GLUING CONSTRUCTIONS IN POISSON GEOMETRY

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1. Introduction

Most examples of Poisson manifolds come through symplectic geometry (symplectic manifolds, symplectic fiber bundles), Lie theory (linear Poisson structures, Poisson Lie groups, Poisson structures on homogeneous spaces), reduction from symplectic, Poisson, Dirac, quasi-Hamiltonian geometry (and also the more flexible cases in dimensions 2 and 3). We would like to put Poisson structures on compact manifolds. We are most interested in Poisson structure the closure of whose support is all the manifold (we know we can build non-trivial Poisson structures localized at will).

Trying to construct on a given closed manifold a (non-symplectic) Poisson structure the closure of whose support is M is basically hopeless. What seems feasible is to construct some of this structures in large families of manifolds (not fixed a priori) by adapting gluing techniques from differential topology. In that way at least we may be able to explore aspects of whether the topology of a manifold may obstruct the existence of certain kind of Poisson structures.

2. Gluing constructions in differential topology.

In differential topology gluing to manifolds M, M' to produce a new one roughly means finding **gluing data** $\phi\colon U \to U'$, $U \subset M, U' \subset M'$ open, to construct

$$M\#_\phi M' := M \coprod M'/U \overset{\Phi}{\sim} U'$$

Observe that not any U will do, because we may get a non-Hausdorff manifold: Glue for example two copies of $\mathbb{R}^2$ by a diffeomorphism of the unit open disks. In all our examples of gluing constructions to be described, we ask (i) U to be diffeomorphic to $K \times (-1, 1)$, where K is a submanifold (often compact), (ii) the closure $\bar{U}$ is a submanifold diffeomorphic to the $K \times [-1, 1)$, so only points near one end are added, and (iii) the diffeomorphism Φ is a product map (ϕ, f) with $\phi\colon K \to K'$, and $f\colon (-1, 1) \to (-1, 1)$ orientation reversing, i.e. for example

$$\Phi(x, t) = (\phi(x), -t)$$

What makes each of the instances we are to present different is the way in which the gluing data is obtained, which one would like to be as "economic" as possible, also to be able to draw results about uniqueness of the diffeomorphism type of $M\#_\Phi M'$ (or uniqueness up to some choices).

Remark 1. *One can weaken the diffeomorphism type of U, by requiring it to have two ends so that the closure only adds a submanifold of points near one of the ends, and Φ interchanges them. The advantage of asking (U, Φ) to be a product as above is that the diffeomorphism type of $M\#_\Phi M'$ will only depend on the isotopy class of ϕ (though we also fixed the product structure on U), rather than on just the diffeomorphism Φ, as it would be the case for general (U, Φ).*

Remark 2. *There is a second way of understanding a gluing with gluing data as above: We are removing $K \times (0,1)$, $K' \times (0,1)$ to obtain manifolds with boundaries K, K' respectively, and then identifying the boundaries via ϕ.*

Example 1. *Connected sum: If we have two manifolds M, M' and we fix points p, p' one on each, then the connected sum amounts to doing the following: choose about each point coordinates $x_1, \ldots, x_m$ and $x'_1, \ldots, x'_m$. Then U, U' are the punctured unit ball $B^m(1)\backslash\{0\} \simeq S^{m-1} \times (-1,1)$. The map ϕ is the restriction to any level sphere of a orthogonal map from one coordinate set to the other. Notice that there are two possible isotopy classes, according to whether the orientation is preserved or reversed. This is actually the only indeterminacy. The choice of coordinates does not affect, since any two such balls are isotopic. If for example the two manifolds are oriented, one chooses the linear map so that the resulting manifold is oriented extending the restricted orientation. Since the group of diffeomorphisms of a manifold acts transitively on points for oriented manifolds the connected sum is uniquely defined.*

Recall that a particular example of connected sum is the complex/symplectic blow up. At the level of differential topology blowing up a point $x \in X^n$ gives $X^n \# -\mathbb{CP}^n$, the minus meaning a reversal of the (complex) orientation.

