# On the Baum-Connes Conjecture

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## 1 Introduction

In the early 1980s Paul Baum and Alain Connes conjectured a link between the $K$-theory of the reduced $C^*$-algebra of a group and the $K$-homology of the corresponding classifying space of proper actions of that group.

This statement, also known as the Baum-Connes Conjecture, is not verified in its full generality even 20 years after its formulation. Yet, no counterexamples to the conjecture are currently known.

In this paper we will describe the connections between the Baum-Connes Conjecture for groups and the notions of proper actions and uniform embeddability.

## 2 Uniform Embeddability

We start with some notions of coarse geometry.

**Definition 2.1.** A map $f$ between two metric spaces is called proper if the preimage of any bounded set under $f$ is bounded.

For example, $f(x) = x^2$ is a proper map from $\mathbb{R}$ to $\mathbb{R}$, while $g(x) = \sin(x)$ is not. Note that even though our examples employ continuous functions, the map $f$ in the definition does not have to be continuous.

**Definition 2.2.** A proper map $f$ is called coarse if for any $\delta > 0$ there exists an $\epsilon > 0$, such that $f$ maps any two points with mutual distance less than $\delta$ to the points not more than $\epsilon$ apart.

It is easy to see that $f(x) = x^2$ from our previous example is not coarse, while $h(x) = ax + b$ is indeed coarse. It should be mentioned that the composition of coarse maps is again coarse.

**Definition 2.3.** Two metric spaces $X$ and $Y$ are called coarsely equivalent if there exist coarse maps

$$f : X \to Y \text{ and } g : Y \to X$$

and some number $d$, such that both compositions

$$g \circ f : X \to X \text{ and } f \circ g : Y \to Y$$

do not map any point more than distance $d$ away from itself.

For example, any space is coarsely equivalent to itself (we can take $f$ and $g$ to be identity maps, and $d$ to be zero), and if any two spaces are coarsely equivalent to the third one, they are coarsely equivalent to each other (to see this, we can take the compositions of coarse maps through the third space to produce two maps, required by the definition, and then use their coarseness to argue why their compositions do not map any point far away from itself). Thus the coarse equivalence is an equivalence relation.

Another example, which is quite relevant to our discusion, comes from the word metric on groups. Consider a finitely generated group $\Gamma$ and define the (word) distance between any two elements $g$ and $h$ of $\Gamma$ to be the smallest number of generators, needed to express the product $g^{-1}h$. Of course, this distance depends on the set of generators we choose, but if we take a finitely generated group and equip it with two different metric structures coming from the word distances with respect to two different generating sets, then the results will always be coarsely equivalent.

Now we are ready to formulate the notion of the uniform embeddability.

**Definition 2.4.** We say that a metric space $X$ is uniformly embeddable into a metric space $Y$ if there is a map $f : X \to Y$ which is a coarse equivalence between $X$ and its image in $Y$.

Revisiting our example with a finitely generated group, we say that such group $\Gamma$ is uniformly embeddable into a metric space $Y$ if there exists a map $f : \Gamma \to Y$, such that for every finite set $F \subseteq \Gamma$ there is some number $d = d(F)$ with the property

$$\gamma_1^{-1}\gamma_2 \in F \implies dist(f(\gamma_1), f(\gamma_2)) < d$$

and for every $d$ there is some subset $F = F(d) \subseteq \Gamma$ with the property

$$dist(f(\gamma_1), f(\gamma_2)) < d \implies \gamma_1^{-1}\gamma_2 \in F.$$

Of course, this coincides with Definition 2.4 if we think of $\Gamma$ as of a metric space equipped with the word metric.

# 3 Proper Actions

Let us start with a group $\Gamma$ which acts on a metric space $X$. This action is called *proper* if the preimage of any compact set under the mapping

$$(\gamma, x) \mapsto (\gamma.x, x), \qquad \gamma \in \Gamma, x \in X$$

is compact.

It is easy to see that for finite groups any action is proper, but if we take the trivial action of $\mathbb{Z}$ on any space $X$, then this action will not be proper, since the preimage of any compact set which meets the diagonal of $X$ will include the entire group.

Among all spaces on which $\Gamma$ acts properly, there is a special one (defined up to homotopy):

**Definition 3.1.** A metric space $E\Gamma$ is called universal for $\Gamma$, if $\Gamma$ acts properly on it, and for any other space $X$ on which $\Gamma$ acts properly, there exists a $\Gamma$-equivariant continuous map $X \to E\Gamma$, unique up to $\Gamma$-equivariant homotopy.

*Remark* 3.1. A more precise definition of the universal $\Gamma$-space in its full generality requires the consideration of arbitrary Hausdorff spaces and then will involve the nuances of paracompactness. All technical details can be found in [8].

Just to have some examples in mind, for $\Gamma$ finite we can take $E\Gamma$ to be a one-point space, and if $\Gamma$ acts properly on a tree, then we can take that tree to represent $E\Gamma$.

# 4 Reduced Group $C^*$-algebra

In this section we will define the reduced group $C^*$-algebra. To start with, let us recall that a normed algebra which is complete is called a Banach algebra.

The algebra of bounded complex-valued functions on some set with pointwise operations and sup-norm is a good example of a Banach algebra.

For our purposes, we need to restrict our attention to the very special class of Banach algebras.

**Definition 4.1.** A Banach algebra $A$ with involution is called a $C^*$-algebra if it satisfies the $C^*$-identity:

$$||a||^2 = ||a^*a||, \qquad a \in A$$

The standard example of a $C^*$-algebra is the algebra of bounded operators on some Hilbert space, or any $C^*$-subalgebra of it. In fact, we can think of any $C^*$-algebra as of a subalgebra of the algebra of all bounded operators on some Hilbert space.

