ADJOINT SYSTEMS ON SURFACES

by

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Introduction.

Let D be a curve (not necessarily irreducible) on a surface S and ω_D its dualizing sheaf. In this note we study some properties of both the adjoint system $|K+D|$ and ω_D. These properties can be useful in various situations where Reider's method (see [R]) cannot be applied and can be used, for instance, to simplify the proofs of the birationality of φ_{3K} for minimal surfaces of general type with $p_g = 0$ of [B-C] and [M].

0. Notation and conventions.

(0.1) S will be a smooth projective surface over $\mathbf{C}$ and a *curve* D on S will be any effective non-zero divisor on S. A point x on D is a *reduced point* if every component of D containing x appears in D with multiplicity 1. We denote by $\sim$ the numerical equivalence for divisors on S.

We will say that an invertible sheaf $\mathcal{L}$ on a curve D is *nef* if deg $\mathcal{L}_{|\Gamma} \geq 0$, for every component Γ of D.

An irreducible curve Γ will be an *hyperelliptic curve in a generalized sense* if Γ has a g^1_2, i.e. an invertible sheaf $\mathcal{L}$ such that deg $\mathcal{L} = 2$ and $h^0(\Gamma, \mathcal{L}) \geq 2$. Recall that if $p_a(\Gamma) \geq 2$ then Γ has a unique g^1_2.

1. The main theorems.

In [F] we find the following theorem:

(1.1) **Theorem.** *Let D be a curve on S and $x \in SingD$. Let $p : \tilde{S} \to S$ be the blow-up at x and E be the exceptional curve. Setting $D' = p^*D - E$ and $D'' = p^*D - 2E$, the following two conditions are equivalent:*

(i) x is a base point of $|K+D|$.

(ii) The restriction map $H^0(D', \mathcal{O}_{D'}) \to H^0(D'', \mathcal{O}_{D''})$ is not surjective.

Using exactly the same method of proof we can obtain the following :

(1.2) **Theorem.** *Let D be a curve on S and $x, y \in SingD$, with $x \neq y$. Let $p : \tilde{S} \to S$*

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be the blow-up at x and y and E_1, E_2 be the corresponding exceptional curves. Setting $D' = p^*D - 2E_1 - E_2$ and $D'' = p^*D - 2E_1 - 2E_2$, the following two conditions are equivalent:

(i) x, y are not separated by $|K + D|$.

(ii) The restriction map $H^0(D', \mathcal{O}_{D'}) \to H^0(D'', \mathcal{O}_{D''})$ is not surjective.

(1.3) **Theorem.** *Let D be a curve on S and x a point of D such that $mult_D(x) \geq 3$ and such that x is not a base point of $|K + D|$. Let $p : \tilde{S} \to S$ be the blow-up at x, E be the exceptional curve and set $D' = p^*D - 2E$ and $D'' = p^*D - 3E$. If r is the rank at at x of the differential of φ_{K_S+D}, then $r = 2 - (h^0(D'', \mathcal{O}_{D''}) - h^0(D', \mathcal{O}_{D'}))$.*

The main ingredients of the proofs of (1.1), (1.2) and (1.3) are the following facts:

Fact 1: Given a curve D on a surface S one has:

$$h^1(S, \mathcal{O}_S(K + D)) = \alpha(D) + h^0(D, \mathcal{O}_D) - 1$$

where $\alpha(D) = \dim \ker \{H^1(S, \mathcal{O}_S) \to H^1(D, \mathcal{O}_D)\}$ (cf. [B], pg 183).

Fact 2: Given $\sigma : \tilde{S} \to S$ a birational morphism one has

$$\alpha(D) = \alpha(\tilde{D})$$

for any curve $\tilde{D}$ on $\tilde{S}$ with $\sigma_*(\tilde{D}) = D$ (see lemma (2.3) of [F]).

Here we give an abreviated proof of (1.2) and leave the proof of (1.3) for the reader.

