TOPOLOGY OF DIFFERENTIABLE MANIFOLDS

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Contents

1. Introduction

Let us say a few words about the two key concepts in the title of the course, **topology** and **differentiable manifolds**.

1.1. **Topology.** It studies topological spaces and continuous maps among them, i.e. the category TOP with objects topological spaces and arrows continuous maps.

Definition 1. *Let* $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ *be topological spaces,* $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ *continuous is a homeomorphism if there exists* $g\colon (Y, \mathcal{T}_Y) \to (X, \mathcal{T}_X)$ *such that* $g \circ f = 1_X, f \circ g = 1_Y$.

In other words, a homeomorphism is an invertible arrow in TOP. Being homeomorphism is an equivalence relation $\simeq$.

Main aim of topology: Study orbits = equivalence classes of TOP under $\simeq$, in particular:

- When two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are homeomorphic, i.e. when two objects in TOP belong to the same $\simeq$-orbit.
- According to Klein's view point understanding the topology of $(X, \mathcal{T}_X)$ should amount to studying those properties of $(X, \mathcal{T}_X)$ invariant under homeomorphism, i.e. studying the invariants of the homeomorphism type of a topological space which are nothing but the properties of the $\simeq$-orbit. For example compactness, separation properties...

From now on all our maps will be assumed to be continuous. A very good strategy is given $(X, \mathcal{T}_X)$ to study maps from and to very simple topological spaces, e.g. the interval $I = [0, 1]$. For example according to Urysohn's lemma being normal amounts to have "enough" maps to I.

Studying the orbits of $\simeq$ is extremely difficult in general. It is often convenient to consider a weaker equivalence relation, **homotopy**.

def:homotmaps **Definition 2.** *We say that* $f, g\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ *are homotopic, and denote it by* $f \sim g$, *if there exist*

$$H\colon X \times I \to Y,$$

such that $H_0 = f, H_1 = g$, *where* $H_t(x) := H(x, t)$.

In other words, two maps are homotopic if they can be joined by a "continuous path of maps" (see remark 12).

Definition 3. $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ *is a homotopy equivalence if there exists* $g\colon (Y, \mathcal{T}_Y) \to (X, \mathcal{T}_X)$ *such that* $g \circ f \simeq 1_X, f \circ g \simeq 1_Y$.

Homotopy equivalence is weaker than homeomorphism, hence two topological spaces not in the same $\sim$-orbit cannot be homeomorphic.

Definition 4. *A property of the* $\sim$*-orbit of* $(X, \mathcal{T}_X)$ *is an invariant of its homotopy type.*

Compactness is an invariant of the homeomorphism type but not of the homotopy type (for example $\mathbb{R}^n$ is homotopic to $\{p\}$).

Invariants come often as **functors**.

We have similar definitions for pairs (X, A) where A is a subset of X (relative theory). So we consider the category of pairs $A \hookrightarrow X \in$ TOP, and the arrows are subset preserving (or equivalently they are pairs of arrows making the obvious diagram commute). Then we have invertible ones (homeomorphisms of pairs), and hence $\simeq$-orbits. We also have the notion of homotopy of maps between pairs (the maps must preserve the subsets) giving rise to $\sim$-orbits.

Exercise 1. *Define homeomorphism and homotopy of pairs.*

Notice how we can combine the idea of homotopy and the study of maps from simple topological spaces as follows:

Definition 5.

$$\pi_0(X) = \{f\colon \{p\} \to X\}/\sim \stackrel{1:1}{\leftrightarrow} \{\text{path} - \text{connected components of X}\}$$

$$\pi_1(X,x) := \{f\colon (I,\partial I) \to (X,x)\}/\sim,$$

$$\pi_n(X,x) := \{f\colon (I^n,\partial I^n) \to (X,x)\}/\sim$$

For $n \geq 2$, $\pi_n(X,x)$ is abelian and very difficult to compute.

Being more precise, we have functors from pointed topological spaces to groups

$$(X,x) \mapsto \pi_n(X,x)$$

(so that a homotopy equivalence between pointed spaces induces an isomorphism between the corresponding homotopy groups).

Other invariants: Homology and cohomology theories. **Algebraic topology** is the subject dealing with the study of these (algebraic) invariants for topological spaces. Topological spaces can be very complicated; techniques from algebraic topology specially suited for **CW-complexes** (see chapter 0 in [2]).

1.2. Manifolds.

Definition 6. *An atlas on $(X, \mathcal{T}_X)$ of class C^r, $r \in \bar{\mathbb{N}}$ ($\bar{\mathbb{N}} := \mathbb{N} \cup \infty$), is defined by the following data:*

- *$\{U_i\}_{i\in I}$ an open cover of X.*
- *$\varphi_i\colon U_i \to \mathbb{R}^{m_i}$ mapping U_i homeomorphically onto an open subset of $\mathbb{R}^{m_i}$.*
- *For each non-empty overlap $U_i \cap U_j$,*

  $$\varphi_{ij} := \varphi_i \circ \varphi_j^{-1}\colon \varphi_j(U_i \cap U_j) \subset \mathbb{R}^{m_j} \to \varphi_i(U_i \cap U_j) \subset \mathbb{R}^{m_i}$$

  is a diffeomorphism of class C^r.

  We call φ_i^{-1} a **chart** *for U_i, and the components of φ* **coordinates** *on U_i.*

There is a notion of C^r-map between topological spaces endowed with C^r-atlas.

Then we can define the notion of two C^r-atlases being compatible by asking the identity map to be C^r w.r.t. to one atlas in the domain and the other in the target. There is also a partial order for compatible atlases -where an atlas precedes a second one if all charts of the former are charts of the latter- and among compatible ones there is a unique maximal atlas.

Definition 7. *A C^r-manifold structure on $(X, \mathcal{T}_X)$ is given by a maximal atlas. Equivalently, it is given by any C^r-atlas, as any such is associated to as unique maximal one.*

*A C^r-***manifold** *is a topological space endowed with a C^r-manifold structure.*

C^0-manifolds are often called **topological manifolds***, and C^∞-manifolds* **smooth manifolds***. Also for us a* **differentiable manifold** *will stand for a C^r-manifold, for some $r \geq 1$.*

In other words, by a C^r-manifold structure we make $(X, \mathcal{T}_X)$ look locally like some $\mathbb{R}^m$, up to C^r-diffeomorphism.

Remark 1. *There are more kinds of manifold structures:*

- *Analytic structures (C^ω).*
- *Piecewise linear structures (PL).*

So we have the categories MAN^r, $r \in \bar{\mathbb{N}}$, and the same fundamental questions as for TOP apply, i.e. determine when two objects are isomorphic, and describe properties up to isomorphism.

Differentiable topology: Study of differentiable manifolds and differentiable maps, i.e. of MAN^r, $r \geq 1$.

The study of topological manifolds and differentiable manifolds uses very different techniques. For the latter we have tools coming from calculus, which simplifies a lot the answering of certain basic questions.

For example we have the following result.

Proposition 1. *Any connected C^r-manifold can be assigned a positive number, its dimension. It is defined as the dimension of the domain vector space of any chart.*

Proof. To check that dimension is well defined, we just need to show that if we have $U \subset \mathbb{R}^m$ an open subset and

$$\varphi \colon U \to \mathbb{R}^n$$

sending U C^r-diffeomorphically into an open subset, then $n = m$.

If $r \geq 1$, then the differential at any point must be invertible, and therefore the result follows.

It the topological case, it follows from **Brouwer's invariance of domain** theorem that asserts that if $f \colon U \subset \mathbb{R}^m \to \mathbb{R}^m$ is injective and continuous, then it is open (i.e. a homeomorphism).

Now we can assume w.l.o.g. that n<m, then the composition $f = i \circ \Phi$, $i \colon \mathbb{R}^n \hookrightarrow \mathbb{R}^m$ is continuous and injective, and therefore open according to Brouwer's, but that is a contradiction.

□

In particular, we see that the only information that we can extract by looking at arbitrarily small neighborhoods of a point in a connected manifold M is its dimension (i.e. it is the unique local invariant). Therefore, **characteristic information of a manifold can only be seen at the global level.**

We can easily get on the same topological manifold different -but diffeomorphic- C^r-manifolds structures, $r \geq 1$.

Example 1. *In $\mathbb{R}$ we consider two different smooth structures. The first one is the canonical one coming from the vector structure. The second one is given by the global chart*

$$\begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ t & \mapsto & t^3 \end{array}$$

The identity from the canonical smooth structure to the latter is a homeomorphism which is not even C^1, because if so $x \to x^{1/3}$ should be C^1.

More generally, given M, N C^s-manifolds, $s \geq 1$, and $\phi \colon M \to N$ a C^r-diffeomorphism $r < s$ (which is not of class C^{r+1}), we can push the C^s-structure on M to one on N (and viceversa), obtaining a new one which is not the same as the original one, but C^r-diffeomorphic. In particular, if for any C^s-manifold M we can find a self C^r-diffeomorphism (see remark 20), then we can change the original structure into a different one, which is diffeomorphic.

In the previous example we spoke of C^r-functions between C^s-manifolds, for $r < s$. To make this precise notice that any C^s-atlas on $(X, \mathcal{T}_X)$ canonically induces a C^r-atlas in $(X, \mathcal{T}_X)$, by just considering the C^s-transition functions as C^r-transition functions. One sees that this gives rise to the following lem:forgetful

Lemma 1. *There are for each s, r, $r < s$* **forgetful functors**

$$\mathcal{F}^{r,s} \colon \mathrm{MAN}^s \to \mathrm{MAN}^t,$$

so that

$$\mathcal{F}^{r,s} \circ \mathcal{F}^{s,t} = \mathcal{F}^{r,t}$$

A fundamental question in the topology of manifolds is to study the fibers of $\mathcal{F}^{r,s}$. One of the main results we will prove is theorem 1 below. But before, we will make the following assumption which will be valid from now on (and will also be strengthened): all manifolds are to be Hausdorff and paracompact.

thm:diffsmooth

Theorem 1. *Let $1 \le r < s$. Then to any C^r-manifold structure on M we can assign a C^s-structure to M which is both compatible with the initial C^r-structure and unique up to C^s-diffeomorphism. In other words, at the level of objects, we have maps $U^{s,r} \colon \mathrm{MAN}^r \to \mathrm{MAN}^s / \simeq$ which invert $\mathcal{F}^{r,s}$.*

So in many cases questions about C^r-manifolds, $r \ge 1$, reduce to questions about smooth manifolds.

As we see from theorem 1 the fibers -at least at the level of diffeomorphism classes- of $\mathcal{F}^{r,s}$, $r, s \ge 1$ have exactly one element. On the other hand, the fibers of $\mathcal{F}^{0,\infty}$ can exhibit very different behaviors, and the following results are known:

1. In dimensions 0,1,2 and 3, the fiber is exactly one orbit. In other words, every topological manifold of the aforementioned dimensions admits -up to diffeomorphism- exactly one smooth structure (compatible with the topological structure).
2. In dimensions ≥ 5 a compact topological manifold admits at most a finite number of smooth structures. It can also admit no smooth structures. In the case of the spheres S^n, $n \ge 5$, these numbers are known: for example it is a famous result that the 7-sphere admits 28 different (non-diffeomorphic) smooth structures.
3. In dimension 4 the fiber can be empty (for certain compact simply connected manifolds), finite and infinite (and even non-countable!). If the manifold is open, then the fiber is non-empty. Being compact does not imply that only a finite number of smooth structures exist.
4. The Euclidean space $\mathbb{R}^n$ admits exactly one smooth structure if $n \neq 4$. $\mathbb{R}^4$ admits uncountable many smooth structures.

Other natural questions:

- If M, N are homotopic, are they C^r-diffeomorphic? (i.e. how much information algebraic topology can give us?).
- An embedding $f \colon M \hookrightarrow N$ is a C^r-map which is an immersion (if $r \ge 1$) and a homeomorphism onto its image. Given M, N, can we embed M in N?. If so, in "how many ways"?.

  For example, we may define an equivalence relation between embeddings f, f' by requiring the existence of $g \colon (N, f(M)) \to (N, f'(M))$ a diffeomorphism. When $N = S^3, M = S^1$, we are led to **knot theory**.
- Do (homotopic) invariants for manifolds have extra properties?. Do homotopy type invariants have C^r-versions/ C^r-constructions?.
- The same question applies for constructions of general topology (i.e., how much we can get by working with differentiable functions rather than with continuous ones).

2. More definitions and basic results

Definition 8. *Given a continuous function $f \colon U \subset \mathbb{R}^m \to \mathbb{R}^n$, U open, a property $\mathcal{P}(f)$ is called*

- *local if for any open cover* $\{U_i\}_{i\in I}$ *of* U $\mathcal{P}(f)$ *holds iff* $\mathcal{P}(f_{|U_i})$ *holds for all* $i \in I$;
- C^r*-diffeomorphic invariant if* $\mathcal{P}(f)$ *holds iff* $\mathcal{P}(\Psi \circ f \circ \Phi)$ *holds, for*
$$\Phi\colon U \subset \mathbb{R}^m \to \mathbb{R}^m\,,\ \Psi\colon V \subset \mathbb{R}^n \to \mathbb{R}^n, f(U) \subset V,$$
any C^r*-diffeomorphisms.*

Given $A \subset \mathbb{R}^m$, *a property* $\mathcal{P}(A)$ *is*
- *local if for any open cover* $\{U_i\}_{i\in I}$ *of* A $\mathcal{P}(A)$ *holds iff* $\mathcal{P}(A \cap U_i)$ *holds for all* $i \in I$;
- C^r*-diffeomorphism invariant if* $\mathcal{P}(A)$ *holds iff* $\mathcal{P}(\Phi(A))$ *holds, for*
$$\Phi\colon U \subset \mathbb{R}^m \to \mathbb{R}^m$$
any C^r*-diffeomorphism,* $A \subset U$.

Notice that local properties are inherited by the restriction to open subsets .

Definition 9. *A subset* $A \subset \mathbb{R}^m$ *is a* C^r*-submanifold (of* $\mathbb{R}^m$*) if there exists open subsets* $\{U_i\}_{i\in I}$ *with* $A \subset \cup_{i\in I} U_i$, *and* **adapted coordinates** $\varphi_i\colon U_i \to \mathbb{R}^m$ *of class* C^r*-such that*
$$A \cap U_i = \Phi^{-1}(\mathbb{R}^n)\,,\ n \le m$$
The **codimension** *of* A *(connected) in* M *(connected) is* $\dim M - \dim A$.

ex:submanifold **Exercise 2.** *Show that a* C^r*-submanifold inherits a* C^r*-manifold structure so that the inclusion* $i\colon A \hookrightarrow \mathbb{R}^m$ *is a* C^r*-embedding (here you can use that being a subset of* $\mathbb{R}^m$, *the topology on* A *will be both Hausdorff and second countable, and thus paracompact; see also the beginning of subsection 2.4).*

Example 2. *The sphere*
$$S^{m-1} := \{x \in \mathbb{R}^m \mid |x|^2 = 1\}$$
is a submanifold of $\mathbb{R}^m$. *At each* $x \in S^{m-1}$ *define the hyperplane*
$$H_x := \{y \in \mathbb{R}^m \mid \langle y, x\rangle = 0\}$$
The map
$$\begin{array}{rcl}\varphi_x\colon \mathbb{R}^m & \longrightarrow & H_x \times \mathbb{R}\\ y & \longmapsto & (y - \langle y, x\rangle, \langle x, x\rangle - 1)\end{array}$$
has invertible differential at x, *so by the inverse function theorem in a neighborhood of* x *it defines a diffeomorphism. It is then clear that such a restriction is an adapted chart.*

Definition 10. *A* C^r*-map,* $r \ge 1$, $f\colon U \subset \mathbb{R}^m \to \mathbb{R}^n$ *is called an*
- *immersion if for all* $x \in U$, $Df_x\colon \mathbb{R}^m \to \mathbb{R}^n$ *is an injective (linear) map.*
- *submersion if for all* $x \in U$, $Df_x\colon \mathbb{R}^m \to \mathbb{R}^n$ *is a surjective (linear) map.*

Lemma 2. *The following properties are local* C^r*-invariant by diffeomorphism.*
- *For* $f\colon U \subset \mathbb{R}^m \to \mathbb{R}^n$, *being of class* C^r *in* U.
- *Being a submanifold.*
- *If* $r \ge 1$, *then being an immersion, submersion are local properties. Moreover, by the implicit function theorem being a submanifold is equivalent to the existence of local submersions* $f_i\colon U_i \to \mathbb{R}^n$ *so that* $A \cap U_i = f_i^{-1}(0)$.

Proposition 2. *Local properties invariant by* C^r*-diffeomorphism can be transferred to* C^r*-manifolds, where they become also local (recall that any open subset of a manifold carries a canonical manifold structure) and invariant by diffeomorphism.*

Indeed, for $f\colon U \subset M \to \mathbb{R}^n$, *we say that* $\mathcal{P}(f)$ *holds if for some (and hence any) cover of* U *by charts* $\{(U_i, \varphi_i)\}_{i\in I}$, $\mathcal{P}(f \circ \varphi_i^{-1})$ *holds for all* $i \in I$.

If now we have $f\colon M \to N$, then it is defined to hold if for some C^r-atlas $\{V_j, \psi_j)\}_{j\in J}$ of N (and hence for any), $\psi_j \circ f\colon M_j \to \mathbb{R}^n$ holds for $j \in J$, where $M_j := f^{-1}(\psi_j^{-1}(\mathbb{R}^n))$.

Definition 11. *For M, N C^r-manifolds, the set of C^r-maps from M to N is denoted by $C^r(M,N)$. We also denote by $\mathrm{Emb}^r(M,N), \mathrm{Imm}^r(M,N), \mathrm{Sub}^r(M,N), \mathrm{Diff}^r(M,N)$ the subsets of embeddings, immersions, submersions and diffeomorphisms.*

2.1. **Submanifold vs. embedding.** Recall that a differentiable map $f\colon M \to N$ is an embedding if it is an immersion ($r \geq 1$) and a homeomorphism onto its image. Bijectivity is a global property, but being open is local.

Any submanifold $A \subset N$, inherits a manifold structure from the adapted charts (this is essentially exercise 2 together with the definition of a submanifold of a manifold); once more, paracompactness of A follows from results of general topology (subsection 2.4).

pro:embsub

Proposition 3.

(1) *If $A \subset N$ is a submanifold then the inclusion $i\colon A \to N$ is an embedding.*
(2) *If $f\colon M \to N$ is an embedding $r \geq 1$, then $f(M)$ is a submanifold.*

Proof. Regarding point 1, bijectivity is clear and openness holds because the manifold structure on A is for the underlying inherited topology. Regularity is C^r-local and C^r-diff. invariant, so it can be checked in adapted charts, where it is clear (it reduces to exercise 2).

To prove point 2 we need to construct adapted charts about any point in $f(M)$. Use the embedding property to find charts U_i, V_i so that $f(U_i) = V_i \cap f(M)$, so it is enough to show that $\psi_i(f(U_i)) \subset \mathbb{R}^n$ is a submanifold, therefore the situation reduces to having an embedding in Euclidean spaces, and this follows from the implicit function theorem. □

rem:enbsameimage

Remark 2. *Therefore, an embedding $f\colon M \to N$ gives a diffeomorphism from M so the submanifold $f(M)$. There can be different embeddings $f, g\colon M \to N$ so that $f(M) = g(M)$. One sees that given one embedding f any other g with that property is of the form $g = f \circ \phi$, with ϕ a diffeomorphism of M.*

Remark 3. *Once more there are differences between the topological case and the differentiable one regarding the issue of submanifolds and embeddings. In the differentiable setting we can use infinitesimal information (being an immersion), which is transformed into local information (adapted charts). In the topological case the situation is much more involved.* **Alexander's Horned sphere** *is an example of embedding $f\colon S^2 \to \mathbb{R}^3$ whose image is not a submanifold.*

ssec:tangent

2.2. **The tangent bundle of a C^r-manifold, $r \geq 1$.** For each $x \in M$, and $c\colon (-\epsilon, \epsilon) \to M$, $c(0) = x$, a C^r**-curve through** x, we can make sense of $\frac{d}{dt}c(t)_{|t=0}$: we say that c, d C^r-curves through x are equivalent if for some local coordinates φ about x ,

$$\frac{d}{dt}\varphi \circ c_{|t=0} = \frac{d}{dt}\varphi \circ d_{|t=0} \qquad (1)$$

eq:equivcurves

Exercise 3. *Show that if equation 1 holds for some local coordinates φ, then it holds for any local coordinates having x inside its domain.*

Definition 12. *T_xM is the set of equivalence classes of such curves $[c]_x$.*

One easily proves

Lemma 3. *T_xM carries a canonical vector space structure.*

def:tgbundle **Definition 13.** *The* **tangent bundle** *of M is*

$$TM := \coprod_{x\in M} T_xM,$$

with $\pi\colon TM \to M$ the obvious projection, and with the following C^{r-1}-manifold structure:

The cover is $\{\pi^{-1}(U_i)\}_{i\in I}$ and the coordinate chart is defined

$$\begin{array}{rcl} D\varphi_i^{-1}\colon \varphi_i(U_i)\times\mathbb{R}^m & \longrightarrow & \pi^{-1}(U_i) \\ (x,v) & \longmapsto & [\varphi_i^{-1}(x+tv))]_{\varphi_i^{-1}(x)} \end{array}$$

To be more precise, TM carries in principle no topology, but for the bijections $\{(\pi^{-1}(U_i), D\varphi_i^{-1})\}_{i\in I}$ there is a unique topology making them into a C^{r-1}-atlas.

One easily observes:

- The manifold structure does not depend on the atlas, so it is induced by the manifold structure on M.
- $\pi\colon TM\to M$ is a C^{r-1}-surjective submersion.
- Each vector space T_xM is sent by $D\varphi_i$ to $\mathbb{R}^n$ with its canonical vector space structure (and the C^{r-1}-manifold structure only considers compatible atlases which are linear on tangent spaces, i.e. it carries a C^{r-1}-**vector bundle structure**). Another way to say this is that the vector structure is compatible with the differentiable structure in the sense that if P is a C^{r-1}-manifold, $s_1, s_2 \in C^{r-1}(P, TM)$ with $\pi\circ s_1 = \pi\circ s_2$, $\lambda_1,\lambda_2\in C^{r-1}(P,\mathbb{R})$, then
$$\lambda_1 s_1 + \lambda_2 s_2 \in C^{r-1}(P, TM)$$
- The **0**-section $\{0_x \mid x\in M\}$ is a submanifold of TM, or by proposition 3 M embeds as the **0**-section
$$\begin{array}{rcl} M & \longrightarrow & TM \\ x & \longmapsto & 0_x \end{array}$$
- For a C^r-map $f\colon M\to N$, there is an induced differential map (of class C^{r-1}) commuting with the projections
$$\begin{array}{rcl} Df\colon TM & \longrightarrow & TN \\ [c]\in T_xM & \longmapsto & [f\circ c]\in T_{f(x)}N \end{array}$$
The differential is a C^{r-1}-map which restricts to a linear map
$$Df_x\colon T_xM \to T_{f(x)}N,\ \forall x\in M$$
We will also use the notation $f_* := Df$.
In particular, if $i\colon A\hookrightarrow M$ is a submanifold, using adapted charts $Di\colon TA \longrightarrow TM$ is easily seen to be an embedding, so that for each $x\in A$ the vector space T_xA is linearly sent to a linear subspace of T_xM.

Observe also that $f\colon M\to N$, $r\geq 1$, is an immersion (resp. submersion) iff

$$Df_x\colon T_xM\to T_{f(x)}N$$

is injective (resp. surjective) for all $x\in M$.

def:vfield **Definition 14.** *A map $s\in C^{r-1}(M,TM)$ with $\pi\circ s = \mathrm{Id}$ is called a (C^{r-1})* **vector field** *(or a section, a name also valid for any vector bundle).*

Note that the set of vector fields $\mathfrak{X}^{r-1}(M)$ is a module over $C^{r-1}(M)$.

2.3. **Transversality and submanifolds.**

Definition 15. *Let $f\colon M \to N$ a C^r-map, $r \geq 1$, and $A \subset N$ a submanifold. We say that f* **is transversal to** *A if for every x so that $f(x) \in A$,*

$$T_{f(x)}N = f_* T_x M + T_{f(x)}A$$

Notice that for $f\colon M \to N$, $x \in N$ is a **regular value** iff f is transversal to x.

Proposition 4. *Let $f\colon M \to N$ a C^r-map, $r \geq 1$, transversal to $A \subset N$. Then $f^{-1}(A)$ -if non-empty- is a submanifold of M of codimension the codimension of A in N.*

Proof. Go to charts, reduce it to the case of a point by projecting, and apply the known statement coming from the implicit function theorem. □

Example 3.

(1) *In $\mathbb{R}^m$ the unit sphere is $f^{-1}(0)$, for $f(x) = |x|^2 - 1$, with 0 a regular value.*
(2) *In $\mathbb{R}^2$ the canonical* **open book decomposition** *$\mathcal{B}_0$ is the partition given by the origin and all the open rays emanating from the origin. Each subset in the partition is a submanifold. An open book decomposition on an arbitrary manifold is defined to be a map $f\colon M \to \mathbb{R}^2$ which is transverse to all submanifolds of $\mathcal{B}_0$, and contains 0 in its image. By pulling back each of them, we get another partition of M by submanifolds. The projection $(x_1, x_2, x_3) \mapsto (x_1, x_2)$ restricted to the unit sphere S^2, gives an open book decomposition. An open book decomposition for S^3 produces a* **fibred link** *$f^{-1}(0)$.*

ssec:funct

2.4. **Topology with C^r-functions.**

Our manifolds are being assumed to be Hausdorff and paracompact. A stronger assumption is to require second countability rather than paracompactness, which we take from now on. They are almost equivalent, because any connected component of a Hausdorff paracompact manifold is second countable. Notice also that most of our results are to be proved working on a single connected component on a manifold. Observe for example that by defining a submanifold by just requiring it to have adapted charts, we recover the Hausdorff and paracompactness properties: indeed, each connected component of the manifold is second countable, hence the intersection of the submanifold with that component is Hausdorff and paracompact, and therefore the submanifold is Hausdorff and paracompact. In any case the reader is invited to verify which of our results hold when only paracompactness is assumed.

Being our manifolds second countable, we have a countable basis $\{W_l\}_{l\in\mathbb{N}}$. We wonder how "good" it can be chosen to be.

lem:nicebasis

Lemma 4. *Given any open cover $\{U_i\}_{i\in I}$ we can find $\{Z_j\}_{j\in\mathbb{N}}$ a countable basis so that*

(1) *Z_j is subordinated to U_i.*
(2) *Each $\bar{Z}_j$ is in the domain of local coordinates (V_j, φ_j) and $\varphi_j(Z_j) = B(0,1)$.*

Proof. The topology of M for each point a basis of compact neighborhoods, since for any point we can use local coordinates and transfer the closed balls centered at the image.

Keep only those W_l such that

- W_l is in some $U_{i(j)}$,
- W_l has compact closure.

It is still a basis. Indeed, regarding the first condition, given any open subset U', and $x \in U$, consider $U'_i := U' \cap U_i$. If $x \in U'_i$, since W_l is a basis we have

$x \in W_{l(i)} \in U_i$. Regarding the second if $x \in U'$, then $x \subset K \subset U'$, and some $W_{l(K)}$ must be contained in int$K$.