Now let $\Gamma$ be any countable group, and let $l^2(\Gamma)$ denote the Hilbert space of square-summable complex-valued functions on $\Gamma$. The standard orthonormal basis of $l^2(\Gamma)$ consists of functions $\delta_g$, which attain value 1 on element $g$ and 0 on every other group element. Given any $\gamma \in \Gamma$, we can define a linear operator $\lambda(\gamma)$ on $l^2(\Gamma)$ by requiring it to map $\delta_g$ into $\delta_{\gamma g}$. This operator is in fact bounded and thus we have a representation $\lambda : \Gamma \to \mathcal{B}(l^2(\Gamma))$, which is called the left regular representation.

The $C^*$-algebra generated by the image of $\Gamma$ in $B(l^2(\Gamma))$ under this representation, is called the reduced $C^*$-algebra of $\Gamma$, and is denoted by $C^*_\lambda(\Gamma)$.

Since $C^*_\lambda(\Gamma)$ is in particular a Banach algebra, we can talk about its $K$-theory in the usual topological sense.

Now we have all the ingredients to formulate the Baum-Connes Conjecture.

# 5 The Baum-Connes Conjecture

Let $\Gamma$ be a countable group, as before. One can define a morphism (also known as the "assembly" map, see [1])

$$\mu : KK^\Gamma(E\Gamma) \to K(C^*_\lambda(\Gamma))$$

from the $K$-homology of the classifying space $E\Gamma$ of proper actions of $\Gamma$ to the $K$-theory of the reduced $C^*$-algebra of $\Gamma$.

Paul Baum and Alain Connes introduced the following conjecture about this morphism:

**Conjecture 5.1.** *The assembly map $\mu$ is an isomorphism.*

This statement is known as the Baum-Connes Conjecture.

*Remark* 5.1. Some authors actually consider two ingredients of the conjecture separately: the injectivity and surjectivity.

In fact, even the injectivity part of the conjecture is a rather difficult problem. It is known, however, that the injectivity of the Baum-Connes assembly map implies the Novikov's higher signature conjecture [2].

There are different approaches to the understanding of the Baum-Connes Conjecture. Here we will discuss only some of them, directly referring to the notions of proper action and uniform embeddability. Let us start by mentioning the following result:

**Theorem 5.1 (Oyono-Oyono, [6], cf. [7]).** *Let $\Gamma$ be a discrete countable group acting on a tree. Then the Baum-Connes Conjecture holds for $\Gamma$ if and only of it holds for all the isotropy groups (which are of course subgroups of $\Gamma$) of the action on vertices of the tree.*

In particular, suppose that such action is proper. Then, since the isotropy subgroups are all finite, and the Baum-Connes Conjecture is trivially true for them, we can deduce that the Conjecture holds for the entire group $\Gamma$.

Does any countable group admit a proper action on a tree? Look, for example, at the following subgroup of $SL(2, \mathbb{Z}[\pi, \pi^{-1}])$:

$$\left\{ \begin{pmatrix} \pi^n & p(\pi, \pi^{-1}) \\ 0 & \pi^{-n} \end{pmatrix}, \qquad n \in \mathbb{Z}, \qquad p \in \mathbb{Z}[x, y] \right\}$$

Obviously this group is countable, but contains a free abelian subgroup $\mathbb{Z}^\infty$ of infinite rank, and therefore cannot act properly on a finite-dimensional space.

If we allow infinite-dimensional spaces, then we can mention

**Theorem 5.2 (Higson, Kasparov, [5]).** *If a group $\Gamma$ admits a metrically proper isometric action on a Hilbert space, then the Baum-Connes Conjecture holds for $\Gamma$.*

This result influenced Yu to use the coarse geometry machinery to prove that the coarse version of the Baum-Connes Conjecture holds for any bounded geometry metric space which is uniformly embeddable into Hilbert space [9].

It is known (see, for example, [4]) that if a discrete group $\Gamma$ uniformly embeds into a Hilbert space, then the Baum-Connes assembly map is injective. This allows one to prove

**Theorem 5.3 (Guentner, Higson, Weinberger, [3]).** *For any field $k$ and any natural number $n$ the injectivity portion of the Baum-Connes Conjecture holds for any countable subgroup of $GL(n, k)$.*

A refined argument in [3] also shows by reducing to Theorem 5.2 that in the case of a subgroup of $GL(2, k)$ the full Baum-Connes Conjecture holds true.

# References


[1] P. Baum, A. Connes, N. Higson, Classifying space for proper action and $K$-theory of group $C^*$-algebras, Contemp. Math., 167:241-291, 1994.

[2] S. Ferry, A. Ranicki, J. Rosenberg, A history and survey of the Novikov conjecture, Novikov Conjectures, Index Theorems and Rigidity, Vol. I:7-66, 1995.

[3] E. Guentner, N. Higson, S. Weinberger, The Novikov Conjecture for Linear Groups, preprint, 2003.

[4] N. Higson, Bivariant $k$-theory and the Novikov conjecture, Geom. Funct. Anal., 10(3):563-581, 2000.

[5] N. Higson, G. Kasparov, $E$-theory and $KK$-theory for groups which act properly and isometrically on Hilbert space, Invent. Math., 144(1):23-74, 2001.

[6] H. Oyono-Oyono, La Conjecture de Baum-Connes pour les groupes agissant sur les arbres, C.R. Acad. Sci. Paris, t. 326, Série I:799-804, 1998.

[7] M. Pimsner, $KK$-groups of crossed products by groups acting on trees, Invent. Math., 86:603-634, 1986.

[8] A. Valette, Introduction to the Baum-Connes Conjecture, ETH Lecture Notes series, Birkhäuser, (?).

[9] G. Yu, The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space, Invent. Math., 139:201-240, 2000.