Proof of (1.2): Consider D a curve on S and $x, y \in SingD$, with $x \neq y$. By the cohomology exact sequence of

$$0 \to m_x \cdot m_y \cdot \mathcal{O}_S(K_S + D) \to m_y \cdot \mathcal{O}_S(K_S + D) \to \mathbf{C} \to 0$$

we have that x, y are separated by $|K + D|$ if and only if the surjective map

$$\psi : H^1(S, m_x \cdot m_y \cdot \mathcal{O}_S(K_S + D)) \to H^1(S, m_y \cdot \mathcal{O}_S(K_S + D))$$

is an isomorphism. Let p, D' and D'' be as in the statement of the theorem. The map ψ will be an isomorphism if and only if $h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(K_{\tilde{S}} + D')) = h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(K_{\tilde{S}} + D''))$. As x, y are multiple points of D, both D' and D'' are effective divisors on $\tilde{S}$. By fact 2, $\alpha(D') = \alpha(D'')$ and so, using fact 1, one has that ψ is not an isomorphism if and only if $h^0(D', \mathcal{O}_{D'}) \neq h^0(D'', \mathcal{O}_{D''})$. Thus we get the assertion. ♢

2. Auxiliary results

To be able to apply the above theorems we will need the following:

(2.1) **Lemma.** (see [CFM]) *Let D be an m-connected curve on the surface S and let $D = D_1 + D_2$ with D_1, D_2 curves. Then:*

(i) if $D_1 \cdot D_2 = m$, then D_1 and D_2 are $[(m+1)/2]$-connected;

(ii) if D_1 is chosen to be minimal subject to the condition $D_1 \cdot (D - D_1) = m$, then D_1 is $[(m+3)/2]$-connected.

(2.2) (see [CFM]) **Lemma.** *Let D be a curve on a smooth surface S, $\mathcal{L}$ an invertible sheaf on D and let $s \in H^0(D, \mathcal{L})$ with $s \neq 0$. Then:*

(i) either s does not vanish identically on any component Γ of D, implying that $\deg \mathcal{L}_{|D'} \geq 0$ for every curve D' contained in D;

(ii) or D is reducible and if $D_1 < D$ is the biggest divisor such that $s_{|D_1} \equiv 0$, then s, regarded as a section of $\mathcal{L} \otimes \mathcal{O}_{D-D_1}(-D_1)$, is as in (i). Hence deg $\mathcal{L}_{|\Gamma} \geq D_1 \cdot \Gamma$ for every curve $\Gamma \leq D - D_1$, and in particular deg $\mathcal{L}_{|D-D_1} \geq D_1 \cdot (D - D_1)$.

(2.3) **Definition.** Let D be a reducible curve on a smooth surface S, $\mathcal{L}$ an invertible sheaf on D and let $s \in H^0(D, \mathcal{L})$ with $s \neq 0$ such that s vanishes identically on some component of D. Let $Z_s < D$ be the biggest curve such that $s_{|Z_s} \equiv 0$.

We will say that s is *0-maximal* if there is no global section t of $\mathcal{L}$ with $Z_s < Z_t$, $Z_s \neq Z_t$.

(2.4) **Proposition.** *Let D be a 0-connected divisor such that $h^0(D, \mathcal{O}_D) = n \geq 2$. Then D decomposes as $D = D_1 + ... + D_n$, where D_1,..., D_n are curves such that:*

(i) $h^0(D_i, \mathcal{O}_{D_i}) = 1$, for $i \in \{1, ..., n-1\}$,

(ii) $\mathcal{O}_{D_i}(D_{i+1} + ... + D_n) \simeq \mathcal{O}_{D_i}$ for all $i \in \{1, ..., n-1\}$

(iii) $D_i \cap (D_{i+1} + ... + D_n) = \emptyset$ or $D_i \subset (D_{i+1} + ... + D_n)$, for $i \in \{1, ..., n-1\}$.

In particular D is not 1-connected.

Proof. As $h^0(D, \mathcal{O}_D) \geq 2$, there exists a section s in $H^0(D, \mathcal{O}_D)$ vanishing identically on some component of D. Choose a 0-maximal such section s and let $D_1 = D - Z_s$. From the exact sequence

$$0 \to \mathcal{O}_{D_1}(-Z_s) \to \mathcal{O}_D \to \mathcal{O}_{Z_s} \to 0$$

we get

$$0 \to H^0(D_1, \mathcal{O}_{D_1}(-Z_s)) \to H^0(D, \mathcal{O}_D) \xrightarrow{r} H^0(Z_s, \mathcal{O}_{Z_s}).$$

Then s, regarded as a section of $\mathcal{O}_{D_1}(-Z_s)$ is as in (i) of lemma (2.2) and this implies by 0-connectedness of D that $D_1 \cdot Z_s = 0$ and $\mathcal{O}_{D_1}(-Z_s) \simeq \mathcal{O}_{D_1}$. Now remark that $h^0(D_1, \mathcal{O}_{D_1}) = 1$. In fact $h^0(D_1, \mathcal{O}_{D_1}) > 1$ would imply the existence of a global section t of $\mathcal{O}_{D_1}$ vanishing on some component of D_1. But then t regarded as a section of $\mathcal{O}_D$ would vanish on some curve strictly bigger than Z_s and this contradicts 0-maximality of s.