Now each of the W_l we keep is in the domain of local coordinates landing inside a compact neighborhood inside the image of the coordinates. Therefore, we can write it as the union of a countable family of balls whose closure is still in the domain of local coordinates, and the union of all then for the W_l verifying the above two properties is Z_j. □

rem:nicebasis **Remark 4.** *A basis is a notion that belongs to the realm of topological spaces. When we are in the manifold setting we would like all our objects to belong to the differentiable category. It is clear that any open subset of a manifold is a submanifold, but we are often interested in looking at closures of these open subsets. Thus, it makes sense to ask them to be submanifolds (nec. with boundary 18). It will also become clear (construction of partitions of unity and proposition 7) why their being diffeomorphic to closed balls is also very useful.*

It is convenient to think of non-compact manifolds as countable increasing union of compact pieces.

def:exha **Definition 16.** *A* **compact exhaustion** *of M is given by a sequence M_j, $j \in \mathbb{N}$, such that*

$$M_j \text{ compact},\ M_j \subset \mathrm{int} M_{j+1},\ M = \bigcup_{j\in\mathbb{N}} M_j$$

lem:compactexh **Lemma 5.** *A C^r-manifold admits compact exhaustions.*

Proof. Take a countable basis Z_j as in lemma 4.

Then define $M_0 = \bar{Z}_0$. we assume by induction that

$$M_j = \bigcup_{j=1}^{d(j)} \bar{Z}_j,\ d(j) \geq j$$

Add a finite number of $Z_j's$ (not in intM_j) with non-empty intersection with M_j and covering it. Define $d(j+1)$ to be the highest index of them. Then

$$M_{j+1} := \bigcup_{j=1}^{d(j+1)} \bar{Z}_j$$

gives the solution. By construction $M_j \subset \mathrm{int} M_{j+1}$. Also M_{j+1} is compact and $d(j+1) \geq j+1$, the latter implying

$$M = \bigcup_{j\in\mathbb{N}} Z_j \subset \bigcup_{j\in\mathbb{N}} M_j$$

□

lem:nicecover **Lemma 6.** *Given any open cover $\{U_i\}_{i\in I}$ we can find $\{V_l\}_{l\in\mathbb{N}}$ a countable cover so that*

(1) *V_l is subordinated to U_i.*
(2) *Each $\bar{V}_l$ is in the domain of local coordinates (U_l', φ_l') and $\varphi_j'(V_l) = B(0,1)$.*
(3) *V_j is locally finite.*

Proof. We take Z_l a basis as in lemma 4 and a compact exhaustion M_d. Out of the compact exhaustion we build another open cover

$$Q_d := \mathrm{int} M_{d+1} \backslash M_{d-1}$$

We can assume w.l.o.g. that the basis Z_k is subordinated to both U_i and Q_d. The refinement is constructed by carefully adding by induction a finite number of $V_j := Z_{l(j)}$. In the first induction step we take a finite number of open subsets covering M_0. Next, we cover $M_1\setminus\text{int}M_0$, compact, by a finite number of them with non-empty intersection with M_1. The (d+1)-th step amounts to covering $M_{d+1}\setminus\text{int}M_d$ by a finite number of $Z_l's$. Being subordinated to Q_k they must be inside $Q_d \cup Q_{d+1} \cup Q_{d+2}$. Observe that open sets added at the (d+1)-th step only intersect those added at the (d-1)-th,d-th, (d+2)-th and (d+3)-th steps, which are a finite quantity. Therefore, the cover is locally finite. □

Fundamental results on general topology assert that certain properties of the topological space M are equivalent to the existence of very particular functions (Urysohn's lemma, partitions of the unity). Since we have a C^r-manifold structure, it seems feasible that those functions can be taken to be C^r, i.e, the richness of $C^0(M)$ should also appear in $C^r(M)$ (and in general in $C^r(M,N)$).

In manifolds, we have **bump functions**, which we use to transfer function theory results from Euclidean space to manifolds.

lem:bump

Lemma 7. *Given any $r' > r > 0$, $y \in \mathbb{R}^m$, we can find a smooth function $\mu\colon \mathbb{R}^m \to \mathbb{R}$ such that*

(1) $\mu_{|B(y,r)} \equiv 1$, $\mu \le 1$,
(2) $\text{supp}(\mu) \subset B(y,r')$,

Proof. Start with the (flat) function at the origin

$$f(t) = \begin{cases} \exp^{-\frac{1}{t^2}},\ t \ge 0 \\ 0,\ t \le 0 \end{cases} \tag{2}$$

Then we translate. Next we get a function starting with the reflection of the original one, so that the product has support in $(-r',-r)$. Then we integrate to get the right step. Next we do the same with another copy to get the right support.

Now use the radius as coordinate to go to more variables. □

Remark 5. *Notice that we can arrange $\mu^{-1}(1) = \overline{B(y,r)}$.*

rem:uryson

Remark 6. *Notice that since we are dealing with normal spaces, given any to closed subsets $A, B \subset M$ we can find a continuous function $f\colon M \to [0,1]$ such that $f_{|A} \equiv 0$, $f_{|B} \equiv 1$. We would like f to be C^r, and lemma 7 is a step in that direction (the solution appears in exercise 32).*

We can use bump functions to easily produce **C^r-partitions of the unity**: firstly, we take a locally finite cover $\{(U_i,\varphi_i)\}_{i\in I}$ such that the balls $\overline{B(y_i,r_i)} \subset \varphi_i(U_i)$ and $\varphi_i^{-1}(B(y_i,r_i))$ is a cover of M. Then we take bump functions μ_i by pulling back the bump functions as in lemma 7 (with support in U_i) and attaining the value 1 at $\varphi_i^{-1}(B(y_i,r_i))$; then divide each function by the total sum. It we want the partition of the unity subordinated to a given cover $\{V_j\}_{j\in J}$, we may always find a refinement as above, and then use a function $k\colon I \to J$ so that $U_i \subset V_{k(i)}$ to transfer it by taking

$$\mu_j = \sum_{i\in k^{-1}(j)} \mu_i$$

Now we can use C^r-partitions of the unity to build **differentiable metrics**.

Any local coordinates $\varphi = (x_1,\cdots,x_m)$ on $U \subset M$ produce a **frame** of $\mathfrak{X}^{r-1}(U)$. Indeed, if $e_1,\ldots,e_n$ is the basis of $\mathbb{R}^n$ used to decompose φ, we define

$$\frac{\partial}{\partial x_i}(x) := [\varphi^{-1}(\varphi(x)+te_i)]_x$$

If we use other coordinates $\varphi' = (x'_1,\ldots,x'_m)$, then

$$\frac{\partial}{\partial x_i}(x) = \sum_{j=1}^{m} \frac{\partial(\varphi'\circ\varphi^{-1})_j}{\partial x_i}(\varphi(x))\frac{\partial}{\partial x'_i}(x) \tag{3}$$

eq:changecoord

Definition 17. *A C^{r-1}-metric on a C^r-manifold is a C^{r-1}-map $x \mapsto \mathrm{InnPr}(T_xM)$, where $\mathrm{InnPr}(T_xM)$ are inner products, a convex cone of the vector space of symmetric bilinear forms on T_xM. In other words, in coordinates $x_1, \ldots, x_m$ and in the basis $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}$ it is given by a symmetric, strictly positive definite matrix with C^{r-1}-entries. Notice that if we change the basis to another one so that the matrix of change of basis has C^{r-1}-entries, the previous property still holds. In particular this is the case for the matrix of change of coordinates, so being C^{r-1} is coordinate free.*

Lemma 8. *Any C^r-manifold M posses C^{r-1}-metrics.*

Proof. Take a cover made of local coordinates $\{(U_i, \varphi_i)\}_{i\in I}$ with $\{\mu_i\}_{i\in I}$ a C^r-partition of the unity. Now we define a global metric

$$g = \sum_{i\in I} \mu_i g_i,$$

where g_i is a C^{r-1}-metric on U_i, for example the Euclidean one in the canonical basis associated to the coordinates.

Since the supports give a locally finite cover, to know how g acts in a couple of vectors we just need to compute a finite sum, so g is well defined, and clearly symmetric and strictly definitive positive. Regarding regularity, in the basis associated to the local coordinates φ_i, $\mu_i g_i$ is obviously C^{r-1}. By equation 3 so the $\mu_j g_j$ are. □

pro:emb1 **Proposition 5.** *Let M be a compact C^r-manifold, $r \geq 1$. Then it embeds in $\mathbb{R}^q$ for some $q \in \mathbb{N}$.*

Proof. Consider any finite cover $\{(U_i, \varphi_i)\}$, $i = 1, \ldots, d$, and bump functions μ_i by pulling back a bump fuction μ, $i = 1, \ldots, d$, with $\mu^{-1}(1) = \overline{B(0,1)}$, and so that $\varphi_i^{-1}(\overline{B(0,1)})$ cover M.

Define the function

$$\begin{array}{rcl} f\colon M & \longrightarrow & \mathbb{R}^{m(d+1)} \\ x & \longmapsto & (\mu_1(x)\varphi_1(x), \mu_1(x), \ldots, \mu_d(x)\varphi_d(x), \mu_d(x)) \end{array}$$

Observe that f is well defined and injective. Since M is compact it is an homeomorphism over its image. □

We state Sard's theorem.

thm:sard **Theorem 2.** *Let $f\colon M \to N$ be a C^r-map, $r > \max\{0, m-n\}$. Then the subset of regular values is dense.*

Remark that if $m < n$ then $z \in N$ is regular iff $z \notin f(M)$.

thm:emb2 **Theorem 3.** *Let M be a compact C^r-manifold. If $r \geq 2$ then*

(1) *For any n$\geq$ 2m it immerses in $\mathbb{R}^n$;*
(2) *For any n$\geq$ 2m+1 it embeds in $\mathbb{R}^n$.*

Proof. We consider the embedding case. Start with any embedding in $\mathbb{R}^q$ as in proposition 5. The idea is to reduce the dimension of the target Euclidean space by composing the embedding with an orthogonal projection onto a hyperplane H, so the composition is still an embedding.

One easily check that $\pi_H\colon M \to H$ is injective iff $\pm v_H$ -the unit vectors orthogonal to H- do not belong to the image of the C^r-map

$$\begin{array}{rcl} L\colon M \times M \backslash \Delta & \longrightarrow & S^{q-1} \\ (x, y) & \longmapsto & \dfrac{(x-y)}{|x-y|} \end{array}$$

Therefore, if $q \geq 2m+1$, we are in the hypothesis of Sard's theorem (we actually only need $r \geq 1$) and we can deduce that W -the complement of ImL- is dense, so any of those directions is adequate to project and still get an injective map.

Regarding the infinitesimal result, notice that inside of the tangent bundle $T\mathbb{R}^q$ we have the unit sphere bundle $S(T\mathbb{R}^q) = \mathbb{R}^q \times S^{q-1}$. Another way to define it is to consider

$$\begin{array}{rcl} |\cdot|^2 \colon T\mathbb{R}^q & \longrightarrow & \mathbb{R} \\ v & \longmapsto & |v|^2 \end{array}$$

This map is C^{r-1} and the value 1 is regular, and its inverse image is the sphere bundle. Now if $M \hookrightarrow \mathbb{R}^q$, we can see $TM \hookrightarrow T\mathbb{R}^q$, and define the sphere bundle as

$$S(TM) := \{v \in TM \mid |v|^2 = 1\}$$

Since 1 is a regular value of the restriction of $|\cdot|^2$ to TM -which is C^{r-1}- it follows that $S(TM)$ is a submanifold.

The infinitesimal condition about being an immersion is equivalent to $\pm v_H$ being in $W^{(1)} := S^{q-1} \setminus \mathrm{Im} L^{(1)}$, where

$$\begin{array}{rcl} L^{(1)} \colon S(TM) & \longrightarrow & S^{q-1} \\ (x, u_x) & \longmapsto & u_x \end{array}$$

□

Sard's theorem implies the density of $W^{(1)}$. Since it is also open, we conclude that $W \cap W^{(1)}$ is dense thus proving the theorem.

rem:approx **Remark 7.** *Theorem 3 becomes an approximation theorem as follows:*

First notice that in theorem 3 what is relevant are the directions $\pm v_H$ along which we project. And we can do it over H or any other hyperplane transverse to the line containing $\pm v_H$.

Given $f \colon M \to \mathbb{R}^n$ a C^r-map, then for any $\epsilon > 0$ we can find $g \colon M \to \mathbb{R}^n$ an embedding such that $|f - g| \leq \epsilon$. To do that, embed it in $\mathbb{R}^q$ with and consider the product map $H \colon M \to \mathbb{R}^n \times \mathbb{R}^q$. Notice the original map is $\pi_{\mathbb{R}^n} \circ H$, so our task is approximating this linear projection by a linear map giving an embedding; but now we write it as a composition of projection onto (coordinate) hyperplanes. Then we use the density part result in Sard's theorem (and the observation that we can project onto any transverse subspace, in particular onto the fixed coordinate hyperplanes).

Exercise 4. *Given an embedding $M \hookrightarrow N$, use adapted charts to show that any differentiable metric on TN induces a differentiable metric on TM.*

Remark 8. *In theorem 3 the Euclidean metric induces a metric on M, and we compute the sphere bundle w.r.t. it. For any differentiable metric the norm square is differentiable and the sphere bundle is a differentiable manifold.*

2.5. **Manifolds with boundary.** Recall that for any subset $B \subset \mathbb{R}^m$ we define a C^r-function

$$f \colon B \to \mathbb{R}^n$$

as a function with a C^r-extension defined in a (open) neighborhood of B in $\mathbb{R}^m$.

def:manboundary **Definition 18.** *A structure of a C^r-**manifold with boundary** on $(X, \mathcal{T}_X)$ is given by an C^r-atlas of charts modeled on $H^m_+ = \mathbb{R}^{m-1} \times [0, +\infty)$. In other words, we want $\{(U_i, \varphi_i)\}_{i \in I}$ so that*

- *$\{U_i\}_{i \in I}$ is an open cover of M.*

- *$\varphi_i\colon U_i \to H^m_+$ maps U_i homeomorphically onto an open subset of H^m_+.*
- *$\varphi_{ij}\colon \varphi_j(U_i \cap U_j)\colon H^m_+ \to \varphi_i(U_i \cap U_j) \subset H^m_+$ is a C^r-diffeomorphism.*

Remark 9.

- *For each transition function φ_{ij}, there must be a C^r-extension Φ_{ij} to an open set of $\mathbb{R}^m$. The same should happen for φ_{ji}, but the extensions need not be inverse of each other. In any case one can refine the cover so that this property holds, so that can be assumed in the definition.*
- *The boundary ∂M is well defined. In the differentiable case the contrary would imply the existence of a differentiable map*
$$f\colon (B(0,1),0) \to (H^m_+, 0)$$
which is a homeomorphism over its image and has invertible differential at the origin. Then take c a curve through 0 which represents a direction transverse to $Df^{-1}(T_0\partial H^m_+)$. Then $f \circ c'_m(0) \neq 0$, but on the other hand 0 is a minimum for $f \circ c_m$, and that leads to a contradiction; in the topological case one uses invariance of domain.
- *The definition of C^r-map, tangent bundle, regular point for a map, regular value (for target manifold without boundary) go through. For example if $x \in \partial M$, then a C^r-curve through x is a C^r-map*
$$c\colon ([0,\epsilon),0) \to (M,x)$$
By definition this means that on any chart has some extension to a C^r-map. Obviously, the vector represented does not depend on the extension, so the equivalence class will be well defined.
- *For a positive half space, a (closed) ball is the intersection of an Euclidean (closed) ball with the positive half space (so depending on the intersection with the boundary there are to homeomorphism types). The existence of countable basis with open sets with closure diffeomorphic to closed balls (lemma 4) equally holds in our more general setting. Similarly, one has compact exhaustions, nice locally finite countable covers as in lemma 6 and differentiable partitions of the unity. Using them one proves the embedding result in Euclidean space given by proposition 5.*

Exercise 5. *Show that ∂M inherits a C^r-manifold structure (with empty boundary) and the inclusion*
$$i\colon \partial M \to M$$
is a C^r-embedding.

From now on manifold will boundary will be referred to as manifold, and we will specify the absence of presence of boundary if necessary.

def:bsubmnfd **Definition 19.** *A submanifold A of H^m_+ is a manifold (with or without boundary) so that it carries adapted charts. In other words, for points in the interior of the half space the chart about it extends to a diffeomorphism of $\mathbb{R}^m$ sending $U_i \cap A$ to the corresponding open set of H^n_+ (perhaps a boundary point). For points in the boundary of H^m_+, the point in A will also be a boundary point and the chart again extends to a local diffeomorphism of H^m_+ so that $\partial A \cap U_i$ goes to $H^n_+ \subset H^m_+$, with $H^n_+ \pitchfork \partial H^m_+$.*

Exercise 6. *Show that the notion of a submanifold of H^m_+ is local and C^r-diffeomorphism invariant (this time the diffeomorphism $\Psi\colon U \subset H^m_+ \to H^m_+$), and therefore it can be transferred to manifolds.*

thm:regbound **Theorem 4.** *Let $f\colon M \to N$ a C^r-map, $r \geq 1$, so that both $f, f_{|\partial M}$ are transversal to $A \subset N$, $\partial A = \emptyset$. Then $f^{-1}(A)$ is a submanifold of M with boundary $f^{-1}_{|\partial M}(A)$.*

Proof. Apply the implicit function theorem with a bit of care, noticing that

$$\ker Df_x \pitchfork \partial H^m_+,$$

so the new coordinates can be chosen to be (f', p), where

- f' is the composition of f with a projection which in adapted coordinates has image a subspace complementary to A,
- p is the projection onto $x + \ker Df_x$ parallel to a subspace inside of ∂H^m_+.

□

rem:moresubmnfd **Remark 10.** *There are two more situations for which $A \subset H^m_+$ could be considered a submanifold; namely when it does not meet the requirements of definition 19, but $A \subset \mathbb{R}^m$ is a submanifold. The two possibilities are*

- *$\partial A = \emptyset$ and $A \cap \partial H^m_+ \neq \emptyset$. By theorem 4 intersection points must be tangent to the boundary.*
- *$\partial A = \emptyset$ and $A \cap \partial H^m_+ \neq \emptyset$. Again, intersection points cannot be transverse to the boundary.*

In any case, since A is a submanifold of $\mathbb{R}^m$ one can always obtain adapted coordinates for A, but in which the image of H^m_+ is not a positive half space anymore unless a neighborhood of x in A is entirely contained in the boundary.

Exercise 7. *Prove the assertions of remark 10.*

We do not have a full analog of 3.

Exercise 8. *Prove that for manifolds M, N possibly with boundary, if $r \geq 1$ then for a C^r-embedding $f\colon M \to N$ the subset $f(M)$ of N is a C^r-submanifold as in definition 19 iff $f \pitchfork \partial M$. Show that if $x \in N$ is such that $f(x) \in \partial N$ but $Df_x(T_xM) \subset T_{f(x)}\partial N$ (i.e. there is not transversality to ∂N at x), then $f(M)$ is in the situation of remark 10, meaning that we get the adapted charts described there.*

ex:boundfunc **Exercise 9.**

- *On a differentiable manifold with non-empty boundary, use a suitable partition of unity to construct $\zeta\colon M \to \mathbb{R}^+$ a differentiable function such that:*
  1. *$\zeta^{-1}(0) = \partial M$.*
  2. *For each $x \in \partial M$, $v_x \in T_xM \setminus T_x\partial M$ (v_x direction transverse to the boundary), $D\zeta_x(v_x) > 0$.*
     Hint: As usual, see how one would solve the problem for the model H^m_+. There the natural choice is the coordinate x_m that defines $H^m_+ = x_m^{-1}([0,\infty))$. It has the right properties near the boundary. Then one can use a partition of unity μ_1, μ_2 subordinated to
     $$x_m^{-1}([0,1)), x_m^{-1}((1/2,\infty))$$
     to produce the desired result
     $$\mu_1 x_m + \mu_2$$
     To go to manifolds one can use a locally finite family $U_i \subset M$, $i \in I$, of open subsets covering ∂M, and extend the result in H^+_m by using a partition of the unity subordinated to $U_i, i \in I, M \setminus \partial M$.
- *Use the function ζ as an extra coordinate to prove an analog of proposition 5 of the following kind: If M is compact, $\partial M \neq \emptyset$, for q large enough there exist $f\colon (M, \partial M) \to (H^q_+, \partial H^q_+)$ an embedding so that $f \pitchfork \partial H^q_+$ (in other words $x \in \partial M$ is sent to ∂H^q_+, and vectors in $T_xM \setminus T_x\partial M$ are sent to vectors transverse to $T_{f(x)}\partial H^q_+$).*

thm:emb2bd

Theorem 5. *Let M be a compact C^r-manifold. If $r \geq 2$ then it embeds in $\mathbb{R}^q$, for any $q \geq 2m+1$ and it has immersions in $\mathbb{R}^q$, for any $q \geq 2m$. Moreover, we obtain similar results for embeddings of $(M, \partial M)$ in $(H^q_+, \partial H^q_+)$, so that M embeds transverse to ∂H^m_+.*

Proof. Notice that when we construct $M \times M$, we get a **manifold with corners**, i.e. modeled on $[0, +\infty)^d \times \mathbb{R}^{m-d}$ (in this case $d = 2$). Sard's theorem also holds for them, but we will see other proofs of the theorem later (see remark 21). □

2.6. **1-dimensional manifolds.** Let $r \geq 1$. A C^r-diffeomorphism

$$f\colon (0,1) \to f(0,1) \subset \mathbb{R}$$

is called **orientation preserving** if $f' := \frac{d}{dt} f > 0$. Recall also that f is a diffeomorphism iff it is an immersion (f' never vanishes), because it is injective and open (it locally sends open intervals into open intervals, and those are a basis for the topology).

pro:1dimext

Proposition 6. *Let $f\colon (0,a) \cup (b,1) \to (0,a') \cup (b',1)$, $0 < a < b < 1$, $0 < a' < b' < 1$ be a C^r-diffeomorphism, sending the first interval to the first and the second to the second and preserving the (canonical) orientation of both. Then for any $\epsilon > 0$ there exist $F\colon (0,1) \to (0,1)$ a C^r-diffeomorphism such that*

$$F_{|(0,a-\epsilon)} = f, \; F_{|(b+\epsilon)} = f$$

Proof. $f'\colon (0,a) \cup (b,1) \to (0,\infty)$ is a C^{r-1}-function. Given any $\delta > 0$, we can use a partition of unity μ_1, μ_2, μ_3 subordinated to $(0,a), (a-\epsilon, b+\epsilon), (b,1)$ and define

$$g_\delta = \mu_1 f_{|(0,a)} + \mu_2 \delta + \mu_3 f_{|(b,1)}$$

Then consider $F_\delta(t) := \int_0^t g(s)ds$, $s \in (0,1)$ (where the integral might be improper). It is clear that F_δ has the required regularity in $(0,1)$. We can adjust δ so that $F_\delta(b+\epsilon) = f(b+\epsilon)$. □

cor:1dimext

Corollary 1. *Given $f\colon (0,a) \to (0,b)$, $0 < a, b \leq 1$ an orientation preserving C^r-diffeomorphism, we can find F a C^r-diffeomorphism of $(0,1)$ extending f in $(0, a-\epsilon)$. We can also arrange it to satisfy $F \geq f$ in $(0,a)$.*

thm:1man

Theorem 6. *Let M be a C^r-manifold connected and of dimension 1. Then M is C^r-diffeomorphic to*

1. *S^1 if $\partial M = \emptyset$ and M compact.*
2. *$(0,1)$ if $\partial M = \emptyset$ and M not compact*
3. *$[0,1]$ if $\partial M \neq \emptyset$ has M compact.*
4. *$[0,1)$ if $\partial M \neq \emptyset$ has M compact.*

Proof. We will proof the second point. Let $\{(U_i, \varphi_j)\}_{j \in \mathbb{N}}$ be a C^r-atlas so that each open is diffeomorphic to an open interval. We can assume w.l.o.g. that no interval in the cover is contained in a finite union of other intervals.

We start by choosing U_0, and then at each step we consider U_{j+1} so that

$$W_j := U_{j+1} \cap V_j \neq \emptyset, \; V_j := \bigcup_{i=0}^{j} U_i$$

We assume that we have

$$\phi_j\colon V_j \to (0,1)$$

a diffeomorphism. We want to show the existence of

$$\phi_{j+1}\colon V_{j+1} \to (0,1)$$

so that

$$\phi_{j+1|V_{d(j)}} = \phi_{j|V_{d(j)}},$$

where $d\colon \mathbb{N} \to \mathbb{N}$ is order preserving, and for each m there exists $j(m)$ such that $d(j) \geq m$, for all $j \geq j(m)$ (d is proper w.r.t. the topology whose closed sets are finite subsets). If that it the case, then

$$\phi = \lim_{j\to\infty} \phi_j \colon M \to (0,1)$$

is clearly a diffeomorphism over its image. It may in principle fail to be onto, but that is not important (in any case, by carefully choosing the stepwise extensions we can arrange it to be onto).

We claim that W_j is an interval which both inside U_{j+1} and V_j does not separate.

The intersection W_j is an open subset, and hence a union of intervals (inside both U_{j+1}, V_j). Assume an interval $I \subset W_j$ separates inside U_{j+1}. Then inside U_{j+1} it has two accumulation points y, z not in W_j. Similarly inside V_j it must have at least an accumulation point w not in W. Then we can find $\{x_n\}_{n\in\mathbb{N}} \subset W_j$ a sequence such that

- it has no accumulation point in W_j, and
- $x_n \to w$.

But then $x_n \subset \bar{I} \cap U_{j+1} \subset U_{j+1}$ compact, and hence has an accumulation point that must be either y, z. But any of them differs from w, and that contradicts the Hausdorffness assumption. So W_j has at most two connected components. Use the charts ϕ_j and φ_{j+1} to induce orientations on U_{j+1} and V_j. If W_j has two connected components, we can assume w.l.o.g that the change of charts is $f\colon (0,a) \cup (b,1) \to (0,a') \cup (b',1)$, $0 < a < b < 1$, $0 < a' < b' < 1$, sending the first interval to the first and the second to the second. We claim both orientations must be reversed, because otherwise we contradict Hausdorfness. But if both orientations are reversed, using proposition 6 one easily proves

$$U_{j+1} \cup V_j \simeq S^1$$

This is not possible: indeed, the next interval U_{j+2} must once more intersect in a open set with an accumulation point in U_{j+2} and not in the intersection. Then we can select a sequence in S^1 which converges to this point not in S^1; but by compactness it must have another limit in S^1.

Therefore the intersection is one interval that does not separate. We can w.l.o.g. suppose that the change of chart $\varphi \circ \phi_j^{-1}$ is given by

$$f\colon (0,a) \to (b,1) \simeq (0,1)$$

preserving the orientation, so corollary 1 gives the desired result. □

Remark 11. *We have exactly the same result for the topological case. The reason is that even though there is no differential, homeomorphisms can be characterized as maps which are strictly monotone, and also make sense of being orientation preserving or reversing. That allows us to easily prove a topological analog of proposition 6 (even better, since we can actually extend wothout modifying), and therefore the proof of theorem 6 goes through with minor modifications. In particular we deduce that each topological (Hausdorff and paracompact) 1-dimensional manifold admits a unique compatible smooth structure.*

Exercise 10. *Describe a smooth manifold structure on* $\mathrm{Gr}(k,m)$ *(resp.* $\mathrm{Gr}_{\mathbb{C}}(k,m)$*) the Grassmannian of k-planes in $\mathbb{R}^m$. (resp. complex k-planes in $\mathbb{C}^m$). Notice that in particular one gets smooth manifold structure on $\mathbb{RP}^m$ (resp. $\mathbb{CP}^m$).*

Exercise 11. *Let* $\mathrm{Gl}(m,\mathbb{R})$ *be the group of invertible* $m\times m$ *matrices. Show that it inherits a manifold structure as an open subset of some Euclidean space. Prove that the subgroup of symmetric and orthogonal and special orthogonal matrices are submanifolds of* $\mathrm{Gl}(m,\mathbb{R})$*, and compute their dimension.*

Show that the group of unitary matrices is a submanifold of the group of invertible $m\times m$ *complex matrices, and compute its dimension.*

Exercise 12. *Let* $\Omega\subset\mathbb{R}^m$ *be a compact m-dimensional differentiable submanifold with non-empty boundary. Suppose that there exist* $x_0\in\mathrm{int}\Omega$ *such that for every* $x\in\Omega$ *the segment* $[x,x_0]$ *is inside of* Ω *and (after prolonging it a bit near* x*) transverse to* $\partial\Omega$*. Show that*

$$\partial\Omega\simeq S^{m-1}$$

.

