivative (or just d if there is no risk of confusion). Recall that one can pullback vertical forms by fiber bundle morphisms and that any known relation involving Lie derivatives also holds for vertical vector fields and forms (it holds fiberwise and defines a smooth section of the corresponding bundle).

Let us denote the cohomology groups of the complex $(\Omega^*_{\text{fib}}(P \to Q), d_\pi)$ by $H^k_{\text{fib}}(P \to Q)$. We have the corresponding forgetful maps $f\colon \Omega^k(P) \to \Omega^k_{\text{fib}}(P \to Q)$, and $f\colon H^k(P) \to H^k_{\text{fib}}(P \to Q)$.

It is straightforward to check that a Poisson structure Λ_P on P compatible with the fibration $\pi\colon P \to Q$ is determined by a closed non singular vertical 2–form $\omega_P \in \Omega^2_{\text{fib}}(P \to Q)$ (and hence $[\omega_P] \in H^2_{\text{fib}}(P \to Q)$). We will call ω_P *the Poisson 2-form* (or just *the Poisson form*) of Λ_P.

There are some results about the cohomology $H^*_{\text{fib}}(P \to Q)$ that will be used later. We start by recalling that for a closed manifold with a metric, Hodge theory allows one to obtain for any k-form α a unique decomposition:

$$\alpha = d\beta \oplus \delta\eta \oplus \rho$$

where β is coexact, η exact and ρ harmonic, and all three are images of α by smooth operators. Moreover we also have relative Hodge theory for a pair (N, K), where N is a compact manifold and K a closed set (i.e, for forms with support contained in $N - K$). This implies that we also have the above results for a compact manifold N with non empty boundary and forms with support in the interior on N (to show it just double the manifold and apply relative Hodge theory).

When $\pi\colon P \to Q$ is a locally trivial fibration and P closed, we can also apply Hodge theory to get the same decomposition of eq. (3.1) above for vertical k-forms. We observe that any metric on P restricts to a metric on each fiber and there we can apply the usual Hodge theory. After gluing what we construct in this way fiberwise, we get again smooth projection operators because in a trivialization we are just working in a fiber with a smooth family of forms and metrics. If P is compact and $\partial P \neq \emptyset$, the relative Hodge theory $(P, \partial P)$ (where we use forms whose support do not intersect ∂P) also holds because on each trivialization (that we use to see that the construction is smooth) the boundaries are setwise identified. As a consequence, we see that for $\pi\colon P \to Q$ locally trivial and P closed (resp. compact with $\partial P \neq \emptyset$), a vertical closed form (resp. a closed form whose support does not intersect ∂P and hence vanishing in a neighborhood of the boundary) is exact if and only if it is fiberwise exact (resp. exact with potential form vanishing in a neighborhood of the boundary). The result also implies that for a smooth (compact) family of exact vertical k-forms one can find a smooth family of vertical $(k-1)$-forms whose exterior derivative is the initial family (we would be working with vertical forms on the direct sum of our initial bundle and the trivial bundle with rank the number of parameters of the family). If all the k-forms of the family vanished in a neighborhood of the boundary, the $k-1$ forms will also vanish in that neighborhood.

3.2. Transversal Poisson fibred submanifolds. A smooth Poisson submanifold [**56**] of a Poisson manifold (M, Λ_M) is defined as a triple (P, Λ_P, j) where $j\colon (P, \Lambda_P) \to (M, \Lambda_M)$ is a Poisson morphism embedding P into M. Besides these, there are submanifolds of a Poisson manifold which inherit a Poisson structure (the foliation induces a foliation by symplectic submanifolds which fits into a Poisson structure) and where the natural inclusion map is not a Poisson morphism. We will consider submanifolds of Poisson manifolds from this more general perspective. Thus, *a Poisson submanifold* of a given Poisson manifold (M, Λ_M) will be a submanifold intersecting the leaves in symplectic submanifolds and inheriting a Poisson structure (necessarily unique) from Λ_M (these are the natural generalization of the symplectic submanifolds of a symplectic manifold). In particular we will be dealing with a special class of Poisson submanifolds compatible with a given fibration.

Definition 3.1. *Let (M, Λ_M) be an n-dimensional Poisson manifold of rank d, (P, Λ_P) a Poisson manifold where P is compact and fibers over the $(n-d)$-dimensional manifold Q, and Λ_P is compatible with the fibration. An embedding $j\colon P \to (M, \Lambda_M)$ is said to embed (P, Λ_P) as a transversal Poisson fibers submanifold of (M, Λ_M) if:*

i. $j(P)$ is contained in the regular set of (M, Λ_M).

ii. $j(P)$ cuts transversally the symplectic leaves of (M, Λ_M).

iii. $j(P)$ inherits a Poisson structure from (M, Λ_M) that coincides with Λ_P.

The existence of such a submanifold implies that the symplectic leaves of (M, Λ_M) are nicely arranged in a neighborhood of the submanifold. To be more precise:

Lemma 3.2. *If $j\colon P \to (M, \Lambda_M)$ embeds the fibers Poisson manifold $(P, \Lambda_P) \to Q$ in M as a codimension r transversal Poisson fibers submanifold of (M, Λ_M), then its normal bundle, with the induced Poisson structure, is also a fibers Poisson manifold over Q.*

Proof. For each $x \in P$ let $S_M(j(x))$ be the symplectic leaf of Λ_M passing through the point $j(x)$, thus $\Lambda_{M|S(j(x))}$ is the inverse of a symplectic form $\omega_M(x)$ on $T_{j(x)}S(j(x))$ and $T_{j(x)}(j(P) \cap S(j(x)))^{\perp_{\omega_M}}$, the symplectic orthogonal of $T_{j(x)}(j(P) \cap S(j(x)))$, is a symplectic r-plane transversal to $T_{j(x)}j(P)$, so we can take it as model for the normal bundle $\nu(P)$ of the embedding. Moreover, for each leaf $S_P \subset P$, the restriction of this model of normal bundle is the corresponding model for the embedding of that leaf $S_P \subset S_M$. In fact, one can choose any compatible almost complex structure for the regular set of (M, Λ_M) and consider the leafwise associated metric. $T_{j(x)}(j(P) \cap S(j(x)))^{\perp_{\omega_M}}$ is then the orthogonal complement of $T_{j(x)}(j(P) \cap S(j(x)))$ with respect to this metric. We can use this leafwise metric to identify the normal bundle with a small enough tubular neighborhood of $j(P)$. This open set inherits a Poisson structure which can be pulled back to the normal bundle (hence the Poisson structure depends on the almost complex structure, but different choices give isomorphic structures). The local triviality of the fiber bundle $\tilde{\pi}\colon \nu(P) \to Q$ follows from that of the associated sphere bundle, which is a compact manifold (and the projection a surjective submersion). $\square$

4. The main construction: Poisson surgery

Let (M, Λ_M) be an n-dimensional Poisson manifold of rank d and let (P, Λ_P) be a fibers compact $(n-2)$-dimensional Poisson manifold over the $(n-d)$-dimensional manifold Q (Λ_P is compatible with the fiber bundle structure $\pi\colon P \to Q$). Suppose we have two disjoint embeddings $j_a\colon P \to M$, $a = 1, 2$, that both embed (P, Λ_P) as a transversal Poisson fibers submanifold of (M, Λ_M). Assume that the normal bundles ν_a (using the model provided by Lemma 3.2 and considering the orientation induced by the Poisson bracket) have opposite Euler class. After identifying ν_a with a tubular neighborhood V_a of $j_a(P)$, any orientation reversing identification $\psi\colon \nu_1 \to \nu_2$ allows us to get a diffeomorphism $\varphi\colon V_1 - j_1(P) \to V_2 - j_2(P)$ preserving the orientation of the fibers (the disks) as the composition of ψ with the diffeomorphism $h(x) = x/\|x\|^2$ that turns each punctured normal fiber inside out.

Definition 4.1. *Let $\#_\psi M$ denote the smooth, foliated manifold, obtained from $M - (j_1(P) \cup j_2(P))$ by identifying $V_1 - j_1(P)$ with $V_2 - j_2(P)$ via the composition $h \circ \psi$. If M is a disjoint union $M_1 \coprod M_2$ and j_a maps P into M_a, the manifold will be called the normal connected sum of M_1 and M_2 along P (via $h \circ \psi$) and will be also denoted by $M_1 \#_\psi M_2$.*

It is easy to check that the diffeomorphism type (as a foliated manifold) it is determined by (j_1, j_2) and the orientation reversing identification $\psi\colon \nu_1 \to \nu_2$ (up to fiber preserving isotopy). Once one of these identifications has been chosen, the remaining possibilities are $[P, S^1] \cong H^1(P; \mathbb{Z})$.

4.1. Topological remarks. If we are given an orientation μ_M on (M, Λ_M), it determines in a neighborhood of $j_a(P)$, together with the Poisson structure, an orientation on P, and this one together with the restricted Poisson form ω_P, an orientation on Q. It is clear that if the orientations on Q obtained in this way from each neighborhood V_a are the same, then μ_M induces an orientation on $\#_\psi M$.

There are some very well know results about the topology of $M_1 \#_\psi M_2$ (see the remarks by Gompf [**24**]). First of all, $\#_\psi M$ is (oriented) cobordant to M. This is seen after identifying in the cobordism $M \times [0, 1]$ neighborhoods of $j_1(P)$ and $j_2(P)$ in the level $\{1\}$, and then rounding corners to get the cobordism manifold X.

Hence, the Pontrjagin numbers ($\#_\psi M$ oriented) behave additively, and in the even dimensional case the formuli for the Euler characteristic and signature are, respectively,

$$\chi(M_1 \#_\psi M_2) = \chi(M_1) + \chi(M_2) - 2\chi(P).$$

$$\sigma(M_1 \#_\psi M_2) = \sigma(M_1) + \sigma(M_2), \ (\#_\psi M \text{ oriented})$$

As it is the case for symplectic manifolds, if $\#_\psi M$ is oriented the surgery construction is compatible (choosing an appropriate framing) with spin structures, and we can conclude:

Lemma 4.2. *If M admits a spin structure and $H^2(P; \mathbb{Z})$ has no $\mathbb{Z}_2$–torsion, then there is a choice of ψ such that $\#_\psi M$ admits a spin structure extending the one on M.*

PROOF. See [**24**], proposition 1.2. □

4.2. Remarks regarding the foliation of $\#_\psi M$. If we start from a regular transversally orientable manifold M, then $\#_\psi M$ is also regular transversally orientable (orientability in regular Poisson manifolds is equivalent to transversal orientability), and its Godbillon-Vey class $GV(\#_\psi M, \Lambda)$ can be computed in terms of the one of M. In particular:

Lemma 4.3. *Let M be transversally orientable. Then $GV(M, \Lambda_M) = 0$ if and only if $GV(\#_\psi M, \Lambda) = 0$.*

PROOF. We know that be can remove disjoint fibers neighborhoods (closed) W_a of $j_a(P)$ such that we have an inclusion $i\colon M - (W_1 \cup W_2) \to \#_\psi M$ and both ends of $M - (W_1 \cup W_2)$ fiber over P (with fiber an annulus). The condition $GV(M, \Lambda_M) = 0$ implies, by naturality, the vanishing of the Godbillon-Vey class of $M - (W_1 \cup W_2)$. Since its ends are fibers, one can choose a representant β of the class vanishing on these ends and conclude the existence of a form γ vanishing on the ends and whose exterior derivative is β. Finally, extending β and γ to forms $\tilde{\beta}$ and $\tilde{\gamma}$ defined on M we obtain $d\tilde{\gamma} = \tilde{\beta}$, where $[\tilde{\beta}] = GV(M, \Lambda_M)$. The other direction is proven similarly. □

4.3. Constructing the Poisson form on $\#_\psi M$. Our final aim is to put a Poisson structure on $\#_\psi M$. To do that, we have to modify slightly the previous construction. Since we have to construct a symplectic structure on each resulting leaf, it is more convenient to use instead of the normal bundles (whose fibers have infinite area), the bundles ν_a^0 of disks of radius $\pi^{-1/2}$. We will compose ψ (that can be assumed to preserve the area of each fiber) with the map

$$i(x) = \left(\frac{1}{\pi\|x\|^2} - 1\right)^{1/2} x\,,$$

which turns each punctured disk inside out.

We notice that V_1, V_2 and Y, the image of $(V_1 \cup V_2) \times [0,1]$ in X (the cobordism between M and $\#_\psi M$), are locally trivial fiber bundles over Q. Any closed form $\omega \in \Omega^k_{\mathrm{fib}}(V_1 \cup V_2 \to Q)$ satisfying $j_1^*\omega = j_2^*\omega$ induces a form $\Omega_{\tilde{V}} \in \Omega^k_{\mathrm{fib}}(\tilde{V} \to Q)$, up to a choice of a compactly supported exact k-form $d\alpha, \alpha \in \Omega^{k-1}_{\mathrm{fib}}(\tilde{V} \to Q)$, where $\tilde{V} \subset \#_\psi M$ is the image of $V_1 \cup V_2$ in $\#_\psi M$. The way to get a representant $\Omega_{\tilde{V}}$ of this family is by retracting disjoint neighborhoods of $j_a(P)$ (containing V_a^0, the image of ν_a^0) onto $j_a(P)$ and extending this map to a smooth retraction $\rho\colon M \to M$ isotopic to the identity, which coincides with the identity out of a compact set of $V_1 \cup V_2$, preserves the fibers of V_a and commutes with $\hat{\jmath}_2 \circ \psi \circ \hat{\jmath}_1^{-1}$ on V_1 and V_2). This k-form is the restriction to $\tilde{V}$ of the one induced on Y by $\rho^*\omega$. Two different choices of the retraction will give raise to two k-forms whose difference will be a compactly supported element of $\Omega^k_{\mathrm{fib}}(\tilde{V} \to Q)$. To see that this compactly supported closed form is exact, it is enough to check it fiberwise. The procedure is the one described by Gompf. We recall that when we round corners to get the cobordism manifold Y, we can think of having added some new levels (i.e., we now have a map $p_2 \colon Y \to [0, 1+\epsilon])$ so that the level set $1+\epsilon$ is $\tilde{V}$, where the circumferences of radius $\pi^{-1/2}$ are identified. As we go from $1+\epsilon$ to 1 we identify circumferences of smaller and smaller radius until we reach the level set 1 where $j_1(P)$ and $j_2(P)$ are identified. The level sets corresponding to the values smaller than 1 are diffeomorphic to $V_1 \cup V_2$. Given another retraction ρ', to evaluate the difference of the k-forms $\rho^*\omega_{|\tilde{V}} - \rho'^*\omega_{|\tilde{V}}$, we homotope ("pushing down") the corresponding smooth oriented k-manifold $M_k \subset p_2^{-1}(1+\epsilon]$ (possibly singular) such that it lies in $p_2^{-1}([0,1])$ and in the level 1 is contained in $j_1(P) \times \{1\}$, cut open Y and project $(V_1 \cup V_2) \times [0,1] \to V_1 \cup V_2 \times \{0\}$, and in the zero level set integrate $\rho^*\omega - \rho'^*\omega$ over the correspondent manifold with boundary M'_k. But since the retractions were homotopic to the identity, both $\rho^*\omega$ and $\rho'^*\omega$ represent the same homology class as ω. This, together with the fact that $j_1^*\omega = j_2^*\omega$, implies that $\int_{M'_k} \rho^*\omega - \omega = 0 = \int_{M'_k} \rho'^*\omega - \omega$. Hence their difference integrates to 0.

Now we will see that this construction works in the Poisson category.