Since $D_1 \cdot Z_s = 0$, either $D_1 \cap Z_s = \emptyset$ or D_1 and Z_s have common components. Suppose that $D_1 \cap Z_s \neq \emptyset$. Then we can write $D_1 = C + A$, $Z_s = C + B$, where C, A, B are effective divisors such that $C \neq 0$ and A, B have no common components. Remark that this implies that $A \cdot B \geq 0$. Suppose that $A \neq 0$. From $D_1 \cdot Z_s = 0$, we have $C^2 + C \cdot A + C \cdot B + A \cdot B = 0$

and from 0-connectedness of D, we have $C \cdot (D - C) = C^2 + C \cdot A + C \cdot B \geq 0$. Therefore $A \cdot B = 0$ and so A and B are disjoint. As $\mathcal{O}_A(-Z_s) \simeq \mathcal{O}_A$, we get $\mathcal{O}_A(-C) \simeq \mathcal{O}_A$, but this implies that $h^0(D_1, \mathcal{O}_{D_1}) > 1$, a contradiction. Therefore $D_1 \subset Z_s$.

Since $h^0(D, \mathcal{O}_D) = n$, $Im\ r$ is $n-1$ dimensional and therefore $h^0(Z_s, \mathcal{O}_{Z_s}) \geq n-1$. Now by lemma (2.1) Z_s is still 0-connected and we can apply an inductive reasoning to obtain the assertion. ♢

(2.5) (see [CFM]) **Proposition.** *Let D be a reducible m-connected curve ($m \geq 1$) on a surface S and let $\mathcal{L}$ be a nef invertible sheaf on D such that deg $\mathcal{L}_{|\Delta} = 0$ for some component Δ of D. Let n:=deg $\mathcal{L}$. Then one has:*

(i) if $n < m$, then $h^0(D, \mathcal{L}) \leq 1$, and, if $h^0(D, \mathcal{L}) = 1$, there is a section s of $\mathcal{L}$, which does not vanish identically on any component of D;

(ii) if $n = m$ then $h^0(D, \mathcal{L}) \leq 2$.

Furthermore, the following are equivalent:

(ii_1) $h^0(D, \mathcal{L}) = 2$;

(ii_2) D can be decomposed as $D = A + B$ with A, B curves such that $A \cdot B = m$, $\mathcal{L} \otimes \mathcal{O}_A(-B) \simeq \mathcal{O}_A$, $\mathcal{L}_B \simeq \mathcal{O}_B$, and there is a section $s \in H^0(D, \mathcal{L})$ that does not vanish identically on any component of D.

The following lemma is useful to apply the results in the next section:

(2.6) **Lemma.** *Let M be a curve on a surface S such that $M^2 \geq 1$ and $\mathcal{O}_S(M)$ is nef. Then:*

(i) every $M' \in |M|$ is 1-connected.

(ii) If $M = A + B$ is a decomposition of M with A, B curves such that $A \cdot B = 1$ then only the following possibilities can occur:

(p_1) $A^2 = -1$ or $B^2 = -1$

(p_2) $A^2 = 0$ or $B^2 = 0$

or

(p_3) $A^2 = B^2 = 1$, $A \sim B$ and $M^2 = 4$.

