Open book decompositions for almost contact manifolds

[1]David Martínez, [2]Vicente Muñoz and [3]Francisco Presas

[1]*Department of Mathematics, Universidad Carlos III de Madrid*
[2]*Department of Mathematics, Universidad Autónoma de Madrid*
[3]*Department of Mathematics, Stanford University*
emails: dmtorres@math.uc3m.es, vicente.munoz@uam.es, fpresas@math.stanford.edu

1. Introduction

Let C be a $(2n+1)$-dimensional manifold. C is called *almost contact* if its tangent bundle admits a reduction to $U(n) \times \mathbb{R}$. As an example, any oriented real hypersurface in an almost complex manifold is almost contact. A (cooriented) contact structure in C is a 1-form α such that $\alpha \wedge (d\alpha)^n$ is non-zero all over the manifold. C is a contact manifold if it admits such a contact structure. This implies that $\xi = \ker \alpha$ is a symplectic bundle with symplectic form given by the restriction of $d\alpha$ to ξ. In particular, ξ admits an almost complex structure and therefore the manifold C is almost contact.

The almost contact condition is the only known homotopical restriction for an odd dimensional manifold to be contact. The following conjecture has remained unproved for 20 years.

Conjecture 1.1. *An almost contact manifold always admits a contact structure.*

The problem was fully understood in the 3-dimensional case by Martinet [16]. Moreover, it has been only recently shown that some classical almost contact manifolds, as the high dimensional tori, admit a contact structure [4]. Geiges and Thomas have pursued the study of some high dimensional almost contact manifolds [8, 9], but a general program to answer Conjecture 1.1 is still lacking.

In this article we propose a geometrization of the almost contact condition in order to understand better this conjecture. More specifically, we shall call a $(2n+1)$-dimensional manifold C *quasi-contact* if it admits a closed 2-form ω such that ω^n is a non-zero $2n$-form all over the manifold. The following lemma is just an application of Gromov's h-principle.

Lemma 1.2. *Given an almost contact manifold C and given $\gamma \in H^2(C, \mathbb{R})$, there exists a quasi-contact structure ω in C such that $[\omega] = \gamma$.*

Proof. If C is almost contact then $C\times\mathbb{R}$ is almost complex. We understand C as the hypersurface $C\times\{1\}\subset C\times\mathbb{R}$. A 2-form τ is called a compatible almost symplectic form in an almost complex manifold (M,J) if $\tau(p)$ is symplectic for any point $p\in M$ and $\tau(\cdot,J\cdot)$ is a Riemmanian metric in M (but note that τ might be not closed). It is well know that, once we fix an almost complex structure, the space of compatible almost symplectic forms is non-empty and contractible. So given an almost complex structure J in $C\times\mathbb{R}$, we pick up an almost symplectic structure λ.

Recall now the following Gromov's theorem

Theorem 1.3. *Let M be an open symplectic manifold. Let τ_0 be an almost symplectic 2-form and let $a\in H^2(M,\mathbb{R})$. There exists a family of almost-symplectic forms τ_t on M such that $\tau_0=\tau$ and τ_1 is a symplectic form which represents the class a.* □

For a proof, see Theorem 7.34 in [15]. Using this result we can find a homotopy going from λ to a symplectic structure ω in $C\times\mathbb{R}$ representing the class $\pi^*(\gamma)$, where $\pi: C\times\mathbb{R}\to C$. Now, it is very simple to check that the restriction of ω to $C\times\{1\}$ gives a quasi-contact structure in C in the homology class γ. □

Once we have a quasi-contact structure in an almost contact manifold C, it is possible to adapt the techniques carried out in [12] to develop an *asymptotically CR geometry*. This will be the topic of Section 3. The first main result that we shall prove is

Theorem 1.4. *Given a $(2n+1)$-dimensional quasi-contact manifold (C,ω) such that $[\omega]$ admits a lift to an element of $H^2(C;\mathbb{Z})$ and a class $\beta\in H^2(C;\mathbb{Z})$, then for k large enough there exists a quasi-contact submanifold N which is Poincaré dual to $k[\omega]+\beta$. Moreover, the natural inclusion $i:N\to C$ induces isomorphisms in homology (resp. homotopy) groups up to order $n-2$ and an epimorphism in order $n-1$.*

From a topological point of view the most powerful case happens when ω is exact, which can always be arranged thanks to Lemma 1.2. However, from a geometrical point of view the choice of a different quasi-contact structure can be very useful. For instance, it gives submanifolds with non-exact quasi-contact structures.

Following the ideas of Giroux and Mohsen in the contact case [10, 11], we give a result about the existence of open book decompositions in quasi-contact manifolds. For this we need

Definition 1.5. *An open book decomposition for (C,ω) consists of the following data:*

- *A codimension 2 smooth submanifold, called the binding, $B\subset C$.*

- *A submersion* $f : C - B \to S^1$ *satisfying that for any point* $p \in B$*, there exist coordinates* $(z_1, \dots, z_n, t) \in \mathbb{C}^n \times \mathbb{R}$ *around* p *such that* B *is defined by* $z_1 = 0$ *and* f *is the map* $(z_1, \dots, z_n, t) \mapsto \arg(z_1)$.

A quasi-contact structure ω of the form $\omega = d\alpha$ is called an *exact* quasi-contact structure. An exact quasi-contact manifold is then a pair $(C, d\alpha)$, where the primitive α is understood as fixed. A transverse field for an exact quasi-contact manifold is a vector field $X \in \Gamma(TM)$ such that $i_{X(p)}\alpha(p) > 0$ whenever $\alpha(p) \neq 0$.