Theorem 4.4. *Let (M, Λ_M) be an n-dimensional Poisson manifold of rank $d \geq 2$ and let (P, Λ_P) be a compact $(n-2)$-dimensional Poisson manifold such that Λ_P is compatible with the fiber bundle structure $\pi \colon P \to Q$, where Q is a $(n-d)$-dimensional manifold. Let $j_a \colon (P, \Lambda_P) \to (M, \Lambda_M)$, $a = 1, 2$, be two disjoint embeddings of (P, Λ_P) as a transversal Poisson fibers submanifold of (M, Λ_M). Suppose that there is an orientation reversing isomorphism of the normal bundles $\psi \colon \nu_1 \to \nu_2$. Then $\#_\psi M$, the normal connected sum along the normal bundles of $j_a(P)$, can be given a canonical Poisson structure Λ, characterized as follows:*

Given disjoint identifications $\hat{\jmath}_a \colon \nu_a \to V_a$ of normal bundles with tubular neighborhoods V_a of $j_a(P)$ that send fibers into leaves, if we denote by $\tilde{V}$ the image of $V_1 \cup V_2$ in $\#_\psi M$, $\tilde{V}$ is a locally trivial fiber bundle with base space Q. Then, there exists a unique fiber isotopy class of Poisson forms on $\tilde{V}$ containing elements ω satisfying the following characterization:

(1) *Let $\Omega_{\tilde{V}}$ be any of the 2-forms induced in $\tilde{V}$ by ω_M (as shown in the previous paragraph). Then $\omega - \Omega_{\tilde{V}} \in \Omega^2_{fib}(\tilde{V} \to Q)$ (which is closed) has compact support and is exact (it does not depend on the representative).*
(2) *The identification $\hat{\jmath}_1 : \nu_1 \to V_1 \subset M$ can be chosen in such a way (i.e., isotopic (rel. $j_1(P)$) by an isotopy with compact support) that the Poisson 2-form ω_M is $SO(2)$-invariant on $V_1^0 = \hat{\jmath}_1(\nu_1^0)$, with ν_1^0 the open disk bundle of radius $\pi^{-1/2}$, and on the closure of each fiber of V_1^0 it is symplectic with area t_0 independent of the fiber (we can isotope the initial embedding into the second one fixing the complement of each disk of radius $r > \pi^{-1/2}$). The forms $(1-s)\omega_M + s\pi^*\omega_P$, $0 \leq s < 1$, are all Poisson on the closure of V_1^0.*
(3) *There is a closed vertical 2-form ζ with compact support in $V_2^0 = \hat{\jmath}_2(\nu_2^0)$, with ν_2^0 the open disk bundle of radius $\pi^{-1/2}$, such that for all $t \in [0, t_0]$ the form $\omega_M + t\zeta$ is Poisson on both $V_1 \cup V_2$ and $j_2(P)$.*

(4) *There is a map $\chi\colon \nu_2 \to \nu_2$ (preserving the disks) isotopic to the identity by an isotopy with support in ν_2^0, such that outside of a compact subset K of V_1^0, the map $\varphi = \hat{\jmath}_1 \circ \psi \circ i \circ \chi \circ \hat{\jmath}_2\colon V_1^0 - j_1(P) \to V_2^0 - j_2(P)$ (where $\hat{\jmath}_1$ is as in point 2) is Poisson with respect to the Poisson form $\tilde{\omega}_M = \omega_M + t_0\zeta$ on M (i.e, we modify the embedding $\hat{\jmath}_2$ to $\chi \circ \hat{\jmath}_2$). The manifold $\#_\psi M$ is obtained from $(M - (K \cup j_2(P)), \tilde{\omega}_M)$ by gluing via φ (it follows that ω equals ω_M on the image on $\#_\psi M$ of the complement of $V_1^0 \cup V_2^0$).*

Moreover, different choices of embeddings of the normal bundles are connected by an isotopy that preserves the isotopy class described above.

Finally, the form ω depends smoothly on ω_M, ω_P (and hence on j_1, j_2) and it can be constructed with each $V_a, a = 1, 2$ lying inside any preassigned neighborhood of $j_a(P)$.

Now we will devote the next paragraphs to prove Theorem 4.4.

4.4. The contraction operator. We recall that $\nu(P)$ is an $SO(2)$-bundle. Let $\tau_s\colon \nu(P) \to \nu(P)$, $0 \le s \le 1$ denote the multiplication by s on each disk and let X_s denote the corresponding vector field. Since X_s is a vertical vector field with respect to the fiber bundle structure $\nu(P) \to Q$, we can define the operator $I\colon \Omega^k_{\mathrm{fib}}(\nu(P) \to Q) \to \Omega^{k-1}_{\mathrm{fib}}(\nu(P) \to Q)$ by

$$I(\rho) = \int_0^1 \tau_s^*(i_{X_s}\rho)ds$$

As usual, if ρ is closed and $j^*\rho = 0$, then $dI(\rho) = \rho$. It is also true that I commutes with any action preserving the $SO(2)$-bundle structure.

Corollary 4.5. *Let ω_1, ω_2 be two Poisson forms on $\nu(P)$ compatible with the fiber bundle structure $\nu(P) \to Q$ verifying $j^*\omega_1 = j^*\omega_2$ and inducing the same orientation on $\nu(P)$. There exist U_1, U_2 neighborhoods of P in $\nu(P)$ and an isomorphism $\phi\colon \nu(P) \to \nu(P)$ isotopic (rel. P) to the identity, by an isotopy with compact support, such that $\phi\colon U_1 \to U_2$ verifies $\phi^*\omega_2 = \omega_1$. If both forms already coincide over a compact subset C of P, we may assume the isotopy to have support on a preassigned neighborhood of the closure of $P - C$.*

The isomorphism ϕ can be chosen to depend smoothly on ω_1 and ω_2. In fact, if we are given smooth families $\omega_{1,r}, \omega_{2,r}$, $b \le r \le c$, coinciding on a fixed neighborhood of a given compact set C, and construct isomorphisms (as in the proof that follows) ϕ_b, ϕ_c verifying $\phi_b^*\omega_{2,b} = \omega_{1,b}$ and $\phi_c^*\omega_{2,c} = \omega_{1,c}$, then there exists a smooth family ϕ_r that verifies $\phi_r^*\omega_{2,r} = \omega_{1,r}$ on a fixed neighborhood of P and equals the identity in the chosen neighborhood of the closure of $P - C$.

Proof. As in the Darboux-Weinstein theorem proof, we consider the vertical closed 2-form $\eta = \omega_1 - \omega_0$ and the family $\omega_t = \omega_0 + t\eta$ (also vertical closed 2-forms). We can find a small neighborhood of P in which the ω_t are non-degenerate (because on P both forms induce the same orientation on the normal disk and because of the compactness of P). There, we know that $\eta = d\alpha$, with $\alpha = I(\eta)$, and we can find a family of vertical vector fields Y_t characterized by the equation $i_{Y_t}\omega_t = -\alpha$. After using a suitable bump function, this 1-parameter family defines a global flow Ψ_t on $\nu(P)$, leaving P stationary. Computing $\frac{d}{dt}(\Psi_t^*\omega_t)$ we conclude that $\Psi_t^*\omega_t$ does not depend on t near P. If the forms coincided in a neighborhood of C, η vanishes on that neighborhood. Regarding families, we see that in the procedure we made a choice of a bump function, and we can smoothly join two such choices. □

Corollary 4.6. *Let (M, Λ_M) be an n-dimensional Poisson manifold of rank d. Let (P, Λ_P) be a regular compact Poisson manifold of dimension $n-2$ which fibers over the $(n-d)$-dimensional manifold Q and such that Λ_P is compatible with the fiber bundle structure. Assume that $j_a\colon (P, \Lambda_P) \to (M, \Lambda_M)$, $a = 1, 2$, embeds (P, Λ_P) as a transversal Poisson fibers submanifold of (M, Λ_M). Suppose that both normal bundles are trivial and let $\psi\colon \nu_1(P) \to \nu_2(P)$ be a bundle isomorphism identifying them and preserving the orientation of the fibers. Then $\#_\psi M$ can be given a Poisson structure Λ.*

Proof. We can identify each normal bundle with $P \times \mathbb{R}^2$ in such a way that each disk $\{z\} \times D^2$ has area form $dx \wedge dy$. We also have isomorphisms $\hat{\jmath}_a\colon P \times D^2_\epsilon \to V_a$, $a = 1, 2$, and $\tilde{\psi}\colon P \times D^2_\epsilon \to P \times D^2_\epsilon$. The main point is that since the normal bundles are trivial, $j_a^*\omega_a + dx \wedge dy$ are Poisson structures that restrict to $j_a^*\omega_a$ on P. Hence, we can find a real number $\delta > 0$, and diffeomorphisms $\tilde{\jmath}_a\colon P \times D^2_\delta \to U_a$ with $\tilde{\jmath}_1^*\omega_1 = j_1^*\omega_1 + dx \wedge dy$, $\tilde{\psi}^*\tilde{\jmath}_2^*\omega_2 = j_1^*\omega_1 + dx \wedge dy$, $U_a \subset V_a$ neighborhoods of $j_a(P)$. Composing $\tilde{\psi}$ with the area preserving map $(r, \theta) \mapsto (\sqrt{\delta^2 - r^2}, -\theta)$ preserves the Poisson structure and hence allows us to define a Poisson structure on $\#_\psi M$. We notice that we could have equally asked our initial fiber bundle morphism ψ to reverse the orientation of the fibers, because by composing with the leafwise reflection $(r, \theta) \mapsto (r, -\theta)$ one can always reverse the orientation of a trivial bundle. □

Remark 4.7: In the above construction, the Poisson structure coincides with Λ_M on $M - (j_1(P) \cup j_2(P))$. But we have to allow perturbations in a neighborhood of one of the embeddings to have uniqueness up to isotopy.

The main obstruction to finally solve the problem posed in Thm. 1 in general is that one cannot put a global Poisson structure on ν_a induced by $j_a^*\omega_a$ and the symplectic structure on the symplectic orthogonals, unless the normal bundle is trivial. We can overcome this difficulty in the following way. We consider ν_a^0, the bundles of open disks of radius $\pi^{-1/2}$, and identify the punctured disks by composing i with ψ to get $\mathcal{B}$, an S^2-bundle with structural group $SO(2)$ whose fibers have an $SO(2)$-invariant area form ω_{S^2} that integrates into 1 on each of them. We have two embeddings $i_0\colon P \to \mathcal{B}$, $i_\infty\colon P \to \mathcal{B}$ with $\hat{\jmath}_1 i_0 = j_1$, $\hat{\jmath}_2 i_\infty = j_2$. Let us denote $E^0 = \mathcal{B} - P_\infty$ (resp. $E^\infty = \mathcal{B} - P_0$). Using Thurston's ideas (see [**41**], Thm. 6.3) we can construct a vertical 2-form η restricting to the above defined area form on each fiber: we consider a form β on $q : \mathcal{B} \to P$ representing the Poincar dual of P_0 so that it integrates to 1 on each fiber (sphere transverse to P_0).

It can be chosen to have support in a small neighborhood of P_0, so that it vanishes on P_∞. We take trivializations $h_k\colon q^{-1}(\mathcal{U}_k) \to \mathcal{U}_k \times S^2$ of $\mathcal{B}$ and a partition of the unity ρ_k subordinated to $\{\mathcal{U}_k\}$. Since $h_k^* \pi_{S^2}^* \omega_{S^2} - \beta = d\alpha_k$ on $q^{-1}(\mathcal{U}_k)$, $\eta = f(\beta + d\sum_k (\rho_k \circ q)\alpha_k)$, where f is the forgetful map $f\colon \Omega^2(\mathcal{B}) \to \Omega^2_{\text{fib}}(\mathcal{B} \to Q)$, satisfies the requirements. The result of averaging $\eta - q^* i_0^* \eta$ (both q, i_0 are maps lifting $\mathrm{id}\colon Q \to Q$) under the $SO(2)$-action is a vertical $SO(2)$-invariant 2-form, that we will still call η, such that it restricts to the canonical volume form on each sphere and $i_0^*\eta = 0$. We can even choose η so that $\eta_{|E^0}$ extends over ν_1 to a closed vertical form that is symplectic on the planes (fibers). We only need to pick β with support away of P_∞, so that on the intersection of that neighborhood with $q^{-1}(\mathcal{U}_i)$ ($\mathcal{U}_i$ contractible) α_k can be chosen to be $h_k^* \pi_{S^2}^* \alpha'$, for any α' with $d\alpha' = \omega_{S^2}$ on that neighborhood. In particular, the restriction of the 1-form $\alpha = 1/2(r^2 - \frac{1}{\pi})d\theta \in \Omega^1(\mathbb{R}^2 - \{0\})$ (given in polar coordinates) to the disk of radius $\pi^{-1/2}$ admits an extension to a form α' on $S^2 - \{0\}$ with $d\alpha' = \omega_{S^2}$.

The forms $\omega_t = q^* j_1^* \omega_1 + t\eta$ are non-degenerate for $0 < t \leq t_1$ because, as Thurston observed, $q^* j_1^* \omega_1$ is non-degenerate on the orthogonal to the tangent space of the spheres (which does not depend on t because it is determined by η). For a choice of η extending to ν_1 as described above, the forms ω_t will be symplectic near the closure of $E^0 \cong \nu_i^0$ in ν_1 for $t_1 \leq t$ small enough.

4.5. Comparing the Poisson structures on $\mathcal{B}, E^0$ and E^∞. Now that we have a family of closed non-degenerate 2-forms on $\mathcal{B}$, we would like to compare one of them with the ones defined on $E^0 \cong \nu_1^0$ and $E^\infty \cong \nu_2^0$ that come from ω_1 and ω_2. Following Cor. 4.5, for each t we could find neighborhoods $(\mathcal{W}_0^t, \omega_t)$ of P_0 and $(\mathcal{W}_\infty^t, \omega_t)$ of P_∞ which are Poisson equivalent to some neighborhoods (depending on t) of $(j_1(P), \omega_1)$ and $(j_2(P), \omega_2)$. But nothing guarantees that $\mathcal{B} = \mathcal{W}_0^t \cup \mathcal{W}_\infty^t$, for some t.

Instead, we use Gompf's construction again. On E^0, let $\varphi = I(\eta)$ and define the vertical vector fields Y_t, $0 < t \leq t_1$ by the condition $i_{Y_t}\omega_t = -\varphi$ (also defined in a neighborhood of the closure of E^0, if η was chosen to extend to ν_1). The key property is that these vector fields are $SO(2)$-invariant. For a fixed t_0, the flow Ψ_t, required to be the identity for $t = t_0$, is $SO(2)$-invariant and of course verifies $\Psi_t^* \omega_t = \omega_{t_0}$. In principle, we know that for any $SO(2)$-invariant compact set $K \in E^0$ there exists an interval J of t_0 in $(0, t_1]$ where the flow $\Psi\colon K \times J \to E^0$ is defined. But in can be shown that Ψ is defined on $E^0 \times [t_0, t_1]$. Given any point x on E^0, it determines an $SO(2)$-orbit on its fiber and hence a disk $D(x)$. We define:

$$A(x) = \int_{D(x)} \eta$$

and,

$$A_t(x) = \int_{D(x)} \omega_t\,,$$

where the forms are pulled back to the disk. The map $A\colon E^0 \to [0,1)$ is a smooth $SO(2)$-invariant proper surjection and it is clear that $A_t(x) = tA(x)$. Given $x \in E^0$, $t_0 \in (0, t_1]$ and $K = D(x)$ we obtain a flow as above on $D(x)$.

Let $D(\Psi_t(x))$ be the disk whose boundary is the $SO(2)$-orbit of $\Psi_t(x)$ (it is also $\Psi_t(\partial D(x))$). Then we have:

$$\begin{aligned} tA(\Psi_t(x)) &= A_t((\Psi_t(x))) = \int_{D(\Psi_t(x))} \omega_t \\ &= \int_{\Psi_t(D(x))} \omega_t = \int_{D(x)} \Psi_t^* \omega_t = \int_{D(x)} \Psi_{t_0}^* \omega_{t_0} = t_0 A(x), \end{aligned}$$

So we can conclude that $A(\Psi_t(x)) = \frac{t_0}{t} A(x)$. Since A, which is proper, decreases with the flow lines (with t increasing), these flow lines cannot abandon E^0 and hence Ψ is defined in $E^0 \times [t_0, t_1]$. The inequality $A(\Psi_{t_1}(x)) < \frac{t_0}{t_1}$ implies that choosing t_0 small enough, Ψ_{t_1} sends E^0 into any initially fixed tubular neighborhood of P_0. In particular, we choose t_0 so that $\Psi_{t_1}(E^0) \subset \mathcal{W}_0^{t_1}$. Hence $\hat{\jmath}_1 \Psi_{t_1}$ sends (E^0, ω_{t_0}) into $(\hat{\jmath}_1 \Psi_{t_1}(E^0), \omega_1)$. Actually, the Poisson morphism Ψ_{t_1} extends to a neighborhood of the closure of E^0, for suitably chosen η and thus it can further be extended to a diffeomorphism $\Psi_{t_1} \colon \nu_1 \to \nu_1$ isotopic to the identity by an isotopy with compact support (but only Poisson in a neighborhood of the closure of $E^0 \cong \nu_1^0$) .