(iii) If $M = A + B$ is a decomposition of M with A, B curves such that $A \cdot B = 2$ then only the following possibilities can occur

(p_1) $A^2 = -2$ or $B^2 = -2$

(p_2) $A^2 = -1$ or $B^2 = -1$

(p_3) $A^2 = 0$ or $B^2 = 0$

(p_4) If $1 \leq A^2$ and $1 \leq B^2$ then $M^2 \leq 9$ and we are in one of the following cases:

	A^2	B^2	M^2	
(C_1)	1	1	6	
(C_2)	1	2	7	
(C_3)	1	3	8	
(C_4)	1	4	9	$2A \sim B$
(C_5)	2	2	8	$A \sim B$

Proof. Suppose we have a decomposition $M = A + B$ with $A \cdot B \leq 0$. Since $M^2 > 0$ and M is nef we must have, say, $A^2 \geq -A \cdot B \geq 0$ and $B^2 > -A \cdot B \geq 0$. By the index theorem the only possibility would be $A^2 = A \cdot B = 0$ giving that $A \sim B$ $(modQ)$. But this contradicts $M^2 > 0$ and so M is 1-connected.

The remainder of the proof follows also by application of the index theorem, using the hypothesis that $\mathcal{O}_S(M)$ is nef. ♢

3. The adjoint system $|K + D|$.

In this section we study some properties of the adjoint system $|K + D|$ for a curve D, using the theorems in section 1.

(3.1) **Theorem.** *If D is 1-connected, a multiple point x of D is a base point of $|K+D|$ if and only if D decomposes as a sum of two curves A, B satisfying :*

(i) $A \cdot B = 1$.

(ii) x is a non-singular point of A and $\mathcal{O}_A(x) \simeq \mathcal{O}_A(B)$.

Furthermore if x is a base point of $|K + D|$, the decomposition appearing above is such that $A \cap B = \{x\}$ (and thus x is a node of D) or $A \subset B$.

Proof. By (1.1) x is a base point of $|K+D|$ if and only if the restriction map $H^0(D', \mathcal{O}_{D'}) \to H^0(D'', \mathcal{O}_{D''})$ is not surjective (we keep the same notation as in (1.1)). Since D is 1-connected, D' is also 1-connected and D'' is 0-connected. Furthermore if $D'' = A' + B'$ with $A' \cdot B' = 0$ then $A' = p^*A - E$ and $B' = p^*B - E$, where A, B are curves such that $D = A + B$ and $A \cdot B = 1$ (cf. [B], pg.183). Now one has $h^0(D', \mathcal{O}_{D'}) = 1$ and thus x is a base point of $|K + D|$ if and only if $h^0(D'', \mathcal{O}_{D''}) \geq 2$. Suppose then that this happens. Then, by (2.4), there is a decomposition $D'' = A' + B'$ with $\Gamma \cdot B' = 0$, for every component Γ of A', $\mathcal{O}_{A'}(-B') \simeq \mathcal{O}_{A'}$ and either $A' \subset B'$ or $A' \cap B' = \emptyset$. By the above description of A', B', we obtain A and B as in the statement.

On the other hand if D has a decomposition satisfying (i) and (ii) then $\mathcal{O}_A(\omega_D) \simeq \omega_A \otimes \mathcal{O}_A(x)$ and thus x is a base point of $|\omega_D|$ implying that x is a base point of $|K + D|$. ♢

(3.2) **Theorem.** *If D is 2-connected, two distinct multiple points x and y of D are not separated by $|K + D|$ if and only if D decomposes as a sum of two curves A, B satisfying :*

(i) $A \cdot B = 2$.

(ii) x, y are non-singular points of A and $\mathcal{O}_A(x + y) \simeq \mathcal{O}_A(B)$.

Furthermore if x, y are not separated by $|K + D|$, the decomposition appearing above is such that $A \cap B = \{x, y\}$ or $A \subset B$.

Proof. By (1.2) x, y are not separated by $|K + D|$ if and only if the restriction map $H^0(D', \mathcal{O}_{D'}) \to H^0(D'', \mathcal{O}_{D''})$ is not surjective (we keep the same notation as in (1.2)). From 2-connectedness of D, one has that D' is 1-connected and that D'' is 0-connected. Furthermore if $D'' = A' + B'$ with $A' \cdot B' = 0$ then $A' = p^*A - E_1 - E_2$ and $B' = p^*B - E_1 - E_2$, where A, B are curves such that $D = A + B$ and $A \cdot B = 2$ (cf. [B], pg. 183). Since $h^0(D', \mathcal{O}_{D'}) = 1$, x, y are not separated by $|K + D|$ if and only if

$h^0(D'', \mathcal{O}_{D''}) \geq 2$. Suppose that this happens. Applying proposition (2.4), we obtain A and B as in the statement.