Example 2. *Family connected sum or normal connected sum: Rather than points this time we pick (i) N, N' (compact) submanifolds, and (ii) an isomorphism of the normal bundles; it restricts to an identification of sphere bundles. Then we use it to identify $\nu(N)\backslash N$ with $\nu(N')\backslash N'$ so fibers are identified as in the connected sum. As we mention, the diffeomorphism type depends on the isotopy class of identifications of sphere normal bundles; as a matter of fact one normally fixes $\varphi \colon N \to N'$ (or a isotopy class of diffeomorphisms) and lets ϕ to vary among lifts, so that isotopy classes of such ϕ correspond to homotopy classes of maps to the corresponding orthogonal group.*

Example 3. *Boundary gluing: We expand on remark 2. For two manifolds with boundary M, M', a diffeomorphism $\phi \colon \partial M \to \partial M'$ determines a new manifold*

$$M \coprod M' / \partial M \stackrel{\phi}{\sim} \partial M'$$

What we have is a topological manifold. To put the smooth manifold structure we need to use product structures near the boundaries, $\partial M \times (-1,1), \partial M' \times (-1,1)$ (so we enlarge a bit each manifold), and then glue using the obvious extension of the boundary diffeomorphism to a product, so we are back in our gluing construction setting. The diffeomorphism type only depends on the isotopy class of ϕ, not on the product structures near the boundary. Observe also that the resulting manifold has no boundary.

Example 4. *Partial boundary gluing: There is a variant of the last construction where in the two manifolds one identifies two codimension 0 submanifolds (with boundary) $\phi \colon W \to W'$. With the aid of product structures one enlarges the manifolds to $M \cup (W \times (-1,1))$ (so what one really adds is $W \times (0,1)$), $M' \cup (W' \times (-1,1))$, and glues using for example*

$$(w,t) \mapsto (\phi(w), 1/2 - t),\ t \in (1/2, 1),$$

in such a way that one creates a "neck". What we produce is a manifold with corners, and then corners are canonically smoothened to produce a manifold with boundary (you will never find a proof of how to canonically smoothen corners in a single book; but it is actually true. It is just very tedious to write down).

This is by far the most important example of gluing, because it contains all **handle attachings**. *An n-dimensional k-handle is topologically an n-ball, but* *written as the manifold with corners* $\overline{B^k(1)} \times \overline{B^{n-k}(1)} = \overline{B^k} \times \overline{B^{n-k}}$. *The boundary is* $S^{k-1} \times \overline{B^{n-k}} \cup \overline{B^k} \times S^{n-k-1}$. *The* **core ball** *is* $\overline{B^k} \times \{0\}$*, and its boundary is the* **attaching sphere**. *Given M an n-manifold with boundary, attaching a k-handle is just doing partial boundary sum with the k-handle, where the submanifold of the boundary of the k-handle that we glue is* $S^{k-1} \times \overline{B^{n-k}}$. *What is useful is that we can encode the gluing data as follows: we think of being gluing first the core ball, this being done by prescribing an attaching map of its boundary the attaching sphere* $\varphi\colon S^{k-1} \hookrightarrow \partial M$. *This is giving an embedded parameterized (k-1)-sphere. Next we have to glue* $S^{k-1} \times \overline{B^{n-k}}$ *to a tubular neighborhood of the embedded sphere; that is saying that the normal bundle of the sphere in ∂M is trivial, and moreover we are giving a specific way of trivializing it,* **a framing**.

When we have a Morse function f on a manifold M, and we have x a critical value so that no other critical value lies in $[f(c)-\epsilon, f(c)+\epsilon]$*, then* $f^{-1}((-\infty, f(c)+\epsilon])$ *is the result of attaching to* $f^{-1}((-\infty, f(c)-\epsilon])$ *a k-handle, where k is the index of the critical point (and the stable manifold is the attached core k-ball).*

Therefore via Morse theory we see that any compact manifold is just the result of a finite sequence of handle attachings.

If now W is an m-manifold, and we give an embedded (k-1)-sphere with a framing of its trivial normal bundle, we can attach an (m+1)-dimensional k-handle to the trivial cobordism $W \times [0,1]$ *in* $W \times \{1\}$. *The new boundary component that we get is what we call the result of performing* **surgery** *on the framed (k-1)-sphere. In the previous Morse theory setting,* $f^{-1}(c+\epsilon)$ *is the result of surgery on the stable (k-1)-sphere of* $f^{-1}(c-\epsilon)$.