Let Q *be any inner product on* $\mathbb{R}^m$*. Show that for any positive c,*

$$Q^{-1}(c):=\{v\in\mathbb{R}^m\,|\,Q(v,v)=c\}$$

is a smooth submanifold of $\mathbb{R}^n$ *and it is diffeomorphic to*

$$S^{m-1}:=\{v\in\mathbb{R}^m\,|\,|v|^2=1\}$$

Exercise 13. *Let* M,N *be* C^r*-manifolds. Show that* $M\times N$ *carries a canonical* C^r*-manifold structure, and that* $T(M\times N)\simeq TM\times TN$*.*

Exercise 14. *Let* N *be a a* C^r*-manifold,* M *a topological space, and* $f\colon M\to N$ *a local homeomorphism (i.e. for each* $x\in M$ *there exist* U *a neighborhood so that* $f\colon U\to f(U)$ *is a homeomorphism). Prove that* M *can be given a canonical* C^r*-manifold structure so that* f *becomes a local* C^r*-diffeomorphism.*

Exercise 15. *Prove that a* C^r*-map which is a* C^1*-diffeomorphism it is a* C^r*-diffeomorphism.*

Exercise 16. *Show that*

$$SO(3)\simeq\mathbb{RP}^3$$

It is enough to exhibit a homeomorphism. It is convinient to think of $\mathbb{RP}^3$ *as the ball* $\overline{B(0,\pi)}\subset\mathbb{R}^3$ *with antipodal points on its boundary identified. To find out to which point of* $\mathbb{RP}^3$ *one should send* $A\in SO(3)$*, use that any such map acts by fixing an axis in* $\mathbb{R}^3$ *and rotating in the orthogonal plane.*

Exercise 17.

(1) *Prove that for any* C^1 *manifold* M *every* $f\in C^1(M,\mathbb{R})$ *has at least 2 critical points.*
(2) *For any sphere* S^m*,* $m\in\mathbb{N}$*, find a smooth function with exactly 2 critical points.*

Exercise 18. *Let* M,N *be differentiable manifolds with non-empty boundary. Prove that a* C^1 *map takes regular points in* $M\backslash\partial M$ *into points in* $N\backslash\partial N$*.*

Exercise 19. *Any compact surface embeds in* $\mathbb{R}^5$*. The 2-torus is the surface*

$$\mathbb{T}^2:=S^1\times S^1$$

Exhibit an embedding

$$j\colon\mathbb{T}^2\hookrightarrow\mathbb{R}^3$$

Hint: Embed it as a surface of revolution.

Exercise 20. *Consider* $\mathbb{CP}^m$ *with homogeneous coordinates* $[X_0 : \cdots : X_m]$ *and let* $d_k = \frac{k!}{n!(k-n)!}$

Show that the maps

$$\begin{array}{rcl} f_{d_k} \colon \mathbb{CP}^m & \longrightarrow & \mathbb{CP}^{d_k} \\ [X_0 : \cdots : X_m] & \longmapsto & [X_0 : \cdots : X_m : 0 : \cdots : 0] \end{array}$$

and

$$\begin{array}{rcl} V_k \colon \mathbb{CP}^m & \longrightarrow & \mathbb{CP}^{d_k} \\ [X_0 \cdots : X_m] & \longmapsto & [X_0^k : X_0^{k-1}X_1 : \cdots : X_{m-1}X_m^{k-1} : X_m^k] \end{array}$$

are embeddings

Exercise 21. *For* $f, g \in C^r(M, \mathbb{R})$*,* $r \geq 1$*, prove the Leibniz rule*

$$D(f \cdot g) = Df \cdot g = f \cdot Dg$$

Hint: Write $f \cdot g$ *as a composition (of three functions) and apply the chain rule.*

3. Function spaces

We aim at understanding how continuous maps can be approximated by differentiable ones, and among the latter trying to describe how "good" can "nearby" maps to a given map be. We need a topology which accounts for the degree of differentiability.

For M, N topological spaces, the **weak or compact open topology** has a subbasis

$$\mathcal{B}(K, V) = \{f \mid f(K) \subset V,\ K \subset M \text{ compact},\ V \subset N \text{ open}\}$$

In other words, it is the smallest topology containing as open subsets the $\mathcal{B}(K, V)$ ($C(M, N)$ is union of them, so it is a subbasis); the topology open subsets are unions of finite intersections of subsets $\mathcal{B}(K, V)$. The corresponding topological space is denoted by $C_W(M, N)$ (or $C^0_W(M, N)$).

Exercise 22.

(1) *One can be more economic and use open subsets* $V \subset N$ *belonging to a given basis.*
(2) *When the manifold* M *is compact we can equally use as basis of the topology* $\mathcal{B}(M, V)$*, where* V *may be chosen to run in a fixed basis.*

rem:homotopy **Remark 12.** *If* M *is locally compact and Hausdorff, then a function*

$$F \colon P \to C_W(M, N)$$

is continuous iff

$$\begin{array}{rcl} F^{\text{ev}} \colon P \times M & \longrightarrow & N \\ (p, x) & \longmapsto & F(p)(x) \end{array} \tag{4}$$

is continuous. Therefore, under such conditions a homotopy H *connecting maps* $f, g \colon M \to N$ *is a continuous curve* $I \to C_W(M, N)$ *connecting* f *and* g*. In other words, homotopy classes of maps can be identified with path connected components of* $C_W(M, N)$

$$[M, N] \xleftrightarrow{1:1} \pi_0(C_W(M, N))$$

In the manifold setting an equivalent description of the compact open topology is given as follows: Take $(U, \varphi), (V, \phi)$ charts for M, N respectively, a compact $K \subset U$, f a function so that $f(K) \subset V$, and $\epsilon > 0$, and consider

$$\mathcal{N}(f, K, U, V, \epsilon) = \{g \mid |\phi \circ f \circ \varphi^{-1} - \phi \circ g \circ \varphi^{-1}|_K < \epsilon\} \tag{5}$$ eq:co

This is a sub-basis for the compact open topology: Indeed, if $f \in \mathcal{B}(K,V)$, then we can find $U_1,\dots,U_d$, $V_1,\dots,V_d$ local coordinates so that $\overline{U}_i$ are still the domain of local coordinates, $V_i \subset V$ and $f(K_i) \subset V_i$, with $K_i := K \cap \overline{U}_i$. Then

$$f \in \bigcap_{i=1}^{d} \mathcal{B}(K_i,V_i) \subset \mathcal{B}(K,V)$$

It is also clear that for $\epsilon > 0$ small enough,

$$\mathcal{B}(f,K_i,U_i,V_i,\epsilon) \subset \mathcal{B}(K_i,V_i),$$

so one inclusion follows. The other is proven similarly.

def:rcopen **Definition 20.** *The weak C^r-topology is defined by the subbasis as an equation 5 but using the C^r-norm in Euclidean spaces. The corresponding topological space is denoted $C^r_W(M,N)$.*

The weak C^∞-topology is defined as the initial topology for the inclusions

$$C^\infty(N,M) \hookrightarrow C^r(N,M),$$

i.e. the smallest making continuous all inclusions.

rem:rnorm **Remark 13.** *One way to define the C^r-norm of a function $f\colon A \subset \mathbb{R}^m \to \mathbb{R}^q$ is as follows:*

$$|f|_{C^r(A)} := \max_{x\in A,|I|\le r} \| \frac{\partial^I f}{\partial x^I} \|$$

Since we are measuring always over compact subsets, there are equivalent ways of defining the C^r-norm.

rem:convergence **Remark 14.**

- *$C^r_W(M,N)$ is second countable and has a complete metric.*
- *The weak topology accounts for uniform convergence on compact subsets. Indeed, $f_n \to f$ in $C^r_W(M,N)$, if for any compact, once we break it into smaller compact subsets $K_1,\dots,K_d$ fitting into charts (U_i,φ_i), $i=1,\dots,d$ (also sent into charts), then we have*

$$|\phi_i \circ f_n \circ \varphi_i^{-1} - \phi_i \circ f \circ \varphi_i^{-1}|_{C^r(\varphi_i(K_i))} \to 0,\ i=1,\dots,d$$

  For $r=\infty$ the above condition holds for all $r \in \mathbb{N}$.
- *When M is compact, $N=\mathbb{R}^n$, and $r \neq \infty$, then is a Banach vector space, i.e. the topology is given by a norm for which the vector space operations are continuous (see exercise 25) and Cauchy sequences converge (completeness); for $r=\infty$ or when M is not compact, it is not any more a Banach vector space, but a Frechet one.*

rem:boundck **Exercise 23.** *Check also that definition 20 makes perfect sense for manifolds with boundary as well, so we also have $C^r_W(M,N)$ in this case.*

We can rephrase remark 7 as the following:

Theorem 7. *For $r \geq 2$ and M compact*

1. $\mathrm{Imm^r(M,\mathbb{R}^n)} \subset C^r_W(M,\mathbb{R}^n)$ *is dense for every $n \geq 2m$.*
2. $\mathrm{Emb^r(M,\mathbb{R}^n)} \subset C^r_W(M,\mathbb{R}^n)$ *is dense for every $n \geq 2m+1$.*

If M is non-compact, the weak topology does not measure how "far apart are two functions in the whole manifold".

Definition 21. *The* **strong C^r-topology** *is defined by the basis whose elements are given by taking for $i \in I$ a locally finite family of charts (U_i,φ_i) (it does not* *need to be an atlas), a family $K_i \subset U_i$ of compacts, (V_i, ϕ_i) charts and f so that $f(K_i) \subset V_i$, and $\epsilon_i > 0$, and then considering the subset*

$$\mathcal{N}^r(f, K_i, U_i, V_i, \epsilon_i) = \{g \mid |\phi_j \circ f \circ \varphi_i^{-1} - \phi_j \circ g \circ \varphi_i^{-1}|_{C^r, K_i} < \epsilon_i, \forall i \in I\}$$

The strong C^∞-topology is defined as the initial topology for the inclusions

$$C^\infty(N, M) \to C^r(N, M)$$

Remark 15.

- *When M is non-compact the strong topology has much more opens than the weak one, to the extent of not being metrizable, second countable. Still, it keeps a nice feature which is being a Baire space.*
- *Convergence of a sequence in the strong topology is uniform convergence in the C^r-norm in the whole manifold, and this is made precise as in remark 14 by taking a locally finite family of compacts covering M. Actually the topology being non second countable implies that sequences are not the right tools to detect continuity of functionals. As a matter of fact converging sequences must be constant outside a compact subset; the family $\{U_i\}_{i \in I}$ in the definition has to be locally finite to avoid getting a topology in which convergence would only occur for constant sequences (by taking $K_n = K$, $U_n = U$, $V_n = V$, $\epsilon_n = \frac{1}{n}$).*

For non-compact manifolds, the functions which are "well behaved" w.r.t. the strong topology are the **proper functions**.

Definition 22. *Let M, N (Hausdorff) topological spaces. A function $f\colon M \to N$ is proper if it pullbacks compact subsets to compact subsets.*

Exercise 24. *A proper function is closed.*

Proper functions are very much related to compact exhaustions. If $f\colon M \to \mathbb{R}^+$ is proper then $M_j := f^{-1}([0, j])$ defines a compact exhaustion. To check that $M_j \subset \mathrm{int} M_{j+1}$ observe that $f^{-1}(([0, j+1))$ is open, and

$$M_j \subset f([0, j+1)) \subset \mathrm{int} M_{j+1}$$

More generally

lem:properexh **Lemma 9.** *$f \in C^0(M, N)$ is proper iff it pulls back compact exhaustions to compact exhaustions.*

Proof. That proper functions pull back compact exhaustions to compact exhaustions follows from the above considerations for $N = \mathbb{R}$.

In the other direction, notice that since $N_j \subset \mathrm{int} N_{j+1}$, then the $\mathrm{int} N_{j+1}$ provide an open cover and any compact K is contained in a finite union, and hence in the one with biggest index. Therefore, $f^{-1}(K)$ is a closed contained in a compact and thus compact. □

To construct a proper function we may take U_i, U_i', $i \in I$, open subsets and $\{\mu_i\}_{i \in I}$ partition of the unity so that

- $\overline{U}_i \subset U_i'$, $\forall i \in I$,
- $\{U_i'\}_{i \in I}$ is a locally finite cover,
- μ_i is supported in U_i' and so that $\mu_i > 0$ on $\overline{U}_i$, $\forall i \in I$.

One can define an exhaustion $\{M_j\}_{j \in \mathbb{N}}$ by stepwise adding new $\bar{U}_i$'s as in lemma 5, but noticing that this time we can add just all open subsets with non-empty intersection (being locally finite it is a finite number). Next define $j(U_i)$ to be the step of the exhaustion procedure in which $\overline{U}_i$ is added. Set

$$f = \sum_i j(U_i) \mu_i$$

We claim f is proper. Indeed, if $x \notin M_j$, then if $x \in U_i$, then $j(U_i) > j$, and therefore $f(x) > j$.

We can take $\{\mu_j\}_{j\in\mathbb{N}}$ a partition of the unity for $\mathbb{R}$ so that $\mathrm{supp}\mu_j \subset (j-1, j+1)$. Then we can pull it back to a partition of the unity $f^*\mu_j$, and then construct

$$\chi_j = \sum_{l=0}^{j} f^*\mu_l, \tag{6}$$

eq:charfunct

such that for $M_j := f^{-1}([0,j])$, $\chi_{j|M_j} \equiv 1$, $\mathrm{supp}\chi_j \subset \mathrm{int}M_{j+1}$ (so they are close to be step functions for the subsets of the compact exhaustion, and are useful to locallize problems in these subsets).

ex:ring

Exercise 25. *For any $r \geq 0$, show that*

(1) *for any $f \in C^r(N,Q)$, the map*

$$\begin{array}{ccc} C^r_S(M,N) & \longrightarrow & C^r_S(M,Q) \\ h & \longmapsto & f \circ h, \end{array}$$

is continuous;

(2) *prove that if $f \in C^r(P,M)$ is proper, then*

$$\begin{array}{ccc} C^r_S(M,N) & \longrightarrow & C^r_S(P,M) \\ h & \longmapsto & h \circ f, \end{array}$$

is continuous.

In particular conclude that

- $\mathrm{Diff}^r(M,M)$ *acts on $C^r_S(M,M)$ by homeomorphisms.*
- *For a given submanifold $A \subset M$ so that the embedding is proper, the restriction map*

$$C^r_S(M,N) \to C^r_S(A,M)$$

is continuous (in fact the are submanifolds -even open subsets- for which the restriction map is not continuous).

(3) *Use the previous results to prove that the ring operations*

$$+,\cdot\colon C^r_S(M,\mathbb{R}) \times C^r_S(M,\mathbb{R}) \to C^r_S(M,\mathbb{R})$$

are continuous.

Hint: Use what you know about the product topology to show that

$$\begin{array}{ccc} C^r_S(M,P) \times C^r_S(N,Q) & \longrightarrow & C^r_S(M\times N, P\times Q) \\ (f,g) & \longmapsto & f\times g \end{array}$$

Then apply the first part of the exercise having into account that

$$f+g = \Delta^*(+\circ(f\times g)),$$

where

$$\begin{array}{ccc} +\colon \mathbb{R}\times\mathbb{R} & \longrightarrow & \mathbb{R} \\ (a,b) & \longmapsto & a+b, \end{array}$$

$$\begin{array}{ccc} \Delta\colon M & \longrightarrow & M\times M \\ x & \longmapsto & (x,x) \end{array}$$

rem:nicecompex

Remark 16. *One further comment is that a compact exhaustion is a notion which is entirely defined using the topology, and the manifold structure does not enter at all. One might think whether in the manifold setting the M_j's can be chosen with "better" properties. In particular we would like them to be submanifolds with boundary. This is always possible. One strategy is to take a C^r-proper function. If*

$m > r$ then we can apply Sard's theorem to get the desired result. The general case is not that straightforward (see remark 18).

One can for example use these compact exhaustions to show that $C_W(M, \mathbb{R}^n)$ is a Frechet space by just using the Banach space result for compact manifolds (remark 14).

pro:localop

Proposition 7. *Let $K \subset \mathbb{R}^m$ any compact. Then for $r \geq 1$*

(1) *If $f \in \mathrm{Imm}^r(U, \mathbb{R}^n)$, there exist $\epsilon > 0$ so that if*
$$|f - g|_{C^r(K)} \leq \epsilon$$
then $g_{|K}$ is an immersion.

(2) *If $f \in \mathrm{Sub}^r(U, \mathbb{R}^n)$, there exist $\epsilon > 0$ so that if*
$$|f - g|_{C^r(K)} \leq \epsilon$$
then $g_{|K}$ is a submersion.

(3) *If $f \in \mathrm{Emb}^r(U, \mathbb{R}^n)$, there exist $\epsilon > 0$ so that if*
$$|f - g|_{C^r(K)} \leq \epsilon$$
then $g_{|K}$ is an embedding.

Proof. Points one and two follow from compactness.

Point three is more interesting. If the result does not hold we would have sequences $f_n, a_n, b_n, \epsilon_n$, $\epsilon_n \to 0$, so that
$$|f_n - f|_{C^1(K)} \leq \epsilon_n, \; f_n(a_n) = f_n(b_n)$$
By compactness of K we can assume $a_n \to a, b_n \to b, a_n - b_n/|a_n - b_n| \to v$ Therefore, since $f(a) = f(b)$, $a = b$ and
$$\frac{|f_n(a_n) - f_n(b_n) - Df_n(b_n)(a_n - b_n)|}{|a_n - b_n|} \to 0, \tag{7}$$
eq:derivative

it follows that $Df(b)v = 0$, which contradicts f being an immersion.

Notice that for equation 7 to hold we need the C^1-condition. In one variable, and using the mean value theorem for functions with derivative we have
$$\frac{f_n(a_n) - f_n(b_n)}{a_n - b_n} = f_n'(c_n),$$
and continuity of the derivative implies the result. The property of K being used is (local) convexity of a neighborhood in which f is defined, so K can be indeed any compact. □

We would like to point out the following interesting consequence:

thm:invest

Theorem 8. *Let $f\colon B(0,r) \subset \mathbb{R}^m \to \mathbb{R}^m$ of class C^r so that $Df(x)$ is invertible fora all x in the ball. Then f is invertible in $B(0,r)$.*

Proof. We will show that the map is injective. Let x, y be distinct points in the ball. By the mean value theorem
$$f(x) - f(y) = Df(c)((x - y)/|x - y|)|x - y|,$$
where c belongs to the segment joining x, y. By assumption the r.h.s. never vanishes, and this finishes the proof.

Now we can apply locally about each point the known inverse function theorem to show that the map is open and the inverse has the right regularity. More generally if $n > m$ we equally conclude injectivity from the hypothesis. □

rem:convexity

Remark 17. *In proposition 7 we can be more precise and say that if f is an embedding of the closed ball in Euclidean space, and g is close enough in C^1-norm so that it is an immersion, then g is another embedding.*

Notice also that this explains that to go from a ball to a manifold we only need to care about checking injectivity (and closedness or openness of the map) in points "far apart".

Convexity is the crucial condition, as seen in the following example:

$$\begin{array}{rcl} z^2\colon \overline{B(0,1)}\backslash\{0\} \subset \mathbb{C} & \longrightarrow & \overline{B(0,1)}\backslash\{0\} \subset \mathbb{C} \\ z & \longmapsto & z^2 \end{array}$$

Exercise 26. *Show that theorem 8 and remark 17 hold for any $K \subset \mathbb{R}^m$ which is convex. Deduce then that they hold for intersections of balls with positive half spaces.*

thm:openness

Theorem 9. *For $r \geq 1$, $\mathrm{Imm}^r(N,M), \mathrm{Sub}^r(N,M), \mathrm{Emb}^r(N,M)$ are open in $C^r_S(N,M)$.*

Proof. Take a locally finite open cover $\{U_i\}_{i\in I}$ as in lemma 6, subordinated to local coordinates $\{(U_i', \varphi_i')\}_{i\in I}$ which contain their closure and such that $\varphi_i'(U_i) = B(0,1)$, $\forall i \in I$. We also suppose the existence of $\{V_i\}_{i\in I}$ a cover of N so that $f(U_i') \subset V_i$, $\forall i \in I$.

We will prove the embedding result. It is clear that for appropriate ϵ_i if

$$g \in \mathcal{N}_1 := \mathcal{N}^r(f, \overline{U}_i, U_i', V_i, \epsilon_i),$$

then g is an immersion.

By theorem 8 we know that $g_{|\bar{U}_i}$ is an embedding, so we must impose further conditions to make sure that points (or rather regions) far apart in M (this, measured using a compact exhaustion) are nor sent to the same one.

We next assume w.l.o.g. that V_i (i) are adapted to $f(M)$ and cover it, and (ii) have compact closure in the domain of ϕ_i, the latter sending V_i to $B(0,1) \subset \mathbb{R}^n$, $\forall i \in I$.

One remark is that by talking of adapted charts we are using proposition 3, which was proven for manifolds without boundary. Recall that when the boundary is not empty, we were dealing had a more restricted notion of submanifold, so proposition 3 does not go through; in any case, and according to remark 10, the embedding condition still grants the existence of local coordinates sending $f(M)$ locally to $\mathbb{R}^n$, but not necessarily into H^m_+. Still, these charts are enough for our purposes.

Notice we can start with such a family of open sets with the last two properties, and then we ask $\{\overline{U}_i\}_{i\in I}$ to be subordinated to $\{f^{-1}(V_i)\}_{i\in I}$ (and perhaps repeat many times each V_i so that indices sets match).

The last assumption is the existence of $\{Z_i\}_{i\in I}$ open cover so that $\overline{Z}_i \subset U_i$ diffeomorphic to a closed ball.

Now we claim we can separate the subsets $f(\overline{Z}_i)$, $f(M\backslash U_i)$, i.e. they are contained respectively in open subsets A_i, B_i with empty intersection. Notice that inside of M the closed subsets $\overline{Z}_i$ and $M\backslash U_i$ can be separated by open subsets C_i, D_i. Since f is an embedding, $f(C_i)$, $f(D_i)$ are open subsets in $f(M)$ with empty intersection. In particular also $f(C_i)$ and $f(D_i) \cap V_i$, both inside V_i. These are open subsets for the subspace $\mathbb{R}^m$ in the adapted chart. Clearly, if we thicken them a bit along the remaining n-m coordinateswe get A_i, B_i' open subsets so that

$$\overline{A_i} \subset V_i \,,\, \overline{A_i} \cap B_i' = \emptyset$$

Finally we can find V_i' a closed ball inside V_i containing A_i. Then we define $B_i = B_i' \cup (N\backslash V_i')$.

Next we claim that if we define

$$\mathcal{N}_0 := \{g \in C^r(M,N) \,|\, g(\overline{Z}_i) \subset A_i,\ g(M\backslash U_i) \subset B_i\},$$

then $\mathcal{N} := \mathcal{N}_1 \cap \mathcal{N}_0$ is a neighborhood of f in $C^r_S(M,N)$.

Assuming this for a second, we will prove that any $g \in \mathcal{N}$ is an embedding.

Firstly, the map is injective: let $x \in M$ belong to some $\overline{Z}_i$; recall that then $g(x) \in A_i$. If $y \in U_i$, then since $g \in \mathcal{N}_1$ we get $g(x) \neq g(y)$. If $y \in M\backslash U_i$, then $g(y) \neq A_i$, so $g(x) \neq g(y)$. Regarding the openness of g, a system of neighborhoods for the induced topology about $g(x)$ consists of the intersection of a system of neighborhoods in N about $g(x)$. Since $g(x) \in A_i \subset V_i'$, we can take neighborhoods inside V_i'. In particular, since $V_i' \cap B_i = \emptyset$, the intersection is contained in U_i. So we can assume that our manifold is in fact U_i, but for it we know that the induced topology is the given one on U_i, and this proves the desired result.

We still need to check that $\mathcal{N}$ is a neighborhood of f in $C^r_S(M,N)$. If M compact just cover each $M\backslash U_i$ by finite number of $\{W_{ij}\}_{j\in J_i}$ diffeomorphic to open balls, with closure subordinated to some charts $\{(U_{ij},\varphi_{ij})\}_{j\in J_i}$, the latter sent into some other charts $\{(V_{ij},\phi_{ij})\}_{j\in J_i}$, and so that

$$\bigcup_{j\in J_i} \overline{W}_{ij} \subset f^{-1}(B_i)$$

Then for $\epsilon > 0$ small enough

$$\mathcal{N}^r(f,\overline{W}_{ij},U_{ij},V_{ij},\epsilon)) \cap \mathcal{N}_1 \subset \mathcal{N}_0$$

Notice that we clearly make sure that $g(M\backslash U_i) \subset B_i$. To get $g(\overline{Z}_i) \subset A_i$ we just need to adjust the ϵ_i's in the definition of $\mathcal{N}_1$.

In the non-compact case take $\{Y_l\}_{l\in\mathbb{N}}$ an open cover as in lema 6. It can be chosen subordinated to $Q_d := \mathrm{int}M_{d+1}\backslash M_{d-1}$, where $\{M_d\}_{d\in\mathbb{N}}$ is a compact exhaustion. By induction we may assume that on the d-th step we add a finite number of $\overline{Y}_l$ to cover each $M_d\backslash\mathrm{int}M_{d-1}$.

For all those Y_l added at the d-th step, we find a finite number of of $W_{lij} \subset U_{lij}$, $j \in J_{li}$, subordinated to Q_d, whose union covers $f^{-1}(B_i) \cap \overline{Y}_l$, fix an appropriate ϵ_d, so that it selects functions sending the union of the $W_{lij} \cap \overline{Y}_l$ into B_i. Notice that in this way $\{U_{lij}\}_{j\in J_{li}}$ is a locally finite open cover, so

$$\mathcal{N}^r(f,\overline{W}_{lij},U_{lij},V_{lij},\epsilon_{d(lij)}) \cap \mathcal{N}_1 \subset \mathcal{N}_0$$

is open.

Just remark that we obtain a subset of $\mathcal{N}_0$, because for $x \notin U_i$, x must belong to some Y_l, and thus $x \in f^{-1}(B_i) \cap \overline{Y}_l$, so we deduce $g(x) \subset B_i$. □

Exercise 27. *Prove theorem 9 for submersions.*

lem:propopen

Lemma 10. $\mathrm{Prop}^r(M,N)$ *is open in* $C^r_S(M,N)$.

Proof. Take a compact exhaustion N_j, $j \in \mathbb{N}$, and pull it back with f to a compact exhaustion M_j (lemma 9). Then take the subset

$$\mathcal{N} = \{g \in C^r(M,N) \mid g(M_j\backslash\mathrm{int}M_{j-1}) \subset \mathrm{int}N_{j+1}\backslash N_{j-2}\},$$

which is open. By construction for any $g \in \mathcal{N}$, $g^{-1}(N_j) \subset M_{j+2}$, and then by lemma 5 the result follows.

Exercise 28. *Prove that* $\mathcal{N}$ *as defined in lemma 10 is open in* $C_S(M,N)$. *Show also that proper maps are closed in the strong topology.*

Since f is proper d is onto. Therefore, functions in $\mathcal{N}$ are proper, since we can take k as in lemma 9 to be the biggest right inverse of d. □

thm:diffopen

Theorem 10. *Assume that M, N are manifolds without boundary. Then $\mathrm{Diff}^r(M,N)$ is open in $C^r_S(M,N)$, for $r \geq 1$. If there is non-empty boundary then the diffeomorphisms are open $C^r((M,\partial M),(N,\partial N))$ with theinduced topology.*

Proof. If M, N connected and without boundary then

$$\mathrm{Diff}^r(M,N) = \mathrm{Emb}^r(M,N) \cap \mathrm{Sub}^r(M,N) \cap \mathrm{Prop}^r(M,N)$$

If we only put the first two subsets, we may obtain diffeomorphism of Euclidean space onto open balls of finite radius, thoughts as elements in $C^r(\mathbb{R}^m, \mathbb{R}^m)$. The failure to be surjective is measured by the failure to be proper.