On the other hand, if D has a decomposition satisfying (i),(ii) and (iii), then $\mathcal{O}_A(\omega_D) \simeq \omega_A \otimes \mathcal{O}_A(x+y)$ and thus by the Riemman-Roch theorem, x and y are not separated by $|\omega_D|$ implying that x and y are not separated by $|K+D|$. ♢

(3.3) **Theorem.** *Suppose that D is 2-connected and x is a reduced point of D such that $mult_D(x) \geq 3$. Let r be the rank at at x of the differential of φ_{K_S+D}.*

If $r = 1$, then D has a decomposition $D = A + B$ where A, B are curves such that $A \cdot B = 2$, $A \cap B = \{x\}$ and x is a non-singular point of A, x is a double point of B (not a disconnecting node).

If $r = 0$, then D has a decomposition $D = A_1 + A_2 + A_3$ where A_1, A_2, A_3 are curves such that:

(i) $A_1 \cdot A_2 = A_1 \cdot A_3 = A_2 \cdot A_3 = 1$
(ii) $A_1 \cap A_2 = A_1 \cap A_3 = A_2 \cap A_3 = \{x\}$
(iii) x is a non-singular point of A_1, A_2 and A_3.

Proof. By (1.3), $r = 1$ if and only if $h^0(D'', \mathcal{O}_{D''}) - h^0(D', \mathcal{O}_{D'}) = 1$ (we keep the same notation as in (1.2)). From 2-connectedness of D, one has that D' is 1-connected and that D'' is 0-connected. Furthermore if $D'' = A' + B'$ with $A' \cdot B' = 0$ then $A' = p^*A - 2E$ and $B' = p^*B - E$, where A, B are curves such that $D = A + B$ and $A \cdot B = 2$ (cf. [B], pg. 183). Now again applying (2.4) we get (a).

The proof of (b) goes exactly the same way using the fact that $h^0(D'', \mathcal{O}_{D''}) = 3$ in this case. ♢

(3.4) **Theorem.** *Suppose that D is 2-connected and x is a point of D such that $mult_D(x) \geq 3$. Let r be the rank at x of the differential of φ_{K_S+D}.*

If $r = 1$, then D has a decomposition $D = A + B$ where A, B are curves such that $A \cdot B = 2$, $A \cap B = \{x\}$ or $A \subset B$ and x is either a non-singular point of A or a double point of A.

If $r = 0$ then D has a decomposition $D = A_1 + A_2 + A_3$ where A_1, A_2, A_3 are curves such that:

(i) $A_1 \cdot A_2 = A_1 \cdot A_3 = A_2 \cdot A_3 = 1$
(ii) $\mathcal{O}_{A_1}(2x) \simeq \mathcal{O}_{A_1}(A_2 + A_3)$ and $\mathcal{O}_{A_2}(x) \simeq \mathcal{O}_{A_2}(A_3)$
(iii) x is a non-singular point of A_1, A_2
(iv) $A_1 \cap (A_2 + A_3) = \{x\}$ or $A_1 \subset (A_2 + A_3)$ and $A_2 \cap A_3 = \{x\}$ or $A_2 \subset A_3$.

Proof. Again as in the other theorems in this section it suffices to apply both (1.3) and (2.4). ♢

4. Behaviour of ω_D on smooth points of D.

In this section we study the behaviour of ω_D. Let us point out that these results also follow from general results still unpublished obtained by F. Catanese for Gorenstein curves.

(4.1) **Theorem.** *Let D be a 1-connected curve on a surface S and let x be a smooth point of D. Then x is a base point of $|\omega_D|$ if and only if either $D \simeq \mathbf{P}^1$ or D is reducible, the unique component Γ of D to which x belongs is a non-singular rational curve and D decomposes as a sum $D = \Gamma + F_1 + ... + F_n$ satisfying*

(i) $F_1, ..., F_n$ are curves such that $\Gamma \cdot F_i = 1$, for every $i \in \{1, ..., n\}$.

(ii) $F_i \cdot F_j = 0$ for $i \neq j$,

(iii) $\mathcal{O}_{F_i}(F_k) \simeq \mathcal{O}_{F_i}$ for all k<i.

Furthermore if x is a base point of $|\omega_D|$, then Γ is a fixed component for $|\omega_D|$.