The restriction of each ω_t to P_∞ induces also a Poisson structure, but in general $i_\infty^* \omega_t \neq j_2^* \omega_2$. But we can modify ω_2 in a neighborhood of $j_2(P)$ (ω_2 has not been involved in all the previous work) so that the above equality holds: we choose $\mu \colon \mathcal{B} \to \mathcal{B}$ an $SO(2)$-equivariant map lifting $id \colon P \to P$ such that μ fixes a neighborhood of P_∞ and collapses a neighborhood of P_0 to P_0. The composition of the restriction of $\hat{\jmath}_2^{-1}$ to V_2^0 with μ can be extended to a map λ from a closed neighborhood U_2 of V_2^0 in V_2 (a neighborhood of ∂U_2 is sent to P_0). We can then modify the Poisson structure of $(U_2, \omega_2) \subset (V_2, \omega_2)$ (without modifying the symplectic foliation), by adding to ω_2 a closed vertical 2-form ζ such that $\omega_2 + \zeta$ is non degenerate (and hence Poisson) and ζ vanishes in a neighborhood of ∂U_2 in U_2. If we call $\zeta = \lambda^* \eta$, then there exists $t_2 > 0$ (by the compactness of P) such that for all $0 \le t \le t_2$, $\tilde{\omega}_M = \omega_2 + t\zeta$ is non degenerate. To solve the problem we just need to pick our previous t_0 smaller than t_2 (and use of course $\tilde{\omega}_M = \omega_M + t_0 \zeta$). So we can glue to define a Poisson form ω on $\tilde{V}$ that satisfies all the requirements of theorem 4.4. To be more precise, we can find a map $\chi \colon E^\infty \to E^\infty$ isotopic to the identity by an isotopy (rel. P_∞) with support in ν_2^0 and Poisson, with respect to the forms ω_{t_0} and $\omega_M + t_0 \zeta$, in a neighborhood U_∞ of P_∞ (the map can actually be extended to a diffeomorphism of ν_2 isotopic to the identity). We glue using the map $\hat{\jmath}_2 \circ \chi \circ i \circ \psi \circ \Psi_{t_1}^{-1} \circ \hat{\jmath}_1^{-1} \colon V_1^0 \to V_2^0$, where Ψ_{t_1} and χ are thought as diffeomorphisms of the normal bundles (instead of having domain in the sphere bundle $\mathcal{B}$). The embeddings we finally use are $\hat{\jmath}_1 \circ \Psi_{t_1}$ and we modify $\hat{\jmath}_2$ by composing on the right with $\chi \colon \nu_2 \to \nu_2$. The only condition that needs to be checked is that the difference $[\omega - \Omega_{\tilde{V}}]$ (which by construction has compact support) is exact. As we saw ,it can be checked fiberwise. Thus, it is enough to show that

$$\langle \omega - \Omega_{\tilde{V}}, F \rangle = 0, \, \forall F \in H^2(\tilde{N}, \mathbb{Z}),$$

for all the fibers $\tilde{N}$ of $\tilde{V} \to Q$. This time we will not write the proof of equation 4.5 because it is, word by word, what Gompf showed ([**24**] pag. 547-548).

Concerning uniqueness, for any smooth family of Poisson forms $\omega_t \in H^2_{\text{fib}}(\tilde{V} \to Q)$, $t \in [0,1]$, such that the forms $\omega_t - \Omega_{\tilde{V}}$ are exact and compactly supported, the forms $\omega_t - \omega_0$ are exact in compactly supported cohomology (we can find common compact set W of $\tilde{V}$ containing all the supports). Hence we can find a family of compactly supported 1-forms α_t with $\frac{d}{dt}\omega_t = \frac{d}{dt}(\omega_t - \omega_0) = d\alpha_t$ and apply Moser's theorem to show that there is an isotopy with support in $W \subset \tilde{V}$ pulling back all the forms of the family to ω_0. The isotopy class of the form constructed using the described procedure is fixed. A different choice of $t \leq t_0$ can be absorbed using the parametrized version of Corollary 4.5. Equally, for any other choice $\hat{\eta}$ the family $\eta_s = s\eta + (1-s)\hat{\eta}$ is valid for the construction and we can again apply the same corollary to the family $\Psi_{s,t}$. Any other choices can be connected by smooth families, and the same happens when we change the embeddings of the normal bundles (preserving the foliations) and the choice of ψ (preserving the fiber bundle structure) by isotopic identifications.

Any Poisson 2-form ω verifying the four conditions of theorem 4.4 is isotopic to one constructed using the described procedure. We use ψ to recover the sphere bundle $\mathcal{B}$ and the modified embeddings to put in $\mathcal{B}$ a $SO(2)$-invariant Poisson form ω_{t_0} that agrees with ω_M on V_1^0 and with $\tilde{\omega}_M$ near $j_2(P)$, and that is also the result of applying the construction of the theorem with $\eta = \frac{1}{t_0}(\omega_{t_0} - q^*\omega_P)$ and $t_1 = t_0$. SO(2)-invariance implies that the fibers are ω_{t_0}-orthogonal to P_∞, so η is actually non-degenerate on the fibers at P_∞. Non degeneracy of ω_t at P_∞, $(t \leq t_0)$ follows from condition 3, applied first to TP_∞. We can extend η to ν_1 after shrinking the embedding $\hat{\jmath}_1 \colon \nu_1 \to M$ (rel. E^0)(non-degeneracy is an open condition). If we apply the construction to the embedding of condition 2 (shrinked (rel. E^0), if necessary), when $t = t_0$ we get the same embedding ($\Psi_{t_0} = id$) because it was already Poisson. The same happens for the second embedding (the correction χ equals the identity), provided we chose the given ζ defining $\tilde{\omega}_M$, rather than setting $\zeta = \lambda^*\eta$. Hence, the gluing map equals φ^{-1} near $j_2(P)$. The only price to pay is that ζ may not be $\lambda^*\eta$, for λ extending the restriction of $\hat{\jmath}_2^{-1}$ to V_2^0 (but we have that $j_2^*\zeta = i_\infty\eta$, and ζ can be assumed to vanish outside $\hat{\jmath}_2(\mathcal{B} - P_0) = V_2^0$). We will show that ω and ω', constructed using $\zeta' = \lambda^*\eta$, are isotopic (by an isotopy fixing the complementary of a compact set in $\tilde{V}$). It will be enough to show that the Poisson forms constructed using $\zeta_s = s\zeta' + (1-s)\zeta$, satisfy the condition 1 of the theorem. But that can be proven using the ideas that proved $\langle \Omega_{\tilde{V}} - \omega, F\rangle = 0 \; \forall F \in H^2(\tilde{N}; \mathbb{Z})$ (see [**24**] pag. 549). □

5. The modular class of $\#_\psi M$

Let (M, Λ_M) a Poisson manifold that we assume for simplicity to be orientable. An important invariant of the Poisson structure is the *modular class* [**57**]. Roughly speaking, it measures up to which extent the Poisson manifold admits a measure transverse to the leaves invariant by all the hamiltonian vector fields. The modular class belongs to the first group of Poisson cohomology of (M, Λ_M) (see [**54**]). For each volume form μ, a vector field (derivation) representing the modular class is defined by the formula

$$\phi_\mu \colon f \mapsto div_\mu X_f,$$

where X_f is the Hamiltonian vector field associated to f and div_μ the divergence with respect to μ.

A Poisson manifold with vanishing modular class is called unimodular. It is clear from what we said that a orientable Poisson manifold is unimodular if and only if there exists a volume form invariant by all the hamiltonian vector fields. Since (at least in the regular set) a volume form is the wedge product of the leafwise Liouville volume form (which is invariant by the hamiltonian vector fields) and a transverse volume form, the invariance of this transverse volume form is equivalent to the invariance of the whole form (and that is why we spoke about measuring the existence of an invariant transverse volume form).

Now let us assume that $\#_\psi M$ is oriented.

Proposition 5.1. *If $(\#_\psi M, \Lambda)$ is unimodular then (M, Λ_M) is also unimodular, but the converse is not true.*

Proof. We first notice that if we have an oriented Poisson manifold (N, Λ_N) and an open set U such that $(U, \Lambda_{N|U})$ is unimodular, then (N, Λ_N) will be unimodular if any of the invariant volumes on $(U, \Lambda_{N|U})$ can be extended to an invariant volume on (N, Λ_N). We will see that there are cases where (N, Λ_N) is unimodular but not all the invariant volumes on a certain open set can be extended to be invariant on (N, Λ_N). It is worth noticing that when (N, Λ_N) is a Poisson fibers manifold and U cuts each leaf in an open connected set (non-empty), then any invariant volume form in $(U, \Lambda_{N|U})$ extends to a unique invariant form on (N, Λ_N) [**57**]. It follows easily that in a general Poisson manifold (N, Λ_N), if we take a closed set V contained in an open one U, such that U (connected) is fibers and V intersects each fiber in a non-empty set whose complement (in the fiber) is connected, then (N, Λ_N) is unimodular if and only if $(N - V, \Lambda_{N|N-V})$ is unimodular. As a consequence, any perturbation of the Poisson bivector on V that preserves the foliation does not affect the unimodularity (resp. non-unimodularity) of (N, Λ_N). Hence, the unimodularity of $(\#_\psi M, \Lambda)$ implies the unimodularity of (M, Λ_M). If we start with (M, Λ_M) unimodular, since V_a fibers over Q, any invariant volume on (M, Λ_M) will determine a couple of volume forms on Q. It is clear that $(\#_\psi M, \Lambda)$ will be unimodular if and only if we are able to find an invariant volume form such that the induced volume forms on Q are the same. Though in general this not true (and we will end up the proof of the proposition constructing counterexamples), we will describe now some situations where this occurs.

Definition 5.2. *Let (M, Λ_M), (P, Λ_P) and $j_1 \colon (P, \Lambda_P) \to (M, \Lambda_M)$ be as in Theorem* **??**. *Assume that $j_1(P)$ has trivial normal bundle. Then once we have fixed a trivialization ψ of the normal bundle, we can apply our* *construction to the disjoint union of* (M, Λ_M) *with* (M, Λ_M). *We denote the resulting manifold by* $(M\#_\psi M, \Lambda_M\#\Lambda_M)$

Corollary 5.3. *Let* (M, Λ_M), (P, Λ_P) *be as in the above definition . Then* (M, Λ_M) *is unimodular if and only if* $(M\#_\psi M, \Lambda_M\#\Lambda_M)$ *is unimodular.*

To construct counterexamples we begin by proving the following lemma:

Lemma 5.4. *There exist Poisson fibers manifolds (actually symplectic bundles) with open sets having invariant volume forms which do not extend to invariant volume forms on the whole manifold.*

Proof. The idea is to start with our fibers open set, and then glue some of the fibers into a single one (so we are putting restrictions on the volume form we pull back from the base space). We consider the Poisson fibers manifold $S^{2n-1} \times D^2 \to S^{2n-1}$, where D^2 is the corresponding closed unit disk with its usual symplectic form (have in mind the case $n = 1$). For each point of S^{2n-1} we consider its image by the antipodal map and identify the boundaries of the corresponding fibers via a reflection (say, on the y-axis) $r_y \colon S^1 \to S^1$. The resulting manifold is a symplectic bundle over $\mathbb{R}P^{2n-1}$ with fiber the sphere with the usual area form (it can also be constructed by considering $S^{2n-1} \subset \mathbb{R}^{2n} \subset \mathbb{R}^{2n+1}$, taking a closed tubular neighborhood of fixed radius of $S^{2n-1} \subset \mathbb{R}^{2n+1}$ and identifying its boundary using the antipodal map and then rescaling the area form). If we remove all the equators we obtain the initial open disk bundle. In this open set, the invariant volume forms come from volume forms on S^{2n-1}, but only the ones invariant under the action of the antipodal map on S^{2n-1} extend to invariant volume forms on the whole manifold.

There is a third way of constructing these manifolds, starting from the final Poisson manifold, which gives much more examples. We choose $(Q, G, (F, \omega), \rho)$ where Q is a compact manifold, G is a normal subgroup of $\pi_1(Q)$ of finite index and ρ is a representation of $K = \pi_1(Q)/G$ in the group of symplectomorphisms of (F, ω) such that there are points in F with trivial stabilizers. Q_G, the cover of Q associated to the subgroup G is a principal K-bundle, so we can construct the associated bundle to the chosen representation ρ by symplectomorphisms. Our resulting manifold M is a symplectic bundle and hence a Poisson manifold, but as a bundle, since it has discrete structural group, it has the unique lifting property. Thus, if on the fiber over the base point x_0 of Q, we pick a point z with trivial stabilizer, the lifting to z of all the homotopy classes of paths based on x_0 gives us an embedding of Q_G in M (transverse to the fibers). On the fiber over x_0, the points close to z have trivial stabilizer which implies that the normal bundle to Q_G is trivial. We can even take as a tubular neighborhood the result of pushing a small disk around z using the unique lifting property, which gives us a symplectic subbundle. It is clear that the invariant volumes on a small tubular neighborhood of Q_G that extend to invariant volume forms on the whole manifold are those which come from K-invariant volume forms on Q_G. $\square$

Now we are ready to finish the proof of proposition 5.1:

To construct the counterexample we take two copies of any of the symplectic fibrations $(Q, G, (F, \omega), \rho) \to Q$ of lemma 5.4 (with F a surface) and consider in both the same embedding of Q_G. Now we fix a volume form μ on Q_G that descends to Q. Then we pick a point $z \in Q_G$ and consider a diffeomorphism $f : Q_G \to Q_G$ homotopic to the identity (rel. z) which is the identity in a neighborhood of the remaining points of the orbit of z and which does not preserve μ in z. We identify both embeddings of Q_G in M via f and perform the fibers connected sum using any framing ψ to obtain manifold which is non unimodular. If it was, an invariant volume form would induce a volume form $e^h\mu$ on Q_G both invariant by the action of K and the action of K conjugated by f, but this cannot happen at the point z. □

6. Poisson manifolds with arbitrary fundamental groups

Using the previous results we can prove the theorem for Poisson manifolds stated in the introduction that extends results for symplectic manifolds.

Proof of theorem 1.2. As it was remarked in the introduction, we only need to prove the case $n = 5$, $d = 4$ because Gompf already showed it for n even and $d \neq n - 1$ (multiplying one of its manifolds by an sphere of the appropriate dimension), and the odd higher dimensional cases follow from the 5-dimensional one (by multiplying by simply connected symplectic manifolds of the appropriate dimension in the case of a codimension 1 symplectic foliation).

We first recall Gompf's proof: one starts with a closed symplectic manifold $T^2 \times \Sigma_g$ such that G can be obtained by collapsing some elements of its fundamental group. The symplectic form is chosen so that these elements are the simple curves of some trivially embedded symplectic tori. The key step is that the manifold which is glued along each one of this tori is a rational elliptic surface (along one of its regular fibers), and the resulting fundamental group, which does not depend on the chosen framing, is the old one with the homotopy of these tori killed. It is worth recalling the topology of this rational elliptic surfaces. They are diffeomorphic to $\mathbb{C}P^2 \overset{9}{\#} (-\mathbb{C}P^2)$ and an example can be constructed by blowing up the nine points of $\mathbb{C}P^2$ where two generic cubics intersect. We get in this way a fibration $p\colon \mathbb{C}P^2 \overset{9}{\#} (-\mathbb{C}P^2) \to \mathbb{C}P^1$ whose fibers are the pencil of cubics generated by the two given ones. The general fiber is a smooth cubic (topologically a torus) and we also have 12 singular fibers which topologically are a sphere with a self intersection point (the result of collapsing a non-separating regular curve of the generic fiber). It is easy to check that the complement of a regular fiber is simply connected. Roughly speaking, the complement fibers over a disk so we only have to care about the fiber. Following [**33**] , we see that this complement can be constructed starting from $D^2 \times T^2$, $T^2 =< a > \times < b >$. Extending the fibration to a bigger disk (in $\mathbb{C}P^1$) containing a singular fiber, amounts to gluing a two handle (with some framing) over either a or b (we have 12 singular fibers, and 6 of the disks go over a and 6 over b). The last step is to glue a neighborhood of the regular fiber over ∞. Hence, any curve contained in a fiber is trivial in $p^{-1}(\mathbb{C}P^1 - \{0, \infty\})$.