The importance of handle attachings and surgery is that we can use it to modify the homotopy groups of a manifold; if a homotopy class can be represented by an embedded sphere with trivial normal bundle, by attaching a handle along it it becomes trivial in the cobordism manifold, and a bit of diagram chasing tells us what happens in the new boundary of the cobordism (for a most beautiful illustration of this see [5]*).*

3. Gluings for regular Poisson structures

If one wants to make gluing constructions compatible with Poisson structures, the one should choose gluing data so that $\Phi\colon U \to U'$ is a Poisson morphism for the induced Poisson structures. That means that the induced Poisson structure should be such that among all the possible gluing maps in the isotopy class, at least one is a Poisson diffeomorphism, and possibly many. At this point we should agree that this is more feasible if the Poisson structure is regular (though some gluings with non-regular gluing data are possible).

Regular Poisson geometry is a blend of foliation theory and symplectic geometry. Let's start with gluings in foliation theory.

3.1. Gluings in foliation theory.

Roughly speaking, the gluings which are compatible correspond to choices of $U = K \times (-1,1)$ for which the boundary $\partial U = K \times \{-1\} \simeq K$ is either a leaf or it is transverse to the foliation. Let us start by the first case, which we may call tangential gluing.

3.1.1. *Tangential gluing:* Because the ∂U is a leaf we are working with codimension 1 foliations. Because the gluing map is a product and should be compatible with the foliations induced on $U = K \times (-1,1)$, $U' = K' \times (-1,1)$, the obvious thing is to assume is that the induced foliations coincide with the product structures that we chose. Thus we are considering the situation of boundary gluing as in section 2 such that the foliations near the boundaries are also given by products structures. Normally rather than imposing the foliation to be a product one looks for sufficient conditions so that this is true; typically we assume $H^1(\partial M;\mathbb{Z})$ to be torsion, so by Reeb-Thurston stability the product foliation condition holds. As a matter of fact it is more interesting to apply the partial boundary gluing but so that the result has no boundary: we pick $N \subset F$ with torsion H^1; clearly $\nu(N) = \nu_{\mathcal{F}}(N) \times (-1,1)$ with the product structure being the induced foliation; we consider as new manifolds with boundary $\nu_{\mathcal{F}}(N) \times (-1,0]$, $\nu_{\mathcal{F}}(N) \times [0,1)$, and look for a diffeomorphism $\phi\colon \nu_{\mathcal{F}}(N) \to \nu_{\mathcal{F}}(N)$ which is the identity near the end $\nu_{\mathcal{F}}(N)$. The new foliated manifold we construct is

$$M\backslash(\nu_{\mathcal{F}}(N)\times(-1,1))\coprod(\nu_{\mathcal{F}}(N)\times(-1,1)\coprod\nu_{\mathcal{F}}(N)\times(-1,1))/\nu_{\mathcal{F}}(N)\times(-1,1)\xrightarrow{\Phi}\nu_{\mathcal{F}}(N)\times(-1,1)$$

Otherwise said, we cut M open along $\nu_{\mathcal{F}}(N)$ and glue back rather than with the identity, with ϕ. The result is a new foliated manifold. This kind of gluing can change drastically the topology.

Example 5. *Pick $(M^3,\mathcal{F})$ foliated by surfaces and co-oriented. Take N to be an embedded circle $\alpha \overset{\varphi}{\simeq} S^1$ in a leaf. The leafwise normal bundle is the cylinder*

$$S^1 \times (-1,1)$$

Use as ϕ a Dehn twist (a compactly supported map that sends each segment $\{\theta\} \times (-1,1)$ to a curve that wraps around the cylinder once, so if we add the boundary of the cylinder the homology class relative to the boundary changes from zero to a generator, meaning that the Dehn twist cannot be isotopic to the identity relative to the boundary). The new manifold is as follows: to the trivial cobordism $M \times [0,1]$ glue a 2-handle $\overline{D^2} \times \overline{D^2}$ so that the core disk $\overline{D^2} \times \{0\}$ attachs to $\alpha \times \{1\}$ via φ, and the framing we use is one of the boundary components of the leafwise normal bundle of α (plus the co-orientation). Our new manifold is the result of surgery on α with its canonical framing.