Proof. Suppose x is a base point of $|\omega_D|$. Then $H^0(D, \omega_D) \simeq H^0(D, \omega_D \otimes \mathcal{I}_{x,D})$. By Serre duality and the Riemman-Roch theorem we have that $h^0(D, \mathcal{O}_D(x)) = 2$. If D is irreducible we have the first part of the assertion. Otherwise by (2.5) we obtain a decomposition of D as a sum of two curves A and F_1 such that $x \in A$, $A \cdot F_1 = 1$ and $\mathcal{O}_A(x) \simeq \mathcal{O}_A(F_1)$.

Now, using (2.2) and 1-connectedness of D it is easy to check that the restriction map $H^0(D, \mathcal{O}_D(x)) \to H^0(A, \mathcal{O}_A(x))$ is injective and therefore $h^0(A, \mathcal{O}_A(x)) \geq 2$. If A is irreducible, A must be a non-singular rational curve and we get a decomposition as in the statement of the theorem. If A is not irreducible, then, by (2.1), A is still 1-connected. Since the number of components of D is finite, by an inductive reasoning we can obtain a decomposition of D as in the statement.

Conversely suppose that D has a decomposition as above . Since Γ is a non-singular rational curve and $\Gamma \cdot (D - \Gamma) = n$, we have $h^0(\Gamma, \omega_D) = n - 1$ and $h^1(\Gamma, \omega_D) = 0$. By (2.1) and the hypothesis (i) and (ii), every F_i is 1-connected. Now it is easy to verify that then (iii) implies that $h^0(F_1 + ... + F_n, \mathcal{O}_{F_1+...+F_n}) = n$.

Consider the long exact sequence obtained from the exact sequence

$$0 \to \omega_{F_1+...+F_n} \to \omega_D \to \mathcal{O}_\Gamma(\omega_D) \to 0$$

Notice that, by Serre duality and 1-connectedness of D, one has $h^1(D, \omega_D) = h^0(D, \mathcal{O}_D) = 1$. Now by a dimension count one sees that the restriction map $H^0(D, \omega_D) \to H^0(\Gamma, \omega_D)$ is the zero map and thus Γ is a fixed component of $|\omega_D|$.

The rest of the assertion is now clear. ♢

(4.2) **Theorem.** *Let D be a 2-connected curve with $p_a(D) \geq 2$ on a surface S and let x, y be two smooth points of D (possibly $x = y$). Then, if x, y are not separated by $|\omega_D|$ one of the following occurs:*

(a) D is an irreducible hyperelliptic curve and $|\mathcal{O}_D(x + y)|$ is the unique g^1_2 on D

or

(b) D is reducible, x,y belong to the same component Γ of D, which is an hyperelliptic curve (in the generalized sense), and D decomposes as a sum $\Gamma + F_1 + ... + F_n$ satisfying:

(i) $F_1, ..., F_n$ are curves such that $\Gamma \cdot F_i = 2$, for every $i \in \{1, ..., n\}$.

(ii) $\mathcal{O}_\Gamma(F_i) \simeq \mathcal{O}_\Gamma(x + y)$, for every $i \in \{1, ..., n\}$.

(iii) $|\mathcal{O}_\Gamma(x + y)|$ is a g^1_2 on Γ.

(iv) $F_i \cdot F_j = 0$, for $i \neq j$.

(v) $\mathcal{O}_{F_i}(F_k) \simeq \mathcal{O}_{F_i}$ for all k<i.

or

(c) $x \neq y$, and $x \in \Gamma$, $y \in \Delta$ where $\Gamma, \Delta \leq D$ are non-singular rational irreducible curves. Furthermore either $D = \Gamma + \Delta$, or $D = \Gamma + \Delta + F_1 + ... + F_n$ where $F_1, ..., F_n$ are curves satisfying (iv) and (v) of (b) and such that $\Gamma \cdot F_i = \Delta \cdot F_i = 1$, for every $i \in \{1, ..., n\}$.

Proof. Remark first that x, y are separated by ω_D if and only if $h^0(D, \omega_D \otimes \mathcal{I}_{x,D}) \neq h^0(D, \omega_D \otimes \mathcal{I}_{x,D} \otimes \mathcal{I}_{y,D})$. Since D is 2-connected and thus, by (4.1) neither x or y are base points of $|\omega_D|$, we obtain then, by the Riemann-Roch theorem and Serre duality, that $h^0(D, \mathcal{O}_D(x+y)) = 2$.