To get our Poisson 5-manifold M with $\pi_1(M) \cong G$, we consider one of Gompf's manifolds (M_G, ω_{M_G}) with $\pi_1(M_G) = G$. We can also assume that $M_G = N_G \# \mathbb{C}P^2 \overset{9}{\#} (-\mathbb{C}P^2)$ and that the fiber removed is $p^{-1}(\infty)$. Let $M_1 = M_G \times S^1$ with the product Poisson structure (the vertical 2-form $p_1^* \omega_{M_G}$, that we rename as ω_{M_G}). In M_G, the fiber $p^{-1}(0) = T$ is a trivially embedded symplectic torus with symplectic form ω_0. Now let $M_2 = T \times S^3$ with the product Poisson structure coming for ω_0 and a Poisson structure of S^3 determined by the Reeb foliation and the usual volume form, and let $k \subset S^3$ be the unknot, which is a Poisson submanifold of S^3 transverse to the foliation. We consider the Poisson submanifolds $P_1 = T \times S^1 \subset M_1$, $P_2 = T \times k \subset M_2$. It is clear that both are transversal Poisson fibers submanifolds. Moreover, they are trivially embedded and any identification of k with the factor S^1 of P_1 identifies P_1 and P_2 as Poisson manifolds. Any identification between normal bundles will allow us to construct the corresponding connected sum along the normal directions. In this case, we have canonical framings; the one in P_1 comes from the projection $p\colon \mathbb{C}P^2 \overset{9}{\#} (-\mathbb{C}P^2) \to \mathbb{C}P^1$ and the one in P_2 from the zero-framing of the unknot. Using this framing and $< a, b, s >$ as base of $H_3(T \times S^1; \mathbb{Z})$ (the choice of s depends on the orientation we pick for M_1), any other framing is given by a triple $(l_1, l_2, l_3) \in \mathbb{Z}^3$. We will denote the obtained Poisson manifold by $M_1 \#_{(l_1, l_2, l_3)} M_2$. The computation of its fundamental group is mere routine, but we will do it anyway because this is not quite the manifold we are looking for. As usual, we apply Seifert-Van Kampen's theorem:

Let D_1 be the unit disk contained in $\mathbb{C}P^1$ and $W_2 = k \times D_2$ be a small tubular neighborhood of k in S^3. Let us call $V_1 = p^{-1}(D^1) \times S^1$, $V_2 = T \times W_2$. $M_1 - V_1 = (M_G - p^{-1}(D_1)) \times S^1$ and $\pi_1(M_G - p^{-1}(D_1)$ has the same generators as $\pi_1(M_G)$ and the same relations except from the one that assures that the loop $\hat{\alpha}$, a lift of $\alpha = \partial D_1$, is vanishing. $\pi_1(M_2 - V_2)$ is the free group generated a, b and by the loop $\beta = \partial \bar{D}_2$ generating the homotopy of $S^3 - W_2$. Now we see that the loop s generating the homotopy of S^1 in $(M_G - V_1) \times S^1$ goes to a curve isotopic to $k + l_3 \beta$. The curves $a, b \subset T \times \{x\} \subset M_2 - V_2$ are seen as the correspondent simple curves generating the homology of a fiber over a point in ∂D_1 plus some multiple of $\hat{\alpha}$. Finally, the loops $\hat{\alpha}$ and β are the same.

We probably did not get the desired manifold because we cannot conclude that $\hat{\alpha}$ is contractible, but we turned our initial problem of killing the generator of the homotopy of S^1 in $M_G \times S^1$ into a problem that amounts to kill a curve in a manifold whose topology we know quite well.

In M_G, we consider T_2, be the torus generated by the loops $\hat{\alpha} + a, b$. T_2 is a symplectic torus trivially embedded (the symplectic structure on $p^{-1}(0) \times D^2_{1+\epsilon}$ can be assumed to be the product symplectic structure). Applying Gompf's construction to M_G and a rational elliptic surface along the normal directions of T_2 and a regular fiber we get a symplectic manifold $\tilde{M}_G$. It is clear that $\pi_1(\tilde{M}_G) = \pi_1(M_G)$, but in $\tilde{M}_G$ we have a disk that bounds $\hat{\alpha}$ lying in $\tilde{M}_G - p^{-1}(D_1)$. Thus, if we do the fiber connected sum of $\tilde{M}_G \times S^1$ and $T \times S^3$ along P_1 and P_2 ($T = p^{-1}(0)$ is of course in $\tilde{M}_G$), we get a Poisson manifold $\tilde{M}_1 \#_{(l_1,l_2,l_3)} M_2$ such that $\pi_1(\tilde{M}_1 \#_{(l_1,l_2,l_3)} M_2) \cong G$. It is worth noticing that the diffeomorphism type of $\tilde{M}_1 \#_{(l_1,l_2,l_3)} M_2$ depends at most on l_3. To see that we observe that $M_2 - V_2$ is a tubular neighborhood of $T \times \hat{\beta}$, where $\hat{\beta}$ is a loop in the interior of $M_2 - V_2$ isotopic to β and thus $\partial(M_2 - V_2)$ has an S^1-bundle structure (over $T \times \hat{\beta}$) . Hence the diffeomorphism type of the connected sum is totally determined by the image in $\partial(\tilde{M}_1 - V_1)$ of the S^1-bundle structure of $\partial(M_2 - V_2)$ (because $\tilde{M}_1 \#_{(l_1,l_2,l_3)} M_2$ is the result of collapsing to a point the fibers of the described fibration), and these fibrations are classified by the value of l_3 (the authors do not know whether different values of l_3 yield different diffeomorphism types).

As we already observed, if we use Kummer surfaces instead of rational elliptic ones to construct $\tilde{M}_G$, both $\tilde{M}_1$ and M_2 can be given spin structures. For any such structures, since $H^2(P_i;\mathbb{Z})$ has no torsion, one can find integers $\bar{l}_1, \bar{l}_2, \bar{l}_3$ with $\tilde{M}_1 \#_{(\bar{l}_1,\bar{l}_2,\bar{l}_3)} M_2$ admitting a spin structure extending any given ones. □

Remark 6.1: In the examples above (dimension 5) there are three kinds of symplectic leaves. We have a family parametrized by S^1 which are diffeomorphic to $\tilde{M}_G - T_1$ and hence have G as fundamental group; we have another S^1-family of leaves diffeomorphic to $\mathbb{R}^2 \times T_1$ and both families fill open connected sets separated by a compact leaf $T_1 \times T$, where T is the closed torus of the Reeb foliation of S^3. Any of the non-closed leaves has the closed one as set of accumulation points.

7. An application to the construction of calibrated foliations

The normal connected sum of two calibrated Poisson manifolds is a regular Poisson manifold with codimension 1 leaves. The existence of a lift to a calibrated structure can be studied through a spectral sequence. In our case, we are just going to give sufficient conditions and an effective construction of the lift in that situation.

Theorem 7.1. *Let $(M_a^{2n+1}, \mathcal{F}_a, \omega_a)$, $a = 1, 2$ be two taut foliations of integer type. Let (P^{2n-1}, Λ_P) be a Poisson manifold which is a symplectic bundle over S^1 (connected fibers). Assume that we have two disjoint embeddings $i_a \colon P \to M_a$, $a = 1, 2$ as transversal fibers Poisson submanifolds of $(M_a^{2n+1}, \mathcal{F}_a, \Lambda_a)$ such that:*

1. *$H^2(P;\mathbb{Z})$ has no torsion.*
2. *The normal bundles of the embeddings $\nu_a(P)$ are trivial.*
3. *The positive 2-forms $\omega_{P,a} = i_a^* \omega_a$ define the same cohomology class in $H^2(P;\mathbb{Z})$ (we already know that they define the same leafwise 2-form).*

Then, for a choice of isomorphism φ, then there are Poisson structures Λ defined in the normal connected sum $M_1 \#_\varphi M_2$ that admit a lift to a taut structure of integer type ω.

Proof. Let us recall that when the normal bundles of the embeddings are trivial, the surgery technique works without perturbing the leafwise 2-form. Applying theorem 4.4 without modifying the structures we obtain Λ a leafwise non-degenerate closed 2-form on $M_1\#_\varphi M_2$.

We want to define the lift ω as the curvature of a line bundle with hermitian connection, whose choice it is clear from the conditions we imposed.

Taking any integer lift of ω_a, we have a choice (L_a, ∇_a) of (isomorphism class) of line bundle with hermitian connection such that $iF_a = \omega_a$.

The pullbacks $L_{P,a} = i_a^* L_a$ are isomorphic bundles. The reason is that for both, the curvatures $\omega_{P,a}$ define the same real cohomology class (condition (3)), and since the integer cohomology has no torsion, the $L_{P,a}$ are representatives of the unique isomorphism class of hermitian line bundles with connection associated to the cohomology class $[\omega_{P,1}] = [\omega_{P,2}]$.

The gluing map that defines $M_1\#_\varphi M_2$ identifies a set A_1 which is tubular neighborhood of $i_1(P)$ minus the submanifold (zero section) , with A_2, another tubular neighborhood of $i_2(P)$ minus section the zero, so that when we approach $j_1(P)$ we are leaving $j_2(P)$.

We just want to show that the identification $\varphi\colon A_1 \to A_2$ lifts to a bundle isomorphism $\Psi\colon L_{1|A_1} \to L_{2|A_2}$. Actually the existence of the isomorphism follows from the fact that $L_{P,1}$ and $L_{P,2}$ are isomorphic complex hermitian bundles.

Indeed, we can think of the annulus as a family $S_{a,t}$, $t \in (1, 0)$ of trivial circle bundles over $j_a(P)$ so that φ sends $S_{1,t}$ to $S_{2,1-t}$. The restriction to $S_{a,t} = S^1 \times j_a(P)$ of L_a is isomorphic to the pullback by $p_2\colon S^1 \times j_a(P) \to j_a(P)$ of $L_{P,a}$. Thus, this restrictions are isomorphic and it is straightforward to define then and isomorphism Ψ.

The hermitian bundle $L_1\#_\Psi L_2 \to M_1\#_\varphi M_2$ has two not everywhere defined hermitian connections ∇_1, ∇_2. They overlap for example in the annulus $A_1 \subset M_1\#_\varphi M_2$. Using a bump function β in $M_1\#_\varphi M_2$ that vanishes in $M_1 - A_1$, starts growing a bit after entering in the annulus, reaches the value 1 before leaving it, and keeps it in $M_2 - A_2$. Let us call $\bar{A}_1 \subset A_1$ to the points where its value is neither 0 nor 1.

One first attempt is to consider the hermitian connection $\beta\nabla_1 + (1-\beta)\nabla_2$. Its curvature multiplied by i defines a closed 2-form. It clearly coincides with $\omega_1 \coprod \omega_2$ in the complement of $\bar{A}_1$. Over this second annulus, we would like its leafwise curvature to coincide with $\beta F_1 + (1-\beta)F_2 = -i\Lambda$. This is not true in general because in A_1, $\nabla_1 = \nabla_2 + B$, where B is in principle a non-vanishing complex valued 1-form.

Instead of trying to define a new identification Ψ, we proceed modifying the connection ∇_2 globally in M_2.

Since ∇_1 is hermitian, $B = iC$, where C is a real valued 1-form. Now, we consider the restriction of the bundle and connections to the (symplectic leafs) of A_1. The foliated 1-form $C_{|\mathcal{F}_1}$ is exact, because $\omega_{2|\mathcal{F}_1} + idC_{|\mathcal{F}_1}$ –the leafwise curvature of (L_2, ∇_2) thought over A_1 after the identification– is $\Lambda = \omega_{1|\mathcal{F}_1} = \omega_{2|\mathcal{F}_2}$ in A_1.

Hence we can find on each leaf a potential for $C_{|\mathcal{F}_1}$. It is easy to make a choice on each leaf so that the resulting function is smooth in A_1. For example, we take a "section" $\tilde{P}$ of A_1 (a copy of P that cuts each leaf of A_1 once), and choose the unique potential function f vanishing at $\tilde{P}$. The next step is to extend f, defined in a subset of M_2, to a function g defined everywhere in M_2. We will probably need to modify it in the points close to $j_2(P)$, and we do it so that $g_{|\bar{A}_1} = f$.

Define in L_2 the hermitian connection $\tilde{\nabla}_2 = \nabla_2 - idg$. It is clear that $\nabla = \beta\nabla_1 + (1-\beta)\tilde{\nabla}_2$ is a hermitian connection on $L_{1\#_\Psi}L_2$ whose leafwise curvature coincides with $-i\Lambda$.

$\omega \overset{def}{=} iF_\nabla$ is the sought for closed 2-form dominating the foliation of $M_{1\#_\varphi}M_2$ and restricting to Λ over the leaves. $\square$

CHAPTER III

Global classification of generic multi-vector fields of top degree

1. Introduction

The recent classification by O. Radko [**51**] of generic Poisson structures on oriented surfaces, raises the question of whether it is possible to extend it to higher dimensions.

This classification –though stated in the language of Poisson geometry– relies on general results from differential geometry and the classification of area forms on closed surfaces. The reason is that, in dimension 2, the integrability condition that a bi-vector field must satisfy in order to be Poisson is void. So, for generic Poisson structures on an oriented surface Σ, the difficult problem of classifying solutions of a non-linear PDE reduces to the classification of (generic) sections of the trivial line bundle $\mathfrak{X}^2(\Sigma) \equiv \Gamma(\wedge^2(T\Sigma))$. Then, standard methods from differential geometry apply and the problem is greatly simplified.

In this chapter we show that O. Radko's classification can be extended to higher dimensions for generic multi-vector fields of top degree.

A bi-vector field is a Poisson structure of top degree. More generally, a multi-vector field of top degree is a *Nambu structure* of top degree.

In fact, Nambu structures are natural generalizations of Poisson structures: a *Nambu structure* of degree r, on a manifold M, is a r-multilinear, skew-symmetric bracket,

$$\{\cdot, \ldots, \cdot\}\colon \underbrace{C^\infty(M) \times \cdots \times C^\infty(M)}_{r} \to C^\infty(M),$$

which satisfies the Leibniz rule in each entry, and a Fundamental Identity that naturally extends the Jacobi identity. For top degree structures the Fundamental Identity is void.

In spite of the formal similarities between Nambu structures and Poisson structures, for $r > 2$ the fundamental identity imposes much more restrictive conditions than one would expect from the Jacobi identity. That is, Nambu structures are in a sense harder to find than Poisson structures. On the other hand, Nambu structures are easier to describe.

We are interested in generic Nambu structures of top degree on a compact oriented manifold M. By the Leibniz rule, such a structure is given by a multi-vector field $\Lambda \in \mathfrak{X}^{\mathrm{top}}(M)$ and genericity means that Λ cuts the zero section of the line bundle $\wedge^{\mathrm{top}}TM$ transversally. In particular, the zero locus $\mathcal{H}$ of the multi-vector field Λ is a hypersurface in M. We will show how one can attach to each connected component H^i of $\mathcal{H}$ a numerical invariant, called the *modular period*, which depends only on the germ of Λ at H^i. We construct also a global invariant which measures the ratio between the volumes of the connected components of the complement of $\mathcal{H}$, called the *regularized Liouville volume.* These notions generalize corresponding notions for 2-dimensional Poisson manifolds.