Generalized Dehn twist are compactly supported diffeomorphism defined in T^*S^n, for any n>1. For any $(M^{2n+1},\mathcal{F}^{2n})$ co-oriented and $S^n \overset{\varphi}{\hookrightarrow} F \in \mathcal{F}$ so that $\nu_{\mathcal{F}}(S^n) \simeq T^*S^n$, it is possible modify the foliated manifold by cutting open along $\nu_{\mathcal{F}}(S^n)$ and gluing back using the generalized Dehn twist. It also holds that the resulting manifold is surgery on the parametrized sphere with an associated canonical framing.

3.1.2. *Transverse gluing:* For a transverse gluing we want $U = K \times (-1,1)$ foliated so that $\partial U = \{-1\} \times K$ is transverse to the foliation. To make our life easier we ask $K = K \times \{0\}$ to be transverse to the foliation, and then the flow of the interval coordinate to be tangent to $\mathcal{F}$. There is a second simplification, which is that we will ask the induced foliation on K to be a fibration. The advantage is that we increase our chances of giving normal forms for the extra leafwise symplectic structure to be considered. The drawback is that if codimension 1 transverse submanifolds - other than sphere bundles of the normal bundle of a transverse curve- are scarce in foliations, and even more difficult is to find them so that the induced foliation is a product.

Example 6. *Leafwise family or normal connected sum: We start by taking $N \pitchfork \mathcal{F}$ so that the induced foliation is a fibration $N \to B$. Recall that because $N \pitchfork \mathcal{F}$ we can assume that a tubular neighborhood*

$$\Psi\colon \nu(N) \to V$$

has been chosen in such a way that fibers are sent inside leaves. Otherwise said, the normal bundle $\nu(N)$ is the result of putting together all the leafwise normal bundles for each of the fibers of $N \to B$. We now take $N' \hookrightarrow M'$ with the same properties and assume we have a diffeomorphism $N \to N'$ which lifts to a bundle isomorphism of the corresponding sphere bundles $\phi\colon K \to K'$. Then we can glue

$K \times (-1,1)$ to $K' \times (-1,1)$ as in the family connected sum, and by the choice of tubular neighborhoods we are sending leaves to leaves.

Observe that if the family connected sum is just connected sum with extra compact parameter, the leafwise version is family connected sum for each fiber of $N \to B$ with extra parameter B, which after all is just family connected sum with a bit of care in the choice of tubular neighborhoods.

In general any kind of gluing for manifolds that behaves well for compact parameters (i.e. that is allows consistent choices of gluing maps) has a version for foliations as a transverse gluing.

3.2. **Gluings in symplectic geometry.** In a regular Poisson manifold $(M,\pi) = (M,\mathcal{F},\omega_{\mathcal{F}})$, if we want to arrange a transverse gluing to be compatible with the Poisson structure, we have to make sure that for each fiber of $K \to B$ the corresponding leafwise gluing is symplectic. So we are led to analyze symplectic gluings.

3.2.1. *Blowing up:* The classical complex blowing up has a symplectic version. Perhaps the cleanest way to see it is to realize it as a symplectic quotient (Lerman's symplectic cut [6]). As a matter of fact by blowing up the origin one is looking at the orbit space of

$$\mathbb{C}^n\backslash\{0\} \times \mathbb{C},\ \lambda(z,w) = (\lambda z, \lambda^{-1} w), \tag{1}$$

which is then complex. A diffeomorphic reduced space is obtained by just keeping the Hamiltonian action of S^1 and taking the reduction at any positive level of the moment map; therefore is it also symplectic. There is a (symplectic) dependence on the level set, since this in particular affects at the volume of the resulting manifold.

By Darboux' theorem about point in a symplectic manifold we can find local complex coordinates $z_1,\dots,z_n$ so that the symplectic form is the standard Kähler. This defines the symplectic blowing up of any point. For a symplectic submanifold N one uses a similar strategy: one works in the normal bundle $\nu(N)$; the restriction of the symplectic form has a linearization along the fibers, each of which can be identified with $\mathbb{C}^m$ in such a way that the structural group reduces to $U(n)$ and the symplectic form in the fiber $\mathbb{C}^n$ is the standard one. By theorems of Weinstein [8] (or results on coupling forms by Guillemin et all) if we pick a unitary connection we can find an invariant closed 2-form restricting to the fibers to the standard one. By $U(m)$-averaging near the zero section we can also achieve $U(m)$-invariance (so the symplectic annihilators of the fibers define another $U(m)$-connection, so in particular at the zero section they are the tangent space to N). Then after pulling back the symplectic form from the base we get a very nice model for the symplectic form near the zero section, because it is $U(m)$-invariant and the diagonal S^1-action is Hamiltonian with momentum map one half of the norm square. Then one considers $(\nu(N)\backslash N) \times \mathbb{C}$ with symplectic form $\omega + w_{\mathrm{std}}$, Hamiltonian action as in equation 1, and momentum map $(x,w) \mapsto \frac{1}{2}||x||^2 - \frac{1}{2}||w||^2$.