A proper map is closed, a submersion open, so surjective on connected components.

In the non-connected case a diffeomorphism ϕ acts bijectively on connected components, and one sees that in can be reduced to the connected case.

The presence of boundary makes the previous proof fail because submersions are open for the induced topology in the image, which might not be open itself. Just think of the sequence

$$\begin{array}{rcl} f_n \colon [0,1] & \longrightarrow & [0,1] \\ t & \longmapsto & \dfrac{n}{n-1}t \end{array}$$

This is fixed once we assume that boundaries go to boundaries.

□

ex:concomp

Exercise 29. *Let M, N be C^r-manifolds, $r \geq 1$ so that there is a decomposition on connected components*

$$M = \coprod_{\gamma\in\Gamma} M_\gamma,\ N = \coprod_{\lambda\in\Lambda} N_\lambda$$

Define for each γ, λ,

$$C^r(M,N)_{\gamma,\lambda} = \{f \in C^r(M,N) \mid f(M_\gamma) \subset N_\lambda\},$$

and show that $C^r(M,N)_{\gamma,\lambda}$ are open and close in $C^r_S(M,N)$.

Deduce that $\mathcal{U} \subset C^r_S(M,N)$ is open/closed iff for all γ, λ

$$\mathcal{U}_{\gamma,\lambda} := \mathcal{U} \cap C^r(M,N)_{\gamma,\lambda}$$

is open/closed. Conclude that in theorem 10 there is no loss of generality in assuming that M is connected.

The topological case is more difficult, but rather interesting. In general the set of embeddings and homeomorphisms is not open. For example take an eight and degenerate it to a nodal curve.

Exercise 30. *Use the previous idea of the nodal curve to construct $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ maps arbitrary close to the identity not being homeomorphisms.*

Theorem 11. *For M without boundary a homeomorphism has a neighborhood of surjective maps in $C^0_S(M,M)$.*

Proof. By exercise 25 we can assume w.l.o.g. that the homeomorphism is the identity. Take any $x \in M$, and a charts about it. to make sure that $B(x,1/2)$ is in the image of nearby maps, we impose conditions of proximity in $B(x,1)$. If such a g is not surjective, we get a map from B^n to S^n, which the above conditions force to be homotopic to a retraction. By theorem 14 we can assume the initial map to be C^1. If we knew it is a retraction, we can apply corollary 5 to conclude the desired result. In general □

So putting together what we know, we have for example:

thm:compembed **Theorem 12.** *If M is compact and $n \geq 2m+1$, $r \geq 2$, then* $\mathrm{Emb}^r(M, \mathbb{R}^n) \subset C^r(M, \mathbb{R}^n)$ *is open and dense (and the same result holds for embeddings in H^n_+ transverse to the boundary).*

4. Approximations

We saw in theorem 12 an approximation result, in which certain maps are approximated by others of "better quality". We will see that approximation is also possible gaining regularity. In this section manifolds are assumed to have empty boundary unless otherwise stated.

These are results based on behavior on Euclidean space, that we globalize using partitions of unity and more refined patching arguments.

Recall that if $f \in C^r(\mathbb{R}^m, \mathbb{R}^n)$, $g \in C^s(\mathbb{R}^m, \mathbb{R})$, g compactly supported, then **the convolution of f by g** is

$$g * f(x) = \int g(y) f(x-y) dy \tag{8}$$

eq:convol

So we may think of being averaging f using g as a weight.

If g is non-negative, has compact support and $\int g = 1$, then it is called a **convolution kernel**. It is feasible that convoluting f with a convolution kernel supported near zero produces a function which approximates f. Similarly, since there is an integration involved we expect to gain regularity.

Indeed, if we make the measure preserving change of variables $z = x - y$, then 8 becomes

$$g * f(x) = \int g(x-z) f(z) dz$$

so if g is C^s then so $g * f$ is.

Notice as well that if f is C^r then we can take derivatives inside the integral sign in 8 and then

$$D^l(g * f) = g * D^l f,\ 0 \leq l \leq r$$

Hence, if we are able to approximate in the C^0-topology then we should also get approximation in the C^r-topology.

thm:eucapprox **Theorem 13.**
Let $f\colon U \subset \mathbb{R}^m \to V \subset \mathbb{R}^n$ be a C^r-function, $r \geq 0$. Then it can be approximated in $C^r_S(U, V)$ by functions in $C^s_S(U, V)$, $r \leq s \leq \infty$.

Proof. We can assume w.l.o.g $V = \mathbb{R}^n$, being $C^r_S(U, V)$ open in $C^r_S(U, \mathbb{R}^n)$. If we use a convolution kernel θ we obtain:

$$|(\theta * f - f)(x)| = \int \theta(y) |f(x-y) - f(x)| dy$$

We can play with the support of the kernel. If we are on a compact K by uniform continuity -arranging the the support to be small enough neighborhood of 0- we can make

$$\max_{x \in K, y \in \mathrm{supp}\theta} |f(x-y) - f(x)| \leq \epsilon$$

To go from compacts to the strong topology in U, we fix $h\colon U \to \mathbb{R}^+$ proper and μ_j, $j \in \mathbb{N}$, a partition of the unity of $\mathbb{R}$ with μ_j supported in $(j-1, j+1)$. Define

$$K_j := h^{-1}([j-1, j+1]),\ U_j = h^{-1}((j-1, j+1))$$

One can check that any open set in $C^s_S(\mathbb{R}^m, \mathbb{R}^n)$ containing f contains a neighborhood of the form

$$\mathcal{N}^r(f, K_j, \epsilon_j)$$

Next construct $g_j \in C^s(\mathbb{R}^m, \mathbb{R}^n)$ so that

$$|h^*\mu_j f - g_j|_{C^r(K_j)} < \frac{1}{3}\min\{\epsilon_{j-1}, \epsilon_j, \epsilon_{j+1}\},\ \mathrm{supp} g_j \subset U_j \subset K_j,$$

and define $g = \sum_{j\in\mathbb{N}} g_j$, which is C^s because the supports are locally finite.

Then

$$|f - g|_{C^r(K_j)} \leq |h^*\mu_{j-1} f - g_{j-1}|_{C^r(K_{j-1}\cap K_j)} + \\ + |h^*\mu_j f - g_j|_{C^r(K_j)} + |h^*\mu_{j+1} f - g_{j+1}|_{C^r(K_j \cap K_{j+1})} < \epsilon_j$$

□

thm:dense

Theorem 14. *$C^s_S(M, N)$ is dense in $C^r_S(M, N)$, for all $r \leq s \leq \infty$.*

Proof. To prove this result, we need more than theorem 13; if we want to build an approximation by induction using an appropriate cover, we need to make sure that when we make a new (local) perturbation we still get a solution over the subset associated to the union of previous open subsets of the cover.

thm:releucapprox

Theorem 15. *Let $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$, and let $B, K \subset U$ closed subset with $f \in C^r_S(U, V)$ already C^s in $K \cap B$. Then f can be approximated by a function which equals f on B, and is C^s in K.*

Proof. Use a smooth partition of the unity μ_1, μ_2, $\mu_{1|B} = 1$, and such that the part of the support of μ_1 in $K\backslash B$ are points which are C^s for f.

It is a straightforward consequence of exercise 25 that the map

$$\begin{aligned} G\colon C^r_S(U, \mathbb{R}^n) &\longrightarrow C^r_S(U, \mathbb{R}^n) \\ g &\longmapsto \mu_1 f + \mu_2 g \end{aligned} \tag{9}$$

is continuous.

Therefore any neighborhood of $T(f) = f$ contains a function $T(g)$, where by theorem 14 g can be chosen to be C^s. By construction $T(g)$ satisfies the required conditions. □

To globalize one needs to observe that given $f \in C^r(M, N)$, $U \subset M, V \subset N$ domains of local coordinates with $f(U) \subset V$, for any open subset W with $\overline{W} \subset U$ the map

$$\begin{aligned} T\colon \mathcal{G} &\longrightarrow C^r_S(M, N) \\ g &\longmapsto g_{|U},\ f_{|M\backslash W} \end{aligned} \tag{10}$$

is continuous, where

$$\mathcal{G} := \{g \in C^r_S(U, V) \mid g_{|U\backslash W} = f\}$$

A globalization argument using an appropriate locally finite cover proves the result. Indeed any neighborhood of f contains one of the form $\mathcal{N} := \mathcal{N}(f, \overline{W}_j, U_j, V_j, \epsilon_j)$, where the W_j contain compacts L_j which cover M.

Order the subsets and by induction use theorem 15 and the continuity of the patching map 10 to construct g_j so that

- $g_{j|M\backslash W_j} = g_{j-1}$,
- g_j is C^s in $\bigcup_{d=0}^{j} L_d$, and
- $g_j \in \mathcal{N}$

The function $g(x)$ is $g_{k(x)}(x)$, where $k(x)$ is an integer bigger than any of the indices of subsets U_j containing x. By the local finiteness g matches $g_{k(x)}(x)$ in a neighborhood of x, so it is C^s. It also belongs to $\mathcal{N}$ since the g_j do and at some stage we do not further perturb in U_j. □

Exercise 31. *Show that the map in equation 9 is continuous.*

cor:density **Corollary 2.**

- *For any* $s > r \geq 1$, $\mathrm{Imm}^s(M,N)$ *(resp.* $\mathrm{Sub}^s(M,N), \mathrm{Prop}^s(M,N), \mathrm{Emb}^s(M,N), \mathrm{Diff}^s(M,N)$*) is dense in* $\mathrm{Imm}^r(M,N)$ *(resp.* $\mathrm{Sub}^r(M,N), \mathrm{Prop}^r(M,N), \mathrm{Emb}^r(M,N), \mathrm{Diff}^r(M,N)$*).*
- *If* $r = 0$ *we have density of smooth surjective maps on homeomorphisms.*
- *In particular, to* C^s*-manifolds are* C^s*-diffeomorphic iff they are* C^r*-diffeomorphic,* $r \geq 1$.

In order to raise differentiability of charts, we need something slightly better than the relative approximation result, namely we need to have freedom to "approximate on open sets with fixed boundary conditions", rather than just with conditions in a neighborhood of the boundary.

thm:extension **Theorem 16.** *Let* M *be a* C^r*-manifold,* $0 \leq r < \infty$*,* $W \subset M$*,* $V \subset \mathbb{R}^n$ *open subsets and* $f \in C^r(M,V)$ *with* $f(W) = V'$*. Then there exist* $\mathcal{N}$ *an open neighborhood of* $f_{|W}$ *in* $C^r_S(W,V')$ *so that the map*

$$\begin{aligned} \mathcal{T} \colon \mathcal{N} &\longrightarrow C^r_S(M,V) \\ g &\longmapsto g,\ f_{|M\backslash W} \end{aligned} \tag{11}$$

is continuous.

Proof. Cover $\partial\overline{W}$ with $\{U_i\}_{i\in I}$ a locally finite cover, so that it contains $\{L_i\}_{i\in I}$ compacts also covering the boundary. Take functions $g \in C^r_S(W,V')$ such that for $y \in M\backslash\overline{W}$, $y \in L_i$

$$|f \circ \varphi_i^{-1}(\varphi_i(y)) - g \circ \varphi_i^{-1}(\varphi_i(y))|_{C^r} \leq d(\varphi_i(y), \varphi_i(U_i\backslash W)) \tag{12}$$ eq:rnomdist

This defines $\mathcal{N}$ an open subset. The reason is that we can take a locally finite cover K_j by compacts, and since only a finite number of L_i intersect each K_j, for suitable ϵ_j equation 12 holds for all the relevant i.

It is easy to see that $\mathcal{T}(g)$ defines a C^r-function, because is we subtract f from it we get a function which vanishes at $M\backslash W$, and whose derivatives up to order r go to zero as points approach $M\backslash W$.

Continuity is also straightforward, very much as for equation 10.

□

exe:urysohn **Exercise 32.** *Prove that the methods of theorem 16 allow us to construct for any* $A \subset M$ *a closed subset, a bounded positive function* $f \colon M \to \mathbb{R}^+$ *such that* $f^{-1}(0) = B$*. Even more, given* A, B *disjoint closed subsets we can find* $f \colon M \to [0,1]$ *so that* $f^{-1}(0) = A$*,* $f^{-1}(1) = B$*.*

Hint: If $r < \infty$*, then out of a proper function* $h \colon M\backslash A \to \mathbb{R}^+$ *and a partition of the unity* $\{\mu_j\}_{j\in\mathbb{N}}$ *of* $\mathbb{R}^+$ *as in theorem 13, we can find integers* $n(j)$ *so that following the proof of theorem 16 we conclude that*

$$\sum_{j\in\mathbb{N}} \frac{1}{n(j)} h^*\mu_j$$

can be extended by zeros to A *giving a* C^r*-function. In the smooth case the procedure has to be more elaborate. One needs to pick a locally finite covering* $\{(U_i,\varphi_i)\}_{i\in I}$ *of the boundary of* $M\backslash A$*. It is clear that the difficulty in defining the function on* $M\backslash A$ *tending fast enough to zero near* A *is concentrated in those points inside* $W := \bigcup_{i\in I} U_i \cap (M\backslash A)$*. We cover* W *by* $\{Z_j\}_{j\in\mathbb{N}}$ *open subsets subordinated to* $\{U_i\}_{i\in I}$*, and so that*

$$\varphi_{i(j)}(V_j) = B(x_j, \delta_j) \subset \mathbb{R}^m,$$

where $2\delta_j = d(x_j, \varphi_{i(j)}(M\backslash A))$. *Then if* $\mu\colon \mathbb{R}^m \to [0,1]$ *is the standard step function with support* $B(0,1)$*, one can use the pullback of the step function*

$$\delta_j^j \mu(2\delta_j^{-1}(x - x_j)),$$

and the sum on $j \in \mathbb{N}$ *yields the desired result.*

Now we are ready to prove one of the main results stated in the introduction.

thm:smoothen **Theorem 17.** *Let* M *be a* C^r *manifold,* $r \geq 1$*. Then for any* $s \in \bar{\mathbb{N}}$*,* $s \geq r$*,* M *admits a* C^s*-structure compatible with the* C^r*-structure, which by corollary 2 is unique up to* C^s*-diffeomorphism.*

Proof. We consider open subsets $U' \subset M$ which can be endowed with a C^s-manifold structure so that the inclusion $i\colon U' \hookrightarrow M$ is a C^r-diffeomorphism, and we put the obvious order (inclusion among them is a C^s-diffeomorphism). It is clear that for an ascending chain the union of the open subsets with the union of all charts is an open subset admitting a compatible C^s-structure, and majorating any element in the chain. Then by Zorn's lemma there is maximal one U'. We will extend it by taking a C^r-chart (U, φ) with non trivial overlap and modifying φ suitably.

Let $V = \varphi(U)$, $W = U' \cap U$, $V' = \varphi(W)$. To extend the C^s structure to $U' \cup U$ in a compatible way, we need to substitute the local C^r-coordinates $\varphi\colon U \to V$ by a new C^r-diffeomorphism ψ which is C^s on W. Consider $f := \varphi_{|W} \in C^r_S(W, V')$. Then by theorem 16 we have a neighborhood $\mathcal{N}$ of f in that $C^r_S(W, V')$ whose elements can be patched with $\varphi_{|U\backslash W}$ so that the map

$$\mathcal{T}\colon \mathcal{N} \to C^r_S(U, V)$$

is continuous.

In particular, by corollary 2 we can take elements $g \in \mathcal{N}$ such that g is a C^s-diffeomorphism. Continuity of the patching map 11, together with openness of C^r-diffeomorphisms, implies that g can be chosen so that $\psi := \mathcal{T}(g)$ is a C^r-diffeomorphism, solving thus the problem.

□

rem:diffcompexh **Remark 18.** *The possibility of endowing any* C^r*-manifold,* $r \geq 1$*, with a compatible smooth structure is extremely powerful. For example, we know that we cannot always apply Sard's theorem for* $C^r(M, \mathbb{R})$*, if* $m \geq r$*. We can use a compatible smooth structure, apply it to smooth functions. In this way we show that we always have compact exhaustions so that each* M_j *is a* C^r*-submanifold (with boundary) of* M*.*

So far we have (i) openness results for distinguished classes of maps, like immersions, embeddings, submersions, proper maps and diffeomorphisms, and a density theorem 14 for C^s maps in C^r maps in the strong topology, and other density results for M compact (remark 7). The latter density result can be extended to obtain density of $\mathrm{Imm}^s(M, N), \mathrm{Emb}^s(M, N)$ in $C^r_S(M, N)$, $r \geq 1$, under certain assumptions. One needs:

(1) A **local weak density result**, which is the case $M = U \subset \mathbb{R}^m$, $N = V \subset \mathbb{R}^n$ and the weak topology. It is based in remark 7, which is an approximation result for M compact, f at least C^2, and $n \geq 2m$ and $n \geq 2n + 1$ respectively. In remark 7 compactness is used in two places. To embed the manifold in some Euclidean space -which we do not need in the local setting- and to obtain one dense subset coming from Sard's theorem which is also open, because it is the complement of the image of a compact. We can apply Sard's to any open subset containing a given compact K, and then restrict to the sphere bundle over K to get the desired open dense subset; if f is just C^1, we can approximate it by a C^2-function.

(2) A **local density result** for the strong topology. That can be done by using an strategy similar to theorem 13. One uses an annular partition of the unity μ_j, and approximates by induction the compactly supported functions using the previous point, and taking projections close enough to the identity so that injectivity of the differential in previous strips survives. Once more since we projection decreases support, the modification only affect certain strips, so for each compact in U after a finite number of steps the function is not perturbed there any further. To prove the embedding result properness of f is needed, so it can actually be used to construct the annular partition of unity which will allow inductive perturbations for which global injectivity (as opposed to the one obtained in compacts) is possible.

(3) An **extension result** -which is given via theorem 16 (or in a less clean way by an induction process for a locally finite cover by compacts following the ideas of the previous point)- and which makes possible and induction step in the construction of the solution. One chooses a locally finite covering by charts $U_j' \simeq B(0,1)$, so that $U_j \simeq B(0,1/2)$. Then if g_j approximates f and is an immersion already in $U_1 \cup \cdots \cup U_j$, inside U_{j+1}' we patch the restriction of g_j to $U_{j+1}' \backslash U_{j+1}$ with an approximation of $g_{j|U_{j+1}}$ which by the previous point can be chosen to be an immersion/embedding on U_{j+1}. Then g_{j+1} is the result of adding the restriction of g_j to $M \backslash U_{j+1}'$. Notice as well that if $r=1$ we can choose a compatible C^2 structure and use density of C^2 maps to perform the construction.

(4) A **continuity principle** which grants that $g := \lim g_j$ becomes a solution. For immersions this holds because the property is checked pointwise. For embeddings again properness of f is needed.

In this way one gets the following results:

thm:maindensity

Theorem 18. *For all $s \geq r \geq 1$ we have:*

(1) $\mathrm{Imm}^s(M,N)$ *is dense in* $C^r_S(M,N)$ *if* $n \geq 2m$.
(2) $\mathrm{Emb}^s(M,N)$ *is dense in* $C^r_S(M,N)$ *if* $n \geq 2m+1$ *and M compact. If* $f \in C^r(M,N)$ *is proper then* $\mathrm{Emb}^s(M,N)$ *is dense in a neighborhood of f.*

cor:closedemb

Corollary 3. *For $r \geq 1$ any C^r manifold M embeds as a closed manifold in $\mathbb{R}^n$, $n \geq 2m+1$.*

Proof. Take any proper function and apply theorem 18. To produce the proper function just notice that the composition of proper functions is proper. Then use any proper function $f\colon M \to \mathbb{R}$ and compose is with any linear injective map $\mathbb{R} \to \mathbb{R}^n$. □

rem:anyemb

Remark 19. *One can extend corollary 3 and prove that any C^r manifold M embeds as a closed submanifold of a C^r-manifold N, provided $n \geq 2n+1$. We only need to produce a proper function $h\colon \mathbb{R}^+ \to N$. This can be done by taking a compact exhaustion by submanifolds (remark 18), constructing the dual graph associated to it -one vertex for each connected component of $M_j \backslash \mathrm{int} M_{j-1}$ and one segment for each common boundary connected component- where each vertex is labeled with the obvious natural number. Connectivity of N implies connectivity of the graph. A maximal tree can always be chosen. It cannot be compact, because a maximal tree contains all vertices. We can produce the copy of $\mathbb{R}^+$ by induction. We start with a vertex at height zero. The induction step is as follows. The vertex v_j boundary of s_j the last segment added (in the first step just the vertex) disconnects, giving rise to a finite number of subtrees. At least one of those not containing v_j has to be infinite.*

We then chose v_{j+1} the segment in that tree. The corresponding embedding of $\mathbb{R}^+$ has to be proper, because we only have a finite number of vertices at each height.

rem:differbit **Remark 20.** *One can also show for any $s \geq> r1$ that the inclusion*

$$C^s_S(M,M) \hookrightarrow C^r_S(M,M)$$

does not fill any neighborhood of the identity. Therefore, for any C^r-manifold $r \geq 1$, we can find different C^r-manifold structures which are C^r-diffeomorphic.

rem:boundfunct **Remark 21.** *One has similar results for manifolds with boundary and also to manifold pairs (M,A). The main difference is the use of an appropriate convolution kernel to approach a C^r-map $f \in C^r((U, U \cap H^m_+), (V, V \cap H^n_+))$ in a compact by another such C^s-map.*

Indeed, if we use coordinates (x, x_m) in $\mathbb{R}^{m-1} \times [0,\infty)$, we just consider the map

$$\begin{array}{rcl} \pi\colon \mathbb{R}^m & \longrightarrow & H^m_+ \\ (x, x_m) & \longmapsto & (x, |x_m|) \end{array}$$

Then given $K \subset U \subset H^m_+$, $f \in C^r(U,V)$, we approximate $\pi^* f$ in $\pi^{-1}(K)$ using convolution kernels symmetric w.r.t. the x_m-coordinate to obtain $g \in C^\infty(\pi^{-1}(U), V)$, that we restrict to U. Notice that $\pi^* f$ will not be C^r in $x_m = 0$, but the approximation result holds because the original f extends over the hyperplane, and for that the result holds true using convolution kernels supported in a half ball say.

One then obtains density results for C^s maps in C^r maps. There are similar results for manifold pairs. This time they are based in the use of a convolution kernels plus a further correction which amounts to add the orthogonal projection onto the target submanifold of the domain submanifold.

Regarding the density results in 18, their are proven using the same pattern, following from the local density result, which can be shown as follows:

(1) *Given $f\colon (H^m_+, \partial H^m_+) \to (H^n_+, \partial H^n_+)$, $f = (f_\partial, f_n)$, one approximates $f_\partial(x_1, \ldots, x_{m-1}, 0)$ by g and then defines*

$$G(x_1, \ldots, x_m) = (g + \int_0^{x_m} \frac{\partial f_\partial}{\partial x_m} dx_m, f_n + \epsilon x_m)$$

For ϵ small enough G approximates f in the C^r-norm in the thickening of a closed ball in the boundary. It sends just the boundary to the boundary. By construction it is an embedding in the points of ∂H^m_+, so it is an embedding in a small thickening. Then we can apply theorem 16 to modify G uniformly far from the ∂H^m_+ (and overlapping slightly with the thickening where it is an embedding). Since the image is also uniformly far from ∂H^n_+, one concludes that the resulting map is still injective (due to the small C^0-size of the deformation).

So one gets density of embeddings of M in H^{2m+1}_+, and also of $(M, \partial M)$ in $(H^{2m+1}_+, \partial H^{2m+1}_+)$ transverse to the boundary.

5. Sard's theorem and transversality

def:critpts **Definition 23.** *Let $f\colon M \to N$ be a C^r-map, $r \geq 1$. The set of critical points is*

$$\Sigma_f := \{x \in M \mid \mathrm{Ker} Df_x \neq \{0\}\}$$

The set of critical values is $f(\Sigma_f)$.

We want to understand as much as possible $f(\Sigma_f)$.

Differentiable maps can be very complicated. Let us start in Euclidean space and with the simplest map. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a polynomial. Df is a polynomial of smaller degree and there are two possibilities:

- $Df = 0$, so $\Sigma_f = \mathbb{R}$ and $f(\Sigma_f) = z_0$.
- $Df \neq 0$, and thus Σ_f is a finite collection of points, and so $f(\Sigma_f)$ is.

The upshot is that the set of critical values is very simple because the set of critical points is already very simple.

For a polynomial $f\colon \mathbb{R}^m \to \mathbb{R}$ the set of critical points Σ_f is described as the zero set of a finite number of polynomial equations. It is known that the solution of such systems decomposes into a finite number of smooth varieties, each locally parametrized by analytic functions (the inverse function theorem holds in the analytic setting). Hence

$$\Sigma_f = \coprod_{i=1}^{d} \Sigma_{f,i},$$

each subset being a connected analytic manifold.

If x, y are distinct points in $\Sigma_{f,i}$, then we can joint then by a smooth (analytic) curve $c(t)$ and

$$\frac{d}{dt}(f \circ c(t)) = Df(c(t))c'(t) = 0,$$

so the curve $f \circ c$ is constant. Therefore, the map f shrinks $\Sigma_{f,i}$ to a point.

So (deep) results on real algebraic geometry give a very precise description of Σ_f, out of which the finiteness of $f(\Sigma_f)$ follows.

For a general differentiable function Σ_f will not have a nice structure. For example take any closed subset $A \subset \mathbb{R}$. According to exercise 32 we can find h with $h^{-1}(0) = A$. Define $f(t) := \int_0^t h(x)dx$. Then $\Sigma_f = A$.

At any rate, the general idea is for a given subset $A \subset M$, to estimate the "size" of $f(A)$ using Cauchy's theorem and Taylor's formula.

First observe that out of the Lebesgue measure in Euclidean space, one cannot in principle induce a measure on a given differentiable manifold M, for the former is not C^r-diffeomorphism invariant (the jacobian of the diffeomorphism enters in the formula). At any rate the notion of a subset $A \subset \mathbb{R}^m$ having (Lebesgue) measure zero is C^r-diffeomorphism invariant.

lem:sard1 **Lemma 11.** *Let $f \in C^1(U, V)$, with $U, V \subset \mathbb{R}^m$ open, and let $A \subset M$ have measure zero. Then $f(A)$ has measure zero.*

Proof. We can assume w.l.o.g. that A is inside a compact ball K contained in U (because the countable union of measure zero sets has measure zero). Therefore Df will be uniformly bounded in K. That means

$$|f(x) - f(y)| \leq Df(c)((x-y)/|x-y|)|x-y| \leq C|x-y| \tag{13}$$ eq:taylor

Therefore the image of any ball of radius ϵ will be inside a ball of radius $C\epsilon$, and the result follows. □

cor:sard1 **Corollary 4.** *Let M be a C^r-manifold, $r \geq 1$. Then the notion of a subset having measure zero is well defined. Moreover, if $f\colon M \to N$ is differentiable and (i) both manifolds have the same dimension, then the image of a measure zero set has measure zero; if (ii) $m < n$ then $f(M)$ has measure zero.*

Proof. Having Lebesgue measure zero is a local property, and by lemma 11 C^r-diffeomorphism invariant. Then it gives rise to the corresponding notion for manifolds.