We aim to prove the following main theorem

Theorem 1.6. *Let* $(C, d\alpha)$ *be an exact quasi-contact manifold. Then there exists an open book decomposition* $f : C - B \to S^1$ *for* C*. The fiber* $f^{-1}(\theta)$ *over any* $\theta \in S^1$ *is an open manifold with the homotopy type of an* $(n+1)$*-dimensional CW-complex. Moreover there exists a perturbation* α' *of* α *with* $d\alpha' = d\alpha$ *and a transverse vector field* X *for* $(C, d\alpha')$ *such that* $df(X)$ *is non-zero everywhere off the binding.*

Here we will have to perturb α, but we will not perturb the quasi-contact structure $d\alpha$, which may be considered fixed.

This existence of open books for general odd dimensional manifolds is a well-known result [20, 14, 19] (in fact, getting a slightly stronger control on the topology of the leaves). This theorem can be considered as another proof of this fact in a special case. The advantage here is that the existence of the transverse vector field provides some extra control. Moreover, the geometry of the map is controlled in various different ways. This provides a framework that will be discussed in Section 5.

2. Quasi-contact manifolds

Let C be a smooth closed $(2n+1)$-dimensional manifold endowed with a 2-form $\omega \in \Omega^2(C)$ such that $\omega^n \neq 0$ at every point of C. The non-degeneracy condition implies that at every point $p \in C$ we may find a basis $\{e_1, e_2, \dots, e_{2n}, e_{2n+1}\}$ of T_pC such that ω_p is written as

$$\omega_p = e_1^* \wedge e_2^* + \cdots + e_{2n-1}^* \wedge e_{2n}^*,$$

where $\{e_1^*, \dots, e_{2n+1}^*\}$ is the dual basis. This defines a 1-dimensional distribution H of TC where H_p is the ray spanned by e_{2n+1}. More intrinsically, ω is a map $T^*C \to \bigwedge^{2n+1}(T^*C)$, where $\mu = \bigwedge^{2n+1}(T^*C)$ is a real line bundle over C, i.e., ω is a section of

$$(T^*C)^* \otimes \mu = TC \otimes \mu,$$

that is, a μ-valued vector field. After projectivization, ω defines H. Otherwise said, such distribution H is defined as the kernel of ω. Locally, $H = \langle X \rangle$, where X is a vector field such that $i_X\omega = 0$.

We say that ω is orientable (oriented) if H is orientable (oriented). Also we say that it is coorientable (cooriented) if the $2n$-plane bundle defined as $D = TC/H$ is orientable (oriented). Note that if C is orientable, then the notion of ω being orientable coincides with ω being coorientable.

Let g be a Riemannian metric on C. This defines a metric in H and a $2n$-dimensional distribution $D = H^\perp \subset TC$. As a $2n$-plane bundle, such D is isomorphic to TC/H. In this case, at any point p, we can choose an *orthonormal* basis $\{e_1, \ldots, e_{2n}, e_{2n+1}\}$ such that $\omega = e_1^* \wedge e_2^* + \ldots e_{2n-1}^* \wedge e_{2n}^*$. Note that this gives a reduction of the structure group of C to $U(n) \times \mathbb{R}$.

An ω-compatible almost-complex structure J is an endomorphism $J : TC \to TC$ such that $J|_H \equiv 0$, $D = \operatorname{im} J \subset TC$ is a codimension 1 distribution and $J|_D$ is a $\omega|_D$-compatible almost complex structure. Obviously there is a decomposition $TC = D \oplus H$ and $\omega|_D$ is a symplectic structure on the bundle D. This defines a metric in D by the usual formula $g(u, v) = \omega(u, Jv)$. Alternatively, a metric g is compatible with ω if, setting $D = H^\perp$, $g|_D$ and $\omega|_D$ define an ω-compatible almost-complex structure J. For a metric g, there is a local Reeb vector field, defined up to sign, as a unitary vector field in H.

As an example, we can think of a (cooriented) contact manifold (C, α). In this case $\omega = d\alpha$, $D = \ker \alpha$, $H = \ker d\alpha$. We have a global Reeb vector field defined by the condition $\alpha(R) = 1$.

Now suppose $d\omega = 0$. In this case we are dealing with a quasi-contact manifold (C, ω). Then we have a Darboux theorem

Theorem 2.1. *Locally at $p \in C$, there exist coordinates $(x_1, \ldots, x_{2n}, x_{2n+1})$ such that $\omega = dx_1 \wedge dx_2 + \cdots + dx_{2n-1} \wedge dx_{2n}$.* □

We may write the coordinates as $(z_1, \ldots, z_n, t) \in \mathbb{C}^n \oplus \mathbb{R}$, where $z_1 = x_1 + ix_2, \ldots, z_n = x_{2n-1} + ix_{2n}, t = x_{2n+1}$. The standard quasi-contact model is $\mathbb{C}^n \oplus \mathbb{R}$ with quasi-contact form $\omega_0 = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + \ldots + dz_n \wedge d\bar{z}_n)$. Choosing the product metric $g_0 = |z_1|^2 + \ldots + |z_n|^2 + t^2$, the associated almost complex structure J_0 is the standard complex structure on the $\mathbb{C}^n$ factor and annihilates the $\mathbb{R}$ factor. The associated distribution is the horizontal distribution defined as $D_h = \mathbb{C}^n \oplus \{0\}$.