If D is irreducible we have assertion (a).

Otherwise suppose first that x, y belong to a unique component Γ of D. As D is reducible, we can apply (2.5) and thus we get a decomposition of D as a sum of two curves A and F_1 such that $x, y \in A$, $A \cdot F_1 = 2$ and $\mathcal{O}_A(x+y) \simeq \mathcal{O}_A(F_1)$.

Now, using (1.1) and 2-connectedness of D it is easy to check that the restriction map $H^0(D, \mathcal{O}_D(x+y)) \to H^0(A, \mathcal{O}_A(x+y))$ is injective and therefore $h^0(A, \mathcal{O}_A(x+y)) \geq 2$. If A is irreducible, $A = \Gamma$ must be an hyperelliptic curve (in the generalized sense) and we get a decomposition as in the statement of the theorem. Suppose that A is not irreducible. Suppose that A is still 2-connected. Then, as in the proof of (4.1), we can apply an inductive reasoning and so we get statement (b). Now we show that A is 2-connected. By (1.4) A is 1-connected. Suppose that $A = B + C$, with B, C curves such that $B \cdot C = 1$. Then, by 2-connectedness of D, $B \cdot F_1 \geq 1$ and $C \cdot F_1 \geq 1$. But this contradicts the fact that, for every component $\Gamma' \neq \Gamma$ of A, $\Gamma' \cdot F_1 = 0$.

Suppose now that $x \in \Gamma$, $y \in \Delta$, with $\Gamma \neq \Delta$. Then, applying (2.2) it is easy to see that the restriction maps $H^0(D, \mathcal{O}_D(x+y)) \to H^0(\Gamma, \mathcal{O}_\Gamma(x))$ and $H^0(D, \mathcal{O}_D(x+y)) \to H^0(\Delta, \mathcal{O}_\Delta(y))$ are injective, giving $h^0(\Gamma, \mathcal{O}_\Gamma(x)) \geq 2$ and $h^0(\Delta, \mathcal{O}_\Delta(y)) \geq 2$. This implies that both Γ and Δ are non-singular rational curves. Furthermore if $D \neq \Gamma + \Delta$, we can apply (2.2) and reason as above to obtain the result. ♢

(4.3) **Proposition.** *Suppose D is a 2-connected reducible curve. Then an irreducible component Γ of D is contracted by ϕ_{ω_D} if and only if Γ is a non-singular rational curve such that $\Gamma \cdot (D - \Gamma) = 2$.*

Proof. Let Γ be an irreducible component of D. We have $p_a(\Gamma) \geq 0$ and $\deg_\Gamma \omega_D = 2p_a(\Gamma) - 2 + \Gamma \cdot (D - \Gamma)$. So, if Γ is as above, Γ is contracted by ϕ_{ω_D}. Conversely, suppose that Γ is contracted by ϕ_{ω_D}. As D is 2-connected and therefore $|\omega_D|$ is base point free (by (2.1) and (3.1)) necessarily $\deg_\Gamma \omega_D = 0$. This gives the assertion, since by 2-connectedness of D, the only possibility is $p_a(\Gamma) = 0$ and $\Gamma \cdot (D - \Gamma) = 2$. ♢

References.

[B] E. Bombieri, *Canonical models of surfaces of general type*, Publ. Math. IHES, 42 (1973), 171-219

[B-C] E. Bombieri, F.Catanese, *The tricanonical map of a surface with $K^2 = 2$, $p_g = 0$, in* C.P.Ramanujam - a tribute, Studies in Mathematics, n.8, Tata Institute of Fundamental Research, Bombay, 1978

[C-F-M] C. Ciliberto, P. Francia and M. Mendes Lopes *Remarks on the bicanonical map for* *surfaces of general type*, Math. Z. 224 (1997), no. 1, 137–166.

[F] P. Francia, *On the base points of the bicanonical system*, in Problems in the theory of surfaces and their classification, Symposia Math., Academic Press, 1991, 141-150.

[M] Y.Miyaoka, *Tricanonical maps of numerical Godeaux surfaces*, Invent.Math., 34, (1976), 99-111.

[R] I.Reider, *Vector bundles of rank 2 and linear systems on algebraic surfaces*, Ann.Math., 127, (1988), 309-316

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