Since our manifolds are second countable, lemma 11 implies an analogous result for manifolds. When $m < n$, and assuming w.l.o.g. that our manifolds are open subsets U, V in Euclidean spaces, we have

$$\mathbb{R}^m \xrightarrow{\pi} \mathbb{R}^m \xrightarrow{f} \mathbb{R}^n,$$

and $f(U) = f \circ \pi(i(U))$, where $i(U)$ is the obvious inclusion, which has measure zero. Then by lemma 11 the result follows. □

To prove Sard's theorem we will apply induction on m, n (first fixing n and allowing m to increase). In particular we proved the result for $m - n = 0$. So we need to prove the case $n \leq m$. Let us define

$$\Sigma_f^{(k)} = \{x \in \Sigma_f \mid Df^k(x) \equiv 0\}$$

By using Taylor theorem, if we stay at a point $x \in \Sigma_f^{(k)}$ we obtain

$$\mathrm{Vol}(f(x + I_\epsilon^m)) \leq C\epsilon^{kn},$$

as long as the function in C^{k+1}, where I_ϵ^m is the m-cube of side ϵ centred at the origin. The set of such critical points can always be covered by $C(1/\epsilon)^m$ cubes of radius ϵ, where C does not depend on ϵ. Then

$$\mathrm{Vol}(f(\Sigma_f^{(k)})) \leq C'\epsilon^{kn-m},$$

so if $kn - m > 0$ the volume tends to 0 as ϵ does.

But then one can write

$$\Sigma_f = \Sigma_f^{(k)} \cup \Sigma_f^{(1)} \cup \Sigma_f^{(0)}$$

Since $m \geq n$, we have $k \geq 2$. Assume that $x \in \Sigma_f^{(1)} \backslash \Sigma_f^{(k)}$, then we have some multiindex I with $1 \geq |I| \geq k - 1$, and an index j, so that

$$\frac{\partial}{\partial x^I} f(x) = 0, \ \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^I} f(x) \neq 0 \tag{14}$$ eq:sardhyp

So we can construct the hypersurfaces $H^{I,j}$ of points for which equation 14 holds. We have

$$\Sigma_f \cap H^{I,j} \subset \Sigma_{f_{|H^{I,j}}}$$

But $\dim H^{I,j} < m$, so by induction on m (with n fixed) we have

$$f_{|H^{I,j}}(\Sigma_{f_| H^{I,j}}) = f(\Sigma_{f_{|H^{I,j}}})$$

has measure zero.

Regarding $\Sigma_f^{(0)}$, we can find coordinates in $\mathbb{R}^n$ in which a component of f is one of them, say x_n. Then the result holds for each f_{x_n}, and by Fubini's theorem for f.

cor:noretract **Corollary 5.** *There exist no continuous retraction* $f\colon \overline{B^m(0,1)} \to S^{m-1}$

Proof. We can assume w.l.o.g. that f is smooth: firstly we compose on the right with a self map of the ball shrinking a neighborhood of the sphere into the sphere, so we obtain a retraction which is smooth in a neighborhood of S^{m-1}. Next we use relative approximation to obtain the smoot retraction. By Sard's theorem there is a regular value z for f, and hence $f^{-1}(z) \subset \overline{B^m(0,1)}$ is a compact 1-dimensional submanifold. Then theorem 6 implies that it must be a finite collection of copies of S^1 an closed intervals, so it cannot just have a boundary consisting on 1 point. □

In general, the same proof forbids the existence of a C^1-retraction $M \to \partial M$ for any compact differentiable manifold.

5.1. **Transversality.** Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ be a differentiable (enough to apply Sard's) map. We can think of Sard's theorem the other way around, and rather than saying that we can approximate any value, say $\mathbf{0}$, by regular ones z_n, we can approximate f by $f - z_n$ which are transverse to $\mathbf{0}$.

Let M, N be manifolds, $A \subset N$ a submanifold. Recall that a residual subset of a topological space is one containing an intersection of countably many dense open subsets.

Let

$$\pitchfork_K^r (M, N; A) := \{f \in C^r(M, N) \mid f_{|K} \pitchfork A\},$$

and also denote $\pitchfork_M^r (M, N; A) = \pitchfork^r (M, N; A)$.

thm:transversality

Theorem 19.

1. *$\pitchfork^r (M, N; A)$ is residual in $C_S^r(M, N)$.*
2. *Let A be closed. If $L \subset M$ is closed (resp. compact) then $\pitchfork_L^r (M, N; A)$ is open in $C_S^r(M, N)$ (resp. $C_W^r(M, N)$).*

The proof is based on the following local result:

Lemma 12. *Let K be a compact inside an open set $U \subset \mathbb{R}^m$. Let $V \subset \mathbb{R}^n$ open and $A = P \cap V$, where P is a linear subspace. Then $\pitchfork_K^r (U, V; A)$ is open and dense in $C_W^r(U, V)$*

Proof. We can assume $V = \mathbb{R}^n$, since $C^r(U, V)$ is open in $C_W^r(U, \mathbb{R}^n)$. Transversality is clearly an open condition in the weak C^1-topology. We just need to prove the density result. Take $f \in C^r(U, \mathbb{R}^n)$, which can be assumed to be smooth. By projecting onto $\mathbb{R}^n/A$, we obtain $\tilde{f}$, that we can approximate by maps transversal to zero by adding constants $\tilde{z}_n$; one can clearly choose constants z_n converging to zero and projecting into $\tilde{z}_n$, and then $f - z_n$ is the desired sequence. □

The globalization result needed to prove theorem 19 is left for the reader; one also needs to know that $C_S^r(M, N)$ **has the Baire property**, so that residual subsets are dense.

def:genpos

Definition 24. *Let A, B differentiable submanifolds of M. They are said to be in general position if i_A is transverse to B (or the other way around since the definition is symmetric in A, B).*

cor:denseemb

Corollary 6. *Let $f \in \mathrm{Prop}^r(M, N)$, $A \in N$ submanifold, and $n \geq 2m+1$. Then f can be approximated in $C_S^r(M, N)$ by embeddings transversal to A. In particular for A, B submanifolds of M then A say can be approximated by embeddings transversal to B.*

Proof. Again we can assume f to be smooth, and then approximate by embeddings, and since each is open it can be approximated by a map in $\pitchfork^r (M, N; A)$ which is still an embedding. □

Theorem 19 is very interesting, but not really "economic". Sometimes we are interested in approximating by transverse maps within a much smaller family of maps, typically parametrized by a (finite dimensional) manifold.

Let P be a manifold, and $F\colon P \to C^r(M, N)$ a map.

Theorem 20. *Let M, N by differentiable manifolds without boundary, and $A \subset M$ a submanifold without boundary. Assume that*

1. *F^{ev} is C^r;*
2. *F^{ev} is transversal to A;*
3. *$r > \max\{0, m + a - n\}$.*

Then the set $\{p \in P \mid F_p \pitchfork A\}$ is residual and hence dense. If A is closed and F continuous for the strong topology then it is open.

Proof. By conditions 1 and 2 $F^{\text{ev}-1}(A)$ is a C^r-submanifold of $P \times M$ of dimension $p+m-a$. Restrict the first projection $P \times M \to P$ to this submanifold. The third hypothesis allows us to apply Sard's theorem, to obtain a residual subset of regular values. By construction those are exactly the values $p \in P$ for which $F_p \pitchfork A$.

The second statement follows from the continuity of F and theorem 19 □

6. Tubular neighborhoods, homotopies and isotopies

One of the aims of this section is to further understand the relation between the topological space $C^r_S(M, N)$ and homotopies. The way to do that is using "tubular neighborhoods".

Given a submanifold $i\colon A \to M$, informally speaking the **normal bundle** are those vectors at the points of A which belong to directions "normal to A".

In a more invariant way, inside $\pi\colon TM \to M$ we can consider **the restriction of TM to A**

$$TM_{|A} := \coprod_{a\in A} T_aM$$

- It has a submanifold structure because $TM_{|A} = \pi^{-1}(A)$, and π is a surjective submersion. For such a differentiable structure, the restriction of the projection
$$\pi\colon TM_{|A} \to A$$
is differentiable.
- Each **fiber** $\pi^{-1}(a) = T_aM$ has a vector structure, which is compatible with the differentiable structure as described in subsection 2.2 (so it is a vector bundle). Notice that this property holds because it holds for TM.

We have an embedding $TA \hookrightarrow TM_{|A}$ which is linear on fibers T_aA. Therefore we can consider the quotient (topological) space

$$\nu(A) := \coprod_{a\in A} T_aM/T_aA$$

Lemma 13. *Assume that $\nu(A)$ admits a differentiable structure so that the projection $p\colon TM_{|A} \to \nu(A)$ is a submersion. Then the differentiable structure is unique and makes $\nu(A)$ into a vector bundle over A*

The proof of the uniqueness is straightforward (and it holds for any quotient space of a differentiable manifold). Notice that for such a smooth structure the projection $\tilde{\nu}(A) \to A$ trivially becomes a surjective submersion, and each fiber $\nu_a(A)$ carries a vector space structure (the quotient vector structure) for which the vector space operations are clearly compatible with the smooth structure (because this same property holds for $TM_{|A}$).

We can find such a differentiable structure as follows: consider $f\colon M \to \mathbb{R}^n$ an embedding; if M has non-empty boundary we select an embedding into a positive half space transverse to the boundary. Then we have the following obvious identifications

$$\begin{aligned}
TM &= \{(y,v) \in T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n \mid (y,v) = (f(x), f_*(x)u) \mid u \in T_xM\}\\
TM_{|A} &= \{(y,v) \in T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n \mid (y,v) = (f(a), f_*(x)u),\ a \in A,\ u \in T_xM\}\\
TA &= \{(y,v) \in T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n \mid (y,v) = (f(a), f_*(x)u),\ a \in A,\ u \in T_aA\}
\end{aligned}$$

Now we can use the Euclidean metric to consider the submanifold of $TM_{|A}$

$$TA^{\perp TM} := \{(y,v) \in T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n \mid y = f(a),\ v \perp f_*T_aA\}$$

It is clear that the map

$$q\colon TM_{|A} \to TA \tag{15}$$ eq:normalprojection

projecting orthogonally w.r.t. Euclidean metric each tangent fiber T_aX into T_aA w.r.t. Euclidean metric is differentiable. Then $TA^{\perp TM} = q^{-1}(\mathbf{0})$, and the result follows.

Exercise 33. *Prove that the map in equation 15 is differentiable.*

Notice also that

(i) the parallel projection along TA gives a differentiable map $TM_{|A} \to TA^{\perp TM}$,
(ii) we have an obvious bijection $TA^{\perp TM} \to \nu(A)$, and
(iii) the composition of the two is $p\colon TM_{|A} \to \nu(A)$.

def:nbundle **Definition 25.** *The normal bundle is $q\colon \nu(A) \to A$ with the above differential structure.*

rem:triv **Remark 22.** *The normal bundle admits charts of the form*

$$\Phi_i\colon q^{-1}(U_i) \to \varphi_i(U_i) \times \mathbb{R}^{m-a}$$

*sending the fibers linearly in to the copies of $\mathbb{R}^{m-a}$. Indeed, on a point $a \in A$ take a frame $u_1, \ldots, u_{m-a}$ in T_aM complementary to T_aA. Extend it constantly to $\mathbb{R}^q$. Then project each vector orthogonally into TM and restrict to the points of A. In that way we get local differentiable maps $U_a \to TM_{|A}$ which are right inverses of the projection (***sections***). By continuity, in a neighborhood of a they project onto a basis of $\nu(A)$. The chart Φ_i sends each vector to its coordinates in this basis.*

Remark 23. *Given any C^{r-1}-metric on TM, going to adapted charts one sees that the collection $\coprod_{a\in A} T_aA^{\perp g}$ of vector subspaces is a submanifold of $TM_{|A}$, and the restriction of the projection is a bijection. Therefore, it gives another model of normal bundle.*

What is relevant about the normal bundle is that it models appropriate neighborhoods of A in M. If M has boundary we consider A to be a submanifold with non-empty boundary, $A = \partial M$ or A a submanifold with boundary include in ∂M.

def:tub **Definition 26.** *A tubular neighborhood of $A \hookrightarrow M$ is given by a diffeomorphism $f\colon \nu(A) \to M$ over its image -an open neighborhood $\mathcal{U}_A$ of (the image of) A- sending the zero section to the given embedding of A. Sometimes one identifies a tubular neighborhood with its image $\mathcal{U}_A$.*

We can construct a tubular neighborhood as follows: let us assume that A has no boundary (nor it is the boundary of M) and consider $TA^{\perp TM}$ as model of $\nu(A)$. Take the map

$$\begin{array}{ccc} TM & \longrightarrow & \mathbb{R}^n \\ (y,v) & \longmapsto & y+v \end{array}$$

We want to modify this map to land in M. Notice that since $y + T_yM$ is the tangent space to M, there exists an open neighborhood $V_y \subset M$ such that it is mapped by the orthogonal projection diffeomorphically to an open subset of T_yM: the inverse function theorem holds with C^r-parameters (and because the radius we the map has inverse is given in terms of C^1-data), by collecting the inverses we are defining a differentiable map $F\colon \mathcal{V} \subset TM \to M$, $\mathcal{V}$ a neighborhood of the $\mathbf{0}$ section. We now define

$$f\colon \mathcal{V}_A \subset TA^{\perp TM} \hookrightarrow TM \xrightarrow{F} M$$

This is a differentiable map defined on a neighborhood of the zero section, and restricts to the identity (or the embedding) to the zero section. Even more, its diferential at the zero section restricts to the identity. This implies that f is open and that for each a in the zero section there is a neighborhood in $TA^{\perp TM}$ in which f is a diffeomorphism over its image. A priory, we might not have injectivity because of points far away in $TA^{\perp TM}$, but this trouble does not appear because A is a submanifold for which we have adapted charts.

So we conclude that $f\colon \mathcal{V}_A \to M$ is a diffeomorphism over its image. Now observe that we can find $h \in C^r(A)$ strictly positive so that

$$\begin{array}{ccc} TA^{\perp TM} & \longrightarrow & TA^{\perp TM} \\ (y,v) & \longmapsto & y + h(v) \end{array}$$

sends $TA^{\perp TM}$ in $\mathcal{V}_A$. The map is obviously a diffeomorphism over its image, so by composing it with the previous one we get the desired tubular neighborhood.

rem:retract

Remark 24. *Each tubular neighborhood comes equipped with a retraction*

$$r\colon \mathcal{U}_A \to A$$

This is so because we have such a canonical retraction on $\nu(A)$ sending each fiber to the origin. Moreover, the retraction is homotopic to the identity map, by taking their convex combination (we can add along the fibers).

The previous construction can be applied to a manifold with boundary M, where the submanifold we consider is ∂M. Notice that the normal bundle $\nu(\partial M)$ has 1-dimensional fibers. Even more, using the embedding in $(H^q_+, \partial H^q_+)$ transverse to the boundary we can consider the map

$$\begin{array}{ccc} T\partial M^{\perp TM} & \longrightarrow & \partial M \times \mathbb{R} \\ (a,v) & \longmapsto & (a, \pm|v|) \end{array}$$

where the sign is positive if the vector is outwards pointing and negative otherwise.

If we apply the tubular neighborhood theorem projecting along ∂H^q_+, we get a map

$$f\colon \partial M \times (-\infty, 0] \to \mathbb{R}^m$$

which is a diffeomorphism over its image contained in M.

thm:collar

Theorem 21. *Let M be a differentiable manifold with boundary. Then we can always find a* **collar of the boundary***, i.e. a transverse coordinate t defined in a neighborhood of ∂M. In particular any manifold with boundary admits an enlargement to a manifold M' with empty boundary (so that M enters as a submanifold).*

ex:tubularbound

Exercise 34. *Show how to construct a tubular neighborhood for a submanifold with boundary $(A, \partial A) \hookrightarrow (M, \partial M)$.*

6.1. **Homotopies, isotopies and linearizations.**

def:isotopy

Definition 27. *Let M, N differentiable manifolds. Recall that a homotopy between C^r-maps f, g is a C^r-map*

$$H\colon M \times I \to N$$

such that $H_0 = f$, $H_1 = g$. H is called an isotopy if each H_t is a diffeomorphism.

rem:collarextension

Remark 25. *Notice that due to the existence of collars, if M is compact and without boundary, we can think of an isotopy as a map $\Psi\colon M \times (-\epsilon, 1+\epsilon) \to N$. To enlarge the map (that does not require compactness) we embed N in Euclidean space. Next take a locally finite cover of ∂M with a partition of the unity subordinated to it. Extend -using the collar- each open subset to an open subset U_i in the collar crossing by a small open interval of radius ϵ_i, and pull each function μ_i to it. By definition, we have a function*

$$\Psi_i\colon U_i \to N \subset \mathbb{R}^q,$$

so that on the overlaps in M the functions match. Define

$$M_1 = M \bigcup_{i\in I} U_i \subset M'$$

which is an open subset of M'.

Define $\Theta_1 = \sum_i \mu_i \Psi_i$ on $\cup_{i\in I} U_i$. This is a differentiable function which on a neighborhood of ∂M in M coincides with Ψ, so it gives a function $\Theta_2\colon M_1 \to H^q$ extending Ψ. This function may not land in N. Let $\mathcal{U}_N$ be a tubular neighborhood of N in H^q. By construction $\Theta_2^{-1}(\mathcal{U}_N)$ is a neighborhood M_2 of M in M'. Now according to remark 24 we have a differentiable retraction $r\colon \mathcal{U}_N \to N$. Consider $\Psi'\colon M_2 \to N$ to be $r \circ \Theta_2$. It is clear that we are extending Ψ, and if M is compact then M_2 contains an open subset of the desired form.

lem:locpathc

Lemma 14. *Let M, N be a C^r-manifolds. Then for any $f \in C^r(M,N)$ there exist a neighborhood in $C^0_S(M,N)$ all whose maps are homotopic to f. In particular $C^r_S(M,N)$ is locally path connected.*

Proof. Embed N in H^q. Clearly, we can take a neighborhood $\mathcal{N}$ of f in $C^0_S(M,\mathbb{R}^q)$ such that if $g \in \mathcal{N}$ then the segment $[g(x), f(x)] \subset H^q$ lies inside a tubular neighborhood $\mathcal{U}_N$ of N. Then we have the homotopy

$$\begin{aligned} H\colon M\times I &\longrightarrow N \\ (x,t) &\longmapsto r((1-t)f + tg), \end{aligned}$$

where $r\colon \mathcal{U}_N \to N$ is the retraction. □

cor:diffhomot

Corollary 7. *Let M, N be a C^r-manifolds . Then any $f \in C^0(M,N)$ is homotopic to a C^r-map. Moreover, any to such maps are C^r-homotopic.*

Proof. Any to such maps give rise to a map $H\colon M\times I \to N$ which is C^r in a collar of the boundary (we can make the homotopy stationary near end points). Then we can approximate H by a C^r-function relative to a neighborhood of the boundary. Indeed, if μ is a bump function taking the value 1 near the boundary, we can approximate the homotopy by a C^r-map H', and take as final homotopy

$$r(\mu H + (1-\mu)H')$$

□

6.2. **Linearizations.** Let $f\colon U \subset \mathbb{R}^m \to \mathbb{R}^n$ be a differentiable function.

Since $T_x\mathbb{R}^m, T_y\mathbb{R}^n$ are canonically identified with $\mathbb{R}^m, \mathbb{R}^n$, we can think of $Df(x)$ -after restricting it- as a map in $C^r(U,\mathbb{R}^m)$, and we can canonically homotope one into the other by the convex combination

$$tf + (1-t)Df(x)$$

Even more, we have:

lem:isolin

Lemma 15. *Let $f\colon B(0,r) \subset \mathbb{R}^m \to \mathbb{R}^m$ be a differentiable map with invertible derivative at the origin. Then there exists $r' > 0$ such that $f, Df(0)\colon B(0,r') \to \mathbb{R}^m$ are isotopic.*

Proof. Take $\Psi'(t,x) = tf + (1-t)Df(0)$. Since the derivative is continuous, there exists $r' > 0$ so that the derivative of Ψ_t in $B(0,r')$ is invertible, as the result follows (sometimes we maybe interested in making the isotopy stationary at end points. Then we take a smooth non-decreasing funtion $\chi\colon [0,1] \to [0,1]$ so that $\chi(\epsilon) = 0, \chi(1-\epsilon) = 1$). □

There is a second -also canonical- way of homotoping a map into its derivative, which extends to more general situations.

Let $f\colon \mathbb{R} \to \mathbb{R}$ be an analytic function, and let us also assume for simplicity $f(0) = 0$. Then the previous homotopy between f and $f'(0)$ reads

$$f'(0)x + t\sum_{j\geq 2} \frac{f^{(j)}(0)}{j!}x^j$$

We can also consider

$$f'(0)x + \sum_{j\geq 2} \frac{f^{(j)}(0)}{j!}t^{j-1}x^j, \qquad (16)$$ eq:lin1

which is nothing but

$$\frac{f(tx)}{t} \qquad (17)$$ eq:lin2

Notice that smoothness of the latter equation is not entirely obvious. It becomes so when we use equation 16, or even better when we write $f(x) = xg(x)$, where g is another analytic function. This is possible because we do have the division property for analytic functions.

Fortunately, the division property also holds for differentiable functions.

pro:diffdiv **Proposition 8.** *Let $f\colon B(0,r) \subset \mathbb{R}^m \to \mathbb{R}$ be a C^r-function with $f(0) = 0$. Then there exist C^r-functions $g_1, \dots, g_m$ so that*

(1) $f(x) = x_1g_1(x) + \cdots + x_mg_m(x)$
(2) $g_i(0) = \frac{\partial f}{\partial x_i}(0)$, $i = 1, \dots, m$

Proof. We apply the fundamental theorem of calculus to the restriction of f to the segment $[0, x]$.

Write

$$f(x) = \int_0^1 \frac{d}{ds}f(sx)ds = \sum_i x_i \int_0^1 \frac{\partial f}{\partial x_i}(sx)ds$$

□

Corollary 8. *Let $f\colon B(0,r) \subset \mathbb{R}^m \to \mathbb{R}^n$ be a differentiable function, $f(0) = 0$. Then*

$$\begin{aligned} H\colon B(0,r)\times I &\longrightarrow \mathbb{R}^n \\ (x,t) &\longmapsto \frac{f(tx)}{t} \end{aligned} \qquad (18)$$

is a differentiable homotopy joining f with $Df(0)$. If $Df(0)$ is invertible, the for an appropriate r' is an isotopy when restricted to $B(0,r')\times I$

Let $f\colon \nu(A) \to \nu(A)$ so that $f_{|A} = \mathrm{Id}$. We want to linearize f in the points of A and along normal directions.

Inside $\nu(A)$ we have the submanifold A, the zero section. We can try to repeat the normal bundle construction. First we consider $T\nu(A)_{|A}$, and then we have the space

$$\coprod_{a\in A} T_a\nu(A)/T_aA,$$

which the unique differential structure making the projection a differentiable map. We claim that $\coprod_{a\in A} T_a\nu(A)/T_aA$ is canonically diffeomorphic to $\nu(A)$.

Indeed, consider the injective differentiable map

eq:fibder
$$\begin{aligned} \nu(A) &\longrightarrow T\nu(A)_{|A} \\ (a,u_a) &\longmapsto (a,[su_a]), \end{aligned}$$

where $[su_a]$ is the class of the curve $s \mapsto su_a$ (which is u_a!; in trivializations this map is the identity). When composing with the projection we get a bijection linear on fibers, and hence a diffeomorphism with the normal bundle.

Let $f\colon \nu(A) \to \nu(A)$ be a differentiable map such that $f_{|A} = \mathrm{Id}$. Then we can compute the differential $Df\colon T\nu(A) \to T\nu(A)$, and restrict it to

$$Df\colon T\nu(A)_{|A} \to T\nu(A)_{|A}$$

But notice that by equation 19 $\nu(A)$ enters as a submanifold, so we can restrict the map to obtain $Df\colon \nu(A) \to T\nu(A)_{|A}$. Finally, we can project into the normal bundle which by equation 19 again can be identified with $\nu(A)$ Notice that at each $a \in A$, we have $T_aA \subset T_a\nu(A)$

def:fibder

Definition 28. *The previous map is the so called* **fiber derivative of** f

$$Df^v\colon \nu(A) \to \nu(A)$$

lem:fibder

Lemma 16. *The fiber derivative is computed by the formula*

$$\begin{aligned} Df^v\colon \nu(A) &\longrightarrow \nu(A) && (19)\\ x &\longmapsto \lim_{s\to 0}\frac{f(sx)}{s} && (20)\end{aligned}$$

Proof. We start by noticing that the expression on the r.h.s. of 19 makes sense, since we can multiply due to the vector space structure. If we go to a trivialization (remark 22) the curve $f(sx)$ is given by

$$\begin{aligned} I &\longrightarrow \mathbb{R}^a \times \mathbb{R}^{m-a} && (21)\\ s &\longmapsto (f_1(sx), f_2(sx)) && (22)\end{aligned}$$

with $f_s(0) = 0$.

To compute the fiber derivative, we keep the base point of the curve $f_1(x)$ and the second component of $(f_1'(0), f_2'(0))$, and this coincides with $\lim_{s\to 0}(f_1(sx), f_2(sx)/s) = (f_1(x), f_2'(0))$.

□

prop:isot

Proposition 9. *Let $f\colon \nu(A) \to \nu(A)$ be a differentiable map such that $f_{|A} = \mathrm{Id}$. Then f is canonically homotopic to Df^v. If f is a diffeomorphism, then there is $\mathcal{V}_A$ a neighborhood of A so that the restriction to $\mathcal{V}_A \times I$ is an isotopy.*

Proof. We use the map

$$\begin{aligned} H\colon \nu(A)\times I &\longrightarrow \nu(A)\\ (x,t) &\longmapsto \frac{f(tx)}{t}\end{aligned}$$

When we go to charts, it is the division lemma what grants regularity. Similarly, the existence of $\mathcal{U}_A$ is granted because of trivializations all maps are seen to have invertible derivative at the points of A. □

Now suppose we are given $(f, \nu(A), \mathcal{U}_A)$ a tubular neighborhood of A in M. Let $(g, \nu(A), \mathcal{U}_A)$ be another such. We would like to know whether we could isotope one into the other. We cannot quite do that, but almost. Let $\phi = g^{-1}\circ f\colon \nu(A) \to \nu(A)$. We can assume w.l.o.g. that ϕ, and the isotopy H connecting it with is fiber derivative are defined in the whole $\nu(A)$, otherwise we would shrink $\nu(A)$ into the neighborhood of the zero section in which it is defined. Then

$$g^{-1} \circ H(x, 1-t)$$

is an isotopy -through tubular neighborhoods- between f and another one so that when composed with g^{-1} it gives a linear isomorphism.

7. Degree, intersection number and Euler characteristic

7.1. **Orientations.** In a vector space V, the set of basis $\mathcal{B}(V)$ -after a choice of base point- can be identified with $GL(V,\mathbb{R})$. Therefore, it inherits a topology (a manifold structure) from the latter, because this is seen to been independent from the choice of base point. In particular $\mathcal{B}(V)$ has two connected components.

Definition 29. *Given a differentiable manifold M, an orientation of T_xM is a choice of one connected component $\mathcal{B}^+(T_xM)$ of $\mathcal{B}(T_xM)$. A basis on the chosen component is called positive.*

Definition 30. *A manifold is orientable if a locally constant orientation of all $\mathcal{B}(T_xM)$ is possible. In other words, about each x there has to be coordinates $x_1,\dots,x_m$ so that if $\{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_m}\}$ is positive (resp. negative) at x, then it is positive (resp. negative) at all points in the domain of the chart.*

An orientation of M is such a choice of positive basis.

Take a map $\gamma\colon (I,\partial I)\to(M,x)$ and choose a cover $U_1,\dots,U_s$ so that I can be broken into consecutive intervals $I_1,\dots,I_s$ with $\gamma(I_j)\subset U_j$.

Definition 31. *We say that γ reverses orientations if*

$$\varphi_{s,s_1}\circ\cdots\circ\varphi_{1,2}\left\{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_m}\right\}\in\mathcal{B}(T_xM)$$

belongs to the same connected component as $\{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_m}\}$. We say that is reverses orientations otherwise.

One checks that the previous definition does not depend on the cover by going to a common refinement of two given ones. For a given curve γ and a cover as above, there exists $\mathcal{N}\in C^0((I,\partial I),(M,x))$ so that for each $\gamma'\in\mathcal{N}$, one also has $\gamma'_{|I_j}\subset U_j$. Therefore γ' preserves the orientation iff γ does it. As a consequence reversing or preserving the orientation is a property of the homotopy class of the curve.