A chart $\Psi : B(p, \kappa) \to \mathbb{C}^n \times \mathbb{R}$ provided by Theorem 2.1 will be called a Darboux chart, and it satisfies $\Psi^* \omega_0 = \omega$. We may assume that at the point p, the distribution D coincides with D_h and the almost complex structure J matches the standard almost complex structure J_0. Actually, the angle between D_h and D is $O(|(z, t)|)$. Also the norm of the operator $J - J_0$ is bounded by $O(|(z, t)|)$. We shall call such Ψ *adapted coordinates* at p.

The almost complex structure J also acts on the cotangent space $T^*C = H^* \oplus D^*$. This produces a decomposition of the complexified cotangent bundle as

$$T^*_{\mathbb{C}} C = H^*_{\mathbb{C}} \oplus (D^*)^{1,0} \oplus (D^*)^{0,1},$$

where J acts as 0 in $H^*_{\mathbb{C}}$, as i in $(D^*)^{1,0}$ and as $-i$ in $(D^*)^{0,1}$. The covariant derivative along the D-directions ∇_D is defined as the composition

$$\Gamma(L^{\otimes k}) \xrightarrow{\nabla} \Gamma(T^*C \otimes L^{\otimes k}) \to \Gamma(D^* \otimes L^{\otimes k})$$

This may be further decomposed as $\nabla_D = \partial + \bar{\partial}$ according to the decomposition on $(1,0)$ and $(0,1)$-forms.

Consider finally the case of an exact quasi-contact manifold $(C, d\alpha)$. In this case α defines, whenever it is non-zero, a hyperplane distribution $K = \ker \alpha$. If $\alpha \wedge (d\alpha)^n$ is non-zero at a point p, then $K_p \cap H_p = 0$. Therefore K_p and D_p are isomorphic through orthogonal projection and K_p inherits a symplectic form $d\alpha|_{K_p}$. In the case that $\alpha \wedge (d\alpha)^n = 0$ at p, then $H_p \subset K_p$ and K_p is not symplectic.

Note that, for generic α, there is a finite number of points x where $\alpha(x) = 0$ and there is a codimension 1 submanifold defined by the condition $\alpha \wedge (d\alpha)^n = 0$.

3. Asymptotically holomorphic theory

Let (C, ω) be a quasi-contact manifold such that the cohomology class $[\omega] \in H^2(C; \mathbb{R})$ admits a lift to integer cohomology $H^2(C; \mathbb{Z})$. Then there is a hermitian line bundle $L \to C$ with $c_1(L) = [\omega]$. Put a connection ∇ on L with curvature $F_\nabla = -i\omega$. This bundle is usually called the *prequantum* line bundle. Note that for an exact quasi-contact manifold, the line bundle L is trivial and hence sections of L are smooth complex valued functions.

We choose an ω-compatible almost-complex structure J and a metric g on C such that $g(u, v) = \omega(u, Jv)$ on D. Let k be an integer. We shall denote by g_k the rescaled metric kg on C. Note that the associated quasi-contact structure to g_k and J is given by the 2-form $k\omega$. Therefore the distribution D is the same for all k. If $p \in C$ then take adapted coordinates $\Psi : B_g(x, \kappa) \to \mathbb{C}^n \times \mathbb{R}$ for (C, ω). Such Ψ satisfies that $\Psi^*\omega_0 = \omega$ and it is an isometry at x. Then there are adapted coordinates Ψ_k for the ball $B_{g_k}(x, \kappa)$ defined as $\Psi_k = k^{1/2}\Psi$. Such coordinates satisfy that $|\Psi_k| \leq C_0$, $|\Psi_k^{-1}|_{g_k} \leq C_0$, $|\nabla^r \Psi_k^{-1}|_{g_k} \leq C_r$ in $B_{g_k}(x, \kappa)$, for constants C_r independent of k. It also follows that in the chart Ψ_k the angle between the distribution D and the horizontal distribution D_h is less than $Ck^{-1/2}|(z, t)|$, with $C > 0$ a uniform constant. Also in this chart, the norm of the map $J - J_0$ is $O(k^{-1/2}|(z, t)|)$.

Definition 3.1. *Let E_k be hermitian vector bundles with connections. Let $\{\tau_k\}_{k \in \mathbb{Z}^+}$ be a sequence of sections of E_k. τ_k is said to be C^r-asymptotically holomorphic with positive constants C_D, C_H if the following inequalities are* *satisfied:*

$$\begin{aligned} |\nabla \tau_k|_{g_k} + \cdots + |\nabla^r \tau_k|_{g_k} &\leq C_H \\ |\tau_k|_{g_k} + |\nabla_D \tau_k|_{g_k} \cdots + |\nabla^r_D \tau_k|_{g_k} &\leq C_D \\ |\bar{\partial} \tau_k|_{g_k} \cdots + |\nabla^{r-1} \bar{\partial} \tau_k|_{g_k} &\leq C_H k^{-1/2}. \end{aligned}$$

We shall abbreviate by saying that τ_k is a C^r-A.H.(C_D, C_H) sequence of sections.

Here the covariant derivatives above are taken using the connection induced in both $T^*_{\mathbb{C}} C$ and $D^*_{\mathbb{C}}$. We use different constants C_D, C_H because it is possible to find C^r-A.H. sequences of sections with interesting transversality properties for which the constant C_D governing the *holomorphic* directions is as small as we want, whereas we cannot keep much control on C_H.