Our main result is the following:

Theorem 1.1. *A generic Nambu structure $\Lambda \in \mathfrak{X}^{top}(M)$ is determined, up to orientation preserving diffeomorphism, by the the diffeomorphism type of the oriented pair $(M, \mathcal{H})$ together with its modular periods and regularized Liouville volume.*

For dimension 2 this result recovers the classification of [**51**].

Using Theorem 1.1 we are also able to describe the Nambu cohomology group $H^2_\Lambda(M)$ which determines the infinitesimal deformations of the Nambu structure. On the other hand, we show that for dimension larger than 2, the Nambu cohomology group $H^1_\Lambda(M)$, which determines the outer automorphisms of the structure, is infinite dimensional.

The plan of the chapter is as follows. In section 1, we recall the definition of a Nambu structure of degree r (definition 2.1) and list briefly some of its main properties.

In section 2, we consider generic Nambu structures of degree n on a n-dimensional oriented manifold (definition 3.1). We define, for each hypersurface H where the n-vector field Λ vanishes, a couple of equivalent invariants. They are the modular $(n-1)$-vector field X^H_Λ (definition 3.2) and the modular $(n-1)$-form Ω^H_Λ, which give two equivalent ways of describing the linearization of Λ along H.

In section 4, we introduce the modular period T^H_Λ, which is just the integral (or cohomology class) of the modular $(n-1)$-form, and depends only on the values of Λ on a tubular neighborhood of H. Conversely, we can recover the Nambu structure in a tubular neighborhood of the oriented hypersurface (H, Ω^H_Λ) once the modular period T^H_Λ is specified (proposition 4.4).

The proof of main result is given in section 5 (theorem 5), where we also introduce the regularized Liouville volume.

In section 6, among the possible cohomologies one can attach to a Nambu structure, we consider (i) the group of infinitesimal outer automorphisms and (ii) the group of infinitesimal deformations of the structure. The later will turn out to have as many generators as the numerical invariants above and we will exhibit explicitly a set of generators, which extends the one for 2-dimensional Poisson manifolds (theorem 6.1). On the other hand, we will show that the first cohomology group is infinite dimensional for $n \geq 3$, something to be expected from the local computations of this groups presented in [**44**].

Finally, in section 7, we observe that the correspondence between isotopy classes of generic bi-vectors on $\Sigma = S^2$ and isomorphism classes of weighted signed trees given in [**51**], holds for those generic Nambu structures in S^n for which the zero locus $\mathcal{H}$ only contains spheres (proposition 7.5).

2. Nambu structures

Poisson manifolds $(M, \{\cdot,\cdot\})$ are the phase spaces relevant for Hamiltonian mechanics. For a Hamiltonian system the evolution of any observable $f \in C^\infty(M)$ is obtained by solving the o.d.e.

$$\frac{df}{dt} = \{H, f\},$$

where $H \in C^\infty(M)$ is the Hamiltonian, a conserved quantity for the system (the "energy").

In 1973 Nambu [**48**] proposed a generalization of Hamiltonian mechanics based on a n-ary bracket. The dynamics of an observable $f \in C^\infty(M)$ would be governed by the an analogous o.d.e.

$$\frac{df}{dt} = \{H_1, ..., H_{n-1}, f\},$$

associated to $n-1$ Hamiltonians $H_1, ..., H_{n-1}$, so now we would have $n-1$ conserved quantities.

In order to have the "expected" dynamical properties this bracket had to satisfy certain constraints. These were clarified by Takhtajan [**53**], who gave the following axiomatic definition of a Nambu structure.

Definition 2.1. *A Nambu structure of degree r in a manifold M^n, where $r \le n$, is a r-multilinear, skew-symmetric bracket,*

$$\{\cdot,\dots,\cdot\}\colon \underbrace{C^\infty(M) \times \cdots \times C^\infty(M)}_{r} \to C^\infty(M),$$

satisfying:

(i) *the Leibniz rule:*

$$\{fg, f_1, \dots, f_{r-1}\} = f\{g, f_1, \dots, f_{r-1}\} + \{f, f_1, \dots, f_{r-1}\}g,$$

(ii) *the Fundamental Identity:*

$$\{f_1, \dots, f_{r-1}, \{g_1, \dots, g_r\}\} = \sum_{i=1}^{r} \{g_1, \dots, \{f_1, \dots, f_{r_1}, g_i\}, \dots, g_n\};$$

The Liebniz rule shows that the operator $X_{f_1,\dots,f_{r-1}} : C^\infty(M) \to C^\infty(M)$ which is associated to $r-1$ functions $f_1, \dots, f_{r-1}$ by

$$X_{f_1,\dots,f_{r-1}}(g) = \{g, f_1, \dots, f_{r-1}\},$$

is a derivation and hence a vector field. This is called the *Hamiltonian vector field* associated with $f_1, \dots, f_r$. More generally, the Leibniz identity shows that we have a r-vector field $\Lambda \in \mathfrak{X}^r(M)$ such that

$$\Lambda(df_1 \wedge \cdots \wedge df_r) = \{f_1, \dots, f_r\}.$$

On the other hand, the Fundamental Identity is equivalent to the fact that the flow of any Hamiltonian vector field $X_{f_1,\dots,f_{r-1}}$ is a canonical transformation, i.e., preserves Nambu brackets. Its infinitesimal version reads

$$\mathcal{L}_{X_{f_1,\dots,f_{r-1}}}\Lambda = 0.$$

Obviously, for Nambu structures of top degree the Fundamental Identity becomes void.

Example 2.2: On $\mathbb{R}^n$ we have a canonical, top degree, Nambu structure which generalizes the canonical Poisson structure in $\mathbb{R}^2$. The Nambu bracket assigns to n functions $f_1,\dots,f_n$ the Jacobian of the map $\mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto (f_1(x),\dots,f_n(x))$, so that

$$\{f_1,\dots,f_n\} = \det\left[\frac{\partial f_i}{\partial x_j}\right].$$

More generally, any volume form $\mu \in \Omega^{\mathrm{top}}(M)$ on a manifold M determines a Nambu structure: if $(x^1,\dots,x^n)$ are coordinates on M, so that $\mu = f dx^1 \wedge \dots \wedge dx^n$, then the Nambu tensor field is

$$\Lambda \equiv \frac{1}{\mu} = \frac{1}{f}\frac{\partial}{\partial x^1}\wedge\dots\wedge\frac{\partial}{\partial x^n}.$$

The Fundamental Identity for $r > 2$ is of a more restrictive nature than one would expect from the case $r = 2$, when it reduces to the usual Jacobi identity: if $r > 2$ besides requiring the fulfilment of a system of first order quadratic partial differential equations, the coefficients *must also* satisfy certain system of quadratic algebraic equations. For example, for a constant r-vector field (M a vector space) the system involving first derivatives is automatically satisfied, while the algebraic relations are non-trivial, and in fact coincide with the well-known Plücker equations. Hence, only decomposable r-vectors define constant Nambu structures. Another example of this rigidity is the following well known proposition (see [**53**]):

Proposition 2.3. *Let Λ be a Nambu structure. For any function $f \in C^\infty(M)$, the contraction $i_{df}\Lambda$ is also a Nambu structure.*

This rigidity makes it harder to "find" Nambu structures than Poisson structures. On the other hand, it makes Nambu structures easier to describe. Henceforth, we will assume that $r > 2$ if $n \geq 3$.

First of all, the Hamiltonian vector fields span a generalized foliation for which the leaves are either points, called *singular points*, or have dimension equal to the degree of the structure. Around these *regular points* we have the following canonical form for a Nambu structure (see for example [**55**]):

Proposition 2.4. *Let $x_0 \in M$ be a regular point of a Nambu structure Λ of degree r. There exist local coordinates $(x^1,\dots,x^n)$ centered at x_0, such that*

$$\Lambda = \frac{\partial}{\partial x^1}\wedge\dots\wedge\frac{\partial}{\partial x^r}.$$

For the singular points there are some deep linearization results due to Dufour and Zung [**16**].

3. Generic Nambu structures of top degree

In this section we consider Nambu structures of degree n in a compact orientable n-dimensional manifold M. Notice that in this case the Fundamental Identity is void, so a Nambu structure is just a multi-vector field $\Lambda \in \mathfrak{X}^n(M)$. We will restrict our attention to *generic* Nambu structures:

Definition 3.1. *A Nambu structure $\Lambda \in \mathfrak{X}^n(M)$ is called **generic** if it cuts the zero section of the line bundle $\wedge^n TM$ transversally.*

The generic sections form an open dense set in the Whitney C^∞ topology.

Let us fix a generic $\Lambda \in \mathfrak{X}^n(M)$ once and for all. Its set of zeros, denoted $\mathcal{H}$, is the union of a finite number of hypersurfaces: $\mathcal{H} = \bigcup_{i\in I} H^i$, $\#I < \infty$. Fix one of them and call it H.

Over the points of H there is some linear information attached to Λ, namely the intrinsic derivative $d\Lambda^H \in T^*_H M \otimes \wedge^n TM$. It can be defined as $d\Lambda^H \equiv \nabla\Lambda_{|H}$, where ∇ is any linear connection on $\wedge^n TM$. This is independent of the choice of connection. The intrinsic derivative gives the linearization of the Nambu structure at H: if we view Λ as a section, it is the tangent space to the graph of Λ. It is important to observe that $d\Lambda^H$ never vanishes due to the transversality assumption. Notice that $d\Lambda^H$ is a section of $T^*_H M \otimes \wedge^n TM$, but due to the nature of our (trivial) line bundle it has two equivalent interpretations which we shall now explain.

Fix a volume form Ω in some neighborhood of H in M, so that $d\Lambda^H \otimes \Omega \in T^*_H M$.

Definition 3.2. *The modular $(n-1)$-vector field of Λ along H is the unique $(n-1)$-vector field $X^H_\Lambda \in \mathfrak{X}^{n-1}(H)$ such that $i_{X^H_\Lambda}\Omega = d\Lambda^H \otimes \Omega$.*

This definition does not depend on the choice of Ω: if $\tilde{\Omega}$ is another volume form, then $\tilde{\Omega} = f\Omega$ for some non-vanishing smooth function f, and we find

$$i_{X^H_\Lambda} f\Omega = d\Lambda^H \otimes f\Omega.$$

Notice that since X^H_Λ is tangent to H and no-where vanishing, we can define the *modular $(n-1)$-form* along H to be the dual $(n-1)$-form $\Omega^H_\Lambda \in \Omega^{n-1}(H)$; that is, $\Omega^H_\Lambda(X^H_\Lambda) = 1$. If we fix a vector field Y over H, which is transverse to H, the modular form along H is given by

$$\Omega^H_\Lambda = (-1)^{n-1}\frac{1}{d_Y\Lambda^H \otimes \Omega} j^* i_Y \Omega,$$

where $j\colon H \hookrightarrow M$ is the inclusion. This expression is independent of Y. The modular $(n-1)$-form along H is non-zero, and hence Λ determines an orientation in H. It is clear that to give either of $d\Lambda^H$, X^H_Λ or Ω^H_Λ, determines the others.

Let us relate these definitions with the well-known notion of modular class of a Poisson manifold. For any Nambu structure of degree r in a oriented manifold there is a natural generalization of the modular class of a Poisson manifold [**?**], which we now recall. Again we fix a volume form Ω on M. Then, for any $(n-1)$-functions $f_1, \dots, f_{n-1}$ on M, we can compute the divergence of the corresponding Hamiltonian vector field:

$$(f_1, \dots, f_{n-1}) \mapsto \operatorname{div}^{\Omega}(X_{f_1,\dots,f_{n-1}}) \equiv \frac{1}{\Omega}\mathcal{L}_{X_{f_1,\dots,f_{n-1}}}\Omega.$$

It turns out that this defines a $(n-1)$-vector field $\mathcal{M}_\Lambda^\Omega$ on M. If $\tilde{\Omega} = g\Omega$ is another volume form, where g is some non-vanishing smooth function, we have

$$\mathcal{M}_\Lambda^{\tilde{\Omega}} = \mathcal{M}_\Lambda^\Omega + X_g,$$

where X_g is the $(n-1)$-vector field

$$X_g(f_1, \dots, f_{n-1}) = \{f_1, \dots, f_{n-1}, g\}.$$

One can introduce certain Nambu cohomology groups to take care of this ambiguity so that the cohomology class $[\mathcal{M}_\Lambda^\Omega]$ is well-defined and independent of Ω. This class is called the *modular class* of the Nambu manifold M and is the obstruction for the existence of a volume form on M invariant under hamiltonian automorphisms.

Each volume form Ω determines a modular $(n-1)$-vector field $\mathcal{M}_\Lambda^\Omega$ representing the modular class, and which will depend on Ω. However, at points where the Nambu tensor vanishes all modular vector fields give the same value (see [**30**]). The modular $(n-1)$-vector field X_Λ^H along H, that we have introduced above, is nothing but the restriction of any modular vector field to H. In our case, however, it has the additional properties that it is non-zero and tangent to H.

4. Local characterization of a Nambu structure

In this section we study the local behavior of a generic Nambu structure $\Lambda \in \mathfrak{X}^n(M)$ in a neighborhood of its zero locus. We show that the germ of Λ around a connected component H of its zero locus is determined, up to isotopy, by the modular periods (to be introduced below).

Since the intrinsic derivative is functorial we immediately conclude that

Lemma 4.1. *Given two generic Nambu structures Λ_1 and Λ_2, with zero locus $\mathcal{H}_1$ and $\mathcal{H}_2$, and a diffeomorphism of Nambu structures $\psi\colon (M, \Lambda_1) \longrightarrow (M, \Lambda_2)$ then*

$$\psi_* X_{\Lambda_1}^{\mathcal{H}_1} = X_{\Lambda_2}^{\mathcal{H}_2}, \ \text{and} \ \ \psi_*\Omega_{\Lambda_1}^{\mathcal{H}_1} = \Omega_{\Lambda_2}^{\mathcal{H}_2}.$$

Hence, it follows that a necessary condition for such a map to exist is that the cohomology classes $[\Omega_{\Lambda_2}^{\mathcal{H}_2}]$ and $[\Omega_{\Lambda_1}^{\mathcal{H}_1}]$ correspond to each other.

Now recall that, given a generic Nambu structure Λ, each component H of its zero locus $\mathcal{H}$ has an induced orientation from the n-vector field Λ. Hence, a class in $H_{dR}^{n-1}(H)$ is completely determined by its value on the fundamental cycle H.

Definition 4.2. *The modular period T^H_Λ of the component H of the zero locus of Λ is*

$$T^H_\Lambda \equiv \int_H \Omega^H_\Lambda > 0.$$

In fact, this positive number determines the Nambu structure in a neighborhood of H up to isotopy. To prove that we need the following classical result concerning the classification of volume forms.

Lemma 4.3. *(Moser, [**45**]) Let M be an orientable closed manifold, Ω_1 and Ω_2 two volume forms in M. If $[\Omega_1] = [\Omega_2] \in H^{top}(M)$, there exists a diffeomorphism isotopic to the identity which sends Ω_1 to Ω_2. Moreover, it can be chosen to be the identity in the closure of the complement of the closed set where the two volume forms coincide.*

The above result can be adapted to volume forms in compact manifolds with boundary which coincide in neighborhoods of the boundary components. We can now state and prove the main result in this section.

Proposition 4.4. *Let Λ_1 and Λ_2 be generic Nambu structures in M which share a common component H of their zero locus, and for which the modular periods coincide: $T^H_{\Lambda_1} = T^H_{\Lambda_2}$. Then, there exists a diffeomorphism $\varphi : M \to M$, isotopic to the identity, and neighborhoods U_1 and U_2 of H, such that φ sends (U_1, Λ_1) to (U_2, Λ_2).*

Proof. First we can use Moser's lemma to construct a diffeomorphism $\phi : M \to M$ isotopic to the identity, which maps H to itself and sends $\Omega^H_{\Lambda_1}$ to $\Omega^H_{\Lambda_2}$. Hence, we can assume that $\Omega^H_{\Lambda_1} = \Omega^H_{\Lambda_2}$, and the problem reduces to a global linearization one.