Hence we can define a map

$$\pi_1(M,x)\to\mathbb{Z}_2, \tag{23}$$ eq:orientationcurve

for which $[\gamma]$ is sent to 1 if preserves orientations, and to 0 otherwise. This is clearly a group homomorphism, and its kernel is a normal subgroup of π_1. Therefore, it has a well defined $2:1$ covering space M^{or}.

Corollary 9. *M is orientable iff all homotopy class -at x say- preserve the orientation, or equivalently if the kernel of equation 23 is trivial. This is equivalent to saying that M^{or} is the trivial cover, i.e. just two copies of M.*

M^{or} is always orientable.

Corollary 10. *M is orientable iff there exists an atlas $(U_i,\varphi_i)_{i\in I}$ for which the determinants of the jacobian of all change of coordinates are positive.*

lem:contbasis **Lemma 17.** *If M is orientable and oriented, for any path $\gamma\colon I\to M$ -since the two connected componets of the general linear group are connected- we can always choose basis $b(t)\in\mathcal{B}(T_{\gamma(t)}M)$ varying continuously (in charts the matrix of $b(t)$ w.r.t. the canonical basis have continuous entries). Then $b(0)$ is positive (resp. negative) iff $b(1)$ is positive (negative).*

Definition 32.

- *Let M an oriented manifold. Then ∂M is orientable and carries a canonical orientation by the rule "outward normal first".*
- *Let M,N oriented manifolds. Then the product M,N carries an obvious product orientation.*

7.2. **The degree of a map.** In this subsection we consider M, N compact oriented differentiable manifolds of dimension m. Let $f \in C^r(M,N)$ and z a regular value for f. Then $f^{-1}(z)$ is a finite collection of points $x_1, \dots, x_s$. We define d_i **the local degree** of f at x_i to be 1 if $Df(x_i)$ preserves the orientation and -1 otherwise.

def:localdeg **Definition 33.** *The degree of f at z is the sum of the local degrees at x_i*

$$\deg(f,z) = \sum_i (-1)^{d_i}$$

We want to prove that the degree does not depend neither on the regular value, nor on the homotopy class of f. To do that we show

lem:welldef **Lemma 18.** *Let $f, g \in C^r(M,N)$ be homotopic maps so that z is regular value for both, then*

$$\deg(f,z) = \deg(g,z)$$

Proof. We put on $I \times M$ the product orientation of $-I$ and M, so that

$$\{0\} \times M = M,\ \{1\} \times M = -M,$$

as oriented manifolds.

By corollary 7 we may assume the homotopy H to be differentiable (at least C^2), and we can further approximate it by a map relative to a collar of the boundary with is transverse to z. Since $H_{|\partial(I\times M)}$ is already transversal to z, we conclude that $H^{-1}(y)$ is a 1-dimensional submanifold of $I \times M$. Hence it will have l disjoint copies of S^1, and k copies of I. Take one such interval $\gamma\colon I \hookrightarrow I \times M$. Obviously, the endpoints of the interval are sent to $H^{-1}(y)$. We claim that for both end points the local degrees -w.r.t. the boundary orientation- have opposite sign. Or in other words

$$0 = \deg(H_{|\partial(M\times I)}, z) = \deg(f,z) - \deg(g,z)$$

Cover $\gamma(I)$ by adapted coordinates, so that $\frac{\partial}{\partial x_1}$ is tangent to $\gamma(I)$, and for $i(0)$ is outward pointing. It is clear that $\dot\gamma(t)$ is a negative multiple of $\frac{\partial}{\partial x_1}$. Again using adapted charts we can choose a continuously varying basis of the form $\{-\dot\gamma(t), b(t)\} \in \mathcal{B}(T_{\gamma(t)}M)$, where $b(t)$ can be seen as a base of the normal space. Assume $b(0)$ is possitive for the boundary orientation and the local degree is possitive. That implies that $dH_{\gamma(t)}(b(t))$ must be a positive basis, and in particular the image of $b(1)$. But by lemma 17 $\{\dot\gamma(1), b(1)\} \in \mathcal{B}^+(T_{\gamma(1)}M)$, which is the same as saying that $b(1)$ is a negative basis for the induced orientation of the boundary, so the local degree at $\gamma(1)$ is negative.

Therefore the result follows. □

Definition 34. *Let $f \in C^0(M,N)$, N connected, Then* $\deg f$ *is defined to be the degree for any homotopic differentiable map at any regular point.*

By corollary 7 and the previous lemma, it does not matter which homotopic differentiable map we use, as long as we prove independence on the regular point. To prove the latter we need an isotopy $\Psi\colon N \times I \to N$ starting at the identity, and sending any given w to any given z. Because then we form the homotopy $H_t = \Psi_t \circ f$, and clearly

$$\deg(f,w) = \deg(H_1, z)$$

To prove the existence of the latter isotopy it is enough to do it on Euclidean space, because then in general we can compose a finite number of the local ones to reach the desired point.

For the local case we use the theory of O.D.E. Given any autonomous C^r-system

$$\dot x(t) = Y(x(t)),$$

if the entries of the matrix $Y(x)$ are compactly supported, then a solution exists for all time giving a C^{r+1}-flow (i.e. isotopy starting at the identity)

$$\Psi\colon \mathbb{R}^m \times I \to \mathbb{R}^m$$

For any embedded curve $i\colon I \to \mathbb{R}^m$, it is possible to find $Y\colon \mathbb{R}^m \to \mathbb{R}^m$ so that

$$Y(i(t)) = \frac{d}{dt} i(t),$$

and this solves the problem.

Remark 26. *Notice that the compactness of N was not used anywhere. It is just that if N is not compact, f cannot be onto and the degree is then zero.*

rem:contdeg **Remark 27.** *Another way to say things is that we have a continuous map*

$$\deg\colon C^0(M,N) \to \mathbb{Z}$$

This is because about any point we have a small neighborhood which is path connected (lemma 14) and hence homotopy invariance is equivalent to continuity.

ex:circle **Example 4.** *For a map $f\colon S^1 \to S^1$ its* **degree or winding number** *is defined by lifting it to a map $F\colon [0,1]$ and taking $F(1) - F(0)$. It is easy to see that it is our degree Since it is a homotopy invariant, and any map is homotopic to $e^{2\pi i k}$, then it is also our degree.*

ex:spheres **Example 5.** *Let $P\colon \mathbb{C} \to \mathbb{C}$ be a polynomial. Then it continuosly extends to a map $\bar{P}\colon S^2 \to S^2$, where the spheres are oriented by the orientation (any) of the plane. A value (different from zero) z is regular iff the zeroes of $P - z$ are simple. Since the map is holomorphic, local degrees are always one, so $\deg\bar{P}$ is the degree of the polynomial.*

When N is a sphere, the degree classifies path connected components of $C^0(M,S^m)$.

thm:degree **Theorem 22.** *$f, g \in C^0(M,S^m)$, M connected, are homotopic iff $\deg(f) = \deg(g)$.*

Proof. We just need to prove the existence of a homotopy when the degrees are equal. In other words, we need to prove the existence of a map $H\colon M \times I \to S^m$ so that $H_{|M\times\{0\}} = f$, $H_{|M\times\{1\}} = g$.

Since the degree of $H_{|\partial(M\times I)}$ is zero, for a regular value z being $M \times I$ connected we can join the points of in the inverse image of z by arcs $\gamma_i\colon I \to M \times I$ so that the end points are points of opposite local degree. If $m > 1$, then we can assume the maps γ_i to be embeddings. If M is the circle, we can arrange the crossings to be transversal, and change the arcs to avoid intersections.

Let us fix one of them γ, and consider its normal bundle $\nu(I)$ together with a tubular neighborhood $k\colon \nu(I) \to \mathcal{U}_I$

In $T(M \times I)_{|I}$ choose a smooth basis $\{\dot\gamma(t), v(t)\}$, so that $v(0), v(1)$ are made of vectors tangent to the boundary. One can see that a model for $\nu(I)$ is given by $I \times \mathbb{R}^{m1}$ via the map

$$\begin{aligned} \Phi\colon I \times \mathbb{R}^m &\longrightarrow \nu(I) \\ (t, u_1, \ldots, u_m) &\longmapsto (\gamma(t), [u_1 v_1(t), \ldots, u_m v_m(t)]), \end{aligned}$$

which is differentiable because it lifts (in the obvious way) to $TM_{|I}$.

Therefore, we have a diffeomorphism $\Theta\colon I \times \mathbb{R}^m \to \mathcal{U}_I$.

We will extend the map to $\mathcal{U}_I$. We have two maps

$$F = \psi_z \circ f \circ \Theta_0,\; G = \psi_z \circ g \circ \Theta_0\colon \mathbb{R}^m \circ V \to \mathbb{R}^m,$$

where φ_z are local coordinates about z. We can apply lemma 15 to both maps to find an isotopy joining each map with its differential at the origin (perhaps reducing the domain to $U' \subset \mathbb{R}^{m-1}$, a closed ball). Since the end points have opposite sign, the jacobian of both differentials has the same sign, therefore we can join them. The result is a smooth isotopy $\Psi\colon U' \times I$ joining F, G.

We now extend H to $\mathcal{U}'_I := \Theta(U' \times I)$ as follows:

$$\begin{array}{rcl} H\colon \mathcal{U}'_I & \longrightarrow & S^m \\ x & \longmapsto & \psi_z^{-1} \circ \Psi \circ \Theta^{-1}(x) \end{array}$$

We do the same for all intervals (the tubular neighborhoods do not intersect, if we choose them to be a tubular neighborhood $\mathcal{U}$ of the union of intervals), so we get a map

$$H\colon \partial(M \times I) \cup \mathcal{U}' \to V_y \subset S^m$$

Obviously $M \times I \backslash H^{-1}(z)$ is a manifold and hence a normal topological space. In that space we have a map

$$H\colon \partial(M \times I) \cup \mathcal{U}' \backslash H^{-1}(z) \to S^m \backslash \{z\}$$

Since the domain is closed and the target homeomorphic (diffeomorphic) to $\mathbb{R}^m$, by Tietze's extension theorem it extends to $M \times I \backslash H^{-1}(z)$. Clearly, this defines the desired homotopy. □

Remark 28. *When we do not have orientability we have a $\mathbb{Z}_2$-valued degree and a result analogous to theorem 22. This is because a compact 1-dimensional manifold has* 0 (mod2) *boundary points.*

7.3. **Intersection number and Euler characteristic.** Let M be compact oriented manifold, N an oriented manifold and $A \subset N$ a closed oriented submanifold so that $n = m + a$. Let $f\colon M \to N$ a map transverse to A.

Definition 35. *The local intersection number of (f, A) at x_i -denoted $\#(f, A, x_i)$ is ± 1 if $\{f_* b(x_i), v(f(x_i))\}$ belongs to $\mathcal{B}^{\pm}(T_{f(x_i)}N)$, where $b(x_i), v(f(x_i))$ are positive basis of $T_{x_i}M, T_{f(x_i)}A$ respectively.*

The intersection number $\#(f, A; N)$ is the sum of local intersection numbers.

Lemma 18, together with the ideas showing the independence of the degree on the regular value, gives:

Proposition 10. *The intersection number does not depend on the homotopy class of f, nor on the isotopy class of A, meaning that we can substitute A by $\Psi_1(A)$, where Ψ is an isotopy of N starting at the identity (see theorem 11).*

Remark 29. *When f is an embedding, we speak about the intersection number of the submanifolds $(f(M), A)$.*

Remark 30. *The degree* $\deg(f, z)$ *is recovered as*

$$\#(\mathrm{graph} f, M \times \{z\}; M \times N)$$

with the obvious orientations.

rem:doubleor **Remark 31.** *When N has even dimension and A middle dimension, we can compute the intersection number $\#(A, A)$, which is independent of the orientation chosen on A.*

Let M be an oriented compact manifold. Then the product $M \times M$ carries a canonical orientation. It contains Δ, which is a copy of M and hence oriented.

Definition 36. *the Euler characteristic of M is*

$$\chi(M) = \#(\Delta, \Delta; M \times M)$$

Observe that it is independent of the chosen orientation.

Remark 32. *The Euler characteristic is well defined also for non-orientable manifolds. Firstly observe that at the points of Δ the tangent space of $M \times M$ is canonically oriented (take to copies of any basis on T_xM. Secondly to compute a local degree we need a local orientation of $T_{(x,x)}\Delta$, which comes from such local orientation of T_xM. But as noticed in remark 31 a change of the local orientation does the same for the deformed diagonal, so the local degree does not change.*

def:lefschetz **Definition 37.** *Given M a compact manifold and $f\colon M \to M$ the* **Lefschetz number** *is*

$$L(f) = \#(\text{graph} f, \Delta; M \times M)$$

It is a signed count of the fixed points. In particular $\chi(M) = L(\text{id})$.

So again we may regard the Lefschetz number as a continuous map

$$L\colon C^0(M,M) \to \mathbb{Z}$$

Notice that $L \neq \deg$, because $\chi(M) = L(\text{id}) \neq 1$

7.4. **Vector fields.** Let M be a differentiable manifold. Recall from definition 14 that a vector field $X \in \mathfrak{M}$ is a differentiable function

$$X\colon M \to TM$$

so that $\pi \circ X = \text{Id}$ (i.e. it is a **section** of the tangent bundle).

As we mentioned the diagonal embedding $M \to M \times M$ has Δ as image. Either of the projections, say $p_1\colon M \times M \to M$, induces a diffeomorphism from $\nu(\Delta)$ into TM, so we get tubular neighborhoods of the form $f\colon TM \to \mathcal{U}_\Delta \subset M \times M$.

A zero of a vector field is an intersection of $X(M)$ with the $\mathbf{0}$ section of TM. The zero is called transversal in the intersection is so.

Notice that any vector field is homotopic to the zero section by the rescaling homotopy tX. In particular, if a vector field X is transverse to the zero section, then the map

$$f \circ X\colon M \to M \times M$$

is homotopic to the diagonal and transverse to it. Therefore we conclude.

thm:zeroes **Theorem 23.** *If a vector field has only transverse zeros, then its signed count equals $\chi(M)$. In particular it does not depend on the vector field.*

Notice that we can compute $\chi(S^m)$ as follows. We consider a vector field emanating from the north pole to the south pole, going along geodesics. After rescaling and around the north pole, a model for it is the Euler vector field, whose local degree at the origin is 1. For the other zero a model is minus the Euler vector field (recall the local orientation of the sphere does not matter), and therefore the degree is $(-1)^m$. Hence, we conclude

thm:zeroesvfield **Theorem 24.**

(1) *Any vector field on an even sphere must have zeros.*
(2) *Any vector field in the upper hemisphere $E^m_+ \subset S^m$ transverse to the equator must have a zero in the interior (and this is nothing but the non-existence of retractions from balls onto their boundaries).*

Proof. For the second statement we take a collar of the equator and use bump functions to deform the given vector field into the one above computing the Euler characteristic (after averaging we orthogonally project form Euclidean space to the sphere). Then the result follows. □

8. Isotopies and gluings and Morse theory

Recall that a vector field $X \in \mathfrak{X}(M)$ is a **section** $X\colon M \to TM$.

lem:isotvfield

Lemma 19. *Any vector field X with compact support can be uniquely assigned a map*

$$\Psi_t^X\colon M \times \mathbb{R} \to M$$

such that

- $\frac{d}{dt}\Psi_t^X(x) = X(\Psi_t^X(x))$
- $\Psi_0 = \mathrm{Id}$

Moreover, such an isotopy satisfies $\Psi_{t+s} = \Psi_t \circ \Psi_s$. Conversely for such an isotopy (perhaps not defined on the whole real line) we get such a vector field, and this establishes a 1 to 1 correspondence.

Proof. Locally the flow is the one associated to an autonomous equation, so locally for small time the flow exists with the required properties. Since the support is compact the result follows.

The converse is also clear. □

More generally, differentiable 1-parameter family of vector fields is a differentiable map
$X\colon M \times I \to TM$, so that each X_t is a vector field

thm:isotvfield

Theorem 25. *There is a 1 to 1 correspondence between compactly supported isotopies and compactly supported vector fields.*

Proof. The assignment is as follows. Let $\Psi\colon M \times I \to M$ be an isotopy. Since it is compactly supported we can assume it is defined on $M \times (-\epsilon, 1+\epsilon)$ (see remark 25). Moreover, for each t we still have a diffeomorphism. Define

$$\begin{array}{rcl}\tilde{\Psi}\colon M \times I & \longrightarrow & M \times I \\ (x,t) & \longmapsto & (\Psi(x,t),t)\end{array}$$

which is in the hypothesis of lemma 19. Therefore it is in correspondence with a vector field $\tilde{X} = X(x,t) + \frac{\partial}{\partial t}$, and hence with $X(x,t) = X_t(x)$. One checks that the equation

$$X_t(\Psi_t(x)) = \frac{d}{ds}\Psi_s(x)_{|s=t}$$

holds (characterizes the correspondence). To go in the other direction we construct $\tilde{X}$ out of $X_t(x)$, and define $\Psi = p_1 \circ \tilde{\Psi}$. □

Theorem 25 is an extremely powerful tool in differential topology

thm:isotext

Theorem 26. *Let $W \subset M$ open and $\Psi\colon W \times I \to M$. Suppose that we have another open subset $U \subset W$ such that $\Psi^{-1}(U) \subset W$. Then for any compact subset $K \subset U$ there exists an isotopy $\Psi'\colon M \times I \to M$ with $\Psi'_{|K\times I} = \Psi_{|K\times I}$.*

Proof. Since $\Psi^{-1}(U) \subset W$, we can define $X \in \mathfrak{X}(U \times I)$. Next we use a partition of the unity to extend it to $X' \in \mathfrak{X}(M \times I)$ so that

$$X'_{\Psi^{-1}(K)} = X_{\Psi^{-1}(K)},\ X'_{|(M\setminus W)\times I} \equiv 0,$$

and observe that $\Psi^{-1}(K)$ is compact. □

cor:subisot

Corollary 11. *Let $\psi\colon A \times I \to M$ be a differentiable map such that for each t is an embedding, A compact $\psi_0 = \mathrm{Id}$. Then it extends to an isotopy of $\Psi\colon M \times I \to M$.*

Proof. Over the submanifold $\psi(A \times I) \subset M \times I$ the isotopy gives a vector field whose projection is $\frac{\partial}{\partial t}$. Then one can easily extend it to a vector field on $M \times I$ supported on an tubular neighborhood, and then correct by adding the appropriate multiple $\frac{\partial}{\partial t}$. □

8.1. **Gluings.** A basic strategy in the study of manifolds, is breaking them into simple pieces, or rather thinking of them as the result of gluing these, and then deduce consequences out of this.

def:gluing

Definition 38. *Let M, N be differentiable manifolds with non-empty connected boundary, and let $f\colon \partial M \to \partial N$ be a diffeomorphism. Define the topological space*

$$M\#_f N = M \coprod N / x \backsim f(x)$$

$M\#_f N$ carries a canonical topological structure for which the inclusions of M, N are homeomorphisms. Putting a differentiable structure is slightly more subtle, and as a matter of fact only exists up to diffeomorphism.

Lemma 20. *$M\#_f N$ carries a canonical differentiable structure up to diffeomorphism for which the inclusions of M, N are differentiable.*

Proof. We will only prove it in the case of compact boundaries. Choose collars so that we have

$$M'_g = M \coprod_g \partial M \times \mathbb{R},\ N'_\alpha = N \coprod_\alpha \partial N \times \mathbb{R}$$

We glue the collar via the diffeomorphism

$$(x,t) \to (f(x), -t)$$

We must prove that for other collars

$$M'_h = M \coprod_h \partial M \times \mathbb{R},\ N'_\beta = N \coprod_\beta \partial N \times \mathbb{R}$$

we get diffeomorphic manifolds.

Clearly, we can assume only one of the collars differ, because then we would take the composition

$$M'_g \coprod N_\alpha \to M'_g \coprod N_\beta \to M'_h \coprod N'_\beta$$

Let us suppose $\alpha = \beta$. Next observe that it suffices to find a diffeomorphism

$$M \coprod_g \partial M \times \mathbb{R} \to M \coprod_h \partial M \times \mathbb{R}$$

which in the triviallized normal bundle is the identity for $t \gg 1$, because then we can extend it by the identity in $N \backslash \mathcal{V}_{\partial N}$.

We want to start defining the diffeomorphisn $M'_g \to M'_h$ to be the identity on M. That means that a point in $x \in \partial M \times \mathbb{R} \subset M'_g$ is sent to $h^{-1} \circ g(x) \in \partial M \times \mathbb{R} \subset M'_h$. We want to modify this map near the boundary so when read in the trivialized normal bundles it becomes the identity near the zero section, rather than $h^{-1} \circ g(x)$. Notice first that for the composition $h^{-1} \circ g(x)$ to be possible we must have $\mathcal{U}_{\partial M,g} \subset \mathcal{U}_{\partial M,h}$. This can be assumed w.l.o.g. because otherwise we can find a diffeomorphism of $g^{-1}(\mathcal{U}_{\partial M,h})$ into the trivialized normal bundle with does not alter points with positive t-coordinate.

By remark 25, we can extend $h^{-1} \circ g$ to a map defined also in a neighborhood of the zero section. Since its differential at each point of the zero section is invertible, by shrinking the neighborhood -for points of positive t-coordinate- we can assume we have a diffeomorphism. Next we can find another self diffeomorphism of $\partial M \times \mathbb{R}$ with the following properties.

- It preserves fibers, an it is supported is an arbitrary neighborhood off the zero section.
- It is such that the differential of $h^{-1} \circ g$ at the zero section is the identity.

We now apply proposition 9 to the restriction of $h^{-1} \circ g$ to a small tubular neighborhood of the zero section in $\partial M \times \mathbb{R}$. We get H_t an isotopy so that

$$H_1 \circ h^{-1} \circ g = \mathrm{Id}$$

Now we use compactness to extend it to an isotopy H_t of $\partial M \times \mathbb{R}$ which is the identity for $|t| \gg 1$. We modify (and extend) the previous map by considering instead

$$H_1 \circ h^{-1} \circ g \colon \partial M \times \mathbb{R} \subset M'_g \to \partial M \times \mathbb{R} \subset M'_h$$

Observe that points whose t-coordinate is very negative are not in the support of H_1, and therefore the definition coincides with the one sending each point of $M \subset M'_g$ to the same one in M'_h. Hence, we have defined an isomorphism, which it is also the identity for $t \gg 1$, and this finishes the proof. □

def:double

Definition 39. *Let M be manifold with non-empty boundary. Then the double is $M \#_{\mathrm{Id}} M$.*

cor:reflection

Corollary 12. *Any manifold with boundary is the quotient of space of a manifold without boundary endowed with a $\mathbb{Z}_2$-action.*

Proof. Just remind that an action of a (Lie) group on a differentiable manifold M is a differentiable map $G \times M \to M$ satisfying the usual axioms.

In our case, the double $M \#_{\mathrm{Id}} M$ carries an obvious $\mathbb{Z}_2$-action so that the quotient map is differentiable. □

Lemma 21. *If $f, g \colon \partial M \to \partial N$ are isotopic, then $M \#_f N$ and $M \#_g N$ are diffeomorphic.*

Proof. We take a collars, so that a neighborhood of ∂N in N is of the form $\partial N \times [0, -\infty)$. Then we define $\psi \colon M \#_f N$ and $M \#_g N$ as follows:

$$\psi_{|M \cup (N \setminus \partial N \times [0,-\infty))} = \mathrm{Id}$$

Let H be the homotopy between $f^{-1} \circ g$ and the identity, and $\chi \colon [0,1] \to [0, \infty]$ a differentiable monotone function with $\chi_{|[0,\epsilon]} = 0, \chi_{|[1-\epsilon,\infty)} = 1$

Then on the collar we define

$$\psi(x,t) = (H(\chi(-t), x), t)$$

and this completes the proof. □

ex:functcobor

Example 6. *Let M with $\partial M = \emptyset$, and $f \colon M \to \mathbb{R}$ a (proper) function. Let z be a regular value, so $W_z := f^{-1}(z)$ is a hypersurface. Then $M = f^{-1}(-\infty, z] \#_{\mathrm{Id}} f^{-1}[z, \infty)$ (up to diffeomorphism).*

8.2. **Morse functions.** We saw in example 6 how to use regular values of functions to break a manifold into blocks. If the function has additional properties, we can say more about how certain blocks look like.

Let $f \colon M \to \mathbb{R}$ be a differentiable function. We can define ∇f **the gradient (vector field) of** f (w.r.t. the metric induced by the Euclidean one) as follows:

extend f to a function on a collar by pulling back using the retraction, compute the usual gradient, and then orthogonally project onto TM. If is clear that critical points of f coincide with those of the extension, and also with zeros of the gradient.

If z is regular value, then $W_z := f^{-1}(z)$ is a hypersurface transverse to ∇f.

Lemma 22. *If the interval* (z_0, z_1) *contains no critical value then for any* z *in the interval*

$$M_{(z_0,z_1)} := f^{-1}(z_0, z_1) \simeq W_z \times (z_0, z_1)$$

Proof. Define the isomorphism $x \mapsto (y, f(x))$, where y is the point in W_z in the same integral curve of ∇f as x. □

Define $M_{\leq z} := f^{-1}((-\infty, z])$.

Corollary 13. *Suppose that the interval* $[z_0, z_1]$ *contains no critical value. Then there exist an isotopy* $\Psi\colon M_{\leq z_1} \times I \to M_{\leq z_1}$ *starting at the identity, and pushing the closed manifold down (w.r.t. values of* f*) along gradient lines so that* $\Psi_1(M_{\leq z_1}) = M_{\leq z_0}$.

Therefore, it is the critical points of a function what produces changes in the diffeomorphism type, and therefore we would like to have functions whose behavior around critical points is easy to describe.

Recall that the cotangent bundle of a manifold M is

$$T^*M = \coprod_{x \in M} T_x^*M$$

and has a vector bundle structure. Given a cover (U_i, φ), the tangent bundle is given by

$$\coprod \varphi(U_i) \times \mathbb{R}^m / \sim$$

where if $x \in U_i \cap U_j$, we identify

$$(\varphi_j(x), u) \sim (\varphi_i(x), D\varphi_{ij}(\varphi_j(x))u)$$

The cotangent bundle is then

$$\coprod \varphi(U_i) \times \mathbb{R}^{m*} / \sim$$

where we identify

$$(\varphi_j(x), \alpha) \sim (\varphi_i(x), D\varphi_{ij}(\varphi_j(x))^{-1^*} u)$$

Given $f \in C^r(M, \mathbb{R})$, $r \geq 1$, then $Df = df$ defines a section of T^*M. In coordinates $x_1, \ldots, x_n$, denote by $dx_1, \ldots, dx_m$ the basis dual to $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}$.

Then

$$df = \sum_i \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_m} dx_m$$

Observe then that $x \in M$ is a critical point of f iff the graph of df hits the zero section and that point.

Remark 33. *Let* g *be a metric in* TM*. The gradient w.r.t.* g *can be defined as follows: the metric induces a map*

$$\begin{array}{rcl} g^{\#}\colon TM & \longrightarrow & T^*M \\ v & \longmapsto & g(v, \cdot) \end{array}$$

linear on fibers. Non-degeneracy of g *is equivalent to* $g^{\#}$ *being an isomorphism on fibers.*

Then we define $\nabla_g f = g^{\#-1} df$.