We are interested in sections transverse to the zero section along the directions of D. We shall use the following concept.

Definition 3.2. *Let τ be a section of a hermitian bundle $E \to C$ with connection. τ is said to be η-transverse along the directions of D (or just η-transverse) if at every $x \in C$ where $|\tau(x)| \leq \eta$, $\nabla_D \tau(x) : D_x \to E_x$ is surjective and the norm of the smallest right inverse for this map does not exceed $1/\eta$.*

Let $E_k \to C$ be a sequence of hermitian bundles with connection. A sequence of sections τ_k of E_k is uniformly transverse to zero if there are $k_0 \in \mathbb{N}$ and $\eta > 0$ so that for every $k \geq k_0$, τ_k is η-transverse to zero at every $x \in C$.

Now fix a line bundle P over C. One can prove the following

Theorem 3.3. *Let (C, ω) be a quasi-contact manifold and let τ_k be a sequence of C^r-A.H.(C_D, C_H) sections of $P \otimes L^{\otimes k}$ ($r \geq 2$). Given $\delta > 0$ small enough, there exist C'_H, k_0 and η depending on δ, C_D, C_H (but not on k), and a sequence σ_k of sections of $P \otimes L^{\otimes k}$ satisfying:*

1. *σ_k is C^r-A.H.(δ, C'_H).*
2. *For every $k \geq k_0$, $\tau_k + \sigma_k$ is η-transverse to zero.*

Sketch of the proof. The proof of the theorem has three steps:

1. *Existence of reference sections.*

   Definition 3.4. *A sequence τ_k of sections of hermitian bundles E_k has C^r-mixed Gaussian decay with respect to a point $x \in C$ if there exist positive constants $\lambda > 0$, C_a and a polynomial P_r verifying that for all $y \in M$ and $r \geq 0$,*

   $$|\nabla^r_D \tau_k(y)|_{g_k} \leq P_r(d_k(x,y)) \exp(-\lambda d_k(x,y)^2),$$

   $$|\nabla^r \tau_k(y)|_{g_k} \leq C_a P_r(d_k(x,y)) \exp(-\lambda d_k(x,y)^2).$$

   Here d_k is the distance associated to g_k.

Definition 3.5. *A family of reference sections $(\tau^{ref}_{k,x})_{k\in\mathbb{N},x\in C}$, is a family of C^r-A.H.(C_D, C_H) sections of E_k such that:*

(a) *$\tau^{ref}_{k,x}$ has C^r-mixed Gaussian decay with respect to x (the constant C_a being independent of x).*

(b) *There are positive constants c, κ so that $|\tau^{ref}_{k,x}| \geq c$ in $B_{g_k}(x,\kappa)$.*

Lemma 3.6. *Let (C,ω) be a quasi-contact manifold. Then there exist families of reference sections for the sequence of bundles $P \otimes L^{\otimes k}$.*

Proof. First let us suppose that P is trivial. Choose adapted coordinates $(z_1,\ldots,z_n,t) \in \mathbb{C}^n \oplus \mathbb{R}$ around x. Such coordinates Ψ_k are actually defined in a ball $B_{g_k}(x,k^{1/2})$. The standard quasi-contact model for $\mathbb{C}^n \oplus \mathbb{R}$ is given by the horizontal distribution D_h and the standard complex structure J_0 and symplectic form ω_0. Since the curvature of $L^{\otimes k}$ is $k\omega = \Psi_k^*\omega_0$, we can choose a (unitary) trivialization of the bundle such that the connection form $d + A_0$ has $A_0 = \frac{1}{4}\sum_{i=1}^n (z_i d\bar{z}_i - \bar{z}_i dz_i)$. Then as in [5] the section $e^{-|(z,t)|^2/4} = e^{-t^2/4}e^{-|z|^2/4}$ is holomorphic in this standard model. This implies the C^r-A.H. condition when we use the distribution D and the almost complex structure J (recall that the angle between D and D_h is bounded by $Ck^{-1/2}|(z,t)|$). Now this section is multiplied by a cut-off function with support in a ball of radius $k^{1/6}$ and extended by zero. This gives rise to a C^r-A.H. sequence of sections $\tau^{\mathrm{ref}}_{k,x}$ with Gaussian decay with respect to x.

If P is not a trivial bundle, then we take a trivialization of P and multiply it by $\tau^{\mathrm{ref}}_{k,x}$. This gives reference sections for $P \otimes L^{\otimes k}$. □

(2) *Local estimated transversality result.* Reference sections are used to turn the estimated transversality problem for C^r-A.H. sections into the corresponding one for C^r-A.H. functions. The amount of transversality that one can obtain can be expressed in terms of the size of the perturbation δ.