3.2.2. *Normal connected sum.* This is the only possible symplectic version of the family connected sum. Notice that for any symplectic structure on $\overline{B^{2n}}$, $n > 1$, there cannot be a symplectic transformation of the punctured open ball inverting the ends. If so, one would get a symplectic structure on the manifold which results from gluing two copies of the ball using that transformation. This manifold is homeomorphic to the sphere S^{2n}, so cohomological reasons prevent its existence.

For the disk of radius ϵ, and the standard symplectic form, in polar coordinates (r,θ) there is a unique such map of the form $i_\epsilon(r,\theta) = (f(r),-\theta)$, where one gets $f(r)$ by solving the O.D.E. resulting form the symplectomorphism and boundary conditions.

What Gompf did [3] was not only showing that this family gluing parametrized by a symplectic manifold was possible in the symplectic category, but more important he showed how to use it to build an enormous amount of symplectic (mosty 4) manifolds with a wide variety of topological properties. In particular he arranged any finitely presented group to be the fundamental group of a symplectic 4-manifold.

The construction. That the construction works has an easy case. When N enters symplectically in (M, ω) and (M', ω') with codimension two trivial line bundle, then by Weinstein's theorem the trivial disk bundle $N \times D^2(\epsilon)$ with product symplectic form $\omega_N + \omega_{\text{stad}}$ is mapped symplectically to tubular neighborhoods V, V' by maps Ψ, Ψ'. Therefore the composition $\Psi' \circ (\text{Id} \times i_\epsilon) \circ \Psi^{-1}$ is the desired gluing symplectomorphism.

When the normal bundle is not trivial there is more work to do, simply because we do not have a normal form; we just know that all symplectic structures matching at the zero section have symplectomorphic tubular neighborhoods. Firstly the oriented normal bundles must have opposite Chern class in order for the differentiable gluing to produce an oriented manifold compatible with the symplectic orientations. Next one picks one of the normal bundles E and compactifies it to a sphere bundle with structural group S^1; this is just the projectivized bundle

$$G := \mathbb{P}(E \oplus \underline{\mathbb{C}}),$$

the latter summand being the trivialized complex line bundle. Each fiber of $E \oplus \underline{\mathbb{C}}$ carries the diagonal S^1-action, which descends to the usual action on $S^2 = \mathbb{CP}^1$.

A bit of work constructs a symplectic form on G which matches at the zero section s_0 and section at infinity s_∞ the ones given by the respective embeddings. By Weinstein's theorem the symplectic forms near the embeddings coincide with the symplectic form in neighborhoods V_0, V_∞ of s_0, s_∞. If for each fiber the neighborhoods had non-empty intersection (an annulus), we would be done, but this is not in general the case. What Gompf does is sliding an annular neighborhood $V_\infty \backslash V'_\infty$ symplectically towards the zero section to get non-empty intersection Doing this requires a very nice choice of symplectic form on G, namely an S^1-invariant one and such that on each fiber matches Fubini-Study. It is important that all the tools he uses to produce symplectic forms, symplectomorphisms,...do not resort to patching techniques, it is possible to do things globally via homotopy operators...

The main applications. Gompf's contruction is specially interesting for 4-manifolds. A symplectic codimension 2 submanifold is a symplectic surface. Notice that by Moser's theorem the restriction of the symplectic form to the surface is determined by its symplectic area. The first Chern class of the normal bundle is the self intersection of the surface. Plenty of examples are provided by complex submanifolds of Kähler manifolds. For example in $\mathbb{CP}^2$ the (smooth) zero set of a homogeneous polynomial of degree d. The genus of the surface is $\frac{1}{2}d(d-1)$ and the self-intersection d^2, the latter by Bezout's theorem.