We fix a collar $U = [-1, 1] \times H$ of the hypersurface H, with transverse coordinate r. Denoting the Nambu structure by Λ we define $\Lambda_0 = (-1)^{n-1} \frac{\partial}{\partial r} \wedge X^H_\Lambda$. We can write $\Lambda = f\Lambda_0$ for some $f \in C^\infty(U)$ and the linearization of Λ is $\Lambda_1 = r\Lambda_0$. We look for a change of coordinates that only reparametrizes the radial coordinate:

$$\phi\colon U \to U, \qquad (r, x) \mapsto (g(r, x), x),$$

and satisfies $\phi_* f\Lambda_0 = r\Lambda_0$. We obtain an o.d.e. for g whose solutions are $g(x, r) = ke^{\int \frac{1}{f} dr}$, with $k \in \mathbb{R}$. Since f vanishes to order 1 along the radial direction, this o.d.e. has a one parameter family of smooth solutions which fix the hypersurface H and define diffeomorphisms (if $k \neq 0$) in a collar of H. Choosing any solution with $k > 0$ we obtain the desired coordinate change. $\square$

Remark 4.5: The existence of a one parameter solutions for the equation above reflects the fact that for any linear structure $cr\frac{\partial}{\partial r} \wedge X^H_\Lambda$, rescaling the radial coordinate is a canonical transformation. Note also that reflection along H gives a canonical transformation reversing the orientation of the tubular neighborhood. We are interested in transformations isotopic to the identity (and hence preserving the orientation of the tubular neighborhood), so our choice of g above is with $k > 0$.

5. The global description of Nambu structures

In order to have a diffeomorphism between two Nambu manifolds it is necessary to have a diffeomorphism sending the zero locus of one structure to the zero locus of the other, preserving their induced orientations. Assuming that condition to hold, our problem is that of transforming a generic structure Λ_1 into another Λ_2, with common oriented zero set $\mathcal{H} = \bigcup_{i\in I} H^i$.

First of all, we saw in the previous section that if the modular periods of each component coincide, we can find collars U_1^i and U_2^i of the hypersurfaces H^i, and a diffeomorphism isotopic to the identity φ sending $(\mathcal{U}_1, \Lambda_1)$ to $(\mathcal{U}_2, \Lambda_2)$, where $\mathcal{U}_j = \bigcup_{i\in I} U_j^i$.

Second, $\mathcal{H}$ splits M into the maximal leafs of both structures. The restriction of the Nambu structures to each of these components define volume forms. However, their volumes are infinite so one cannot require them to match. Instead, we could try to define ratios of the volumes between the various components which are finite. This raises some accounting problems, so instead observe that for a component H, a volume form Ω defined in a neighborhood of H and the volume form Λ^{-1} define orientations on the complement of H, which match on one side of H and are opposite on the other side. Given any function $h \in C^\infty(M)$ vanishing linearly in the components of $\mathcal{H}$ (its graph is transverse to the zero section and vanishes exactly at $\mathcal{H}$), we let $M^\epsilon(h) = f^{-1}(\mathbb{R} - (-\epsilon, \epsilon))$, with $\epsilon > 0$ small enough so that $M^\epsilon(h)$ contains the complement of the union of collars of the H^i, and we set

$$V_\Lambda^\epsilon(h) = \int_{M^\epsilon(h)} \Lambda^{-1}.$$

Here Λ^{-1} denotes the volume form dual to Λ, and to integrate we use the given orientation of M. The following definition generalizes the one given in [**51**] for the case of 2-dimensional Poisson manifolds.

Definition 5.1. *The regularized Liouville volume of Λ is defined as*

$$V_\Lambda = \lim_{\epsilon \longrightarrow 0} V_\Lambda^\epsilon(h),$$

where h is any function vanishing linearly at $\mathcal{H}$.

One has to check that the limit exists and is finite and that it does not depend on the choice of function h. In fact, we only need to prove the independence on the choice of function, because granted this we can use a function that locally coincides with a radial coordinate in which the n-vector field is linear. For this function the existence of the limit is trivial.

To check the independence on the choice of h, we fix coordinates (r, x) around each component H such that $\Lambda = (-1)^{n-1} r \frac{\partial}{\partial r} \wedge X_\Lambda^H$, and we consider two cases:

(1) Assume $h(r,x) = g(x)r$, with $g(x) \neq 0$ for all $x \in H$. The difference $V_\Lambda^\epsilon(h) - V_\Lambda^\epsilon(r)$ vanishes for all $\epsilon > 0$. This is because on each collar we have the product measure, so the integral is obtained by averaging over the open regions of H where $g < 1$ and $g > 1$, the integrals of the odd function $\pm\frac{1}{r}$, over two intervals symmetric with respect to the origin and not containing it.

(2) Assume now that h vanishes linearly at $\mathcal{H}$. Then, $h - \frac{\partial h}{\partial r}(0,x)r$ vanishes in the radial direction at least to second order at H. The compactness of H implies that for all $x \in H$ and all $\epsilon > 0$, there exist constants k_1 and k_2 such that the absolute value $V_\Lambda^\epsilon(h) - V_\Lambda^\epsilon(\frac{\partial h}{\partial r}(0,x)r)$ is bounded by the average over H of the integral of the function $\pm\frac{1}{r}$ over the segments $[-b_\epsilon, -a_\epsilon] \cup [a_\epsilon, b_\epsilon]$, where $a_\epsilon > k_1\epsilon$ and $b_\epsilon - a_\epsilon < k_2\epsilon^2$. Hence, there exists a constant k (independent of x and r) such that the integral over the segments in the ray through x is bounded by $k\epsilon$; this makes the total integral smaller that $k\epsilon T_\Lambda^H$. Hence when $\epsilon \to 0$ the difference vanishes.

The modular periods and the regularized volume determine the Nambu structure:

Theorem 5.2. *For $j = 1, 2$, let M_j be oriented compact manifolds with generic Nambu structures Λ_j having zero locus $\mathcal{H}_j = \bigcup_{i\in I} H_j^i$. Assume that there exists a diffeomorphism ψ sending $(M_1, \mathcal{H}_1)$ to $(M_2, \mathcal{H}_2)$ and preserving the induced orientations of the zero locus. Then there exists an isomorphism between the two Nambu structures, isotopic to ψ, if and only if the following conditions are satisfied:*

(i) *the modular periods coincide, i.e., $T_{\Lambda_1}^{H_1^i} = T_{\Lambda_2}^{\psi H_1^i}$, $\forall i \in I$,*
(ii) *the regularized volumes match, i.e., $V_{\Lambda_1} = \varepsilon V_{\Lambda_2}$, where $\varepsilon = 1$ if ψ is orientation preserving and $\varepsilon = -1$ if it reverses the orientations on the M_i.*

Proof. As we saw above, if the modular periods of each component coincide, we can find collars U_1^i and U_2^i of the hypersurfaces H^i, and a diffeomorphism isotopic to the identity φ which sends $(\mathcal{U}_1, \Lambda_1)$ to $(\mathcal{U}_2, \Lambda_2)$, where $\mathcal{U}_j = \bigcup_{i\in I} U_j^i$.

Now, $\mathcal{H}$ splits M in the maximal leafs of both structures, whose area with respect to the duals Λ_i^{-1} of any of the n-vectors is infinite. For any such leaf L, we can select a hypersurface H^{i_0} on its boundary and shrink accordingly the size of either $U_1^{i_0}$ or $U_2^{i_0}$ (recall we have canonical transformations doing that) such that one can find compact submanifolds $W_j \subset L$ which are the result of removing from L the corresponding side of the collars of radius, say, 1/2 (the original radius is 1), verifying: (i) the Λ_1^{-1}-volume of W_1 coincides with the Λ_2^{-1}-volume of W_2 and (ii) φ sends W_1 to W_2. Finally, we apply Moser theorem to conclude the existence of a diffeomorphism isotopic to the identity also matching the Nambu structures at L.

Observe that when we modify the size of, say, $U_1^{i_0}$, we are changing the volume of both $L - \mathcal{U}_1$ and $L' - \mathcal{U}_1$, where L and L' are the leaves whose boundary contains H^{i_0}. It follows that we can make our n-vector fields coincide, as well as in the collars of $\mathcal{H}$, in all the maximal leaves but possibly one. To see this we can take the graph dual to the splitting given by $\mathcal{H}$, where each vertex represents a maximal leaf and an edge joining two vertices stands for a hypersurface on its common boundary. This graph is indeed a tree; we fix a vertex v_0 on this tree and consider the graph distance (of vertices) with respect to v_0. We can then proceed by stages, where at each stage we consider all the vertices with the same distance, starting from the furthest way vertices. For those vertices, i.e., maximal leafs, we apply the above reasoning to the hypersurface representing the only edge reaching them (there are no loops). When we are done we erase those vertices and edges connecting to them, so we obtain a smaller tree. We keep on doing that until we reach the vertices at distance one. One can use all but one edge and there will remain only one hypersurface which splits two components. The fact that the regularized volumes match, grants us the matching of the areas of both remaining components for an appropriate collar, and we are done. □

The set of generic Nambu structures has an action of $Diff_0(M)$ (resp. $Diff^+(M)$). Its space of orbits has as many connected components as isotopy classes (resp. oriented diffeomorphism classes) of oriented hypersurfaces $\mathcal{H} = \bigcup_{i\in I} H^i$. Theorem gives an explicit parametrization of each connected component of this moduli space.

6. Nambu cohomology

There are several cohomology theories one can associate to a Nambu manifold (see [**30, 44**]). Here we will be interested in the cohomology associated with the complex

$$0 \longrightarrow \wedge^{n-1}C^\infty(M) \longrightarrow \mathfrak{X}(M) \longrightarrow \mathfrak{X}^n(M) \longrightarrow 0$$

where the first map is $f_1 \wedge \cdots \wedge f_{n-1} \mapsto X_{f_1,\dots,f_{n-1}}$, while the second map is $X \mapsto \mathcal{L}_X\Lambda$. Notice that the associated cohomology groups have simple geometrical meanings:

- $H^0_\Lambda(M)$ is the space of Casimirs of the Nambu structure;
- $H^1_\Lambda(M)$ is the space of infinitesimal outer automorphisms of the Nambu structure;
- $H^2_\Lambda(M)$ is the space of infinitesimal deformations of the Nambu structure.

Computations of Nambu cohomology for *germs* of Nambu structures defined by quasipolynomials (functions vanishing at the origin with finite codimension) were done by Monnier in [**44**]. Here we are interested in *global* Nambu structures with the simplest singularity. These computations can be though of as a infinitesimal version on the classification theorem; in particular, $H^2_\Lambda(M)$ will turn out to be the tangent space of the class of $[\Lambda]$ in the moduli space of generic Nambu structures.

The main result of this section is the following

Theorem 6.1. *Let Λ be a generic Nambu structure on a oriented compact manifold M with zero locus $\mathcal{H} = \bigcup_{i\in I} H^i$. The group $H^2_\Lambda(M)$ has dimension $\#I+1$ and a set of generators is given by*

$$\beta_1(-1)^{n-1}r\frac{\partial}{\partial r}\wedge X_\Lambda^{H^1},\ldots,\beta_{\#I}(-1)^{n-1}r\frac{\partial}{\partial r}\wedge X_\Lambda^{H^{\#I}},\ \Omega,$$

where Ω is a volume form, and each β_i is a bump function supported in a collar of the hypersurface H^i.

We can give a geometric description of the isomorphism $H^2_\Lambda(M)\simeq\mathbb{R}^{\#I+1}$ as follows. Each $\Theta\in\mathfrak{X}^n(M)$ is cohomologous to an n-vector field whose vanishing set contains $\mathcal{H}$ and is generic in a neighborhood of $\mathcal{H}$. Then we can write $[\Theta]=[g\Lambda]$ where g is some smooth function which assumes a constant value c_i in the collar of each U^i. The isomorphism is

$$[\Theta]\longmapsto\left(\frac{T_\Lambda^{H^1}}{T_\Theta^{H^1}},\ldots,\frac{T_\Lambda^{H^{\#I}}}{T_\Theta^{H^{\#I}}},V_\Theta^{\mathcal{H},\Lambda}\right),$$

where:

(a) $T_\Lambda^{H^i}/T_\Theta^{H^i}=c_i$,

(b) $V_\Theta^{\mathcal{H},\Lambda}$ is the regularized integral of $g\frac{1}{\Lambda}$.

The rest of this section is dedicated to the proof of this result, which consists of a Mayer-Vietoris argument: we first compute the groups in the collars and then we glue them using information about the infinitesimal automorphisms in those neighborhoods.

6.1. Computation of $H^2_\Lambda(U)$.

Let us fix $H\subset\mathcal{H}$ and $U=(-1,1)\times H$ a collar.

Proposition 6.2. *$H^2_\Lambda(U)\simeq\mathbb{R}$ and a generator is the linearization $(-1)^{n-1}r\frac{\partial}{\partial r}\wedge X_\Lambda^H$.*

Proof. Any vector field X can be written $X=A\frac{\partial}{\partial r}+X_H$, where $A\in C^\infty(U)$, $X_H\in(-1,1)\times TH$. Defining $\Lambda_0=(-1)^{n-1}\frac{\partial}{\partial r}\wedge X_\Lambda^H$, one has:

$$\begin{aligned}
L_X\Lambda &= A\Lambda_0+rL_X\Lambda_0=\\
&= (A-r\frac{\partial A}{\partial r})\Lambda_0+(-1)^{n-1}r\frac{\partial}{\partial r}\wedge L_{X_H}X_\Lambda^H=\\
&= (A-r\frac{\partial A}{\partial r}+r\operatorname{div}^{\Omega_\Lambda^H}(X_H))\Lambda_0,
\end{aligned}$$

where $\operatorname{div}^{\Omega_\Lambda^H}(X_H)$ is the divergence of X_H with respect to Ω_Λ^H.

We want to show that any n-vector field $f\Lambda_0$ in U is equivalent to a linear one. We first make it vanish at the origin by adding $L_{-f\frac{\partial}{\partial r}}\Lambda$ (so it becomes $r\frac{\partial f}{\partial r}\Lambda_0$) and still call it $f\Lambda_0$. Let us assume for the moment that has at least quadratic vanishing at the origin so we can write $f=r^2g$. Notice that $L_{-r\int gdr}\Lambda=(-r\int gdr+r\int gdr+r^2g)\Lambda_0=f\Lambda_0$. Writing $f=r\hat{f}$, we look at the value $c=\int_{\{0\}\times H}\hat{f}\Omega_\Lambda^H\in\mathbb{R}$, and observe that we can always find $Y\in\mathfrak{X}(H)$ such that $\hat{f}_{|H}+div^{\Omega_\Lambda^H}(Y)=c$. The n-vector field

$\Lambda_1 = f\Lambda_0 + L_Y\Lambda$ has $c\Lambda_0$ as constant linear part at H. Finally, $\Lambda_1 - cr\Lambda_0$ has at least quadratic vanishing at H and hence it is a coboundary.

We still have to show there are no perturbations sending a linear structure to another with different relative period (the constant c above). This amounts to showing that the equation $E_c \equiv A - r\frac{\partial A}{\partial r} - rdiv^{\Omega^H_\Lambda}(X_H) = cr$ has no solutions for $c = 1$. Actually we need also to study the equation E_0 of Nambu infinitesimal automorphisms.