Proposition 3.7. *(Proposition 4.4 in* [13]*) Let B be the unit ball in $\mathbb{C}^n$ and let $f\colon B \times [0,1] \to \mathbb{C}^m$ be a complex valued function. Let $0 < \delta < 1/2$ be a constant and let $\sigma = \delta(\log(\delta^{-1}))^{-p}$, where p is a suitable fixed integer depending only on the dimensions n, m. Assume that $f_t = f(\cdot,t)$ satisfies the following bounds over $B \times [0,1]$:*

$$|f_t| \leq 1,\ |\bar{\partial} f_t| \leq \sigma,\ |\nabla\bar{\partial} f_t| \leq \sigma.$$

Then there exists a smooth curve $w\colon [0,1] \to \mathbb{C}^m$ such that $|w| < \delta$ and the function $f_t - w(t)$ is σ-transverse to zero over a smaller ball $\frac{9}{10}B$. Moreover, if $|\partial f_t/\partial t| < 1$ and $|\partial(\nabla f_t)/\partial t| < 1$, we can choose w such that $|d^i w/dt^i| < \Phi_i(\delta)$, for all $i \in \mathbb{N}$, $d^j w/dt^j(0) = 0$ and

$d^j w/dt^j(1) = 0$ for all $j \in \mathbb{N}$, where Φ_i if a function depending only in the dimensions n, m. □

In this proposition the outcome is σ-transversality with respect to the horizontal distribution D_h. The bound on $|dw/dt|$ implies say, $0.9\,\sigma$-transversality with respect to D, once k is large enough.

(3) *Globalization process.* The globalization process is a standard procedure that allows to perturb an arbitrary sequence of C^r-A.H. sections to obtain uniform transversality (for every k bigger than some k_0). It only relies on the existence of local perturbations by sequences of C^r-A.H. sections with C^r-mixed Gaussian decay for which the amount of local uniform transversality that can be obtained is that of Proposition 3.7 (see for example Proposition 4.2 in [12]).

□

Observe that for the sequence of bundles $P \otimes L^{\otimes k}$, we only need the version of Lemma 3.7 with $m = 1$, i.e., for maps of the form $\mathbb{C}^n \oplus \mathbb{R} \to \mathbb{C}$.

In general, the results of Theorem 3.3 can be also obtained for sequences of hermitian bundles E_k, as long as we have *reference frames*. These are local frames $\tau_{k,j}$ made of C^r-A.H. sections with C^r-mixed Gaussian decay, so that over some g_k-ball of constant radius they can be compared with a unitary frame. I.e., if $f_{k,j}$ is a unitary frame and $\tau_{k,i} = \sum_j a_{ij,k} f_{k,j}$ then the matrix $(a_{ij,k})$ and its inverse (and its derivatives) are bounded uniformly independent of k. The sequence $E \otimes L^{\otimes k}$, where E is any hermitian bundle with connection, admits reference frames (just by multiplying reference sections with local unitary frames).

In order to prove Theorem 1.6 we will need the following estimated transversality result. The statement is as follows:

Theorem 3.8. *Let (C, ω) be a quasi-contact manifold, $p_1, \dots, p_r \in C$ fixed points in C and a constant $\lambda > 0$. Given $\delta > 0$ small enough (compared to λ), there exist C_R, C_H, C'_H, k_0 and η depending on δ (C'_H, η independent of λ ; C_D, C_H depending on λ), and a sequence σ_k of sections of $L^{\otimes k}$ satisfying:*

(1) *σ_k is C^r-A.H.(C_D, C_H).*
(2) *For $k \geq k_0$, σ_k is η-transverse to zero.*
(3) *$\sigma_k(p_j) = 0$ and for any point $x \in B_{g_k}(p_j, \lambda)$ satisfying $\sigma_k(x) \neq 0$, we have that $\left| d\frac{\sigma_k(x)}{|\sigma_k(x)|} \right| \geq 1/2$.*
(4) *σ_k is C^r-A.H.(δ, C'_H) on the open set $C - \bigcup_j B_{g_k}(p_j, \lambda)$.*

The correct understanding of the result is as follows. Property (3) can be obtained on a finite number of balls of g_k-radius arbitrarily big without altering the transversality and the bounds of the asymptotically holomorphic sequence σ_k outside those balls.

Proof. We start by constructing a sequence localized at a neighborhood of the given points. First we modify J around p_j to make it integrable. Choose one of the points p_j. Choose a surjective function s defined in a small neighborhood U of p_j satisfying that $\ker d\alpha(p_j) = \ker ds(p_j)$. Locally s must be understood as a "vertical" coordinate. Define in $U \times \mathbb{R}$ the closed form $\omega = ds \wedge dt + d\alpha$, where t is the a coordinate for $\mathbb{R}$ (we are omitting the obvious pull-backs). It is simple to check that ω is symplectic. We choose an almost complex structure J_s in $U \times \mathbb{R}$ by pulling-back the one in D and extending it to the whole tangent bundle by declaring $J_s(\frac{\partial}{\partial s}) = \frac{\partial}{\partial t}$.

Now trivialize, as in [5, 1], a neighborhood of $U \times \mathbb{R}$ with a Darboux chart $\psi : U \times (-\varepsilon, \varepsilon) \to \mathbb{C}^{n+1}$ that is holomorphic at the origin and has a compact variation of the $\bar{\partial}$-part. We can also assume that $\psi(U \times \{0\})$ is tangent to $\mathbb{C}^n \times \mathbb{R} \subset \mathbb{C}^{n+1}$ at the origin and its tangent spaces get away from this horizontal subspace at a bounded rate. Moreover we can define a sequence of charts $\psi_k = k^{-1/2}\psi$ that are Darboux charts for the symplectic structure in $\mathbb{C}^{n+1}$ given as $k\omega_0$.