One can **symplectically blow down** a symplectic sphere of self intersection -1. Simply observe that $\mathbb{CP}^1 \subset \mathbb{CP}^2$ has self intersection 1 (Bezout). Topologically the blow down is replacing the curve by a point, with is the same thing as making normal connected sum with $\mathbb{CP}^2$ along $\mathbb{CP}^1$. Observe that the only requirement to perform it is the matching of the areas, so one (i) orients the surface so that the area is positive, and (ii) rescales Fubini-Study accordingly.

One can as well blow down a symplectic sphere with self intersection -4, something which has no analog in complex geometry. Just recall that by the degree formula a smooth quadric Q is a sphere, but with self intersection 4. The blowing down by definition is the normal connected sum with $\mathbb{CP}^2$ along a smooth quadric.

It is possible to realize any **finitely presented group as the fundamental group of a symplectic 4-manifold**. If we have g generators and r words (relations), we take the manifold $\Sigma_g \times \mathbb{T}^2$. Let α_i, β_i, $i = 1, \ldots, g$ be a symplectic basis of $H^1(\Sigma)$. We take embedded curves β_i (representing the β_i), and γ_j representing the r words in the α_i. Denote by α, β embedded curves with $\alpha \times \beta = \mathbb{T}^2$. To get the desired group we have to kill the curves representing β_i, γ_j, and α, β. We consider the embedded tori $\alpha \times \beta_i, \alpha \times \gamma_j, \alpha \times \beta$. Equivalently, we must kill the fundamental group of those $g + j + 1$ tori. On can slightly perturb the product symplectic form in $\Sigma_g \times \mathbb{T}^2$ so that the previous tori are symplectic and with triviall normal bundle. Now the crucial step is the existence of a symplectic manifold, the elliptic surface $E(1)$ with the following property: it is a fibration by symplectic tori (some of them singular) so that the complement of a regular fiber is simply connected. Then it follows that each time we make normal connected sum along one of the $g + j + 1$ tori T_l and a fiber of the elliptic surface, by Seifert Van-Kampen the fundamental group of the new symplectic manifold amounts to introduce relations killing the homotopy of T_l.

3.2.3. *Boundary gluings.* We mention -without giving details- that boundary gluings are also possible in symplectic geometry (so we have a notion of symplectic cobordism). What we need is conditions so that the symplectic structure in collars of the boundary is the same. This works under the assumption of product structures given by very special vector fields: either Liouville ($L_X\omega = \omega, L_{X'}\omega' = -\omega'$) or symplectic ($L_X\omega = L_{X'}\omega' = 0$). In the first case we glue a convex contact boundary component to a concave boundary component; in the second we glue along Poisson (actually cosymplectic) boundary components. The advantage is that a contact isomorphism. (resp. Poisson isomorphism) determine a symplectic diffeomorphism of the collars. It is possible to devise a handle attaching compatible with the symplectic structure, but the attaching spheres and framings have to be of a special nature (coisotropic); the new boundary also inherits a contact (resp. Poisson) structure (see [8, 7]).

3.3. **Gluings in Poisson geometry.** We let $(M, \mathcal{F}, \omega)$ be our regular Poisson manifold. If we want to perform a tangential gluing by cutting open along $\nu_{\mathcal{F}}(N)$ and gluing back via ϕ, then it should be an element of $\mathrm{Symp}^{\mathrm{comp}}(\nu_{\mathcal{F}}(N), \omega)$. But that is not enough. Recall that locally the foliation is $\nu_{\mathcal{F}}(N) \times (-1, 1)$, and leafwise symplectic form is the family of symplectic forms ω_t, $t \in (-1, 1)$. After the gluing we still need to have a smooth family, and that is not granted by just asking $\phi^*\omega_0 = \omega_0$. We may lose the smoothness when moving along transverse directions. Exactly as we do to put a smooth structure when gluing along the boundary, a sufficient condition is choosing a product structure so that ω_t is independent of t. But that is equivalent to saying that there exist $\Omega \in \Omega^2(\nu_{\mathcal{F}}(N) \times (-1, 1))$ such that

(1) $\Omega_{\mathcal{F}} = \omega_t$
(2) $d\Omega = 0$

Interestingly enough, under such conditions if N is a lagrangian sphere in a leaf, then its leafwise normal bundle is isomorphic to T^*N, so we can perform a generalized Dehn twist. The good news is that within the (compactly supported) isotopy class of such maps there are symplectic representatives, so **generalized Dehn surgery** provides a gluing construction for regular Poisson manifolds of codimension 1 whose leafwise symplectic form lifts to a closed 2-form [7].