Lemma 6.3. *The equation E_1 has no solutions, and $Z^1_\Lambda(U)$, the space of solutions of E_0, can be identified with the vector space:*

$$Z^1_\Lambda(U) \cong span < r\frac{\partial}{\partial r}, X_H \in (-1,1) \times TH|\, div^{\Omega^H_\Lambda}(X_H(0)) = 0 >$$

Proof of Lemma 6.3. In the equations E_c we can write the term $div^{\Omega^H_\Lambda}(X_H)$ in the form ψ_r, where ψ_r is a smooth family in $r \in (-1,1)$ of functions in $C^\infty(H)$ which satisfy $\int_H \psi_r \Omega^H_\Lambda = 0$, $\forall r \in (-1,1)$. We can think of them as given data in the equations. Then the solutions of E_1 can be explicitly written as:

$$A = kr + r\int \frac{\psi_r - 1}{r} dr, k \in \mathbb{R}$$

Any solution has to be a smooth continuation of the above expression but it cannot exist. Indeed, since ψ_0 has vanishing integral, we can find a point x in H, such that $\psi_0(x) = 0$. Hence, in a small segment $[-\epsilon, \epsilon] \times \{x\}$ the real valued function $r\int \frac{\psi_r(x)-1}{r} dr$ is ,up to a smooth function, $rlnr$ (not even C^1). $\square$

With this we have finished the computation of $H^2_\Lambda(U)$. $\square$

Remark 6.4: Regarding E_0, its solutions are of the form

$$A = kr + r\int \frac{\psi_r}{r} dr,$$

which will be smooth if and only if $\psi_0 = 0$, or the corresponding vector field $X_H(0)$ is divergence free with respect to Ω^H_Λ. Hence the space of solutions Z^1_Λ can be identified with:

$$Z^1_\Lambda \cong span < r\frac{\partial}{\partial r}, X_H \in (-1,1) \times TH|\, div^{\Omega^H_\Lambda}(X_H(0)) = 0 >$$

6.2. From $H^2_\Lambda(U)$ to $H^2_\Lambda(M)$. The remaining step is to piece all the local information. We just showed that in the same radial coordinate in which Λ is linearized in $U^i = (-1,1) \times H^i$, we can find a representant Θ of the cohomology class such that $\Theta_{|U^i} = c_i(-1)^{n-1}r\frac{\partial}{\partial r} \wedge X^{H^i}_\Lambda = c_i\Lambda$, where the relative period $\frac{T^{H^i}_\Lambda}{T^{H^i}_\Theta}$ is c_i, which might be zero. In particular, for a non-vanishing structure all the local invariants vanish because by looking at its dual form it is clear that since it does not vanishes, we can push its graph down (or up) to the zero section to make it vanish in the U^i's. This operation can be made without changing the area (do it randomly and multiply by the ratio of both areas). We also saw that we can restrict our attention to coboundaries X such that $X_{|U^i}$ is a solution of E_0 in the radial coordinates. Now it has to be proven that the global regularized volume with respect to Λ is well defined. But this is equivalent to showing that the regularized volume of $L_X\Lambda$ vanishes, for X infinitesimal automorphism of Λ in U^i. For a function h coinciding with the radial coordinate in each U^i, $M^r(h) = M - \bigcup_{i\in I}(r,r)\times H^i$. We have:

$$\int_{M^r(h)} L_X\Lambda = \int_{M^r(h)} di_X\frac{1}{\Lambda} = \pm\sum_{i\in I}\left(\int_{\{r\}\times H^i} i_X\frac{1}{\Lambda} - \int_{\{-r\}\times H^i} i_X\frac{1}{\Lambda}\right)$$

And for a fixed component H and $X = kr + r\int \frac{div^{\Omega^H_\Lambda}(X_H)}{r}dr\frac{\partial}{\partial r} + X_H$, the function

$I(r) = \int_{\{r\}\times H} i_X\frac{1}{\Lambda}$ equals:

$$\begin{aligned} I(r) &= (-1)^{n-1}\int_{\{r\}\times H}\frac{1}{r}\left(kr + r\int\frac{div^{\Omega^H_\Lambda}(X_H)}{r}dr\right)\Omega^H_\Lambda = &(6.13)\\ &= (-1)^{n-1}\int_{\{r\}\times H}\left(k + \int\frac{div^{\Omega^H_\Lambda}(X_H)}{r}dr\right)\Omega^H_\Lambda &(6.14)\end{aligned}$$

Due to the fact that $div^{\Omega^H_\Lambda}(X_H)(0) = 0$, the above formula defines a smooth function for all $r\in[-1,1]$. Its derivative is easily computed:

$$\begin{aligned}\frac{dI}{dr} &= (-1)^{n-1}\frac{d}{dr}\int_{\{r\}\times H}(k + \int\frac{div^{\Omega^H_\Lambda}(X_H)}{r}dr)\Omega^H_\Lambda = \\ &= (-1)^{n-1}\int_{\{r\}\times H}\frac{div^{\Omega^H_\Lambda}(X_H)}{r}\Omega^H_\Lambda = 0\end{aligned}$$

The vanishing is clear for $r\neq 0$ and follows by continuity. Hence $I(r)$ is constant and $V^{\mathcal{H},\Lambda}_\Theta$ is well defined.

It only remains to show that two n-vectors Θ_1 and Θ_2 with equal linearizations and regularized volume are in the same class. Its difference $\tilde\Theta$ vanishes in a neighborhood of the boundary of $M - \mathring{\mathcal{U}}$. Then the form $\frac{1}{\Lambda}(\tilde\Theta)\cdot\frac{1}{\Lambda}$ has compact support (shrinking a bit the collars if necessary) and vanishing integral, so we can find a compactly supported vector field Y whose divergence is $\frac{1}{\Lambda}(\tilde\Theta)\cdot\frac{1}{\Lambda}$. It follows that $L_{\tilde Y}\Lambda = \tilde\Theta$, where $\tilde Y$ extends Y trivially. The assertion about the basis of $H^2_\Lambda(M)$ follows easily.

6.3. Some comments about $H^1_\Lambda(M)$ and $H^0_\Lambda(M)$. We will focus our attention in what happens in a collar U. There we need the description of the Hamiltonian vector fields in order to draw some results. Given $f\in C^\infty(U)$ we split at each point its derivative $df = \frac{\partial f}{\partial r}dr + d_Hf$. We can express the vector space $B^1_\Lambda(U)$ of Hamiltonian vector fields as follows:

$$B^1_\Lambda(U) = \{(-1)^{n-1}rX^H_\Lambda(d_Hf_1,\ldots,d_Hf_{n-1})\frac{\partial}{\partial r}+\sum_{j=1}^{n-1}(-1)^{n-i}r\frac{\partial f_j}{\partial r}X^H_\Lambda(d_Hf_1,\ldots,\widehat{d_Hf_j},\ldots,d_Hf_{n-1})\},$$

with $f_1, \ldots, f_{n-1} \in C^\infty(U)$.

Hence all Hamiltonian vector fields must vanish along H. Let us denote for each H^i by $\mathfrak{X}_{free}(H_i)$ the vector space of divergence free vector fields in H_i with respect to the volume form $\Omega_\Lambda^{H^i}$. If we call for the moment r_i to the corresponding radial coordinate, we have the following

Corollary 6.5.

(1) $< r_i \frac{\partial}{\partial r_i} > \oplus \mathfrak{X}_{free}(H^i) \subset H_\Lambda^1(U^i)$.

(2) $\bigoplus_{i\in I}(< \phi_i \cdot r_i \frac{\partial}{\partial r_i} > \oplus \phi_i \cdot \mathfrak{X}_{free}(H^i)) \subset H_\Lambda^1(M)$, *where* ϕ_i *are bump functions supported in the collars. For* $n \geq 3$ *this space is clearly infinite dimensional.*

Proof. The assertion about the divergence free vector fields is clear. Regarding the size of the space just notice that it can be identified with closed $n-2$-forms in H^i which contain the exact ones. From the description of $B_\Lambda^1(U)$ we see that the coefficient of $r\frac{\partial}{\partial r}$ contains the factor $X_\Lambda(d_H f_1, \ldots, d_H f_{n-1})$ which cannot be everywhere non-vanishing on each $\{r\} \times H$ by compactness. $\square$

We see that the case $n = 2$ is quite special and in fact one can easily compute $H_\Lambda^1((-1,1) \times S^1)$.

Corollary 6.6. $H_\Lambda^1((-1,1)\times S^1)$ *is spanned by the modular vector field* $X_\Lambda^{S^1}$ *and* $r\frac{\partial}{\partial r}$.

Proof. $X_\Lambda^{S^1}$ trivializes TS^1 so one can write $X_H = gX_\Lambda^{S^1}$, $g \in C^\infty((-1,1)\times S^1)$. One checks that

$$B_\Lambda^1 = \{-rX_\Lambda^{S^1}(d_{S^1}f)\frac{\partial}{\partial r} - r\frac{\partial f}{\partial r}X_\Lambda^{S^1} \mid f \in C^\infty((-1,1) \times S^1)\}$$

$$Z_\Lambda^1 = \{\left(kr + r\int \frac{X_\Lambda^{S^1}(d_{S^1}g)}{r}dr\right)\frac{\partial}{\partial r} + gX_\Lambda^{S^1} \mid g \in C^\infty((-1,1)\times S^1),\ g_{|S^1} = k_1,\ k, k_1 \in \mathbb{R}\}$$

Thus, $gX_\Lambda^{S^1} - k_1 X_\Lambda^{S^1}$ is a cocycle; just take $f = -\int \frac{g-k_1}{r}dr$. $\square$

In general determining the group $H_\Lambda^1(U)$ seems to us a very difficult problem.

Also there seems to be little hope to compute $H_\Lambda^0(M)$. For example for $n = 3$ we see that $X_{f,g} = 0$ implies that df and dg have to be proportional. If we assume f to be a Morse function, g has to be constant on its leaves. So we have as many choices for g as the ring of smooth functions of the leaf space M^3/f. This is a one dimensional space that can be very different for the same manifold (one can construct them from a handle decomposition of the manifold just looking at how the π_0 changes when we add handles).

7. Some special families of Nambu structures

As we have seen, the problem of classifying Poisson structures on a given manifold includes that of the classification of certain arrangements of oriented hypersurfaces (those arrangements that come from the zeros of a function). For M^n one can consider the dual tree to $(M, \mathcal{H})$ and put a plus sign if the orientation of the n-tensor in the corresponding maximal leaf coincides with that of M and minus otherwise. Giving the signs is equivalent to giving the orientation of the hypersurfaces.

For S^2, O. Radko [**51**] defines $\mathcal{G}_k(S^2)$ as the set of generic Poisson structures on S^2 with k vanishing curves. A weighted signed tree is defined as a tree with a plus or minus sign attached to each vertex so that for each vertex, those belonging to the boundary of its star have opposite sign; each edge is weighted with a positive number (the modular period), and a real number (the regularized volume) is assigned to the whole graph. She proves the following:

Theorem 7.1 ([**51**]). *The set $\mathcal{G}_k(S^2)$ up to orientation preserving isomorphisms coincides with the isomorphism classes of weighted signed trees with $k+1$ vertices (the isomorphism has to preserve the real number attached to the graph).*

The result relies on the fact that there is a one to one correspondence between arrangements of k circles in S^2 (in fact up to isotopy) and isomorphism classes of trees with $k+1$ vertices (observe also that every tree can be signed in two ways). One can isotope two arrangements with equivalent graph because, up to isotopy, the circle can sit in S^2 in a unique way splitting S^2 in two disks, and that results admits a well known generalization.

Theorem 7.2 (Smooth Schoenflies theorem). *Any smooth embedding of $j\colon S^{n-1} \hookrightarrow S^n$ bounds an n-dimensional ball and hence splits the sphere into two n-dimensional balls. In particular it is isotopic to the standard one where S^{n-1} sits inside $\mathbb{R}^n \subset \mathbb{R}^n \cup \{\infty\} = S^n$ as the boundary of the euclidean ball of radius one. It also holds for embeddings in $\mathbb{R}^n$.*

As consequence of this result one easily proves the following:

Lemma 7.3. *There is a one to one correspondence between arrangements of k $(n-1)$-spheres in S^n and isomorphism classes of trees with $k+1$ vertices.*

Definition 7.4. *Let us define $\mathcal{G}_k(S^n)$ to be the set of generic Nambu structures in S^n whose vanishing set consist of k spheres.*

Give S^n the usual orientation, so we can put signs in the dual trees. We just proved the following

Proposition 7.5. *The set $\mathcal{G}_k(S^n)$ is, up to isotopy, the same as the equivalence classes of weighted signed trees with $k+1$ vertices.*

Conclusions and future research

In this thesis we have studied different aspects of 3-different geometries with topological character, (2)-calibrated structures, Poisson structures and generic Nambu structures. In our opinion the most interesting results have been those obtained for calibrated compact manifolds. For them we have shown the existence of an approximately holomorphic geometry giving rise to a number of topological constructions compatible with the calibrated structure: Lefschetz pencils, calibrated cycles, determinantal subvarieties and embeddings in $\mathbb{CP}^m$ transverse to certain holomorphic foliations of $\mathbb{CP}^m$.

The local geometry of calibrated manifolds have been analyzed, showing thus the existence of Darboux charts and reference sections for very ample sequences of locally splittable hermitian line bundles. Also some aspects of the A.H. theory for symplectic manifolds (or even a.c. manifolds) have been clarified as those related to the modification of the almost complex structure in the hermitian bundles of r-jets and the construction of the non-linear bundles of pseudo-holomorphic r-jets for maps to projective spaces and their properties.

As well of the mentioned results that have been treated in detail, there is a number of questions and potential applications that show up naturally, essentially associated to the study of calibrated foliations.

When the leaves are complex manifolds it is natural to ask whether it is possible to perturb the sequence of sections so that they become holomorphic. The difference w.r.t. to the even dimensional situation –in which this perturbation exists– is the absence ,in principle, of a natural elliptic operator whose solutions are the holomorphic sections.

We can even consider the previous question for taut foliations in 3-manifolds. For them, every almost complex structure is integrable. Moreover, the results of E. Ghys show that it is possible to develop a meaningful theory for leafwise holomorphic sections (see [**21**]). In particular, E. Ghys proves the existence of holomorphic maps to $\mathbb{CP}^2$ which immerse each leaf; this is exactly the same kind of result that we have been able to obtain but with A.H. maps. A deep understanding of Ghys' methods should share some light about how A.H. sections can be perturbed into holomorphic ones, but maybe using local methods developed in this monograph (that is, being able to construct holomorphic reference sections as Donaldson does in the symplectic case [**12**]).

We also think that a detailed study of the combinatorial aspects of Lefschetz pencils for taut foliation is needed, because we think that these structures are potential tools in the study of taut foliations. Without being precise, the image of the link of singular points endows S^2 with a CW-complex structure. The inverse image of the 1-skeleton (of the image of the link) defines a singular surface (the structure of the self intersections can be understood using the local model). All the information of the taut structure is carried by this surface, because its complement is a collection of tori trivially foliated. Thus a combinatorial characterization of these surfaces would be really helpful. Since any taut foliation admits a Lefschetz pencil structure, we might be able to construct not only the foliation (from the foliated surface plus some piece of combinatorial data) but the foliation with a compatible Lefschetz pencil structure.

There is another kind of potential applications obtained form the theory we have developed for which the starting point are "quasi-contact structures", i.e., (closed) manifolds M^{2n+1} endowed with non-degenerated closed 2-forms ω (and for which we select an auxiliary codimension 1 distribution transverse to the kernel of the 2-form). Next, we construct n-generic maps $\phi_k \colon M \to \mathbb{CP}^n$. By definition, n is the smallest dimension for which the maps have empty base loci. The pullback of ω_{FS} is cohomologous to $k\omega$and we can think of it as a choice of 2-form is such class with interesting dynamical properties. $\phi_k^*\omega_{FS}$ degenerates along an stratified submanifold Σ_k, which is the degeneration locus of ϕ_k. We must have into account that the geometric interpretation is only approximately true. To get the usual interpretations it would be necessary to assume ϕ_k to be holomorphic in a neighborhood of the corresponding strata, something possible for $n = 1$ and possibly for $n = 2$ (for the latter there is a 3-manifold $\Sigma_1(\phi_k)$ of points where the rank drops to 2 and inside it a link $\Sigma_{1,1}(\phi_k)$ of points where the kernel is approximately tangent to $\Sigma_1(\phi_k)$). The new 2-form gives rise to a dynamical system (in principle non-smooth) defined as follows: we consider X_k the vector field in the kernel of $\phi_k^*\omega_{FS}$ with positive coorientation and whose component along the kernel of $k\omega$ has norm 1 in the g_k-metric. It is only defined in the complement of Σ_k. To give a global definition we multiply it by the unique function $g \in C^\infty(M)$ so that $g(k\omega^n) = \phi_k^*\omega_{FS}^n$ over $\mathcal{F}$. The function g tends to zero as the point approaches Σ_k, which gives an extension by zeros of gX_k. in principle the extension might be non-smooth. If we started from a 3-manifold with a calibrated foliation then the vector field is indeed smooth. Still in dimension 3 and for arbitrary distribution D dominated by ω, one checks that even though gX_k might be non-smooth, it is possible to rescale it (using Darboux charts covering Σ_k and gluing with appropriate bump functions) to define a smooth vector field proportional to X_k with the same fixed points. We expect –for any dimension– the existence of smooth rescalings of gX_k (with the same fixed points). It is clear that the existence of normal forms would be extremely helpful and would give a description of the flow near Σ_k. It would be necessary and alternative qualitative study of such flow in a neighborhood of Σ_k. The important property of this flow is that the trajectories either are fixed points, or converge to Σ_k (both when time tends to ∞ and $-\infty$) or are periodic. In dimension 3 this is nothing but information obtained out of a Lefschetz pencil structure (the latter gives also information about the orbit space). Of course if we start for example with a calibrated foliation instead of with a quasi-contact structure it is also true that the non-fixed orbits are transverse to the foliation and also the degeneration locus is transverse to it.