There is a holomorphic bundle with connection $\beta = \frac{1}{4}\sum(z_i d\bar{z}_i - \bar{z}_i dz_i)$ which pulls back to the bundle L in C (through all the previous identifications). As in Lemma 3.6,

$$s_k^{\mathrm{ref}} = e^{-k|(z,t)|^2/4}$$

is a holomorphic section of the bundle $L^{\otimes k}$. Note that s_k^{ref} is real valued considered as a function to $\mathbb{C}$. This means that its argument is always zero. Now we define the holomorphic section

$$s_k^{\mathrm{good}} = e^{\lambda^2/4} \cdot k^{1/2} z_1 e^{-k|(z,t)|^2/4}.$$

In $\mathbb{C}^n \times \mathbb{R}$ we have

$$\frac{s_k^{\mathrm{good}}}{|s_k^{\mathrm{good}}|} = \frac{z_1}{|z_1|}.$$

Therefore, after pull-back, we get the bound required in (3) (the details are just a sequence of tedious pull-backs in the style of [5]). Moreover, the section s_k^{good} is A.H. thanks to the Gaussian decay of s_k^{ref}. It is η-transverse to zero on a ball $B_{g_k}(p_j, \lambda)$ for some uniform η independent of λ, since if $|s_k^{\mathrm{good}}| < \eta$ then $k^{1/2}|z_1| < \eta$ and then

$$|\partial s_k^{\mathrm{good}}| = e^{\lambda^2/4} \cdot (k^{1/2} dz_1)(1 - k|z_1|^2/4) e^{-k|(z,t)|^2/4}$$

is bigger than η. Also note that

$$|s_k^{\mathrm{good}}|_{C^r, g_k} \le C \tag{1}$$

outside the ball $B_{g_k}(p_j, \lambda)$ for some uniform constant C independent of λ.

We repeat this process at each of the points p_j, obtaining a section in a neighborhood of each point. We perturb, as usual, in the rest of the manifold with a perturbation of order δ small compared to λ. When we make a perturbation centered in a point x_i at distance $O(k^{1/6})$ of p_j, we take as reference section $\gamma_i z_1 \tau_{k,x}^{\mathrm{ref}}$ instead of $\tau_{k,x}^{\mathrm{ref}}$, where z_1 is the coordinate of the previous discussion that exists because the point is close to p_j, and γ_i is chosen to satisfy $|\gamma_i z_1(x_i)| = 1$. Such section is A.H. when considered in C and it allows to construct a sequence of sections σ_k satisfying still that $\sigma_k(p_j) = 0$. The transversality and bounds obtained are independent of λ, outside the balls $B_{g_k}(p_j, \lambda)$ because of equation (1). This proves property (4).

It is simple to check that the perturbation does not destroy property (3) in the balls. This is because we have chosen the sections $z_1 \tau_{k,x}^{\mathrm{ref}}$ at points close to the balls and this perturbation vanishes whenever s_k^{good} does. □

4. Applications.

We now prove the two existence results stated in the introduction.

4.1. Existence of codimension 2 quasi-contact submanifolds.

Let us see how Theorem 3.3 implies very easily Theorem 1.4.

Proof of Theorem 1.4. Construct a hermitian line bundle P whose first Chern class is $\beta \in H^2(M;\mathbb{Z})$. Construct a sequence of C^r-A.H. and ε-transverse sections s_k of the bundles $P \otimes L^{\otimes k}$ using Theorem 3.3. The zero sets $Z(s_k)$ are smooth codimension 2 submanifolds that are Poincaré dual to $[\beta + k\omega]$. Moreover the transversality along D implies that D is transverse to $Z(s_k)$ and in particular $D_k(x) = D_x \bigcap T_x Z(s_k) = \ker \nabla_D s_k$. To conclude that $Z(s_k)$ is quasi-contact we just have to show that $D_k(x)$ is a symplectic subspace of D_x. For that we use the following lemma:

Lemma 4.1. [5] *Given a map $f : \mathbb{C}^n \to \mathbb{C}$, the subspace $\ker f$ is symplectic if and only if $|f^{1,0}| > |f^{0,1}|$.* □

In our case we have to check that $|\partial s_k(x)|_{g_k} > |\bar{\partial} s_k(x)|_{g_k}$. This follows, for k large, from the ε-transversality ($|\nabla_D s_k(x)| > \varepsilon$), the bounds of the sequence ($|\bar{\partial} s_k(x)|_{g_k} = O(k^{-1/2})$) and the triangular inequality ($|\partial s_k(x)| \geq |\nabla_D s_k(x)| - |\bar{\partial} s_k(x)|$).

To finish we have to study the homology groups of $Z(s_k)$. The result written in the statement follows if we are able to find a Morse function f_k in $C - Z(s_k)$ such that all the critical points have index at least $n-1$. The natural choice is $f_k = |s_k|^2$. From there, the proof follows word by word the one given in the contact case [12]. □

4.2. Existence of open books.

The complex line $\mathbb{C}$ has an open book decomposition $\mathcal{B}$, where the binding is the origin and the leaves are the rays emanating from it. The strategy to put an open book decomposition in C is to construct an asymptotically holomorphic section τ_k of $L^{\otimes k}$ in general position with respect to $\mathcal{B}$. A condition of transversality for the section will assure that the pull-back of $\mathcal{B}$ is an open book for C. Note that the binding B is going to be the zero set of τ_k and by Theorem 1.4, this is a codimension 2 quasi-contact submanifold.

We must remark that the following proof follows closely the one described by Giroux and Mohsen [10, 11] in the contact case.