Definition 40. $f \in C^r(M, \mathbb{R})$, $r \geq 2$ *is a* **Morse function** *if (the graph of)* df *intersects the zero section of* T^*M *transversely.*

Recall that for $f\colon \mathbb{R}^m \to \mathbb{R}$ of class C^2, **the Hessian** Hf is the symmetric matrix

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j}$$

Lemma 23. *df has transverse intersection with the zero section at x iff in some (and hence in any) coordinates $Hf(x)$ is non-degenerate.*

Proof. In coordinates we get a trivialization of the form $U \times \mathbb{R}^m$, with the zero section $U \times \{0\}$. The transversality at x is equivalent to the projection of $T_x df$ onto $\mathbb{R}^m \times \{0\}$ being surjective. A basis of $T_x df$ is given by

$$\left\{ \frac{\partial}{\partial x_j} + \sum_i \frac{\partial}{\partial x_j} (\frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_m} dx_m) \right\}_{j=1,\ldots,m},$$

and hence the result follows. □

How abundant are Morse functions? Since they are defined by a transversality property, one expects then to be a residual subset of $C^r(M, \mathbb{R})$.

thm:densemorse **Theorem 27.** *Morse functions are and open residual subset of $C^r_S(M, \mathbb{R})$.*

Proof. Openness is clear, because transversality of df to the zero section is a C^2-condition. On a compact on the domain of local coordinates, C^2-close function to a Morse one will be Morse.

Regarding density we only prove the local result. We may assume f to be smooth as well. If $f\colon U \subset \mathbb{R}^m \to \mathbb{R}$, then consider the map

$$\begin{aligned} F\colon \mathbb{R}^m &\longrightarrow C^\infty(U, TU) \\ y &\longmapsto df(x) + \langle x, y\rangle = d(f + \langle x, y \rangle) \end{aligned}$$

Then since F^{ev} is a submersion, we deduce the existence of a dense subset of parameters for which transversality holds. □

A Morse function $f\colon U \subset \mathbb{R}^m \to \mathbb{R}$ with a critical point at zero has a Taylor expansion starting with a non-degenerate quadratic form. The following impressive result says that in some coordinates there are no higher terms.

thm:morsecoord **Theorem 28.** *Let $f\colon U \subset \mathbb{R}^m \to \mathbb{R}$ be a Morse function and zero a critical point. Then there exist coordinates $w_1, \ldots, w_m$ centered at zero so that*

$$f(w) = f(0) + \sum_{i=1}^{\lambda} -w_i^2 + \sum_{i=\lambda+1}^{m} w_i^2$$

where λ is the signature of the Hessian.

Proof. By applying the division property twice (proposition 8), we can write

$$f(x) - f(0) = x^t G x,$$

where $G(0)$ is the Hessian $Hf(0)$.

In one variable we would have

$$f(t) - f(0) = t^2 G(t) = (t\sqrt{G(t)})^2,$$

and $t\sqrt{G(t)}$ is a coordinate about zero because its derivative is $\sqrt{G(0)}$.

In general apply induction, by fixing $x_1, \ldots, x_{m-1}$, and then applying the result to all lines orthogonal to the hyperplane $x_m = 0$. First, a linear change of coordinates is required so that $\frac{\partial^2 f}{\partial x^m \partial x^m}(0) \neq 0$ Then we get a new coordinate w_m so that

$$f(x_1, \ldots, x_{m-1}, w_m) - f(0) = \pm w_m^2 + f(x_1, \ldots, x_{m-1}, 0)$$

□

Remark 34. *So Morse's lemma is a result about the possibility of "taking squares roots" of certain smooth functions. Notice as well that since taking the square root is an analytic operation the lemma holds also in $C^\omega(M)$.*

We will assume that M is compact from now on. We can assume w.l.o.g. that our Morse functions on theorem 27 take different values on critical points. For example use the Morse coordinates of theorem 28, to check that $f(w) + a\mu(x)$, where μ is a bump function and a tends to zero, has only zero as critical point, but the critical value changes to a.

Let us fix Morse coordinates $x_1, \ldots, x_m$ about a critical point of f with index λ, so that

$$f(x) = \sum_{i=1}^{\lambda} -x_i^2 + \sum_{i=\lambda+1}^{m} x_i^2$$

We will assume that in Morse coordinates the metric in the Euclidean one, so the gradient of f is the usual one

$$\nabla f = \sum_{i=1}^{\lambda} -x_i \frac{\partial}{\partial x_i} + \sum_{i=\lambda+1}^{m} x_i \frac{\partial}{\partial x_i}$$

The **stable manifold** $W^{\mathrm{s}}(0)$ (resp. **unstable manifold** $W^{\mathrm{u}}(0)$) to the union of the trajectories of ∇f converging to 0 as t goes to infinity (resp. minus infinity).

It is clear (assuming f to be defined in $\mathbb{R}^m$) that

$$W^{\mathrm{s}} = \mathbb{R}^\lambda \times \{0\},\ W^{\mathrm{u}} = \{0\} \times \mathbb{R}^{m-\lambda}$$

By changing f to $-f$, we exchange stability into unstability. We will only define concept for the former, being those for the latter obtained by applying the definition to $-f$.

rem:noeuler **Remark 35.** *On Morse coordinates the degree of ∇f at the origin is $(-1)^\lambda$. Therefore if c_λ is the number of critical points of index λ, we conclude*

$$\chi(M) = \sum_{\lambda=0}^{m} (-1)^\lambda c_\lambda$$

If we now do the same for $-f$ we get

$$\chi(M) = \sum_{\lambda=0}^{m} (-1)^{m-\lambda} c_\lambda$$

and adding up

$$2\chi(M) = \sum_{\lambda=0}^{m} ((-1)^\lambda + (-1)^{m-\lambda}) c_\lambda$$

and one concludes that if m is odd then

$$\chi(M) = 0$$

Define for $z < 0$, $W^{\mathrm{s}}_z = W^{\mathrm{s}} \cap M_{\geq z}$, which is the ball $B^\lambda(0, \sqrt{z})$. Then

$$S^{\lambda-1}_{\sqrt{z}} := \partial B^{m-\lambda}(0, \sqrt{z})$$

The flow of the gradient vector field pushes stables spheres into stable spheres.

For each $z < 0 < \epsilon$ consider

$$M_{z,\epsilon} := \{x \in \mathbb{R}^m \mid f(x) \leq z\} \cup \{x \in \mathbb{R}^m \mid \sum_{i=\lambda+1}^{m} x_i^2 \leq \epsilon\}$$

The boundary of $M_{z,\epsilon}$ is made of pieces of the hypersurfaces W_z and $\sum_{i=\lambda+1}^m x_i^2 = \epsilon$ -which are transversal to the gradient- and its flow induces a homotopy

$$\Psi\colon M_{-z} \times I \to M_{-z}$$

so that $\Psi_1(M_{-z}) = M_{z,\epsilon}$.

In some sense the information carried by $M_{z,\epsilon}$ "should be the same" as the one carried by M_{-z}. The problem is that the former is not a manifold.

Recall that a manifold with corners was defined by having charts modeled on

$$H^m_{d,\mathrm{cx}} := \mathbb{R}^{m-d} \times (-\infty, 0]^d$$

We will call that for the moment manifold with convex corners (see remark 39). We similarly define a **manifold with concave corners** if the charts take values on

$$H^m_{d,\mathrm{cv}} := \mathbb{R}^m \backslash \mathrm{int} H^m_{d,\mathrm{cx}}$$

Remark 36. *Notice that $H^m_{d,\mathrm{cx}}$ determines a positive cone in $\mathbb{R}^*m$ consisting on forms which are positive on it. By duality, we have positive vector, which are positive combinations in any (positive) basis of vectors in the positive half hyperplanes. If there exists a linear isomorphism interchanging $H^m_{d,\mathrm{cx}}$ with $H^m_{d,\mathrm{cv}}$ in particular it exchanges positive half hyperplanes, sending thus a positive basis into a positive basis. Therefore, it fixes $H^m_{d,\mathrm{cx}}$, so we conclude that there is no diffeomorphism from the convex to the concave model (see again remark 39).*

Remark 37. *The convex model appears when taking products of manifolds with boundary. We will see latter an appearance of the concave (see also lemma 24).*

lem:mcorners **Lemma 24.** *Let $f, g\colon U \subset \mathbb{R}^m \to \mathbb{R}$ be differentiable maps such that zero is a regular value for (f, g). Then the subset*

$$E = \{x \in \mathbb{R}^m \mid f(x) \leq 0 \text{ or } g(x) \leq 0\}$$

carries a canonical structure of manifold with concave corners.

Proof. Call $A = \{f = g = 0\}$, which by hypothesis is a codimension 2 submanifold. About each $a \in A$, take adapted charts $x_1, \ldots, x_{m-2}, x_{m-1}, x_m$. Then the coordinates $x_1, \ldots, x_{m-2}, x_{m-1}, f, g$ locally send E onto an open subset of $H^m_{2,\mathrm{cv}}$. □

Remark 38. *If we define E as an intersection rather than as a union, we get a manifold with convex corners.*

As a consequence of lemma 24 we conclude that $M_{z,\epsilon}$ is a manifold with concave corners.

A standard folklore theorem in differential topology says that "corners can be smoothenned".

thm:smoothing **Theorem 29.** *Let M be a manifold with concave corners. Assume that the boundary is compact. Then there exists a canonical manifold with boundary M' -up to diffeomorphism- such that M embeds in M'.*

Sketch of the proof. We will only proof the case of codimension two corners.

We start by looking at the codimension one part of the boundary, which is a codimension 2 submanifold A. In the general case is a submanifold with corners.

Step 1: (Regular) submanifolds in the boundary of manifolds with corners also have tubular neighborhoods. For each $a \in A$, inside $\nu_a(A)$ -which is a vector space- we have the subset $M_a(A)$ of vector fields that represent curves in M. There exists a linear isomorphism

$$(\nu_a(A), M_a(A) \simeq (\mathbb{R}^d, H^d_{d,\mathrm{cv}})$$

It is clear -going for example to charts a centered at a- that $M(A) \subset \nu(A)$ is a manifold with corners.

More precisely, the claim is the existence of an embedding

$$f\colon (M(A), A) \to (M, A)$$

extending the identity on A. To show this we observe that using the embedding $M \subset \mathbb{R}^n$, we have a map

$$g\colon \mathcal{V}_A \subset \nu(A) \to M$$

which is a diffeomorphism over its image, a neighborhood $\mathcal{U}_A$ of A in M. It is straightforward that $\mathcal{V}_A$ is itself a manifold with concave corners so that $\nu_a(A) \cap \mathcal{V}_A$ is a manifold with concave corners inside $\nu_a(A)$ (recall that by definition near a corner point -in chart- the embedding locally extends to an embedding of a neighborhood of the origin in $\mathbb{R}^m$, so locally there is an embedded piece of $\mathbb{R}^m$ containing a neighborhood of a in M, so we can project over it; this only defines the map for those points sent into M, and our claim follows form the local analysis). Each codimension 1-component at the origin is tangent to a coordinate hyperplane L_i. By using the flow line of the orthogonal vector field we can isotope $\nu_a(A) \cap \mathcal{V}_A$ into another manifold with concave corners one of whose codimension 1 boundary components is L_i. By repeating the procedure d times we find an diffeomorphism (isotopic to the identity) in a neighborhood of $0 \in \nu_a(A)$ into $\nu_a(A)$, such that

$$\nu_a(A) \cap \mathcal{V}_A \to M_a(A)$$

This can be done on a coordinate chart about $a \in A$, and the construction globalizes giving a diffeomorphism

$$\phi\colon \mathcal{W}_A \subset \nu(A) \to \mathcal{W}'_A$$

so that $\phi(\mathcal{V}_A) = \mathcal{W}'_A \cap M(A)$.

Therefore, the existence of a tubular neighborhood follows.

In the general case, one proves the existence of tubular neighborhoods for closed submanifolds in the boundary, by induction starting by the submanifold of greatest codimension and then extending to higher dimensional ones.

Step 2: Smoothen $M(A) \subset \nu(A)$. We select any vector field $X \in \mathfrak{X}(\nu(A))$ defined in a open neighborhood $\mathcal{Z}_A$ of A in $\nu(A)$, such that (i) it is transverse to $\partial M(A) \backslash A$, and (ii) it is outward pointing. Next Define L_X to be the intersection with $\mathcal{Z}_A$ of the collection of all hyperplanes orthogonal to X. This is a hypersurface in $\nu(A)$ and by choosing a smaller tubular neighborhood of A we can assume that X is transverse to L_X in $\mathcal{Z}_A$. In particular each flow line from L_X hits once $\partial M(A)$. Let

$$h\colon L_X \to \mathbb{R}^+$$

be the function which measures the time $h(x)$ to flow from L_X to $\partial M(A)$. Notice that h is differentiable away from A.

We define $M'(A)$ to be a manifold with boundary with the following properties:

- $M'(A) \cap (\nu(A) \backslash \mathcal{Z}_A) = M(A)$.
- There exists a differentiable function $h'\colon L_X \to \mathbb{R}^{>0}$ and $\mathcal{Y}_A$ a tubular neighborhhod of A in L_X such that $h' \geq h$ and $h'_{|L_X \backslash \mathcal{Y}_A} = h_{|L_X \backslash \mathcal{Y}_A} =$.

It is an easy exercise to show that such a $M'(A)$ always exists.

Then we define

$$M' = M'(A) \coprod_f M$$

This is clearly a manifold with the required properties.

In the general case we have to substitute L_X by a more complicated hypersurface (basically do it for each stratum and interpolate correctly to get one with the right properties).

Step 3: Prove uniqueness up to diffeomorphism. We have to understand the effect of different choices.

Firstly, and for any $\mathcal{Z}'_A$ a tubular neighborhood of the zero section, we can push back along the flow lines of X to define an isotopy

$$\Psi' \colon M'(A) \times I \to M'(A)$$

sending $M'(A)\backslash M$ into $\mathcal{Z}_A$ (i.e. we can connect $\mu_A h$ with h', where μ_A is a bump function with arbitrary small support near A). Therefore, we can assume that the gluing occurs in an arbitrarily small tubular neighborhood of A.

Secondly, if we pick another tubular neighborhood $(g, \nu(A), \mathcal{U}_A)$, we obtain a self diffeomorphism of $M(A)$ into itself. One shows that this extends to a diffeomorphism of $\nu(A)$ near the zero section. By the previous observation, we can pick this neighborhood to be the one in which the gluing occurs. As a result, we can assume

$$g^{-1} \circ f \colon \nu(A) \to \nu(A)$$

As usual the idea is to perturb the isomorphism close to the zero section, to have the required properties. Notice that $g^{-1} \circ f(L_X)$ and L_X are tangent at the points of A, and are contained in $M(A)$. As a consequence, we can find an isotopy supported in $M(A)$ sending $g^{-1} \circ f(L_X)$ to L_X in a neighborhood of A in $\nu(A)$. Now we have two vector fields X and $g^{-1} \circ f_* X$ transversal to L_X and $\partial M(A)\backslash A$. It is also clear that we can find an isotopy

$$\Psi'' \colon \nu(A) \times I \to \nu(A)$$

with arbitrary small support sending $g^{-1} \circ f_* X$ to X and being the identity on L_X. Obviously, $\Psi''(M'(A)) \simeq M'(A)$. Notice that after the isotopy the smooth function h'' defining the new boundary may fail so be greater or equal than h, but the inequality holds in A and hence in a neighborhood. So we can connect them as above and isotope $\Psi''(M'(A))$ so that we only get contribution for those in which the vector field matches X, and where $h'' \geq h$, and this finishes the proof. ☐

Now the following result is straightforward

Corollary 14. *The subset E defined in lemma 24 is such that its smoothing E' is diffeomorphic to M_{-z}*

rem:cxcv **Remark 39.** *One can also smoothen convex corners by removing an appropriate tubular neighborhood. Also, concave corners can be turned into concave ones, so that the smoothings are isomorphic (so then we understand the corresponding manifolds with corners as isomorphic).*

ex:cxcv **Example 7.** *Take $B^3(0,1)$ and pull from the equator to produce B^3_{cx} a manifold with convex corners. Similarly press towards the inside to get B^3_{cv}. Both manifolds have as smoothing B^3, so up to diffeomorophism they are the same.*

We want to describe $M_{z,\epsilon}$ as the result of gluing $M_{\leq z}$ with something else.

Let M be a manifold with concave corners of codimension 2 at most (and compact boundary). Let A be the codimension 2 boundary strata, that we suppose has trivial normal bundle, i.e. two everywhere linearly independent sections giving

$$\phi \colon \nu(A) \to A \times \mathbb{R}^2$$

Observe that this is not automatic. Repeat example 7 for $\mathbb{RP}^3$. Then a neigborhood of the equator in the boundary becomes a mobius band with a corner along the circle.

Let us suppose that $A, \partial M$ are connected. Then one checks

that $\partial M\backslash A$ has two connected components. Let W be the union of one of them with A. The normal bundle $\nu(A; \partial M\backslash \mathrm{int} W)$ is canonically isomorphic to

$\nu(W)_{|A}$, and both normal bundles are trivial. One checks the existence of a tubular neighborhood

$$f\colon (\nu(W), \nu(W)_{|A}, A) \to (M, \partial M \backslash \mathrm{int} W, A)$$

which is doing nothing but putting a collar about W. As usual, this collar is unique up to isotopy (an isomorphism if we add the part for $t > 0$).

pro:cornergluing

Proposition 11. *Let M be a manifold with compact boundary, and $Y^{m-1} \subset \partial M$ a submanifold with boundary B. Let N a manifold with compact connected boundary and codimension 2 convex corners, such that the submanidols of corners $A \subset \partial N$ is connected and has trivial normal bundle. Let $f\colon Y \to W$, W the closure of one of the components of $\partial N \backslash A$, be a diffeomorphism. Then on*

$$M \#_f N := M \coprod N / x \sim f(x)$$

there exists a canonical -up to diffeomorphism- structure of manifold with codimension 2 concave corners.

If we glue with g isotopic to f the result is the same. Similarly, if we isotope Y into Y' the result does not change.

Proof. Use a collar on ∂M and W to glue in the obvious way. One proves that up to diffeomorphism there is no dependence on the choice of collars. □

Definition 41. *An m-dimensional k-**handle** is the manifold*

$$h^m_k := \overline{B^k(0,1)} \times \overline{B^{m-k}(0,1)}$$

The **core** *of the k-handle is $\overline{B^k} \hookrightarrow \overline{B^k(0,1)} \times \{0\}$. Its boundary S^{k-1} is the* **attaching sphere**.

The **cocore** *is $\overline{B^{m-k}}$ and its boundary S^{m-k-1} is the* **belt sphere**.

Let M be a manifold with boundary an $i\colon S^{k-1} \hookrightarrow \partial M$ and embedding. Suppose further that $\nu(i(S^{k-1}); \partial M)$ is trivial, and we fix a trivialization

$$\phi\colon S^{k-1} \times \mathbb{R}^{m-k} \to \nu(i(S^{k-1}); \partial M)$$

Choose further a tubular neighborhood

$$f\colon \nu(i(S^{k-1}); \partial M) \hookrightarrow \mathcal{U}_{i(S^{k-1})} \subset \partial M$$

Then we have an isomorphism

$$\begin{array}{rcl} \psi\colon S^{k-1} \times \overline{B^{m-k}(0,1)} \subset \partial h^m_k & \longrightarrow & \partial M \\ x & \longmapsto & f \circ \phi(x) \end{array}$$

def:khandle

Definition 42. *The result of adding a k-handle to M along $i(S^{k-1})$ with framing $f \circ \phi$ is the manifold*

$$M \#_{f \circ \phi} h^m_k$$

Lemma 25. *$M \#_{f\circ\phi} h^m_k$ is independent of the tubular neighborhood and on the isotopy class of ϕ.*

Proof. This follows from the independence of the gluing of isotopies changing the tubular neighborhood, and isotopies of the gluing map. □

Corollary 15. *M_{-z} is the result of attaching a λ-handle to the stable sphere $S^{k-1}_{\sqrt{z}}$ (with its canonical parametrization) and the only framing extending to W^{s}_z. Notice that the core goes to W^{s}_z and the cocore to W^{u}_z.*

Corollary 16. *A compact (connected) manifold M without boundary can be built inductively by attaching a finite number of handles.*

Proof. Just find a Morse function (with critical points corresponding to different critical values). The minimum provides a 0-handle, and then we keep on attaching until we close the manifold with the m-handle -and m-ball- associated to the maximum. □

8.3. **More on k-handles and smoothings.** So far we used the Morse coordinates centered at a critical point x, with very particular metrics, to fully understand the behavior of the function when crossing the critical value $f(x)$. We want this time to understand the global behavior of ∇_f.

Lemma 26. *Every integral curve trajectory of ∇f converges to a critical point when $t \to \pm\infty$, and those limits differ.*

Proof. Just notice that $|\nabla f|^2 = df(\nabla f)$, so $f_*\nabla f$ is an strictly positive multiple of $\frac{\partial}{\partial s}$ away from critical points.

Therefore is $c(t)$ is an integral curve, $f(c(t)$ is strictly growing away from critical points. By compactness, $f(c(t))$ converges to $z \in f(M)$. Also, by compactness

$$|Df_X(v)| \leq C|v|$$

, so since $f_*\nabla f$ converges to zero and $|\nabla f|$ is bounded, we conclude that $|\nabla f(c(t))|$ converges to zero. □

For any critical point x, we define the stable manifold exactly as in the model in Morse coordinates

$$W^{\mathrm{s}}(x) = \{y \in M \mid \lim_{t\to\infty} c_y(t) = x\}$$

lem:hslide1 **Lemma 27.** *Let x be a critical point of signature λ, $z < f(x)$, and suppose that all trajectories in $W^{\mathrm{s}}(x)$ converge for $t \to -\infty$ to critical points with values smaller than z. Then $W^{\mathrm{s}}(x)_z := W^{\mathrm{s}}(x) \cap M_{[z,f(x)]}$ is an embedded submanifold diffeomorphic to $\overline{B^\lambda(0,1)}$.*

Proof. Take a small δ an consider K a (small) compact neighborhood of $W^{\mathrm{s}}(x) \cap M_{[z,f(x)-\delta]}$. Fix also Morse coordinates U_x (with the metric being the Euclidean one). We claim there exists $h \in C^r(M, \mathbb{R}^+)$ so that $-h\nabla f$ is supported in K, and the flow carries the annulus $\overline{B^\lambda(0,2\delta)} \backslash B^\lambda(0,2\delta)$ onto $W^{\mathrm{s}}(x) \cap M_{[z,f(x)-\delta]}$.

The corresponding isotopy extends to a global one Ψ, sending the submanifold $\overline{B^\lambda(0,2\epsilon)}$ onto $W^{\mathrm{s}}(x)_z$, and that finishes the proof. □

In particular the above isotopy gives an embedding

$$S^{\lambda-1}_\delta = S^{\lambda-1} \hookrightarrow W_z$$

as the boundary of $W^{\mathrm{s}}(x)_z$. Even more, for $\epsilon > 0$ small enough it sends

$$S^{\lambda-1}_\delta \times \overline{B^{m-\lambda}(0,\epsilon)} \subset \partial(\overline{B^\lambda(0,\delta)} \times \overline{B^{m-\lambda}(0,\epsilon)})$$

diffeomorphically into a tubular neighborhood of $\partial W^{\mathrm{s}}(x)_z$ in W_z. So we conclude

Corollary 17. *$M_{\leq z} \cup \Psi_1(\overline{B^\lambda(0,\delta)} \times \overline{B^{m-\lambda}(0,\epsilon)}) \subset M$ is*

$$M_{\leq z} \#_\phi h^m_k$$

Moreover, we still have ∇f transverse to the boundary (because that was the case for $\overline{B^\lambda(0,\delta)} \times \overline{B^{m-\lambda}(0,\epsilon)}$).

Remark 40. *Notice that the condition on lemma 27 is equivalent to saying that for all y critical point with $f(y) \in [z, f(x))$, we have $W^{\mathrm{s}}(x) \cap W^{\mathrm{u}}(y) = \emptyset$.*

Given a Morse function f, and values $z_0 < f(y) < f(x) < z_1$ such that no critical points other than y, x lie in $M_{[z_0,z_1]}$, we know

$$M_{\leq z_1} \simeq (M_{\leq z_0} \#_\phi h^m_\lambda) \#_{\phi'} h^m_{\lambda'} \tag{24}$$ eq:2cobord

Suppose for the moment that $W^{\mathrm{s}}(x) \cap W^{\mathrm{u}}(y) = \emptyset$. Then

$$\partial W^{\mathrm{s}}(x)_{z_0} \cap \partial W^{\mathrm{s}}(y)_{z_0} = \emptyset$$

and we may take isotopies as in lemma 27 supported over non intersecting tubular neighborhoods of $W^{\mathrm{s}}(x)_{z_0}$ and $W^{\mathrm{s}}(y)_{z_0}$. If we call Ψ to the composition, we conclude that

$$M_{\leq z_0} \cup \Psi_1(\overline{B^\lambda(0,\delta)} \times \overline{B^{m-\lambda}(0,\epsilon)} \cup \overline{B^{\lambda'}(0,\delta')} \times \overline{B^{m-\lambda'}(0,\epsilon')}) \subset M_{\leq z_1}$$

is a subset diffeomorphic to

$$M_{\leq z_1} \simeq (M_{\leq z_0} \#_\Phi (h^m_\lambda \coprod h^m_{\lambda'})$$

Moreover, once smoothened inside M the gradient is transverse to the boundary. Therefore, we can flow the smoothing, and since we have no more critical points on $M_{[z_0,z_1]}$ we get a diffeomorphism onto $M_{\leq z_1}$.

Corollary 18. *Under the above conditions, $M_{\leq z_1}$ is obtained from $M_{\leq z_0}$ by adding two handles, and the attaching can occur in any order (or both at a time).*

Observe that $W^{\mathrm{s}}(x) \cap (W^{\mathrm{u}}(y) = \emptyset$ is equivalent to picking $f(y) < z < f(y)$ and on W_z verify

$$\partial W^{\mathrm{s}}(x)_z \cap \partial W^{\mathrm{u}}(y)_z = \emptyset$$

Those are spheres of dimension $m-\lambda-1$ and $\lambda'-1$, so if $m-\lambda-1+\lambda'-1 < m-1$, i.e. $\lambda \geq \lambda'$, then we expect them to have empty intersection.

def:ms **Definition 43.** *A metric g is called* **Morse-Smale** *if for all critical points x, y*

$$\partial W^{\mathrm{s}}(x) \cap \partial W^{\mathrm{u}}(y) = \emptyset$$

Proposition 12. *Morse-Smale metrics are dense (and even more).*

Proof. We just do the case where we have to critical points x, y in $M_{(z_0,z_1)}$. If there is intersection we must have $f(x) > f(y)$. That intersection appears as the intersection of $S^{m-\lambda-1}$ and $S^{\lambda'-1}$ in W_z. By an isotopy in W_z with small C^1-size, we can separate the spheres. We assume the isotopy to be the identity near the end points of the interval. Since $M_{[z,z+\delta]} \simeq W_z \times [z, z+\delta]$ using the gradient, we can extend the isotopy to the whose manifold. We are going to substitute ∇f by $\Psi(\nabla_f)$. It is easy to make it the gradient of f w.r.t. a new metric. We just need to declare the new vector field orthogonal to the level hypersurfaces of f, and scale it appropriately. □

cor:perfect **Corollary 19.** *A compact m manifold is the result of attaching handles in increasing dimension.*

Remark 41. *Neglecting the order of suitable handles can be also achieved at the level of Morse functions. In particular corollary 19 translates into the existence of* **perfect Morse functions**, *defined by the property $f(x) = \lambda$ if x is a critical point with index λ.*

prop:cancel **Proposition 13.** *Let M be a (compact) connected manifold. Then it admits a handle decomposition with just one 0-handle and one m-handle (or a Morse function with a unique local minimum and maximum).*

Proof. Let us give a perfect Morse function. If we have two index 0 critical points y_0, y_1, connectivity implies that we must have x and index 1 critical point whose stable manifold is the union of the critical point x, and two trajectories which converge to y_0, y_1 in negative time.