Observe that all the applications of the A.H. theory that we have stated are for calibrated manifolds. But we have deliberately developed a relative theory that can be applied not only for symplectizations, but in more general situations.

It is more or less clear that the relative theory that we have developed has as immediate corollaries the existence of constructions relative to symplectic submanifolds or to calibrated submanifolds of compact symplectic manifolds. For example: given (M, ω) a compact symplectic manifold of integer type any oriented hypersurface Q with a transverse vector field defined in a neighborhood of Q canonically defines a calibrated structure in Q ($D$is defined to be the kernel of $i_X\omega$). We can for example construct relative Lefschetz for the quadruple (M, ω, Q, X), i.e., Lefschetz pencils for (M, ω) whose restriction are Lefschetz pencils for $(Q, D, \omega_{|Q})$ (in fact we only have local models around the base points in $A \cap X$ seen as base points for the Lefschetz pencil in M). This situation is specially interesting when $L_X\omega = q\omega,\ q \in \mathbb{Z}$, because Q is either a contact hypersurface or Poisson (in the latter it is a symplectic bundle because $[\omega]$ is an integer class). Notice that the symplectic manifold can have non-empty boundary and the hypersurface in question its boundary. The obvious example we have in mind is that of symplectic fillings of contact manifolds.

One can also construct relative Lefschetz pencils for triples (M, ω, N), N any symplectic submanifold.

Regarding the contents of chapter II, a surgery construction for Poisson manifolds has been introduced; the main corollary is that the fundamental group does not obstruct the existence of (non-trivial) Poisson structure. Following with the more topological spirit of chapter II, a question we think worth investigating is the following: when the summands of the normal connected sums are integrable Poisson manifolds in the sense of R. L. Fernandes and M. Crainic [**11**] (for example calibrated foliations of integer type) and the corresponding normal connected sum is also integrable (which itself is another interesting question; anyway we know that for calibrated structures the sum is also calibrated and hence integrable), does it correspond to any surgery construction in the symplectic grupoid? Notice that the main tool we have to study this question is the description –more or less manageable– of the symplectic grupoid as a quotient of classes of A-paths (see [**11**] for the corresponding definitions). The interesting observation is that both the symplectic and the Poisson surgery occur along codimension 2-submanifolds. If there is any construction of the grupoid of the normal connected sum as a symplectic surgery for the grupoids and the fibred submanifold, this construction will be a new one because it would use codimension 4 "submanifold".

Finally, in the third chapter we have given a classification of generic Nambu structures in closed orientable manifolds. We think that regarding the classification of geometries without local invariants (or with invariants easy to describe, as is the case of generic Nambu structures), the most interesting questions are those related to poisson structures (even in surfaces), but with more complex singularities. In fact, not only the classification of this structures is important but also that of other objects defined out of them. For example, and even for generic Poisson structures in surfaces it would be of interest to give a description of the space of contravariant connections (see [**18**]), and more explicitly, the existence of geometric structures in this space (recall that the moduli space of covariant connections for symplectic structures is symplectic, or Poisson when the boundary is non-empty).

APPENDIX A

The $\bar{\partial}$-Neumann problem with parameters

The $\bar{\partial}$-Neumann problem, in principle for domains $\Omega \subset \mathbb{C}^n$ with smooth boundary, asks about the solutions and its regularity for the P.D.E.

$$\bar{\partial}\beta = \alpha, \tag{A.1}$$

where α is a (p, q)-form –at least of integrable square– and necessarily $\bar{\partial}$ closed.

The basic reference we will follow in this appendix is chapter 7 of [**35**].

A first observation is that if β is a solution and λ is $(p, q-1)$-form with holomorphic coefficients, then $\beta + \lambda$ is another solution. Thus, in order to obtain a reasonable theory is necessary to make a canonical choice of "good solution" solution. A reasonable choice is in case we have solutions, taking the one orthogonal to the holomorphic functions in Ω.

We will state a existence theorem in which the regularity of the solution in the boundary of Ω will depend on its geometry. We recall that a closed domain Ω is *strictly pseudo-convex* if its Levi form is positive definite in all the points of the boundary (definition 7.4.3 in [**35**]). Let us denote by $H_s^{(p,q)}(\Omega)$ the Hilbert space of (p, q)-forms with coefficients int he corresponding Sobolev space and $\wedge^{p,q}(\Omega)$ the (p, q)-forms with smooth coefficients (also in the boundary). The fundamental result is the following:

Theorem A.1. *(Theorem* 7.9.14 *in* [**35**]*) Let $q \geq 1$, $\alpha \in H_0^{(p,q)}(\Omega)$, where Ω is strictly pseudo-convex. There a unique $\beta \in H_0^{(p,q)}(\Omega)$ exists such that β is orthogonal to the kernel of $\bar{\partial}$ and $\bar{\partial}\beta = \alpha$. If $\alpha \in \wedge^{p,q}(\Omega)$ then $\beta \in \wedge^{p,q-1}(\Omega)$ and the following fundamental estimate holds:*

$$|\beta|_s \leq A_s|\alpha|_s, \ \forall \alpha \in \wedge^{p,q}(\Omega), \tag{A.2}$$

where A_s does not depend on α, and $|\cdot|_s$ is the corresponding Sobolev norm. β will be called from now on the canonical solution.

It is necessary to notice that one of the most delicate points is the behavior of the solution at the boundary, the extra regularity obtained which is less than the one on the interior ("hypoellipticity").

We pretend to deduce from this result and using rather elementary methods a parametric version with estimates for C^h norms.

For our main application Ω is B^{2n}, the unit ball -which is clearly strictly pseudo-convex– and we do not need all the power of the previous result.

Indeed, from $\phi(z, \theta) \in C^\infty(B^{2n}(0, 1+\epsilon) \times S^1)$, we want to construct a function $\phi' \in C^\infty(B^{2n}(0, 1-\epsilon) \times S^1)$ holomorphic for each fixed θ and whose C^h-distance to ϕ is controlled by the corresponding norm of $\bar{\partial}\phi$. To do that we want to solve equation A.1 with $\alpha_\theta = \bar{\partial}\phi(z,\theta)$ and use the *unique solution* β_θ given by A.1 to define $\phi'(z,\theta) = \phi(z,\theta) - \beta_\theta(z)$. Moreover, since we do not ϕ' to be defined in the same domain as ϕ it will be enough to consider $\alpha_\theta = f\partial\phi$, where f is a cut-off function so that α_θ has compact support (in the interior of B^{2n}), so we do not need the delicate analysis at the boundary.

In any case, we will see that from theorem A.1 we easily deduce the following corollary.

Corollary A.2. *Let (P, g) be a compact riemannian manifold of dimension u, $q \geq 1$ and Ω a domain of $\mathbb{C}^n$ strictly pseudo-convex and compact. Let $\alpha(z,t) \in \wedge^{p,q}(\Omega \times M)$ with $\bar{\partial}\alpha_t = 0$. Then a unique $\beta \in \wedge^{p,q-1}(\Omega \times M)$ exists such that β_t is orthogonal to the kernel of $\bar{\partial}$ and $\bar{\partial}\beta_t = \alpha_t$. Moreover, $\forall j \in \mathbb{N}$ positive constants B_j not depending on α exist so that*

$$|\beta|_{C^j} \leq B_j |\alpha|_{C^j}, \tag{A.3}$$

where $|\cdot|_{C^j}$ is the sum of the norms of the supremum for the form and its first j covariant derivatives. We use the product metric with factors the euclidean in Ω and g, and the covariant derivatives are w.r.t. the corresponding Levi-Civita connection.

Proof. Define $\beta(z,t)$ so that β_t is the canonical solution given by theorem A.1 with data α_t. Uniqueness implies that $\beta(z,t)$ is well defined.

We will see that once the smoothness of β has been proven the bounds of A.3 follows easily.

We observe that we can assume P to be an open set of $\mathbb{R}^u$ (with coordinates $t_1, ..., t_u$).

Let $(z', t') \in \Omega \times \mathbb{R}^u$ be any point. To prove continuity of β we apply the triangular inequality to write

$$|\beta(z,t) - \beta(z',t')| \leq |\beta(z,t) - \beta(z',t)| + |\beta(z',t) - \beta(z',t')|.$$

The continuity of β_t is consequence of theorem $A.1$. Thus continuity for β in (z', t') would follow from proving that for any $\epsilon > 0$ an $\delta > 0$ exists so that

$$\sup_{z\in\Omega} |\beta(z,t) - \beta(z,t')| \leq \epsilon, \text{ if } |t - t'| \leq \delta. \tag{A.4}$$

For each $t \in \mathbb{R}^u$ set $\gamma_t(z) := \beta_t(z) - \beta_{t'}(z)$. It is straightforward that $\gamma_t \colon \Omega \to \mathbb{C}$ is orthogonal to the holomorphic functions (because is the difference of two vectors in that subspace) and that $\bar{\partial}\gamma_t = \alpha_t - \alpha_{t'}$. Therefore γ_t is the canonical solution with data $\alpha_t - \alpha_{t'}$, so we can apply the fundamental estimates. In particular and recalling that $|\cdot|_s$ denote the corresponding Sobolev norms, for $s > n$ we have:

$$|\gamma_t|_{C^0} \leq K_s|\gamma_t|_{j+n} \leq K_s A_s |\alpha_t - \alpha_{t'}|_s \leq K_s A_s V_s |\alpha_t - \alpha_{t'}|_{C^{j+n}}. \tag{A.5}$$

The first inequality is deduced from the Sobolev immersion theorem, the second from theorem A.1 and the third is obvious. Finally, $\alpha_t - \alpha_{t'}$, together with all its derivatives (in the coordinates of $\mathbb{C}^n$) vanish for t'. The compactness of Ω and the smoothness of α imply that A.4 holds.

The next step is to prove that $\frac{\partial\beta}{\partial t_m}$, $1 \leq m \leq u$, exists and it is continuous. The obvious candidate is the canonical solution with data $\frac{\partial\alpha}{\partial t_m}$. Denote it by $\dot{\beta}$. Let us fix $t' \in \mathbb{R}^u$ and consider the function $\zeta(z,t) := \beta_{t'} + \int_{t'_m}^{t_m} \dot{\beta}(z, t_1, ..., t_{m-1}, v, t_{m+1}, ..., t_u)dv$. If we show that ζ coincides with β, then from the fundamental theorem of calculus $\frac{\partial\beta}{\partial t_m} = \dot{\beta}$. Since $\dot{\beta}$ itself is the canonical solution, as we just saw it will be continuous. Thus, we would deduce that β is C^1 (the existence and continuity of the partial derivatives along $\mathbb{C}^n$ is guaranteed by theorem A.1).

Since $\dot{\beta}$ is continuous, $\bar{\partial}\int_{t'_m}^{t_m} \dot{\beta}dv = \int_{t'_m}^{t_m} \bar{\partial}\dot{\beta}(z,v)dv$. Hence $\bar{\partial}\zeta = \alpha_{t'} + \int_{t'_m}^{t_m} \frac{\partial\alpha}{\partial t}dv = \alpha_t(z)$. moreover, if F is any holomorphic function,

$$\int_\Omega (\int_{t'_m}^{t_m} \dot{\beta}dv)\bar{F}dw = \int_{t'_m}^{t_m} (\int_\Omega \dot{\beta}\bar{F}dw)dv = 0,$$

because the integrals commute by continuity of $\dot{\beta}$ and F.

Therefore ζ coincides with β because both are the canonical solution with data α.

Regarding the partial derivatives of order 2, those only involving the variables $z_i, \bar{z}_j$ exist and are continuous by theorem A.1. Existence and continuity of $\frac{\partial^2\beta}{\partial z_i \partial t_m}$, $\frac{\partial^2\beta}{\partial \bar{z}_j \partial t_m}$ and $\frac{\partial^2\beta}{\partial t_q \partial t_m}$, with $1 \leq i,j \leq n$, $1 \leq q,m \leq u$, follow from the fact that $\frac{\partial\beta}{\partial t_m}$ is the canonical solution and as we just saw it is C^1. Continuity of the partial derivatives $\frac{\partial^2\beta}{\partial z_i \partial \bar{z}_j}$, $\frac{\partial^2\beta}{\partial \bar{z}_j \partial z_i}$ is a consequence of the inequality corresponding to A.5, but starting form the $|\cdot|_{C^2}$-norm on the leftmost term.

The last possibility is a derivative of the type $\frac{\partial^2\beta}{\partial t_m \partial z_i}$ or $\frac{\partial^2\beta}{\partial t_m \partial \bar{z}_j}$, whose existence and continuity follows from Schwartz's lemma. Recall than in its weakest form it assures that if both $\frac{\partial^2\beta}{\partial z_i \partial t_m}$ (resp. $\frac{\partial^2\beta}{\partial \bar{z}_j \partial t_m}$) and $\frac{\partial\beta}{\partial z_i}$ (resp. $\frac{\partial\beta}{\partial \bar{z}_j}$) exist and are continuous then $\frac{\partial^2\beta}{\partial t_m \partial z_i}$ (resp. $\frac{\partial^2\beta}{\partial t_m \partial \bar{z}_j}$) exists, it is continuous and coincides with $\frac{\partial^2\beta}{\partial z_i \partial t_m}$ (resp. $\frac{\partial^2\beta}{\partial \bar{z}_j \partial t_m}$). The only necessary ingredient is the continuity of $\frac{\partial\beta}{\partial z_i}$ (resp. $\frac{\partial\beta}{\partial \bar{z}_j}$), and this follows from inequality A.5 for norms $|\cdot|_{C^2}$.

Once β has been shown to be C^2, differentiability to higher orders follows by induction using the commutativity of the partial derivatives. Indeed, if we assume that β is C^h and we have a partial derivative of order $h+1$, it can be of three types. The first is that in which there is no derivative w.r.t. a variable t_m, and existence and continuity follow from theorem A.1 and the inequality correspondent to A.5 but starting with $|\cdot|_{C^{h+1}}$ norms. The second possibility is a partial derivative in which a derivative w.r.t some t_m is taken, but not in the last position (not of $n+1$). We can switch the partial derivative to the first place and use that $\frac{\partial\beta}{\partial t_m}$ is of class C^h by induction. The third possibility is where we have a unique t_m in the partial derivative in the last position. Again we apply Schwartz's lemma to the corresponding derivative of order $h-1$ of β.

Thus, we deduce that the canonical solution $\beta\colon \Omega \times \mathbb{R}^u \to \mathbb{C}$ is smooth.

The existence of constants B_j so that A.3 holds is obvious, because whenever we have a partial derivative of order j, it can be written $\frac{\partial^j \beta}{\partial z^a \partial \bar{z}^b \partial t^c}$, where a, b, c representing certain multiindices. It is now enough to consider the $\bar{\partial}$-problem $\frac{\partial \alpha}{\partial t^c}$, and apply the bounds of theorem A.1 together with the ones coming form the appropriate Sobolev embedding to obtain the desired result.

$\square$

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