Proof of Theorem 1.6. We start with an exact quasi-contact manifold $(C, d\alpha)$ and perturb our primitive α by adding to it a small exact 1-form df in order to make the zeroes of α isolated and transverse. We denote these zeroes by $\{p_j\}_{j=1,\dots,r}$.

We apply Theorem 3.8 choosing the zeroes $p_1, \dots, p_r$ as the set of points in the statement. We will fix later the constant λ. We get a sequence of sections s_k which is asymptotically holomorphic and transverse. Since L is topologically trivial, actually s_k are smooth functions on C. We define the binding $B \subset C$ as the zero set $B = Z(s_k)$. By Theorem 1.4, B is a codimension 2 quasi-contact submanifold of C. Now we want to show that the map

$$\begin{array}{rcl} \phi_k : C - Z(s_k) & \to & S^1 \\ p & \mapsto & \dfrac{s_k}{|s_k|} \end{array}$$

is an open book satisfying all the required properties.

First, note that we have in $Z(s_k)$ that $\nabla_D s_k$ has a small right inverse. Moreover we have $|\nabla\nabla s_k|_{g_k} = O(1)$. Therefore applying the implicit function theorem to the set

$$\mathcal{U} = \{x \in C : |s_k(x)| \le \rho\},$$

with $\rho > 0$ small enough (independent of k, though) and to s_k we get that $\mathcal{U}$ is diffeomorphic to a subset of $Z(s_k) \times \mathbb{R}^2$. Also, after this transformation, ϕ_k becomes the standard projection map from $Z(s_k) \times (\mathbb{R}^2 - \{0\})$ to S^1. Note that ρ depends only on η and C'_H and therefore it is independent of λ.

Now let us see that there is a vector field X controlling the open books that we obtain. Define X by the condition of being the dual of α through g and normalized so that it has unit length. Recall that the connection 1-form on $L^{\otimes k}$ is given by $-ik\alpha$, so

$$\nabla s_k = ds_k - ik\alpha s_k.$$

Now the asymptotically holomorphic condition on s_k implies first that $|\nabla s_k|_{g_k} = O(1)$. Therefore, in metric g, we have

$$ds_k(X) - ik\alpha(X)s_k = \nabla_X s_k = O(k^{1/2}).$$

Now, let $p \in C$ be a point satisfying that $|s_k(p)| > \rho$. Suppose first that p is in one of the balls $B_{g_k}(p_j, \lambda)$. Then by property (3) of Theorem 3.8 we get that ϕ_k is surjective. Suppose now that the point satisfies $|s_k(p)| > \rho$ and it is not in any of the balls. Therefore there is a uniform constant $c > 0$ such that $|\alpha| \geq ck^{-1/2}\lambda$ at p, measured in the metric g. Therefore

$$|ik\alpha(X)s_k| \geq ck^{1/2}\lambda\rho.$$

On the other hand we have

$$ds_k(X) = ik\alpha(X)s_k + O(k^{1/2})$$

where the second term is smaller than the first one, if we take λ big enough. This is because the second term does not depend on λ according to property (4) of Theorem 3.8. Now we compute

$$\frac{d\phi_k}{\phi_k}(X) = \frac{s_k d\bar{s}_k - \bar{s}_k ds_k}{|s_k|^2}(X) = -2ik\alpha(X) + O(k^{1/2}).$$

This implies that, for k large, $|d\phi_k(X)| > 0$ at p.

Now we have to extend X to the interior of the balls $B_{g_k}(p_j, \lambda)$. For this we consider the vector field X_j dual to the 1-form $i\frac{d\phi_k}{\phi_k}$ defined in $B_{g_k}(p_j, \lambda) - B$. We interpolate (linearly) between X and X_j in a small annulus. This gives a vector field, that we call X again, which satisfies $d\phi_k(X) > 0$ in $C - \mathcal{U}$. With this gradient like X we perturb α (with exact forms) inside the balls to make X transverse to it.

Finally, it is simple to deform X in a neighborhood of $Z(s_k)$ of g_k-radius ρ by using the transversal form $Z(s_k) \times (\mathbb{R}^2 - \{0\})$ obtained before, in order to get another transverse field which is gradient-like for ϕ_k and transverse to α.

To finish we need to study the topology of a single leaf. The argument is very similar to the one given in the proof of Theorem 1.4. We fix a leaf $\mathcal{L}_\theta = \phi_k^{-1}(\theta)$, $\theta \in S^1$. Recall that at any $p \in \mathcal{L}_\theta$, the tangent space to the leaf is $\ker d\phi_k(p)$. We consider the function $f_k = |s_k|^2$ restricted to $\mathcal{L}_\theta$. We shall conclude once we show that all the critical points have index greater or equal to $n - 1$. We assume that the critical points are generic, i.e., the function f_k is Morse (in the leaf). This can be achieved by a small perturbation of s_k (preserving the estimated transversality and the asymptotic holomorphicity of the sequence). We may also assume that the critical points of f_k are isolated in the leaf and at such points the tangent space to the leaf is symplectic.