Therefore, a small neighborhood of $\overline{W^s \cup y_0 \cup y_1}$ is diffeomorphic to the result of attaching a $I \times \overline{B^{m-1}(0,1)}$ to $\overline{B^m(0,1)} \coprod \overline{B^m(0,1)}$. The attaching sphere is $S^0 = a, b$. We can send a to any point in the boundary of the first ball, and choose any isotopy class of attaching map. Rather than choosing the ball as 0-handle, we choose an upper half ball E^m_+. We can do that because its smoothing is clearly seen to be diffeomorphic to the ball. Then the attaching map $\overline{B^m(0,1)} \to \partial E^m_+$ can be chosen to be go to the hyperplane in the boundary, and to be linear. By changing the orientation in the one handle, it can be assumed to preserve it, and hence we may choose $\rho\mathrm{Id}$, where ρ is very close to 1. Once we smoothen, we easily see that we get a copy of E^m_+. Similarly, we glue to the other one identifying points (up to scaling), and once we smoothen we get the sphere. □

cor:handlebody **Corollary 20.** *Let $m \geq 3$. There is a unique -up to diffeomorphism- way of gluing d 0-handles and k 1-handles so that we obtain a connected oriented m-manifold (with boundary), and in fact only depends on $k - d - 1 \geq 0$.*

Proof. We know that the attaching spheres are pair of points in the disjoint union of d S^{m-1}, so the attaching order does not quite matter. By connectivity there must be at least $d-1$ 1-handles, connecting the balls h^m_0. By proposition **??**

$$h^m_0 \coprod \overset{(d)}{\cdots} \coprod h^m_0 \#(h^m_0 \coprod \overset{(d-1)}{\cdots} \coprod h^m_0) \simeq h^m_0$$

The attaching spheres of the remaining $k-d-1$ 1-handles are $2(k-d-1)$ points. We know that any two such set of points in $S^{m-1} = \partial h^m_0$ are transformed into each other by an isotopy starting at the identity (compose isotopies each for each pair of points, so that the support of the each one does not include the points already moved). Therefore, there is no dependence on the points.

For each 1-handle $I \times \overline{B^{m-1}(0,1)}$, we know that the attaching maps

$$\{0\} \times \overline{B^{m-1}(0,1)} \to S^{m-1},\ \{1\} \times \overline{B^{m-1}(0,1)} \to S^{m-1}$$

are one orientation reversing and one orientation preserving. To extend the isomorphism to the 1-handle we can -if necessary- reflect the 1-handle on the hyperplane $t = 1/2$, so that the 1-handle in both manifolds is glued by isotopic attaching maps, and therefore gives diffeomorphic manifolds. □

def:csum **Definition 44.** *Let M, N connected oriented manifolds without boundary. Consider $M \coprod N \times I$ (oriented so that $M \coprod N \times I$ inherits the given orientation). Attach a 1-handle to $M \coprod N \times \{1\}$ with one point of the attaching sphere going to each connected component, and so that the manifold we obtain is orientable (and connected). Then the* **connected sum** *$M \# N$ is the boundary component different from $M \coprod N \times \{0\}$ with the induced orientation.*

The connected sum is well defined. There is no dependence on when the 1-handle is attached. Also since we must get an oriented manifold, very much as in corollary 20 we see that we get diffeomorphic manifolds, and therefore diffeomorphic boundaries.

Remark 42. *There is other way of defining the connected sum: pick $x \in M, y \in N$ and $\mathcal{U}_x, \mathcal{U}_y$ small (tubular) neighborhoods. Now take diffeomorphisms*

$$(\mathcal{U}_x, x) \to (B(0,1), 0)\ (\mathcal{U}_y, y) \to (B(0,1), 0)$$

one orientation preserving and the other orientation reversing, and then using spherical coordinates identify

$$(\theta, r) \mapsto (\theta, \sqrt{1-r^2})$$

away form the origin.

9. 2 and 3 dimensional compact oriented manifolds

9.1. **Compact, oriented surfaces.** Let M be a compact, connected and oriented surface.

lem:circleiso **Lemma 28.** *Any orientation preserving diffeomorphism of S^1 is isotopic to the identity.*

Proof. We know that such a map has degree one, and hence it is already homotopy to the identity, and we want to improve the quality of the homotopy.

By a rotating we can assume it fixed the identity. Go to the universal cover $\mathbb{R}$ and the isomorphism f appears as $F\colon \mathbb{R} \to \mathbb{R}$ with $F(t+1) = F(t)+1$ and $F(t) \geq t$. Then the convex combination solves the problem. □

Corollary 21. *For an oriented surface, a 2-handle (a disk) can be glued in a unique way up to diffeomorphism so that the orientation extends. Or in other words, every diffeomorphism od S^1 extends to the unit disk (because the identity does extend).*

cor:no2hand **Corollary 22.** *In a handle decomposition (Morse function) of M, the diffeomorphism type is entirely determined by the 0 and 1 handles.*

Definition 45. *The* **genus** *g of M is one half the minimum number of 1-handles in a decomposition. One has*

$$\chi(M) = 2 - 2g$$

Corollary 23.

(1) *The sphere S^2 is characterized by having a Morse function with only a minimum and a maximum as critical points, or by $\chi(S^2) = 2$.*
(2) *The torus $\mathbb{T}^2$ is characterized by having a Morse function with a minimum and a maximum and two saddle points as critical points, or by $\chi(\mathbb{T}^2) = 0$.*

Proof. By proposition 13 we can take a Morse function for M with just one 0-handle and one 2-handle. If $\chi(M) = 0$, then M is just determined by the 0-handle (we just have to cap off the 0-handle with another disk), and then $M = S^2$.

Notice that $\chi(M)$ can never be odd. The reason is that each time we attach a 1-handle, the number of connected components of the boundary increases by 1 (mod 2), an hence the result follows because after attaching all we must obtain just one connected component.

Hence, the second smallest $\chi(M)$ is 2. The manifold M is determined by the way we attach the 1-handles. The attaching spheres go to pair of points x, x' and y, y' in $S^1 = \partial h_0^2$, so the diffeomorphism type of M is determined by the isotopy type of (S^1, x, x', y, y'). Notice that if two points belonging to the same attaching sphere follow each other, after gluing we do not get a boundary with one connected component.

Hence, the surface is unique up to isomorphism, and one checks is it $\mathbb{T}^2$. □

Consider $i\colon S^1 \hookrightarrow M$ an embedding with image A.

Definition 46. *A is said to separate M if $M \backslash A$ has two connected components. It is called nonseparating otherwise.*

When we **cut** M **open along** A we get an oriented surface with two boundary components. It can be capped of uniquely to an oriented surface $\tilde{M}$ by adding a 2-handle an a 0-handle. When we add the two disks we can define a function which has a minimum and a maximum on each, and other critical points in $M\backslash A$. We can clearly adapt it if we glue back to M so that it is still Morse and with the same critical points except for the local maximum and minimum. Therefore

$$\chi(\tilde{M}) = \chi(M) + 2$$

If A is non-separating then $\tilde{M}$ is connected, and has two connected components otherwise.

thm:classsurf

Theorem 30. *M is determined up to diffeomorphism by $\chi(M)$, and*

$$M \simeq \mathbb{T}^2 \# \overset{\chi(M)/2}{\cdots} \# \mathbb{T}^2$$

Proof. We prove it by induction. The surface M is characterized by the isotopy type of $(S^1, x_1, x_1', \dots, x_k, x_k')$, where $k = \chi(M)/2$. We may think of them as pairs of points of the same color, each pair of a different one. Any such pair splits S^1 in two intervals. One checks that it is not possible for an interval to contain just pairs of points, otherwise connectivity of the boundary would not be possible.

Hence, we can find four points x_1, x_1', x_s, x_s' so that $(S^1, x_1, x_1', x_s, x_s')$ is as in the torus. It is easy now to find an embedded S^1 splitting the 0-handle and the union of the two 1-handles form the rest of the surface. Hence we get

$$M = M' \# \mathbb{T}^2,$$

with $\chi(M') = \chi(M) + 2$, and this proves the result. □

rem:clasbound

Remark 43. *Notice in particular that theorem 30 also yields a classification of connected, compact, oriented surfaces with boundary. Indeed, we can cap them uniquely to one without boundary. Notice as well that two surfaces M, M' with diffeomorphic boundary and diffeomorphic capping are diffeomorphic. The reason is that they appear by removing some d disks, but two different embeddings of d closed disks are diffeomorphic (because it is the case for points).*

In particular any surface with two boundary components whose capping is the sphere must be an annulus $S^1 \times I$.

cor:spherecurv

Corollary 24.

(1) *Any differentiable curve in the sphere separates.*
(2) *Two disjoint curves in S^2 split it into two disks and and annulus.*

Proof. Suppose the curve does not separate. Then we get $\tilde{M}$ with $\chi(\tilde{M}) = 4$, but that is not possible according to our classification. So even more we get a diffeomorphism of the sphere sending the curve to the equator, and according to lemma 28 we can further isotope to reach any parametrization inducing the same orientation.

Regarding the second assertion, we know that the first curve A splits in two disks, E^2_+, E^2_-. The second belongs to one of them, so it again splits into two disks. So we get D, E^2_+ and a third subset that we claim it is clearly connected (because it is so when we add E^2_+).

So it is an oriented, connected, surface with two boundary components whose capping is S^2. By remark 43 it must be the annulus. □

Definition 47. *A system of non-separating curves is an embedding $i \coprod_{i=1}^d S^1 \hookrightarrow M$ such that $M\backslash(i \coprod_{i=1}^d S^1)$ is connected.*

It is maximal is no other curve can be added so that the non-separation property remains.

Lemma 29. *Maximal systems of non-separating curves in a surface of genus g have g curves, and given any two there is a self-diffeomorphism of M sending one to the other (neglecting orientations of the curves)*

Proof. Each curve produces a new surface $\tilde{M}$ whose Euler characteristic increases by two. Besides, it appears with two disks whose boundaries are the curves. If we select a different curve we get another surface diffeomorphic to $\tilde{M}$. The diffeomorphism can be arranged to send the disks to the disks. Therefore, it extends to the connected sum along each pair of disks, giving thus a diffeomorphism of the original surface sending one non-separating curve to the other.

By induction one sees that when we have g curves we arrive at a surface whose capping is the sphere. By corollary 24 no further curve can be added. □

9.1.1. *Diffeomorphisms of oriented surfaces.* We start by proving the following fundamental theorem of Waldhausen.

ref:isotsphere

Theorem 31. *Any orientation preserving diffeomorphism of S^2 is isotopic to the identity.*

Proof. Let f be such an isomorphism. We know it has degree 1 so it is homotopic to the identity. But we want to improve this result. We first isotope so that the poles are fixed. Due to the orientability assumption we can further isotope so that we get matching in neighborhoods of the poles. So we are led to prove the result for the annulus $M = S^1 \times [-1, 1]$ and a map which is the identity in a neighborhood of the boundary.

Let us use coordinates θ, t, and assume as well that the subset $\theta \in [-\epsilon, \epsilon]$ is fixed. We will associate to the identity map the vector field $X = \frac{\partial}{\partial t}$ and to the map f $Z = f_*X$. Both are vector field whose trajectories are closed. The advantage is that we can parametrize the annulus as follows

$$\begin{aligned} [0,1] \times [-1,1] &\longrightarrow M \\ (\theta, t) &\longmapsto c_t^Y(\theta) \end{aligned}$$

where c_t^Y is the integral curve through $(0, t)$ of the vector field $Y = X, Z$.

We want to interpolate between X and f_*X. Consider for each t the map

$$\begin{aligned} G_t^Z : S^1 &\longrightarrow S^1 \\ \theta &\longmapsto \frac{\dot{c}_t^Z(\theta)}{|\dot{c}_t^Z(\theta)|} \end{aligned}$$

We claim that this map has degree zero. Indeed, for $t = 1$ is constant, and as t moves we get homotopic maps, so the degree is the same.

For each such map we have a canonical homotopy to the constant one. It is induced form convex combination in the universal cover (contraction, since one lifted map is constant). Therefore we get a smooth family of non-vanishing vector fields $X_s(c_t^Z(\theta))$, so that $X_0 = Z$ and $X_1 = hX$. Of course we can assume $h = 1$.

Let Ψ^{X_s} the corresponding isotopy. The initial idea is define our isotopy

$$\begin{aligned} \Phi : M \times I &\longrightarrow M \\ (\theta, t, s) &\longmapsto \Psi_s(\Psi_0^{-1}(\theta, t)) \end{aligned} \tag{25}$$

The problem is that for the intermediate vector fields, we may get integral curves which are not closed (and if so of the right period), but that can be fixed.

Notice that near $\theta = 0$ all vector fields coincide with X. We claim that every integral curve goes around the cylinder. Indeed, the vector fields do not vanish and have norm uniformly bounded by below. If it stays in the annulus, since it is defined for infinite time it must go back and forth and up and down, so it has infinite vertical and horizontal tangencies. In particular there are accumulation points. Let

x be one such, and pick a box flow. Being an accumulation point means that an integral curve $c(t)$ enters in the interior of the flow box at time t_0, stays until time t_1 then wanders around and goes back in time t_2. But this is not possible. Otherwise we can use a small segment transverse to the flow to close $c_{|[t_0+\epsilon,t_2+\epsilon]}$ into a closed curve c'. Now we now that in particular it bounds a disk. It is routine to slightly modify the integral field to make it tangent to the smoothing of c'. The result is a vector field in a disk tangent to the boundary (and with closed integral curves nearby). By doubling we get one in the sphere with no zeros, but that contradicts theorem 24.

Hence, all integral curves go around the annulus. We get a one parameter family of return maps on the interval. We can isotope them to the identity, and use a collar of $\theta = 0$ to modify the vector fields so that the return map is the identity. So they all have integral curves, and we can scale them to have the same period. Therefore, the isotopy Φ in equation 25 is well defined and solves our problem.

We still need to prove that we can isotope the segment $\theta = 0$ so it is fixed. We observe that any two embeddings of S^1 in S^2 are isotopic. This is because they can be made disjoint, and then by corollary 24 they bound an annulus, so we can use the transverse coordinate to isotope one into the other. In our case we do it with the image by f of a meridian (the full geodesic). Since f is orientation preserving, once we isotore the curves to have the same image, we can use another isotopy so that the parametrization agrees. Then we can arrange things in the poles relatie to the meridian (since the homotopies are convex sums along meridians).

□

cor:no3handle **Corollary 25.**

1. *Any diffeomorphism of* $S^2 = \partial h_3^3$ *extends to* h_3^3.
2. *The only information needed to attach three handle in a three manifold is the degree of the attaching map (i.e., whether it preserves the orientation or not). In particular, for compact orientable manifolds (without boundary) three handles can be attached in a unique way, so they can be neglected.*

In $\mathbb{T}^2$ we consider to curves A, B so that $\#(A, B) = \pm 1$. We know they cannot be isotopic.

Given a diffeomorphism $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ consider the matrix

$$H_f = \begin{pmatrix} \#(A, f(A)) & \#(B, f(A)) \\ \#(A, f(B)) & \#(B, f(B)) \end{pmatrix}$$

Theorem 32. *Two diffeomorphism* f, f' *are isotopic iff* $H_f = H_{f'}$.

Proof. The composition of maps is multiplicative w.r.t. the matrices, so we can assume $f' = \mathrm{Id}$.

Now it is enough to show that we can isotope $f(A) \cup f(B)$ into $A \cup B$, because if that is the case we can arrange the map to be the identity on a tubular neighborhood. What is left is a disk, with a map matching the identity near the boundary, and we proved in theorem 31 that can be isotoped to the identity.

Take a curve $A \subset \mathbb{T}^2$. The torus has $\pi_1(\mathbb{T}^2) = \mathbb{Z} \oplus \mathbb{Z}$, and a basis is given by the curves $A_0 := S^1 \times \{1\}$ and $B_0 := \{1\} \times S^1$. To compute the homotopy class of A in such basis we just have to compute the homotopy class of the map $S^1 \to A \xrightarrow{p_i} S^1$. This is nothing but the degree of the corresponding map. If the curve A is transverse to both elements in the basis, then for both projections 1 is a regular value, and the degree of one projection is the intersection number with the other curve of the basis. In particular the information about the matrix implies that $f(A_0)$ is homotopic to A_0 and $f(B_0)$ to B_0 (we may suppose f fixed the common base point of A_0, B_0). So actually the matrix M_f is also the isomorphism f_* induced on π_1.

We are going to show it for $f(A_0)$. Let $C \to \mathbb{T}^2$ be the cover whose fundamental group is $\mathbb{Z}\langle A_0\rangle$. This is a cylinder in which the preimage of B_0 is a real line.

Since $f(A_0)$ is homotopic to A_0 it lifts to A_0'. We are going to fing periodic isotopies on C so that A_0' is closer to A_0 (a lift of A_0 at the fixed base point). We will call C_k to $S^1 \times [k, k+1] \subset C$. By compactness A_0' reaches a maximal and a minimal height. Assume the maximal height belongs to C_k, $k > 1$. We will inductively push it down by periodic isotopies. The intersection $C_k \cap A_0'$ is a finite number of oriented arcs. We can close each arc I_s with one of the two segments in $S^1 \times \{k\}$. Since we are in the annulus either we get a curve bounding a disk or non-trivial in homotopy. We close with I_s' so that the curve (with corners) bounds a disk. Then we isotope in C_k so that I_s is pushed into I_s'. This can be extended to a periodic isotopy. The result is an isotopy in $\mathbb{T}^2$ so that the lifted curve does not enter in C_k any more. By induction, we may assume that A_0' only enters in the negative part, apart from a small segment I_0 near he base point in which it matches A_0. Now we repeat the process but pushing up. The observation is that when we do it for a segment I_s, we also push the ones of bigger height whose translates appear in the disk. But none of then has height -1, for otherwise it will come form a piece of positive height; the only possibility is the one coming by prolonging I_0, but this is excluded because then then corresponding translation of I_0 would intersect I_s. Therefore, we end up with a curve isotopic to $f(A_0)$ which does not intersect A_0. By cuuting open along A_0, we are in the cylinder and our curve separates (it is non trivial in homotopy), so we can find a transverse coordinate to push it into A_0. the segmentnested ones If we tranaslate the curve A_0' by the $\mathbb{Z}$-acion we must get disjoint copies. We claim that no pieces of the translates enter in the disk region. If that was not the case, we would have a piece I_l coming from $C_{k'}$, $k' < k$. Now we can reach I_l by moving from I. But since A_0' comes from an embedded curve in the torus. Once we have solve it for a curve, we can arrange the other to match B_0 near the base point, and repeat the previous process relative a tiny neighborhood of A_0 in which B_0 does not enter any more. □

Definition 48. *Let M_g be a compact, connected, orientable surface of genus g. Then the mapping class group of M_g is*

$$\mathrm{Diff(M)/Isot}$$

The mapping class group is obviously a group. What we saw for S^2 and T^2, appart from an explicit computation, is that

$$\mathrm{Diff(M)/Isot} = \mathrm{Diff(M)}/\sim$$

We will only state the generalization of the above result.

Theorem 33. *For any closed, connected, orientable surface, two diffeomorphisms are isotopic iff they are homotopic.*

The proof is based on the fact that on such surfaces two curves are isotopic iff they are homotopic. Once we know this result, we can cut open along a maximal system of non-separating curves, and then apply the result for the (punctured) sphere.

9.2. **Compact, oriented three manifolds.**

9.3. **Heegard decompositions.** Let M be a closed, oriented and connected three manifold. Let us fix f a perfect Morse function with one local minimum and one local maximum. Let us consider $M = M_{\leq 3/2} \#_f M_{\geq 3/2}$

Then $M_{\leq 3/2}$ is made of a 0-handle and g-1handles. By corollary 20 up to diffeomorphism it is determined by the number g. We will call it a **genus g handlebody**, and denote it by H_g. Notice that $\partial H_g \simeq M_g^2$.

By reversing the Morse function,

$$M_{\geq 3/2} \simeq H_g'$$

By remark 35 $\chi(M) = 0$, but $\chi(M) = g' - g$, so we conclude that

$$M \approx H_g \#_\phi H_g \tag{26}$$ eq:heegard

where

$$\phi\colon \partial H_g \to \partial H_g$$

is orientation preserving.

Definition 49. *A Heegard decomposition of M is a decomposition as in equation 26*

rem:mcg **Remark 44.** *The importance of the study of mapping class groups of surfaces stem from the fact that in a Heegard decomposition of M, the diffeomorphism class of M only depends on* $[\phi] \in \mathrm{Diff(M_g)/Isot}$.

def:3genus **Definition 50.** *The* **genus** *of M is the minimal genus on a Heegard decomposition of M.*

Remark 45. *The genus can always be artificially augmented by adding pairs of* **cancelling handles** *********Show genus decomposition of the sphere************

thm:genus0 **Theorem 34.** *The only genus 0 zero three manifold is the sphere.*

Proof. This is theorem **??** or corollary 25 that says that two 3-handles can be glued in a unique way up to diffeomorphism to yield and orientable manifold. □

9.4. **Lens spaces.**

Definition 51. *Genus 1 three manifolds are called* **lens spaces**.

Example 8. $\mathbb{RP}^3$ *and circle bundles over* S^2 *are lens spaces.*

Notice that there is a surjective map from $\mathrm{Diff}(\mathbb{T}^2)/\mathrm{Isot} = SL(2,\mathbb{Z})$ to lens spaces.

To understand these spaces better we fix two genus 2 handles bodies $H_{2,a}, H_{2,b} = \overline{B^2(0,1)} \times S^1$

Let

$$H_f = \begin{pmatrix} q & p \\ s & r \end{pmatrix}$$

M_ϕ be the matrix associated to the gluing map. It is best to think that we start with $H_{2,b}$ and then glue $H_{1,a}$ using ϕ. We will do the gluing in two stages:

Firstly we select the disk $D = B^2(0,1) \times \{1\}$. Its boundary in $m \in \mathrm{T}^2$ belongs to the homotopy class of the **meridian**. Notice that if we slightly thicken the disk to $\mathcal{U}_D = D \times [-\epsilon, \epsilon]$, then

$$H_{2,a} \backslash \mathrm{int} \mathcal{U}_D \simeq h_3^3$$

after smoothing. Now according to corollary 25 3-handles can be glued in a unique way, therefore what determines the isomorphism class is the gluing of the 2-handle $\mathcal{U}_D$, which in turn is totally determined by the attaching map restricted to the meridian curve ∂D. In other words, by the pairwise prime pair pair (p,q). We denote the corresponding lens space $L(p,q)$

Proposition 14.

(1) $\pi_1(L(p,q)) = \mathbb{Z}_p$.
(2) $L(p,q) \simeq L(-p,-q),\ L(p,q) \simeq L(p,q+kp),\ k \in \mathbb{Z}$
(3) $L(p,q) \simeq L(p,q')$.

Proof. The first assertion is obvious.

Point two follows because there are certain transformations of $SL(2,\mathbb{Z})$ trnasforming one matrix into the other, and those maps extend to the solid torus. In the first case the isomorphism in question is $-\mathrm{Id}$, obtained by reflecting on each circle of $\mathbb{T}^2$ and extending in the obvious way to $H_{2,b}$. In the second we must transform the longitud curve $l := \{1\} \times S^1$ into $l + km$. But this follows from cutting along the diks, twisting the required number of times, and then gluing back.

The third point is obtained by duality, i.e. gluing in the opposite order, and q' is obtained by just inverting the gluing matrix. □

9.5. **Higher genus.** What we have done for lens spaces also works for higher genus. On a handlebody H_g we can find $m_1, \ldots, m_g$ a maximal family of non-separating curves, each of which bounds a disk, and such that removing an open neighborhood of these disks results in a 3-handle.

Therefore, to determine M^3 we just need to know $f(m_1), \ldots, f(m_g) \subset \partial H_{g,b}$, which we know must be a maximal family of non-separating curves.

Corollary 26. *A Heegard decomposition for M^3 is given by two maximal families of non-separating curves on M^2_g (assumed to bound disks that once removed give a ball in the respective handlebodies).*

Remark 46. *Since we know that homotopic curves are isotopic, the two families can be given up to homotopy.*

10. Exercises

Exercise 35. *Describe a smooth manifold structure on $\mathrm{Gr}(k,n)$ (resp. $\mathrm{Gr}_{\mathbb{C}}(k,n)$) the Grassmannian of k-planes in $\mathbb{R}^n$. (resp. complex k-planes in $\mathbb{C}^n$). Notice that in particular one gets smooth manifold structure on $\mathbb{RP}^n$ (resp. $\mathbb{CP}^n$).*

Exercise 36. *Let $\mathrm{Gl}(n,\mathbb{R})$ be the group of invertible $n \times n$ matrices. Show that it inherits a manifold structure as an open subset of some Euclidean space. Prove that the subgroup of symmetric and orthogonal and special orthogonal matrices are submanifolds of $\mathrm{Gl}(n,\mathbb{R})$, and compute their dimension.*

Show that the group of unitary matrices is a submanifold of the group of invertible $n \times n$ complex matrices, and compute its dimension.

Exercise 37. *Let $\Omega \subset R^n$ be a compact n-dimensional differentiable submanifold with non-empty boundary. Suppose that there exist $x_0 \in \mathrm{int}\Omega$ such that for every $x \in \Omega$ the segment $[x, x_0]$ is inside of Ω and (after prolonging it a bit near x) transverse to $\partial\Omega$. Show that*

$$\partial\Omega \simeq S^{n-1}$$

.

Let Q be any inner product on $\mathbb{R}^n$. Show that for any positive c,

$$Q^{-1}(c) := \{v \in \mathbb{R}^n \,|\, Q(v,v) = c\}$$

is a smooth submanifold of $\mathbb{R}^n$ and it is diffeomorphic to

$$S^{n-1} := \{v \in \mathbb{R}^n \,|\, |v|^2 = 1\}$$

Exercise 38. *Let M, N be C^r-manifolds. Show that $M \times N$ carries a canonical C^r-manifold structure, and that $T(M \times N) \simeq TM \times TN$.*

Exercise 39. *Let N be a a C^r-manifold, M a topological space, and $f\colon M \to N$ a local homeomorphism (i.e. for each $x \in M$ there exist U a neighborhood so that $f\colon U \to f(U)$ is a homeomorphism). Prove that M can be given a canonical C^r-manifold structure so that f becomes a local C^r-diffeomorphism.*

Exercise 40. *Prove that a C^r-map which is a C^1-diffeomorphism it is a C^r-diffeomorphism.*

Exercise 41. *Show that*

$$SO(3) \simeq \mathbb{RP}^3$$

Exercise 42.

(1) *Prove that for any C^1 manifold M every $f \in C^1(M, \mathbb{R})$ has at least 2 critical points.*
(2) *For any sphere S^n find a smooth function with exactly 2 critical points.*

Exercise 43. *Let M, N be differentiable manifolds with non-empty boundary. Prove that a C^1 map takes regular points in $M \backslash \partial M$ into points in $N \backslash \partial N$.*

Exercise 44. *Any compact surface embeds in $\mathbb{R}^5$. The 2-torus is the surface*

$$\mathbb{T}^2 := S^1 \times S^1$$

Exhibit an embedding

$$j\colon \mathbb{T}^2 \hookrightarrow \mathbb{R}^3$$

Hint: Embed it as a surface of revolution.

Exercise 45. *Consider $\mathbb{CP}^m$ with homogeneous coordinates $[X_0 : \cdots : X_m]$ and let $d_k = \frac{k!}{n!(k-n)!}$*

Show that the maps

$$\begin{array}{rcl} f_{d_k}\colon \mathbb{CP}^m & \longrightarrow & \mathbb{CP}^{d_k} \\ [X_0 : \cdots : X_m] & \longmapsto & [X_0 : \cdots : X_m : 0 : \cdots : 0] \end{array}$$

and

$$\begin{array}{rcl} V_k\colon \mathbb{CP}^m & \longrightarrow & \mathbb{CP}^{d_k} \\ [X_0 \cdots : X_m] & \longmapsto & [X_0^k : X_0^{k-1}X_1 : \cdots : X_{m-1}X_m^{k-1} : X_m^k] \end{array}$$

are embeddings

References

[1] Kirby, Robion C. The topology of 4-manifolds. Lecture Notes in Mathematics, 1374. Springer-Verlag, Berlin, 1989.
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