Regarding transverse gluings, any gluing in symplectic geometry that admits a version with parameters (a manifold of parameters, which implies having consistent choices) works in (regular) Poisson geometry. Therefore for any Dirac submanifold $N \hookrightarrow M$, $N \pitchfork \mathcal{F}$ so that the induced foliation is a fibration with compact fibers, we can blow up along its normal directions inside the symplectic leaves. Similarly, if we have N embedding in two Poisson manifolds $(M, \mathcal{F}, \omega_{\mathcal{F}})$, $(M', \mathcal{F}', \omega'_{\mathcal{F}'})$ as above, and so that the two induced structures are Poisson isomorphic and the normal bundles opposite, we can perform normal connected sum.

For example, suppose $(M, \mathcal{F}, \omega_{\mathcal{F}})$ is five dimensional, $\mathcal{F}$ co-oriented, and N^3 embeds as a Dirac submanifold intersecting $\mathcal{F}$ transversely. Assume further that the induced foliation has a sphere. Then by Reeb-Thurston stability the symplectic foliation on F is the product $S^2 \times S^1$. Assume finally that one sphere inside its symplectic leaf has self intersection -1 or -4. Then the same happens for all the family. Then if we take $S^1 \times \mathbb{CP}^2$, with the pullback of Fubini-Study conveniently rescaled on each leaf, we can perform normal connected sum along $S^1 \times \mathbb{CP}^1$, $S^1 \times Q$ respectively, and we can say that we are blowing down the corresponding family of spheres.

We can also construct five dimensional Poisson manifolds with codimension 1 leaves and any finitely presented fundamental group. For any such group G we start with one of Gompf's symplectic manifolds X having G as fundamental group. Then we move onto the Poisson manifold $S^1 \times X$. Observe that the issue is killing a copy of S^1 via a surgery. To do that with normal connected sum we must find another such five dimensional manifold with a transverse curve α which is trivial in homotopy. The answer is given by the Reeb foliation $\mathcal{R}$ of S^3. Then one shows that normal connected sum of $S^1 \times X$ and $\mathbb{T}^2 \times (S^3, \mathcal{R})$ along certain embedding of $S^1 \times \mathbb{T}^2$ does the job. [4]

4. Singular gluings

There is a possibility of performing connected sum for singular Poisson manifolds under very special circumstances. We need to have to singular points so that the linearization is semi-simple of compact type. We use Conn's to find coordinates in which the Poisson structure is linear. We also insist on using an orthonormal basis for the negative of the Killing form. Then we can perform the connected sum using the identity as gluing map for some sphere. Still, a modification is needed. The corresponding Poisson structure is not smooth; we have to use appropriate smooth functions on the quadratic Casimir given by the norm square. The effect is that in a chart as we approach the origin the Poisson structures restricted to each sphere, which in principle varies linearly on the distance, will do it depending on a function that will achieve a critical point, and at that sphere we change chart and start moving far from the origin.

Blowing up and down in 4-manifolds with a generalized complex structure are also examples of singular surgeries (for the underlying Poisson structures) [1, 2].

References

[1] Cavalcanti, Gil R.; Gualtieri, Marco A surgery for generalized complex structures on 4-manifolds. J. Differential Geom. 76 (2007), no. 1, 35–43.
[2] Cavalcanti, Gil R.; Gualtieri, Marco Blow-up of generalized complex 4-manifolds. arXiv:0806.0872
[3] Gompf, Robert E. A new construction of symplectic manifolds. Ann. of Math. (2) 142 (1995), no. 3, 527–595.
[4] Ibort, A., Martnez Torres, D. A new construction of Poisson manifolds. J. Symplectic Geom. 2 (2003), no. 1, 83–107.
[5] Kervaire, Michel A.; Milnor, John W. Groups of homotopy spheres. I. Ann. of Math. (2) 77 1963 504–537.
[6] Lerman, E. Symplectic cuts. Math. Res. Lett. 2 (1995), no. 3, 247–258.
[7] Martínez Torres, D. A higher dimensional generalization of taut foliations. math.DG/0602576
[8] Weinstein, A. A universal phase space for particles in Yang-Mills fields. Lett. Math. Phys. 2 (1977/78), no. 5, 417–420.

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