Now we proceed as follows. We compute the derivative of the function f_k to get

$$\partial f_k = \langle \partial s_k, s_k \rangle + \langle s_k, \bar{\partial} s_k \rangle. \tag{2}$$

At a critical point $x \in \mathcal{L}_\theta$, we have $d(f_k|_{\mathcal{L}_\theta})(x) = 0$. In particular, denoting $N_x = D_x \bigcap T_x\mathcal{L}_\theta$, we know that restricted to N_x, we have $\partial f_k = 0$. We claim that

$$|df_k(x)| = O(k^{-1/2})$$

for all the subspace D_x. We already know this in N_x. Denote by $\langle v_x \rangle$ the (Riemannian) orthogonal subspace to N_x in D_x. We choose v_x to be unitary in g_k-norm. Observe that $Jv_x \in N_x$, since Jv_x is in D_x and it is orthogonal to v_x. In particular, $d_{Jv_x} f_k = 0$. By equation (2), this implies that

$$|\langle \partial_{Jv_x} s_k, s_k \rangle| = |\langle s_k, \bar{\partial}_{Jv_x} s_k \rangle| = O(k^{-1/2}).$$

So $|\partial_{Jv_x} s_k| = |i\partial_{v_x} s_k| = O(k^{-1/2})$ and hence

$$d_{v_x} s_k = \partial_{v_x} s_k + \bar{\partial}_{v_x} s_k = O(k^{-1/2}),$$

always measured in g_k-norm. Finally, this implies that

$$d_{v_x} f_k = O(k^{-1/2}),$$

as claimed.

Since f_k is a real valued function, we have that $|\partial_{v_x} f_k(x)| = |\bar{\partial}_{v_x} f_k(x)| = O(k^{-1/2})$. By equation (2) we get at the critical point $|\langle \partial s_k, s_k \rangle| = |\langle s_k, \bar{\partial} s_k \rangle| = O(k^{-1/2})$. Also since $\partial s_k(x)$ is very small on D_x, the η-transversality guarantees that $|s_k(x)| > \eta$. This implies that $|\partial s_k|_{g_k} = O(k^{-1/2})$.

Another derivative will yield

$$\begin{aligned} \bar{\partial}\partial f_k &= \langle \bar{\partial}\partial s_k, s_k \rangle + \langle \partial s_k, \partial s_k \rangle + \langle s_k, \partial\bar{\partial} s_k \rangle + \langle \bar{\partial} s_k, \bar{\partial} s_k \rangle = \\ &= \langle \bar{\partial}\partial s_k, s_k \rangle + O(k^{-1/2}), \end{aligned}$$

where the term $O(k^{-1/2})$ is measured in g_k-norm.

Now recall that $\bar{\partial}\partial + \partial\bar{\partial}$ is the part $(1,1)$ of the curvature of the bundle $L^{\otimes k}$. So we have

$$\begin{aligned} \bar{\partial}\partial f_k &= -(ikd\alpha)|s_k|^2 - \langle \partial\bar{\partial} s_k, s_k \rangle + O(k^{-1/2}) = \\ &= -(ikd\alpha)|s_k|^2 + O(k^{-1/2}), \end{aligned}$$

at x, for any pair of tangent vectors in $D_x \cap T_x\mathcal{L}_\theta$. Note that $kd\alpha$ is $O(1)$ in g_k-norm.

We have that $\nabla f_k(x) = 0$ on $D_x \cap T_x\mathcal{L}_\theta \cap J(D_x \cap T_x\mathcal{L}_\theta)$. Therefore an easy computation yields that

$$-2i\bar{\partial}\partial f_k(u, Ju) = H_{f_k}(u) + H_{f_k}(Ju)$$

at the point x, for vectors $u \in D_x \cap T_x\mathcal{L}_\theta$ and where H_{f_k} is the Hessian of f_k in the directions of D_x.

Now suppose that the index of the critical point x is less than or equal to $n-2$, i.e., that there is a subspace $V \subset T_x\mathcal{L}_\theta$ in which the Hessian H_{f_k} is positive, such that its dimension is greater or equal to $n+2$. Then $V \cap D_x$ is a subspace of D_x of dimension $n+1$ at least. Therefore $V \cap D_x \cap J(V \cap D_x)$ has dimension at least 2. Let u be a non-zero vector in this intersection. Then $H_{f_k}(u) + H_{f_k}(Ju)$ is positive whereas $-2i\bar{\partial}\partial f_k(u, Ju) = -2kd\alpha(u, Ju)|s_k|^2 + O(k^{-1/2})$ is negative. This is a contradiction.

Hence the index of the critical point is at least $n-1$. Therefore the leaf $\mathcal{L}_\theta$ is homotopy equivalent to an $(n+1)$-dimensional CW-complex. □

5. Further speculations

Theorem 1.6 is a first step in a characterization of the relations between contact and almost contact manifolds. We recall that, according to [10, 11], an open book decomposition recovers a contact structure whenever the leaf is Stein and the monodromy (map generated by the return 1 map of the gradient of ϕ_k) is a symplectomorphism. According to a theorem of Eliashberg [7], given a $2n$-dimensional open manifold with the homotopy type of a n-dimensional skeleton and satisfying an extra homotopical assumption, it is always possible to construct a Stein structure in the manifold. The problem is so to understand the conditions in which the leaves in Theorem 1.6 become homotopic to an n-dimensional skeleton and then approximate the monodromy map of the open book by a symplectomorphism. The classical results about existence of open books [14, 20, 19] are far away from providing a method to control the monodromy map, though. They provide a method of controlling the topology of the leaf, but nothing is said about the monodromy map.

However, in our case the monodromy map is in a sense more controlled. To go further we will need to understand the behavior of 1-parametric families of open books decompositions. This has already been studied in the holomorphic situation [11].

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